Probability in the Engineering and Informational Sciences, **33**, 2019, 146–159.
doi:10.1017/S0269964818000062

FURTHER RESULTS ON QUANTILE ENTROPY IN THE PAST LIFETIME

Guoxin Qiu
Department of Business Administration,
School of Business, Xinhua University of Anhui,
Hefei, 230088, China
and
Department of Statistics and Finance,
School of Management, University of Science and Technology of China,
Hefei, 230026, China
E-mail: qiugx02@ustc.edu.cn

Bounds of the quantile entropy in the past lifetime of some ageing classes are explored firstly. The quantile entropy in the past lifetime of a random variable is shown to be increasing if its expected inactivity time is increasing. Some closure properties of the less quantile entropy in the past lifetime order are obtained under the model of generalized order statistics. Moreover, sufficient conditions are given for a function of a random variable and for a weighted random variable to have more quantile entropy in the past lifetime than original random variable.

Keywords: generalized order statistics, LPQE order, quantile entropy in the past lifetime, reversed hazard rate, weighted distribution

Mathematics Subject Classification (2001): Primary 60E15, Secondary 60N05

1. INTRODUCTION

Let X be a non-negative random variable with probability density function (pdf) f, cumulative distribution function (cdf) F and survival function $\bar{F} = 1 - F$, respectively. Ruiz and Navarro [33] defined the inactivity time of X as the random variable $X_{(t)} = [t - X | X \le t]$. The expected inactivity time of X is given by

$$m(t) = \mathrm{E}X_{(t)} = \frac{1}{F(t)} \int_0^t F(x)\,\mathrm{d}x, \quad t > 0.$$

Denoting the reversed hazard rate of X by $\widetilde{r} = f/F$, we have $m(t)\widetilde{r}(t) = 1 - m'(t)$, where $m'(t) = \frac{\mathrm{d}}{\mathrm{d}t} m(t)$. For different properties of the expected inactivity time, one may refer to Kayid and Ahmad [22], Ahmad and Kayid [1], Ahmad, Kayid, and Pellerey [2], Li and Xu [25], and Badia and Berrade [5]. To measure the uncertainty contained in inactivity time or past lifetime of X, Di Crescenzo and Longobardi [12] introduced the past entropy as

$$\overline{H}_t(X) \triangleq H(X_{(t)}) = -\int_0^t \frac{f(x)}{F(t)} \log \frac{f(x)}{F(t)}\,\mathrm{d}x, \quad t > 0. \qquad \mathbf{(1.1)}$$

 0269-9648/18 $25.00

It should be noted that as $t \to \infty$, $\overline{H}_t(X)$ become the well-known Shannon entropy (Shannon [35]) given by $H(X) = -\int_0^\infty f(x)\log f(x)\,\mathrm{d}x$. Readers can refer to Di Crescenzo and Longobardi [12,13], Nanda and Paul [29,30], Kundu, Nanda, and Hu [23] and Kundu, Nanda, and Maiti [24] for more works on the past entropy. By means of $\widetilde{r}$, we can rewrite (1.1) as

$$\overline{H}_t(X) = 1 - \frac{1}{F(t)}\int_0^t f(x)\log\widetilde{r}(x)\,\mathrm{d}x, \quad t > 0. \tag{1.2}$$

Quantile function $F^{-1}(u) = \inf\{t : F(t) \geq u\}$, $0 < u < 1$, is equivalent to the cdf in modeling statistical data (Gilchrist [16], Nair and Sankaran [27]). Putting $t = F^{-1}(u)$ in (1.1), Sunoj, Sankaran, and Nanda [36] introduced the following quantile entropy in the past lifetime for X,

$$\overline{H}_{F^{-1}(u)}(X) = \log u - \frac{1}{u}\int_0^u \log f(F^{-1}(x))\,\mathrm{d}x, \quad u \in (0,1).$$

Basically, $\overline{H}_{F^{-1}(u)}(X)$ gives the expected uncertainty contained in the conditional density about the predictability of an outcome of X until $100u\%$ point of distribution (Sunoj, Sankaran, and Nanda [36]). Using $\overline{H}_{F^{-1}(u)}(X)$, Sunoj, Sankaran, and Nanda [36] not only introduced the definitions of decreasing (increasing) quantile entropy in the past lifetime [DPQE (IPQE)] class of life distributions and less quantile entropy in the past lifetime (LPQE) order, but also explored some properties of them. Recently, after giving an equivalent definition of the LPQE order, Kang [21] surveyed some closure and reversed closure properties of the LPQE order under several stochastic models.

The objective of this paper is to explore further properties of DPQE (IPQE) classes of life distributions and LPQE order. Section 2 gives the bounds of the quantile entropy in the past lifetime for some ageing classes. Section 3 gives relationships between DPQE and IPQE classes of life distributions and its expected inactivity time. Closure properties of LPQE order under the model of generalized order statistics are discussed in Section 4. Section 4 also gives sufficient conditions for a weighted random variable to have more quantile entropy in the past lifetime than original random variable.

Throughout this paper, all random variables are implicitly assumed to be non-negative absolutely continuous. The term "increasing" ("decreasing") means non-decreasing (non-increasing).

2. PRELIMINARIES

2.1. Definitions and lemmas

In this paper, we need some concepts of non-parametric ageing classes, including increasing (decreasing) failure rate [IFR (DFR)], increasing (decreasing) failure rate in average [IFRA (DFRA)], new better (worse) than used [NBU (NWU)] and increasing (decreasing) expected inactivity time [IEIT (DEIT)]. One may refer to Barlow and Proschan [7] and Marshall and Olkin [26] for their definitions, properties, and applications. The following relationships hold among the ageing classes mentioned above:

$$\text{IFR} \subseteq \text{IFRA} \subseteq \text{NBU}, \quad \text{DFR} \subseteq \text{DFRA} \subseteq \text{NWU}.$$

Let Y be another random variable with pdf g. X is said to be smaller than Y in the likelihood ratio order, denoted by $X \leq_{\mathrm{lr}} Y$, if $g(x)/f(x)$ is increasing in $x \geq 0$. For the definitions of other stochastic orders, such as hazard rate order ($\leq_{\mathrm{hr}}$), usual stochastic order ($\leq_{\rm st}$), convex transform order ($\leq_{\rm c}$), star order ($\leq_{*}$), super-additive order ($\leq_{\rm su}$) and dispersive order ($\leq_{\rm disp}$), one may refer to Shaked and Shanthikumar [34]. It is well-known that

$$X \leq_{\rm lr} Y \Longrightarrow X \leq_{\rm hr} Y \Longrightarrow X \leq_{\rm st} Y; \quad X \leq_{\rm c} Y \Longrightarrow X \leq_{*} Y \Longrightarrow X \leq_{\rm su} Y.$$

$X \leq_{\rm st} Y$ if and only if $\mathrm{E}\phi(X) \leq \mathrm{E}\phi(Y)$ for all increasing functions ϕ. Theorem 4.B.2 of Shaked and Shanthikumar [34] states that if $X \leq_{\rm st} Y$, then $X \leq_{\rm su} Y$ (and therefore $X \leq_{*} Y$ or $X \leq_{\rm c} Y$) implies $X \leq_{\rm disp} Y$. Lemma 2.1 below generalizes this theorem.

LEMMA 2.1 (Ahmed et al. [3]): *Let X and Y be two random variables with pdfs f and g, respectively, such that $f(0) \geq g(0) > 0$. If $X \leq_{\rm su} Y$ ($X \leq_{*} Y$ or $X \leq_{\rm c} Y$), then $X \leq_{\rm disp} Y$.*

DEFINITION 2.2 (Sunoj, Sankaran, and Nanda [36]): *X is said to have*

(1) *decreasing (increasing) quantile entropy in the past lifetime* [DPQE (IPQE)] *if $\overline{H}_{F^{-1}(u)}(X)$ is decreasing (increasing) in $u \in (0, 1)$;*
(2) *less quantile entropy in the past lifetime than Y, denoted by $X \leq_{\rm LPQE} Y$, if $\overline{H}_{F^{-1}(u)}(X) \leq \overline{H}_{G^{-1}(u)}(Y)$ for all $u \in (0, 1)$.*

2.2. Bounds of quantile entropy in the past lifetime for some ageing classes

It was shown in Theorem 2.2 of Kang [21] that $X \leq_{\rm LPQE} Y$ if $X \leq_{\rm disp} Y$. Combining this fact and Lemma 2.1, we have the following Lemma 2.3. This lemma can also be viewed as the generalization of Theorem 2.3 of Kang [21].

LEMMA 2.3: *If $X \leq_{\rm su} (\leq_{*}, \leq_{\rm c}) Y$ and $f(0) \geq g(0) > 0$, then $X \leq_{\rm LPQE} Y$.*

Theorem 4.8.11 of Shaked and Shanthikumar [34] declares that $X \in$ IFR (IFRA, NBU) if and only if $X \leq_{\rm c} (\leq_{*}, \leq_{\rm su}) Z$, where Z has the exponential distribution with failure rate $\lambda\,(\lambda > 0)$. By Lemma 2.3, we have $X \leq_{\rm LPQE} Z$ if the $f(0) \geq \lambda > 0$. Noting that the quantile entropy in the past lifetime of Z is given by

$$\overline{H}_{F_Z^{-1}(u)}(Z) = 1 + \log\frac{u}{\lambda} + \frac{1-u}{u}\log(1-u), \quad u \in (0,1),$$

we have the following proposition by Definition 2.2 (2).

PROPOSITION 2.4: *If $X \in$ IFR (IFRA, NBU) and $f(0) \geq \lambda > 0$, then*

$$\overline{H}_{F^{-1}(u)}(X) \leq 1 + \log\frac{u}{\lambda} + \frac{1-u}{u}\log(1-u), \quad u \in (0,1).$$

Example 4 of Shaked and Shanthikumar [34] states that X has a decreasing pdf f if and only if $U \leq_{\rm c} X$, where $U \sim \mathcal{U}(0, 1)$. Since the quantile entropy in the past lifetime of U is given by $\log u$, we have the following proposition by analogy with Proposition 2.4.

PROPOSITION 2.5: *If X has a decreasing pdf f such that $f(0) \leq 1$, then*

$$\overline{H}_{F^{-1}(u)}(X) \geq \log u, \quad u \in (0,1).$$

In view of $\overline{H}_{F^{-1}(u)}(X) \to H(X)$ as $u \to 1$, we obtain the following corollaries 2.6 and 2.7 by letting $u \to 1$ in Propositions 2.4 and 2.5, respectively.

COROLLARY 2.6: *If* $X \in$ IFR (IFRA, NBU) *and* $f(0) \geq \lambda > 0$, *then* $H(X) \leq 1 - \log \lambda$.

COROLLARY 2.7: *If* X *has a decreasing pdf* f *such that* $f(0) \leq 1$, *then* $H(X) \geq 0$.

Another interested application of the fact that $X \leq_{\text{disp}} Y$ implies $X \leq_{\text{LPQE}} Y$ is illustrated in the following Proposition 2.8.

PROPOSITION 2.8: *Let* $W_k = X_{k:n} - X_{k-1:n}$ *be the* k*-th sample spacing between order statistics* $X_{1:n} \leq X_{2:n} \leq \cdots \leq X_{n:n}$ *of a random sample of* n *observations on* X *with failure rate* r. *Denote the cdf of* W_k *by* F_{W_k}. *If* $r(x) \leq \lambda$ *for all* $x \geq 0$, *then*

$$\overline{H}_{F_{W_k}^{-1}(u)}(W_k) \geq 1 + \log \frac{u}{(n-k+1)\lambda} + \frac{1-u}{u} \log(1-u), \quad u \in (0,1).$$

PROOF: Let Y have an exponential distribution with failure rate λ. Denote the order statistics and the k-th sample spacing of Y by $Y_{1:n} \leq Y_{2:n} \leq \cdots \leq Y_{n:n}$ and $V_k = Y_{k:n} - Y_{k-1:n}$, respectively. Recall that $X \geq_{\text{disp}} Y$ if $X \geq_{\text{hr}} Y$ and X or Y is DFR. Thus, if the failure rate of Y is denoted by r_Y, then $r(x) \leq \lambda = r_Y(x)$ for all $x \geq 0$. This implies $X \geq_{\text{hr}} Y$ and hence, $X \geq_{\text{disp}} Y$. Since the distribution of V_k is exponential with failure rate $(n-k+1)\lambda$, it follows from Theorem 4.2 (ii) of Rojo and He [32] that $W_k \geq_{\text{disp}} V_k$. Hence,

$$\begin{aligned}\overline{H}_{F_{W_k}^{-1}(u)}(W_k) &\geq \overline{H}_{F_{V_k}^{-1}(u)}(V_k) \\ &= 1 + \log \frac{u}{(n-k+1)\lambda} + \frac{1-u}{u} \log(1-u), \quad u \in (0,1),\end{aligned}$$

where F_{V_k} is the cdf of V_k. ■

3. DPQE AND IPQE CLASSES

A random variable X is said to have increasing (decreasing) uncertainty of life [IUL (DUL)] if $\overline{H}_t(X)$ is increasing (decreasing) in $t \geq 0$ (Nanda and Paul [29]). Since

$$\overline{H}_{F^{-1}(u)}(X) = \overline{H}_t(X)|_{t=F^{-1}(u)},$$

and $F^{-1}(u)$ is an increasing function, we see that $\overline{H}_{F^{-1}(u)}(X)$ and $\overline{H}_t(X)$ have the same monotone properties. That is, the IUL (DUL) class of life distributions is equivalent to the IPQE (DPQE) class of life distributions. For this reason, we only consider the properties of IUL (DUL) class of life distributions in this section. The following Theorem 3.1 was borrowed from Di Crescenzo and Longobardi [12] to give the relationship between IUL class and decreasing reversed hazard rate (DRHR) class of life distributions.

THEOREM 3.1: *If* X *is* DRHR, *then* X *is* IUL.

Result 2.6 of Chandra and Roy [9] states that if X is DRHR, then X is IEIT. The following Theorem 3.2 shows that the condition of Theorem 3.1 can be weakened.

THEOREM 3.2:

(1) *If* X *is* IEIT, *then it is* IUL.
(2) *If* X *is* DUL, *then it is* DEIT.

https://doi.org/10.1017/S0269964818000062 Published online by Cambridge University Press

Proof: For Part (1), it follows from (1.2) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overline{H}_t(X) &= -\frac{1}{F^2(t)}[f(t)F(t)\log\widetilde{r}(t) - f(t)\int_0^t f(x)\log\widetilde{r}(x)\,\mathrm{d}x] \\ &= \frac{\widetilde{r}(t)}{F(t)}\int_0^t f(x)\log\widetilde{r}(x)\,\mathrm{d}x - \widetilde{r}(t)\log\widetilde{r}(t) \\ &= \widetilde{r}(t)[1-\overline{H}_t(X)-\log\widetilde{r}(t)]. \end{aligned} \tag{3.1}$$

Recall that the exponential distribution achieves maximal entropy among all the continuous distributions with given mean. That is, if $m(t) < \infty$, then for all $t \geq 0$,

$$\overline{H}_t(X) \leq 1 + \log m(t).$$

Therefore, we have from (3.1) that

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\overline{H}_t(X) &\geq \widetilde{r}(t)[-\log m(t) - \log\widetilde{r}(t)] \\ &= -\widetilde{r}(t)\log[m(t)\widetilde{r}(t)] \\ &= -\widetilde{r}(t)\log[1-m'(t)] \\ &\geq 0, \end{aligned} \tag{3.2}$$

where the last inequality follows from the assumption that $m(t)$ is increasing. This completes the proof of Part (1). For Part (2), since X is DUL, we have $\frac{\mathrm{d}}{\mathrm{d}u}\overline{H}_t(X) \leq 0$. It follows from (3.2) that $m'(t) \leq 0$ for all $t \geq 0$. Thus, X is DEIT.

Next, we give an example to show the usefulness of Theorem 3.2. ■

Example 3.3: Let X be a random variable with cdf F given by

$$F(x) = \begin{cases} 0, & x \leq 0, \\ \dfrac{x}{2}, & 0 \leq x < 1, \\ \dfrac{x^2+2}{6}, & 1 \leq x < 2, \\ 1, & x \geq 2. \end{cases}$$

X is not DRHR as shown in Example 3.2 of Nanda and Paul [29]. Hence, we cannot say that X is IUL by Theorem 3.1. However, the expected inactivity time of X is given by

$$m(t) = \begin{cases} \dfrac{t}{3}, & 0 \leq x < 1, \\ \dfrac{t^3+6t-4}{3(t^2+2)}, & 1 \leq x < 2, \\ t - \dfrac{17}{18}, & x \geq 2. \end{cases}$$

$m(t)$ is obviously increasing. We can say that X is IUL by Theorem 3.2 (1). In fact, as shown in Example 3.2 of Nanda and Paul [29], we can also prove that X is IUL by using (1.1).

A random variable X is said to have increasing reversed variance residual life (IRVR) if the variance of $X_{(t)}$ is increasing in $t > 0$. Nanda et al. [31] proved that if X is IEIT, then

X is also IRVR. That is, the IEIT property is stronger than the IRVR property. One may wonder whether the condition in Theorem 3.2 (1) can be relaxed to be IRVR. The next example gives a negative answer.

Example 3.4: Let X be a random variable with cdf

$$F(x) = \begin{cases} 0, & x \le 0, \\ \dfrac{x^2}{16}, & 0 \le x < 1, \\ \dfrac{x^4 - 2x + 2}{16}, & 1 \le x < a, \\ 1, & x \ge a, \end{cases}$$

where $a \approx 2.06338$ is the unique positive root of the equation $x^4 - 2x - 14 = 0$. As shown in Example 2.1 of Nanda et al. [31], X is IRVR. However,

$$\overline{H}_{1.269}(X) = -0.09989 > -0.1006 = \overline{H}_{1.32}(X).$$

X is not IUL.

Order statistics can be used in many fields, including statistical inference, goodness-of-fit tests, reliability, and quality control. For example, in reliability theory, order statistics are used for statistical modeling. Let $X_1, X_2, \cdots, X_n$ be a random sample of size n from a population X, and $X_{1:n} \le X_{2:n} \le \cdots \le X_{n:n}$ be the corresponding order statistics. Then $X_{k:n}$ represents the lifetime of an $(n-k+1)$-out-of-n system. It is to be noted that $X_{1:n}$ represents the lifetime of a series system, whereas $X_{n:n}$ represents that of a parallel system. Kundu, Nanda, and Hu [23] showed that IUL class life distributions are preserved under the formation of series systems. However, as shown in the following example, DUL class life distribution is not preserved under the formation of series systems.

Example 3.5: Let X have an inverted exponential distribution with cdf $F(x) = \mathrm{e}^{-\lambda/x}$, $x > 0$. It is easy to verify that X is DUL. But, if we let $\lambda = 1.2$, then

$$\overline{H}_{0.2}(X_{1:2}) = 0.00822 < 0.20570 = \overline{H}_{1.2}(X_{1:2}) > \overline{H}_{2.2}(X_{1:2}) = -0.66476.$$

$X_{1:2}$ neither is DUL nor is IUL.

If the reversed hazard rate of a random variable is not decreasing but is increasing, we say this random variable has increasing reversed hazard rate (IRHR). The following theorem shows that $\overline{H}_t(X_{n:n})$ is decreasing in n if X is IRHR.

Theorem 3.6: *If X is* IRHR, *then $\overline{H}_t(X_{n:n})$ is decreasing in $n \ge 1$.*

Proof: It follows from (1.2) that

$$\begin{aligned}\overline{H}_t(X_{n:n}) &= 1 - \frac{1}{F^n(t)} \int_0^t nF^{n-1}(x) f(x) \log[n\widetilde{r}(x)]\,\mathrm{d}x \\ &= 1 - \log n - \frac{1}{F^n(t)} \int_0^t nF^{n-1}(x) f(x) \log \widetilde{r}(x)\,\mathrm{d}x\end{aligned}$$

https://doi.org/10.1017/S0269964818000062 Published online by Cambridge University Press

$$= 1 - \log n - \int_0^t g_{n:n}^t(x) \log \widetilde{r}(x)\,\mathrm{d}x$$
$$= 1 - \log n - \mathrm{E}[\log \widetilde{r}(X_{n:n}) | X_{n:n} \le t], \tag{3.3}$$

where

$$g_{n:n}^t(x) = \frac{nF^{n-1}(x)f(x)}{F^n(t)}, \quad x \le t,$$

is the pdf of $[X_{n:n}|X_{n:n} \le t]$. Thus,

$$\frac{g_{n:n}^t(x)}{g_{(n-1):(n-1)}^t(x)} = \frac{n}{n-1}\frac{F(x)}{F(t)}$$

is increasing in $x \in [0, t]$. This implies $[X_{(n-1):(n-1)}|X_{(n-1):(n-1)} \le t] \le_{\mathrm{lr}} [X_{n:n}|X_{n:n} \le t]$, and hence, $[X_{(n-1):(n-1)}|X_{(n-1):(n-1)} \le t] \le_{\mathrm{st}} [X_{n:n}|X_{n:n} \le t]$. If $\widetilde{r}$ is increasing, then,

$$\mathrm{E}[\log \widetilde{r}(X_{(n-1):(n-1)})|X_{(n-1):(n-1)} \le t] \le \mathrm{E}[\log \widetilde{r}(X_{n:n})|X_{n:n} \le t].$$

That is, $\mathrm{E}[\log \widetilde{r}(X_{n:n})|X_{n:n} \le t]$ is increasing in n. Hence, $\overline{H}_t(X_{n:n})$ in (3.3) is decreasing in n. This completes the proof. ■

4. LPQE ORDER

Sunoj, Sankaran, and Nanda [36] proved that the LPQE order is closed under the linear transformations. Kang [21] proved that the LPQE order is closed under non-linear transformations and other stochastic models. In this section, we first give sufficient conditions for a function of a random variable to have more (less) quantile entropy in the past lifetime than itself. As a consequence, result related on the accelerated life model is obtained. In Section 4.2, we display the closure and the reversed closure properties of the LPQE order under the generalized order statistics model. In Section 4.3, the weighted distributions are considered. We give general conditions under which the weighted distribution has more quantile entropy in the past lifetime than the original distribution.

The following two lemmas will play an important role in the proofs of the main results in this section.

Lemma 4.1 (Kang [21]): *$X \le_{\mathrm{LPQE}} Y$ if and only if, for all $t \ge 0$,*

$$\int_0^t f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))}\,\mathrm{d}x \ge 0.$$

Lemma 4.2 (Barlow and Proschan [7]): *Let $\mu(x)$ be a measure on the interval (a, b), not necessary non-negative, where $-\infty \le a < b \le +\infty$, and $h(x) \ge 0$ be a decreasing function defined on (a, b). If $\int_a^t \mathrm{d}\mu(x) \ge 0$ for all $t \in (a, b)$, then $\int_a^b h(x)\,\mathrm{d}\mu(x) \ge 0$.*

4.1. Monotone transformations

Theorem 4.3:

(1) *If ϕ is a non-negative increasing function with $\phi'(x) \ge 1$ for all $x \ge 0$, then $X \le_{\mathrm{LPQE}} \phi(X)$;*

https://doi.org/10.1017/S0269964818000062 Published online by Cambridge University Press

(2) *If* ϕ *is an increasing function with* $\phi(0) = 0$ *and* $\phi'(x) \le 1$ *for all* $x \ge 0$*, then* $\phi(X) \le_{\mathrm{LPQE}} X$.

PROOF: Denote the cdf and pdf of $\phi(X)$ by $F_{\phi(X)}$ and $f_{\phi(X)}$, respectively. Then,

$$\int_0^t f(u) \log \frac{f(u)}{f_{\phi(X)}(F^{-1}_{\phi(X)} \circ F(u))}\,\mathrm{d}u = \int_0^t f(u) \log \frac{f(u)}{f_{\phi(X)}(\phi(u))}\,\mathrm{d}u$$

$$= \int_0^t f(u) \log \phi'(u)\,\mathrm{d}u$$

$$\ge 0,$$

where the inequality comes from the assumption that $\phi'(x) \ge 1$ for all $x \ge 0$. This completes the proof of Part (1) by Lemma 4.1. The proof of Part (2) is similar.

The usefulness of Theorem 4.3 is illustrated by the following example. ■

Example 4.4: Let X have the exponential cdf $F(x) = 1 - \mathrm{e}^{-\lambda x}$, $x \ge 0$, $\lambda > 0$, and let

$$\phi(x) = \frac{1}{\gamma} \log\left(1 + \frac{\lambda\gamma}{\beta} x\right), \quad x \ge 0, \gamma > 0, \lambda \le \beta.$$

Then, $\phi(X)$ has the Gompertz–Makeham cdf $F_{\phi(X)}(x) = 1 - \exp[-\beta(\mathrm{e}^{\gamma x} - 1)/\gamma]$, $x \ge 0$. It follows from Theorem 4.3 (2) that $\phi(X) \le_{\mathrm{LPQE}} X$, which implies

$$\overline{H}_{F^{-1}_{\phi(X)}(u)}(\phi(X)) \le \overline{H}_{F^{-1}(u)}(X) = 1 + \log \frac{u}{\lambda} + \frac{1-u}{u} \log(1-u), \quad u \in (0,1).$$

Making use of a time-dependent scale transformation function $W(x)$ as bridge, Cox and Oakes [10] put forward the following accelerated life model (ALM) to study the relationship between F and G,

$$F(x) = G(W(x)),$$

where $W(x)$ is strictly increasing with $W(0) = 0$, and $W(x) \to \infty$ as $x \to \infty$. In general, assume that $W(x)$ is continuous and differentiable in $[0, \infty)$,

$$W(x) = \int_0^x w(t)\,\mathrm{d}t, \quad w(t) \ge 0, x \in [0, \infty).$$

Therefore, if G is viewed as the cdf of a life Y for an item functioning in a baseline (reference) environment, then F defines a cdf for another random variable, say, X, who can be regarded as the lifetime for the item functioning in a severe environment.

For ALM, it is interesting to investigate the role of $W(x)$ or $w(x)$. For example, through investigating the role of $W(x)$ or $w(x)$ in establishing the ageing properties of X via the ageing properties of Y, Finkelstein [14] obtained some non-parametric properties of IFR, IFRA, and NBU classes of life distributions. Next, we give sufficient condition under which X has less quantile entropy in the past lifetime than Y.

THEOREM 4.5: *If* $W(x) - x$ *is increasing in* x*, then* $X \le_{\mathrm{LPQE}} Y$.

PROOF: Obviously, we see from above discussion that $X = \phi(Y)$, where $\phi = W^{-1}$. Thus, Theorem 4.5 can be proved by Theorem 4.3 (2). ■

Remark 4.6: It was shown in Di Crescenzo and Longobardi [12] that if X is IUL, then $\phi(X)$ is also IUL for all non-negative increasing convex function ϕ with $\phi(0) = 0$. For ALM, if X is IUL and $W(x)$ is convex, then Y is also IUL.

https://doi.org/10.1017/S0269964818000062 Published online by Cambridge University Press

4.2. Generalized order statistics (GOSs)

The concept of GOSs was introduced by Kamps [19,20] as a unified approach to a variety of models of ordered random variables. GOSs have been of interest in the past 10 years because they are more flexible in reliability theory, statistical modeling, and inference, see Gajek and Okolewski [15] and Cramer, Kamps, and Rychlik [11].

Formally, uniform GOSs are defined via some joint pdf on a cone of the $\mathbb{R}^n$. GOSs based on an arbitrary cdf F are defined by means of the quantile function of F.

Definition 4.7 (Kamps [19]): *Let $n \in \mathbb{N}$, $k \geq 0$, $m_1, \cdots, m_{n-1} \in \mathbb{R}$, and*

$$M_r = \sum_{j=r}^{n-1} m_j, \quad 1 \leq r \leq n-1, \quad M_n = 0,$$

be parameters such that for all $r \in \{1, \cdots, n\}$,

$$\gamma_{r,n} = k + n - r + M_r \geq 1,$$

and let $\widetilde{m} = (m_1, \cdots, m_{n-1})$ if $n \geq 2$ ($\widetilde{m}$ arbitrary if $n = 1$). If the random variables $U_{(r,n,\widetilde{m},k)}$ possess a joint pdf of the form

$$f_{U_{(1,n,\widetilde{m},k)},\cdots,U_{(n,n,\widetilde{m},k)}}(u_1, u_2, \cdots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_{j,n}\right) \left(\prod_{i=1}^{n-1} (1-u_i)^{m_i}\right) (1-u_n)^{k-1}$$

on the cone $0 \leq u_1 \leq u_2 \leq \cdots \leq u_n < 1$ of $\mathbb{R}^n$, then they are called uniform GOSs. Now, let F be an arbitrary cdf. The random variables $X_{(r,n,\widetilde{m},k)} = F^{-1}(U_{(r,n,\widetilde{m},k)})$, $r = 1, \cdots, n$, are called the GOSs based on F.

Throughout this subsection, we will consider the special case of GOSs, that is $m_1 = m_2 = \cdots = m_{n-1} = m$, and in this time, GOSs will be denoted by $X_{(r,n,m,k)}$. If F is absolutely continuous with pdf f, Lemma 3.3 of Kamps [20] states that for each $r = 1, \cdots, n$, the marginal pdf of the r-th GOS $X_{(r,n,m,k)}$ based on F is given by

$$f_{(r,n,m,k)}(x) = \varphi_{r:n}(F(x)) f(x), \tag{4.1}$$

where

$$\varphi_{r:n}(x) = \frac{c_{r-1,n}}{(r-1)!}(1-x)^{\gamma_{r,n}-1} g_m^{r-1}(x), \quad x \in (0,1),$$

$$g_m(x) = \begin{cases} \frac{1}{m+1}[1-(1-x)^{m+1}], & m \neq -1, \\ -\log(1-x), & m = -1, \end{cases}$$

and $c_{r-1,n} = \Pi_{i=1}^r \gamma_{i,n}$. Note that the corresponding marginal cdf of the r-th GOS based on F is

$$F_{X_{(r,n,m,k)}}(x) = \phi_{r:n}(F(x)), \tag{4.2}$$

where

$$\phi_{r:n}(u) = 1 - c_{r-1,n}(1-u)^{\gamma_{r,n}} \sum_{j=0}^{r-1} \frac{1}{j! c_{r-j-1,n}} g_m^j(u), \quad u \in (0,1).$$

Let $m = 0$ and $k = 1$, (4.1) reduces to the pdf of the r-th ordinary order statistic of a random sample from F. Let $m = -1$ and $k \in \mathbb{N}$, then the model of k-record values is obtained. Choosing the parameters appropriately, several other models of ordered random variables are seen to be particular cases, see Kamps [20], Balakrishnan, Cramer, and Kamps [6] and Belzunce, Mercader, and Ruiz [8]. The following Theorem 4.8 gives the closure properties of the LPQE order under GOSs models.

THEOREM 4.8: *Let* $\{X_{(r,n,m,k)}, r = 1, 2, \cdots, n\}$ *and* $\{Y_{(r,n,m,k)}, r = 1, 2, \cdots, n\}$ *be* GOSs *based on cdfs* F *and* G*, respectively.*

(1) *If* $X_{(n,n,m,k)} \leq_{\text{LPQE}} Y_{(n,n,m,k)}$ *for* $k = 1$, $m \geq -1$*, then* $X \leq_{\text{LPQE}} Y$.

(2) *If* $X \leq_{\text{LPQE}} Y$*, then* $X_{(1,n,m,k)} \leq_{\text{LPQE}} Y_{(1,n,m,k)}$ *for* $k \geq 0$, $m \in \mathbb{R}$.

PROOF: We note from (4.2) that, for all $1 \leq r \leq n$,

$$G^{-1}_{Y_{(r,n,m,k)}} \circ F_{X_{(r,n,m,k)}}(x) = (G^{-1} \circ \phi^{-1}_{r:n}) \circ (\phi_{r:n} \circ F(x)) = G^{-1} \circ F(x). \qquad \textbf{(4.3)}$$

(1) It follows from $X_{(n,n,m,k)} \leq_{\text{LPQE}} Y_{(n,n,m,k)}$ for $k = 1$, $m \geq -1$ and Lemma 4.1 that

$$\begin{aligned}
&\int_0^t f_{(n,n,m,k)}(x) \log \frac{f_{(n,n,m,k)}(x)}{g_{(n,n,m,k)}(G^{-1}_{(n,n,m,k)} \circ F_{(n,n,m,k)}(x))}\,\mathrm{d}x \\
&= \int_0^t \varphi_{n:n}(F(x)) f(x) \log \frac{\varphi_{n:n}(F(x)) f(x)}{g_{(n,n,m,k)}(G^{-1} \circ F(x))}\,\mathrm{d}x \\
&= \int_0^t \varphi_{n:n}(F(x)) f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))}\,\mathrm{d}x \\
&\geq 0.
\end{aligned}$$

Since $k = 1$, $m \geq -1$, we have $\gamma_{n,n} = 1$ and hence, $\varphi^{-1}_{n:n}(x) = \frac{(n-1)!}{c_{n-1,n}} g_m^{1-n}(x)$ is decreasing in x. Thus, we have from Lemma 4.2 that $\int_0^t f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))}\,\mathrm{d}x \geq 0$, $t \geq 0$, which is equivalent to $X \leq_{\text{LPQE}} Y$ by Lemma 4.1.

(2) The assumption $X \leq_{\text{LPQE}} Y$ is equivalent to $\int_0^t f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))}\,\mathrm{d}x \geq 0$, $t \geq 0$. Since $\varphi_{1:n}(F(x)) = \gamma_{1,n}(\bar{F}(x))^{\gamma_{1,n}-1}$ is decreasing in x. It follows from Lemma 4.2 that

$$\begin{aligned}
&\int_0^t f_{(1,n,m,k)}(x) \log \frac{f_{(1,n,m,k)}(x)}{g_{(1,n,m,k)}(G^{-1}_{(1,n,m,k)} \circ F_{(1,n,m,k)}(x))}\,\mathrm{d}x \\
&= \int_0^t \varphi_{1:n}(F(x)) f(x) \log \frac{\varphi_{1:n}(F(x)) f(x)}{g_{(1,n,m,k)}(G^{-1} \circ F(x))}\,\mathrm{d}x \\
&= \int_0^t \varphi_{1:n}(F(x)) f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))}\,\mathrm{d}x \\
&\geq 0, \quad t \geq 0,
\end{aligned}$$

which is equivalent to $X_{(1,n,m,k)} \leq_{\text{LPQE}} Y_{(1,n,m,k)}$. ■

Remark 4.9: Theorem 4.8 (1) reduces to Theorems 3.2 of Kang [21] when $m = 0$. Theorem 4.8 (2) reduces to Theorem 3.1 of Kang [21] when $m = 0$, $k = 1$.

Let $\{X_i, i \geq 1\}$ be a sequence of independent and identically distributed random variables. An observation X_n is called an upper record value if $X_n > X_i$, for every $i < n$. Note

https://doi.org/10.1017/S0269964818000062 Published online by Cambridge University Press

that X_1 is a trivial record. The times at which record values appear are given by the random variables $\{T_n,\, n \geq 1\}$ and is defined recursively by $T_1 = 1$, $T_n = \min\{j : j > T_n - 1,\, X_j > X_{T_n - 1},\, n > 1\}$. The sequence of record values corresponding to $\{X_i,\, i \geq 1\}$ is then defined by $U_n = X_{T_n}$, $n \geq 1$. Here U_n is known as n-th upper record value. An analogous definition deals with lower record values. For a detailed discussion on record values, one may refer to Arnold, Balakrishnan, and Nagaraja [4]. Since the n-th upper record value can be regarded as the special case of GOS when $m = -1$ and $k = 1$, we have the following corollary.

Corollary 4.10: *Let U_n^X and U_n^Y be the n-th upper record value based on random variables X and Y, respectively. Then, $U_n^X \leq_{\mathrm{LPQE}} U_n^Y$ if and only if $X \leq_{\mathrm{LPQE}} Y$.*

It follows from Theorem 4.8 that $X_{(1,n,m,k)} \leq_{\mathrm{LPQE}} Y_{(1,n,m,k)}$ if $X_{(n,n,m,k)} \leq_{\mathrm{LPQE}} Y_{(n,n,m,k)}$ for $k = 1$, $m \geq -1$. One may wonder whether the restriction, $k = 1$, can be relaxed. Actually, the following theorem gives more.

Theorem 4.11: *Let $\{X_{(r,n,m,k)},\, r = 1, 2, \cdots, n\}$ and $\{Y_{(r,n,m,k)},\, r = 1, 2, \cdots, n\}$ be* GOSs *based on cdfs F and G, respectively. For all $k \geq 0$, $m \geq -1$, if $X_{(r_1,n,m,k)} \leq_{\mathrm{LPQE}} Y_{(r_1,n,m,k)}$, $1 < r_1 \leq n$, then $X_{(r_2,n,m,k)} \leq_{\mathrm{LPQE}} Y_{(r_2,n,m,k)}$, $1 \leq r_2 \leq r_1$.*

Proof: By (4.1), we have

$$\begin{aligned}\frac{\varphi_{r_2:n}(F(x))}{\varphi_{r_1:n}(F(x))} &= \frac{c_{r_2-1,n}}{c_{r_1-1,n}} \frac{(r_1-1)!}{(r_2-1)!} [\bar{F}(x)]^{\gamma_{r_2,n}-\gamma_{r_1,n}} [g_m(F(x))]^{r_2-r_1} \\ &= \frac{c_{r_2-1,n}}{c_{r_1-1,n}} \frac{(r_1-1)!}{(r_2-1)!} [\bar{F}(x)]^{(r_1-r_2)(m+1)} [g_m(F(x))]^{r_2-r_1}.\end{aligned}$$

Case 1: $m > -1$. Since $r_1 \geq r_2$, $[\bar{F}(x)]^{(r_1-r_2)(m+1)}$ is decreasing in x. The increasing property of $g_m(x) = \frac{1}{m+1}[1-(1-x)^{m+1}]$, $x \in (0, 1)$ guarantees $[g_m(F(x))]^{r_2-r_1}$ is decreasing. Thus, $\varphi_{r_2:n}(F(x))/\varphi_{r_1:n}(F(x))$ is decreasing in x.

Case 2: $m = -1$. The increasing property of $g_{-1}(x) = -\log(1-x)$, $x \in (0, 1)$, guarantees $[g_{-1}(F(x))]^{r_2-r_1}$ is decreasing. Thus,

$$\frac{\varphi_{r_2:n}(F(x))}{\varphi_{r_1:n}(F(x))} = \frac{c_{r_2-1,n}}{c_{r_1-1,n}} \frac{(r_1-1)!}{(r_2-1)!} [g_{-1}(F(x))]^{r_2-r_1}$$

is decreasing in x as well.

Now, $\varphi_{r_2:n}(F(x))/\varphi_{r_1:n}(F(x))$ is decreasing in x for all $m \geq -1$. The assumption $X_{(r_1,n,m,k)} \leq_{\mathrm{LQE}} Y_{(r_1,n,m,k)}$ is equivalent to that, for all $t \geq 0$,

$$\begin{aligned}&\int_0^t f_{(r_1,n,m,k)}(x) \log \frac{f_{(r_1,n,m,k)}(x)}{g_{(r_1,n,m,k)}(G^{-1}_{(r_1,n,m,k)} \circ F_{(r_1,n,m,k)}(x))}\, \mathrm{d}x \\ &= \int_0^t \varphi_{r_1:n}(F(x)) f(x) \log \frac{\varphi_{r_1:n}(F(x)) f(x)}{g_{(r_1,n,m,k)}(G^{-1} \circ F(x))}\, \mathrm{d}x \\ &= \int_0^t \varphi_{r_1:n}(F(x)) f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))}\, \mathrm{d}x \\ &\geq 0.\end{aligned}$$

https://doi.org/10.1017/S0269964818000062 Published online by Cambridge University Press

It follows from Lemma 4.2 that

$$\begin{aligned}
&\int_0^t f_{(r_2,n,m,k)}(x) \log \frac{f_{(r_2,n,m,k)}(x)}{g_{(r_2,n,m,k)}(G^{-1}_{(r_2,n,m,k)} \circ F_{(r_2,n,m,k)}(x))} \, \mathrm{d}x \\
&= \int_0^t \varphi_{r_2:n}(F(x)) f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))} \, \mathrm{d}x \\
&= \int_0^t \frac{\varphi_{r_2:n}(F(x))}{\varphi_{r_1:n}(F(x))} \varphi_{r_1:n}(F(x)) f(x) \log \frac{f(x)}{g(G^{-1} \circ F(x))} \, \mathrm{d}x \\
&\geq 0.
\end{aligned}$$

This completes the proof by Lemma 4.1. ■

4.3. Weighted distributions

The concept of weighted distributions is widely used for studies in reliability, biometry, survival analysis, forestry, ecology, and several other fields. Weighted distributions arise when the observations generated from a stochastic process are recorded with some weighted function. See, for example, Jain, Singh, and Bagai [18], Gupta and Kirmani [17] or Nanda and Jain [28]. Denote the failure rate of X by r. Then the failure rate for the weighted random variable X_w associated with X and to a positive real function w is defined by

$$r_w(t) = \frac{w(t)}{\mathrm{E}[w(X) | X \geq t]} r(t). \tag{4.4}$$

Next, we study the sufficient conditions to have the order $X \leq_{\mathrm{LPQE}} (\geq_{\mathrm{LPQE}}) X_w$ using the fact that $X \leq_{\mathrm{disp}} Y$ if $X \leq_{\mathrm{hr}} Y$ and X or Y is DFR.

Proposition 4.12: *If $w(t)$ is increasing (decreasing) in t, and if X or X_w is* DFR, *then $X \leq_{\mathrm{LPQE}} (\geq_{\mathrm{LPQE}}) X_w$.*

Proof: If $w(t)$ is increasing (decreasing) in t, then

$$\mathrm{E}[w(X) | X \geq t] = \frac{1}{\bar{F}(t)} \int_t^\infty w(x) f(x) \, \mathrm{d}x \geq (\leq) \frac{1}{\bar{F}(t)} \int_t^\infty w(t) f(x) \, \mathrm{d}x = w(t).$$

It follows from (4.4) that $X \leq_{\mathrm{hr}} (\geq_{\mathrm{hr}}) X_w$ and hence $X \leq_{\mathrm{disp}} (\geq_{\mathrm{disp}}) X_w$. Thus, the desired results are proved by Theorem 2.2 of Kang [21].

The length biased distribution is an important case of weighted distributions when $w(t) = t$. They represent sampling procedures where the sampling probabilities are proportional to the samples values. Denoting the length biased random variable associated with X by X_{lb}, we have the following corollary from Proposition 4.12. ■

Corollary 4.13: *If X or X_{lb} is* DFR, *then $X \leq_{\mathrm{LPQE}} X_{lb}$.*

The equilibrium distribution of a renewal process is another important particular case of weighted distributions when $w(t) = 1/r(t)$. Denote equilibrium distributed random variable associated with X by X_e. Applying Proposition 4.12 again, we have the following corollary.

Corollary 4.14: *If X is* DFR, *then $X \leq_{\mathrm{LPQE}} X_e$.*

The proportional odds family (POF), also known as tilt parameter family, is a semi-parametric family useful in the study of survival and reliability data (Marshall and Olkin [26], p.242). The proportional odds random variable associated with X, denoted by X_p, is defined by the cdf

$$F_{X_p}(t) = \frac{F(t)}{\theta + (1-\theta)F(t)}, \quad t \geq 0,$$

for $\theta > 0$, where θ is the proportional constant. Thus, the failure rate of X_p is given by

$$r_{X_p}(t) = \frac{1}{1 + (\theta - 1)\bar{F}(t)} r(t), \quad t \geq 0.$$

For this model, we have the following proposition by analogy with Proposition 4.12.

Proposition 4.15: *If* $\theta \geq 1$ $(0 < \theta \leq 1)$, *and if* X *or* X_p *is* DFR, *then* $X \leq_{\mathrm{LPQE}} (\geq_{\mathrm{LPQE}}) X_p$.

Acknowledgements

This work was supported by NNSF of China (Nos: 11371340, 11471303) and Key Program of Education Department of Anhui and Xinhua University (Nos: ZK2018A0680, 2017rw002).

References

1. Ahmad, I.A. & Kayid, M. (2005). Characterizations of the RHR and MIT orderings and the DRHR and IMIT classes of life distributions. *Probability in the Engineering and Informational Sciences* 19: 447–461.
2. Ahmad, I.A., Kayid, M., & Pellerey, F. (2005). Further results involving the MIT order and the IMIT class. *Probability in the Engineering and Informational Sciences* 19: 377–395.
3. Ahmed, A.N., Alzaid, A., Bartoszewicz, J., & Kochar, S.C. (1986). Dispersive and superadditive ordering. *Advances in Applied Probability* 18: 1019–1022.
4. Arnold, B.C., Balakrishnan, N., & Nagaraja, H.N. (1998). *A first course in order statistics*. New York: Wiley.
5. Badia, F.G. & Berrade, M.D. (2008). On the reversed hazard rate and mean inactivity time of mixtures. In T. Bedford, J. Quigley, & L. Walls (eds.), *Advances in mathematical modeling for reliability*. Glasgow, Scotland: 5th International Conference on Mathematical Methods in Reliability, pp. 103–110.
6. Balakrishnan, N., Cramer, E., & Kamps, U. (2001). Bounds for means and variances of progressive type II censored order statistics. *Statistics and Probability Letters* 54: 301–315.
7. Barlow, R. & Proschan, F. (1981). *Statistical theory of reliability and life testing, probability model*. New York: Holt, Rinehart and Winnston.
8. Belzunce, F., Mercader, J.A., & Ruiz, J.M. (2005). Stochastic comparisons of generalized order statistics. *Probability in the Engineering and Informational Sciences* 19: 99–120.
9. Chandra, N.K. & Roy, D. (2001). Some results on reversed hazard rate. *Probability in the Engineering and Informational Sciences* 15: 95–102.
10. Cox, D.R. & Oakes, D. (1984). *Analysis of survival data*. New York: Chapman and Hall.
11. Cramer, E., Kamps, U., & Rychlik, T. (2002). On the existence of moments of generalized order statistics. *Statistics and Probability Letters* 59: 397–404.
12. Di Crescenzo, A. & Longobardi, M. (2002). Entropy-based measure of uncertainty in past lifetime distributions. *Journal of Applied Probability* 39: 434–440.
13. Di Crescenzo, A. & Longobardi, M. (2004). A measure of discrimination between past lifetime distributions. *Statistics and Probability Letters* 67: 173–182.
14. Finkelstein, M.S. (2001). A note on some aging properties of the accelerated life model. *Reliability Engineering and System Safety* 71: 109–112.
15. Gajek, L. & Okolewski, A. (2000). Sharp bounds on moments of generalized order statistics. *Metrika* 52: 27–43.
16. Gilchrist, W. (2000). *Statistical modelling with quantile functions*. Boca Raton, Florida: Chapman and Hall/CRC.

17. Gupta, R.C. & Kirmani, S.N.U.A. (1990). The role of weighted distributions in stochastic modeling. *Communications in Statistics–Theory and Methods* 19: 3147–3162.
18. Jain, K., Singh, H., & Bagai, I. (1989). Relations for reliability measures of weighted distributions. *Communications in Statistics–Theory and Methods* 18: 4393–4412.
19. Kamps, U. (1995a). *A concept of generalized order statistics*. Stuttgard: Teubner, B.G.
20. Kamps, U. (1995b). A concept of generalized order statistics. *Journal of Statistical Planning and Inference* 48: 1–23.
21. Kang, D. (2015). Further results on closure properties of LPQE order. *Statistical Methodology* 25: 23–35.
22. Kayid, M. & Ahmad, I.A. (2004). On the mean inactivity time ordering with reliability applications. *Probability in the Engineering and Informational Sciences* 18: 395–409.
23. Kundu, C., Nanda, A.K., & Hu, T. (2009). A note on reversed hazard rate of order statistics and record values. *Journal of Statistical Planning and Inference* 139: 1257–1265.
24. Kundu, C., Nanda, A.K., & Maiti, S.S. (2010). Some distributional results through past entropy. *Journal of Statistical Planning and Inference* 140: 1280–1291.
25. Li, X. & Xu, M. (2006). Some results about MIT order and IMIT class of life distributions. *Probability in the Engineering and Informational Sciences* 20: 481–496.
26. Marshall, A.W. & Olkin, I. (2007). *Life distributions. Structure of non-parametric, semi-parametric, and parametric families. Springer Series in Statistics*. New York: Springer.
27. Nair, N.U. & Sankaran, P.G. (2009). Quantile based reliability analysis. *Communications in Statistics–Theory and Methods* 38: 222–232.
28. Nanda, A.K. & Jain, K. (1999). Some weighted distributions results on univariate and bivariate cases. *Journal of Statistical Planning and Inference* 77: 169–180.
29. Nanda, A.K. & Paul, P. (2006a). Some properties of past entropy and their applications. *Metrika* 64: 47–61.
30. Nanda, A.K. & Paul, P. (2006b). Some results on generalized past entropy. *Journal of Statistical Planning and Inference* 136: 3659–3674.
31. Nanda, A.K., Singh, H., Misra, N., & Paul, P. (2003). Reliability properties of reversed residual lifetime. *Communications in Statistics–Theory and Methods* 32: 2031–2042.
32. Rojo, J. & He, G.Z. (1991). New properties and characterizations of the dispersive ordering. *Statistics and Probability Letters* 11: 365–372.
33. Ruiz, J.M. & Navarro, J. (1996). Characterizations based on conditional expectations of the double truncated distribution. *Annals of the Institute of Statistical Mathematics* 48: 563–572.
34. Shaked, M. & Shanthikumar, J.G. (2007). *Stochastic orders*. New York: Springer.
35. Shannon, C.E. (1948). A mathematical theory of communication. *The Bell System Technical Journal* 27: 379–423.
36. Sunoj, S.M., Sankaran, P.G., & Nanda, A.K. (2013). Quantile based entropy function in past lifetime. *Statistics and Probability Letters* 83: 366–372.