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# *On a group of 1440 birational transformations of four variables that arises in considering the projective equivalence of double sixes.*

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The lines of a double-six will here be represented by the usual notation

$$a,\ b,\ c,\ d,\ e,\ f,$$
$$a',\ b',\ c',\ d',\ e',\ f',$$

where two lines whose symbols are in the same line or same column of this scheme are non-intersectors and all other pairs of lines intersect. Any six of the lines, no two of whose symbols are in the same column, and just three are in the same row, are generators of a quadric, and the actual position in space of each of the other six is determined by the two points in which it intersects this quadric.

A point on the quadric is determined by the parameters of the two generating lines that pass through it, and these parameters may be chosen so that to three arbitrarily assigned generators of either system the values 0, 1, $\infty$ of the parameter correspond.

Hence if $S$ is the quadric of which $a, b, c, d', e', f'$ are generators, the parameters $\lambda$, $\mu$ of the generators of $S$ may be chosen so that:

$$a,\ b,\ c \text{ are } \lambda = 0,\ \lambda = 1,\ \lambda = \infty,$$
$$d',\ e',\ f' \text{ are } \mu = 0,\ \mu = 1,\ \mu = \infty,$$

$a'$ meets $S$ where $\lambda = 1,\ \mu = u_1$ and $\lambda = \infty,\ \mu = U_1$,

$b'$ ,, ,, $\lambda = \infty,\ \mu = u_2$ ,, $\lambda = 0,\ \mu = U_2$,

$c'$ ,, ,, $\lambda = 0,\ \mu = u_3$ ,, $\lambda = 1,\ \mu = U_3$,

$d$ ,, ,, $\lambda = v_1,\ \mu = 1$ ,, $\lambda = V_1,\ \mu = \infty$,

$e$ ,, ,, $\lambda = v_2,\ \mu = \infty$ ,, $\lambda = V_2,\ \mu = 0$,

$f$ ,, ,, $\lambda = v_3,\ \mu = 0$ ,, $\lambda = V_3,\ \mu = 1$.

Now each of the lines $a', b', c'$ meets each of the lines $d, e, f$. This gives nine equations connecting the $u$'s, $U$'s, $v$'s and $V$'s. It



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is readily found that one of these equations is a consequence of the other eight, and that these give

$$u_3 = U_1 U_2 \cdot \frac{(U_1-1)(U_2-1)-(u_1-1)(u_2-1)}{u_1u_2(U_1-1)(U_2-1)-U_1U_2(u_1-1)(u_2-1)},$$

$$U_3 = u_1 u_2 \cdot \frac{(U_1-1)(U_2-1)-(u_1-1)(u_2-1)}{u_1u_2(U_1-1)(U_2-1)-U_1U_2(u_1-1)(u_2-1)},$$

$$v_1 = \frac{(U_1-u_1)(U_2-1)}{(U_1-1)(U_2-1)-(u_1-1)(u_2-1)},$$

$$V_1 = \frac{(U_1-u_1)(u_2-1)}{(U_1-1)(U_2-1)-(u_1-1)(u_2-1)},$$

$$v_2 = \frac{(U_1-u_1)u_2}{U_1U_2-u_1u_2},$$

$$V_2 = \frac{(U_1-u_1)U_2}{U_1U_2-u_1u_2},$$

$$v_3 = \frac{(U_1-u_1)U_2(u_2-1)}{u_1u_2(U_1-1)(U_2-1)-U_1U_2(u_1-1)(u_2-1)},$$

$$V_3 = \frac{(U_1-u_1)(U_2-1)u_2}{u_1u_2(U_1-1)(U_2-1)-U_1U_2(u_1-1)(u_2-1)}.$$

This is, of course, only one form of solution of the nine equations. It corresponds to the supposition that besides $a$, $b$, $c$, $d'$, $e'$, $f'$ the two lines $a'$, $b'$ are taken arbitrarily. There will be five other forms of solution corresponding to the suppositions that $b'$, $c'$; $c'$, $a'$; $d$, $e$; $e$, $f$; or $f$, $d$ respectively are taken arbitrarily. Since the double-six is determined uniquely when either $b'$, $c'$ or $e$, $f$ are given in addition to $a$, $b$, $c$, $d'$, $e'$, $f'$, the last four of the above equations must determine $u_1$, $U_1$, $u_2$, $U_2$ uniquely in terms of $v_2$, $V_2$, $v_3$, $V_3$.

If the notation $ab'$ (sequence of symbols immaterial) is used to denote the point of intersection of the lines $a$ and $b'$, the values of $\mu$ at the four points $d'b$, $e'b$, $f'b$, $a'b$ are $0, 1, \infty, u_1$; so that $u_1$ is the cross-ratio of the two pairs of points $a'b$, $e'b$; $d'b$, $f'b$. The symbols $U_1$, $u_2$ and $U_2$ have similar significations. Hence when $u_1$, $U_1$, $u_2$, $U_2$ are given, the cross-ratio of each such two pairs of points on any line of the double-six is uniquely determined.

Now consider the permutation

$$\begin{pmatrix} a, & b, & c, & d, & e, & f, & a', & b', & c', & d', & e', & f' \\ a_1, & b_1, & c_1, & d_1, & e_1, & f_1, & a_1', & b_1', & c_1', & d_1', & e_1', & f_1' \end{pmatrix}$$

of the twelve symbols, where $a_1$, $b_1$, $c_1$, $d_1$, $e_1$, $f_1$ are $a$, $b$, $c$, $d$, $e$, $f$ in some altered sequence: $a_1'$, $b_1'$, $c_1'$, $d_1'$, $e_1'$, $f_1'$ are $a'$, $b'$, $c'$, $d'$, $e'$, $f'$ in an altered sequence; and

$$\begin{pmatrix} a', & b', & c', & d', & e', & f' \\ a_1', & b_1', & c_1', & d_1', & e_1', & f_1' \end{pmatrix}$$

gives the same permutation of $a', b', c', d', e', f'$ that

$$\begin{pmatrix} a, & b, & c, & d, & e, & f \\ a_1, & b_1, & c_1, & d_1, & e_1, & f_1 \end{pmatrix}$$

gives of $a, b, c, d, e, f$.

Let $S_1$ be the quadric containing $a_1, b_1, c_1, d_1', e_1', f_1'$. The parameters of its generators may be chosen so that

$$a_1, \; b_1, \; c_1 \text{ are } \lambda = 0, \; \lambda = 1, \; \lambda = \infty,$$
$$d_1', \; e_1', \; f_1' \text{ are } \mu = 0, \; \mu = 1, \; \mu = \infty.$$

Then if

$a_1'$ meets $S_1$ where $\lambda = 1$, $\mu = u_1'$ and $\lambda = \infty$, $\mu = U_1'$,
and $b_1'$ ,, $S_1$ ,, $\lambda = \infty$, $\mu = u_2'$ ,, $\lambda = 0$, $\mu = U_2'$,
$u_1', U_1', u_2', U_2'$ are rational functions of $u_1, U_1, u_2, U_2$. Also since the double-six is uniquely determined by $u_1', U_1', u_2', U_2'$, it follows that $u_1, U_1, u_2, U_2$ can be uniquely determined in terms of $u_1', U_1', u_2', U_2'$.

Hence the equations

$$u_1' = f\,(u_1, U_1, u_2, U_2), \qquad u_2' = g\,(u_1, U_1, u_2, U_2),$$
$$U_1' = F\,(u_1, U_1, u_2, U_2), \qquad U_2' = G\,(u_1, U_1, u_2, U_2),$$

specify a birational substitution of the four symbols; and each permutation of the twelve original symbols of the kind specified gives rise to such a birational substitution on $u_1, U_1, u_2, U_2$. Moreover it is clearly the case that to the product of two of the permutations there corresponds the product of the two birational substitutions to which they give rise.

Hence the set of 720 birational substitutions is a representation (which has not hitherto been recognised) of the symmetric group of degree six.

There is some interest in shewing how the generating substitutions of the group in this form may be determined. The specifica-

| | $a$ $(\lambda=0)$ | $b$ $(\lambda=1)$ | $c$ $(\lambda=\infty)$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a'$ | × | $1, u_1$ | $\infty, U_1$ | — | — | — |
| $b'$ | $0, U_2$ | × | $\infty, u_2$ | — | — | — |
| $c'$ | $0, u_3$ | $1, U_3$ | × | — | — | — |
| $d'$ $(\mu=0)$ | $0, 0$ | $1, 0$ | $\infty, 0$ | × | $V_2, 0$ | $v_2, 0$ |
| $e'$ $(\mu=1)$ | $0, 1$ | $1, 1$ | $\infty, 1$ | $v_1, 1$ | × | $V_3, 1$ |
| $f'$ $(\mu=\infty)$ | $0, \infty$ | $1, \infty$ | $\infty, \infty$ | $V_1, \infty$ | $v_3, \infty$ | × |

tion of the twelve $u$'s, $U$'s, $v$'s, $V$'s originally introduced is conveniently given by the diagram, where the symbols written in each compartment give the values of $\lambda$, $\mu$ at the corresponding point.

If the quadric $S'$, determined by $a'$, $b'$, $c'$, $d$, $e$, $f$, is used in the place of the quadric $S$, the diagram would be

| | $a$ | $b$ | $c$ | $d$ $(\mu=0)$ | $e$ $(\mu=1)$ | $f$ $(\mu=\infty)$ |
|---|---|---|---|---|---|---|
| $a'$ $(\lambda=0)$ | × | $0, U_2''$ | $0, u_3''$ | $0, 0$ | $0, 1$ | $0, \infty$ |
| $b'$ $(\lambda=1)$ | $1, u_1''$ | × | $1, U_3''$ | $1, 0$ | $1, 1$ | $1, \infty$ |
| $c'$ $(\lambda=\infty)$ | $\infty, U_1''$ | $\infty, u_2''$ | × | $\infty, 0$ | $\infty, 1$ | $\infty, \infty$ |
| $d'$ | — | — | — | × | $v_1'', 1$ | $V_1'', \infty$ |
| $e'$ | — | — | — | $V_2'', 0$ | × | $v_2'', \infty$ |
| $f'$ | — | — | — | $v_3'', 0$ | $V_3'', 1$ | × |

Now it is a known property of a double-six that there is a quadric, with respect to which $a, a'$; $b, b'$; ...; $f, f'$ are polar lines. Reciprocation with respect to this replaces the four points

$$b, a', \quad d, a', \quad e, a', \quad f, a'$$

by the four planes

$$[b', a], \ [d', a], \ [e', a], \ [f', a],$$

where $[b', a]$ denotes the plane containing $b'$ and $a$. These four planes are met by the line $c$ in the four points

$$b', c, \quad d', c, \quad e', c, \quad f', c,$$

so that this set of points is projective with the set

$$b, a', \quad d, a', \quad e, a', \quad f, a'.$$

With the notation here used this gives

$$U_2'' = u_2,$$

and in the same way the relations

$$u_2'' = U_2, \quad U_1'' = u_1, \quad u_1'' = U_1, \quad \dots \quad V_1'' = v_1, \quad v_1'' = V_1, \quad \dots$$

are proved. Hence the second of the above diagrams is

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a'$ | × | $0, u_2$ | $0, U_3$ | $0, 0$ | $0, 1$ | $0, \infty$ |
| $b'$ | $1, U_1$ | × | $1, u_3$ | $1, 0$ | $1, 1$ | $1, \infty$ |
| $c'$ | $\infty, u_1$ | $\infty, U_2$ | × | $\infty, 0$ | $\infty, 1$ | $\infty, \infty$ |
| $d'$ | — | — | — | × | $V_1, 1$ | $v_1, \infty$ |
| $e'$ | — | — | — | $v_2, 0$ | × | $V_2, \infty$ |
| $f'$ | — | — | — | $V_3, 0$ | $v_4, 1$ | × |

Part of the diagram, corresponding to the permutation

$$\begin{pmatrix} a, & b, & c, & d, & e, & f, & a', & b', & c', & d', & e', & f' \\ b, & c, & d, & e, & f, & a, & b', & c', & d', & e', & f', & a' \end{pmatrix},$$

is

| | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| $a'$ | × | $0, \infty$ | $1, \infty$ | $\infty, \infty$ | — | — |
| $b'$ | — | × | $1, u_1'$ | $\infty, U_1'$ | — | — |
| $c'$ | — | $0, U_2'$ | × | $\infty, u_2'$ | — | — |
| $d'$ | — | — | — | × | — | — |
| $e'$ | — | $0, 0$ | $1, 0$ | $\infty, 0$ | × | — |
| $f'$ | — | $0, 1$ | $1, 1$ | $\infty, 1$ | — | × |

Comparing this with the two that have been obtained above, it follows that

$$\left.\begin{matrix} u_1', 0, & 1, & \infty \\ u_2, & 1, \infty_1, & U_1 \end{matrix}\right\}, \quad \left.\begin{matrix} U_1', 0, & 1, & \infty \\ 1, & v_2, & V_3, & 0 \end{matrix}\right\}, \quad \left.\begin{matrix} u_2', 0, & 1, & \infty \\ \infty, & v_2, & V_3, & 0 \end{matrix}\right\}, \quad \left.\begin{matrix} U_2', 0, & 1, & \infty \\ U_3, & 1, & \infty, & u_1 \end{matrix}\right\}$$

are respectively projective ranges.

Hence

$$u_1' = \frac{u_2 - 1}{u_2 - U_1}, \quad U_1' = \frac{V_3(1 - v_2)}{(V_3 - v_2)}, \quad u_2' = \frac{V_3}{V_3 - v_2}, \quad U_2' = \frac{U_3 - 1}{U_3 - u_1}.$$

When the values of $v_2$, $V_3$ and $U_3$ are entered in this it becomes

$$\left.\begin{aligned} u_1' &= \frac{u_2 - 1}{u_2 - U_1} \\ U_1' &= \frac{U_2 - 1}{U_2 - u_1} \\ u_2' &= \frac{(U_2 - 1)(U_1 U_2 - u_1 u_2)}{U_1 (U_2 - u_1)(U_2 - u_2)} \\ U_2' &= \frac{(u_2 - 1)(U_1 U_2 - u_1 u_2)}{u_1 (u_2 - V_1)(U_2 - u_2)} \end{aligned}\right\} \quad \ldots\ldots(\mathrm{A}).$$

The substitution that corresponds to the permutation

$$\begin{pmatrix} a, & b, & c, & d, & e, & f, & a', & b', & c', & d', & e', & f' \\ b, & a, & c, & d, & e, & f, & b', & a', & c', & d', & e', & f' \end{pmatrix}$$

is found readily in the form

$$u_1' = U_2, \quad U_1' = u_2, \quad u_2' = U_1, \quad U_2' = u_1 \quad \ldots\ldots(\mathrm{B});$$

and this, together with the substitution just obtained, generate the group of 720 birational substitutions on four symbols which gives a representation of the symmetric group of degree six.

It has been seen above that, to the permutation,

$$\begin{pmatrix} a, & b, & c, & d, & e, & f, & a', & b', & c', & d', & e', & f' \\ a', & b', & c', & d', & e', & f', & a, & b, & c, & d, & e, & f \end{pmatrix}$$

of the twelve symbols, which does not occur among the 720 permutations already considered, and which is permutable with every one of them, corresponds the substitution

$$u_1' = U_1, \quad U_1' = u_1, \quad u_2' = U_2, \quad U_2' = u_2 \quad \ldots\ldots(\mathrm{C}),$$

on the four symbols. Hence every substitution of the group of birational substitutions generated by (A) and (B) is permutable with the substitution (C); and (A), (B) and (C) generate a group of order 1440, which is simply isomorphic with the direct product of the symmetric group of degree six and an operation of order two; and in it (C) is a self-conjugate substitution of order two.

Suppose now that $A, B, C$ are any three non-intersecting lines and $D', E', F'$ any three lines which meet each of $A, B, C$.

Two arbitrarily given double-sixes can, by suitable collineations, be changed into two, six of whose lines are $A, B, C, D', E', F'$. These two are completely defined by the values

$$u_1,\ U_1,\ u_2,\ U_2$$

and

$$u_1',\ U_1',\ u_2',\ U_2'$$

of the parameters that have been introduced. Then the necessary and sufficient condition that the two double-sixes should be projectively equivalent is that $u_1', U_1', u_2', U_2'$ and $u_1, U_1, u_2, U_2$ should be connected by one of the substitutions of the group generated by (A), (B) and (C).