CONTINUOUS FUNCTIONS OF BOUNDED nth VARIATION

by GAVIN BROWN
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1. Introduction

Let n be a positive integer. We give an elementary construction for the nth variation, $V_n(f)$, of a real valued continuous function f and prove an analogue of the classical Jordan decomposition theorem. In fact, let $C[0, 1]$ denote the real valued continuous functions on the closed unit interval, let A_n denote the semi-algebra of non-negative functions in $C[0, 1]$ whose first n differences are non-negative, and let S_n denote the difference algebra $A_n - A_n$. We show that S_n is precisely that subset of $C[0, 1]$ on which $V_n(f) < \infty$. (Theorem 1).

This paper can be regarded as completing our work in (**2**) where we discussed S_n as a Banach algebra in its cone norm. We show that our nth variation norm coincides with the cone norm on S_n and combine this representation theorem with the main results of (**2**) in the statement of Theorem 2.

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2. Notation

$$A_n = \{f \in C[0, 1] : \Delta_h^r f(x) \geqq 0,\ 0 \leqq r \leqq n,\ 0 \leqq x \leqq x + rh \leqq 1\},$$

where

$$\Delta_h^0 f(x) = f(x),\ \Delta_h^1 f(x) = f(x+h) - f(x),\ \Delta_h^n f(x) = \Delta_h^1(\Delta_h^{n-1} f(x)).$$

$$S_n = A_n - A_n = \{f - g : f, g \in A_n\}. \quad A_\infty = \bigcap_{n=1}^{\infty} A_n \text{ and } S_\infty = A_\infty - A_\infty.$$

It is well known (cf. (**4**)) that

$$A_\infty = \left\{f \in C[0, 1] : f(t) = \sum_{r=0}^{\infty} \alpha_r t^r,\ \alpha_r \geqq 0,\ \sum_{r=0}^{\infty} \alpha_r < \infty\right\}.$$

It follows that S_∞ comprises those functions in $C[0, 1]$ with absolutely convergent Taylor series.

3. nth variation

Suppose that $0 < x \leqq 1$, then we say that a subdivision D of $[0, x]$ is n-*admissible* if D is of the form $\{x_r\}_{r=0}^{m}$, where $m \geqq n$, $0 = x_0 < x_1 \ldots < x_m \leqq x$,

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$x_r - x_{r-1} = h_D$, $r = 1, \ldots, m$, and $x - x_m < h_D$. Note that the *mesh length* h_D completely determines D.

Definition. For each $f \in C[0, 1]$ we define the nth *variation of f over* $[0, 1]$ by

$$V_n(f) = \sup_D \sum_{r_n=0}^{m-n} \sum_{r_{n-1}=0}^{r_n} \cdots \sum_{r_1=0}^{r_2} |\Delta_{h_D}^n f(x_{r_1})|,$$

where D ranges over all n-admissible subdivisions of $[0, 1]$.

If $V_n(f) < \infty$, we say that f has *bounded* nth *variation* on $[0, 1]$.

For the remainder of this section we suppose that f belongs to $C[0, 1]$ and has bounded nth variation on $[0, 1]$.

If $0 < x \leqq 1$, and D is an n-admissible subdivision of $[0, x]$, let

$$t_D = \sum_{r_n=0}^{m-n} \cdots \sum_{r_1=0}^{r_2} |\Delta_{h_D}^n f(x_{r_1})|, \quad p_D = \sum_{r_n=0}^{m-n} \cdots \sum_{r_1=0}^{r_2} (\Delta_{h_D}^n f(x_{r_1}))^+,$$

$$n_D = \sum_{r_n=0}^{m-n} \cdots \sum_{r_1=0}^{r_2} (\Delta_{h_D}^n f(x_{r_1}))^-.$$

(If λ is a real number, $\lambda^+ = \max(\lambda, 0)$, $\lambda^- = -\min(\lambda, 0)$.) We define

$$t(x) = \sup_D t_D, \; p(x) = \sup_D p_D, \; n(x) = \sup_D n_D,$$

where, in each case, the sup is over all n-admissible subdivisions of $[0, x]$. If $x = 0$, define $p(0) = n(0) = t(0) = 0$. Note that $t(1) = V_n(f)$.

Interchanging the order of summation in the above gives a neater formula:

Lemma 1.

$$t_D = \sum_{r=0}^{m-n} \binom{m-r-1}{n-1} |\Delta_{h_D}^n f(x_r)|, \quad p_D = \sum_{r=0}^{m-n} \binom{m-r-1}{n-1} (\Delta_{h_D}^n f(x_r))^+,$$

$$n_D = \sum_{r=0}^{m-n} \binom{m-r-1}{n-1} (\Delta_{h_D}^n f(x_r))^-.$$

Lemma 2.

(i) $t_D = p_D + n_D$,

(ii) $p_D - n_D = f(x_m) - \sum_{r=0}^{n-1} \binom{m}{r} \Delta_{h_D}^r f(0)$.

Proof. Part (i) is obvious. Let us consider (ii). We have

$$p_D - n_D = \sum_{r_n=0}^{m-n} \cdots \sum_{r_1=0}^{r_2} \Delta_{h_D}^n f(x_{r_1}).$$

The right hand side equals

$$\sum_{r_n=0}^{m-n} \cdots \sum_{r_2=0}^{r_3} (\Delta_{h_D}^{n-1} f(x_{r_2+1}) - \Delta_{h_D}^{n-1} f(0)),$$

and repeated application gives

$$p_D - n_D = f(x_m) - \sum_{r=0}^{n-1} \binom{m-n+r}{r} \Delta_{h_D}^r f(x_{n-r-1}).$$

The difference operator satisfies

$$\Delta_{h_D}^r f(x_{n-r-1}) = \sum_{t=0}^{n-r-1} \binom{n-r-1}{t} \Delta_{h_D}^{t+r} f(0),$$

so that

$$p_D - n_D = f(x_m) - \sum_{s=0}^{n-1} \alpha_s \Delta_{h_D}^s f(0),$$

where

$$\alpha_s = \sum_{r=0}^{s} \binom{m-n+r}{r}\binom{n-r+1}{s-r}.$$

By comparing coefficients of z^s in the identity

$$(1-z)^{-m+n-1}(1-z)^{-n+s} \equiv (1-z)^{-m+s-1},$$

we see that $\alpha_s = \binom{m}{s}$, $s = 0, \ldots, n-1$. It follows that

$$p_D - n_D = f(x_m) - \sum_{s=0}^{n-1} \binom{m}{s} \Delta_{h_D}^s f(0),$$

as required.

The next two lemmas constitute the crucial step in the argument.

Lemma 3. *If D' is obtained from D by bisecting each subinterval of D (and adding a new right hand end-point if necessary) then*

$$t_{D'} \geqq t_D,\ p_{D'} \geqq p_D,\ n_{D'} \geqq n_D.$$

Proof. It will suffice to consider p_D. Since the addition of a right hand end-point will not decrease p_D we can suppose no such addition is made.

Suppose, then, that $D = \{x_r\}_{r=0}^{m}$, $D' = \{y_r\}_{r=0}^{2m}$, where $x_r - x_{r-1} = h$, $r = 1, \ldots, m$, $y_r - y_{r-1} = \frac{1}{2}h$, $r = 1, \ldots, 2m$, and, in particular, $y_{2r} = x_r$, $r = 0, 1, \ldots, m$.

Write

$$\lambda_r = (\Delta_h^n f(x_r))^+, \ \mu_r = (\Delta_{\frac{1}{2}h}^n f(y_r))^+.$$

Then by Lemma 1,

$$p_D = \sum_{r=0}^{m-n} \binom{m-r-1}{n-1} \lambda_r, \tag{1}$$

$$p_{D'} = \sum_{r=0}^{2m-n} \binom{2m-r-1}{n-1} \mu_r. \tag{2}$$

We define

$$q_D = \sum_{r=0}^{m-n} \binom{m-r-1}{n-1}\left(\sum_{t=0}^{n} \binom{n}{t} \mu_{2r+t}\right). \tag{3}$$

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The coefficient of μ_{2s} in $q_{D'}$ is not greater than

$$\sum_{0 \leqq 2t \leqq n} \binom{m-s+t-1}{n-1}\binom{n}{2t}.$$

But by equating coefficients of $z^{2m-2s-n}$ in the identity

$$(1-z)^{-n} \equiv (1-z^2)^{-n}(1+z)^n,$$

we obtain

$$\sum_{0 \leqq 2t \leqq n} \binom{m-s+t-1}{n-1}\binom{n}{2t} = \binom{2m-2s-1}{n-1}.$$

Similarly the coefficient of μ_{2s+1} in q_D is not greater than

$$\sum_{0 \leqq 2t+1 \leqq n} \binom{m-s+t-1}{n-1}\binom{n}{2t+1} = \binom{2m-2s-2}{n-1}.$$

It follows from (2) that

$$p_{D'} = \sum_{r=0}^{2m-n} \binom{2m-r-1}{n-1} \mu_r \geqq q_{D'}. \tag{4}$$

However,

$$\sum_{t=0}^{n} \binom{n}{t} \mu_{2r+t} = \sum_{t=0}^{n} \binom{n}{t} (\Delta^n_{\frac{1}{2}h} f(y_{2r+t}))^+ \geqq \left(\sum_{t=0}^{n} \binom{n}{t} \Delta^n_{\frac{1}{2}h} f(y_{2r+t}) \right)^+.$$

$$\left(\sum_{t=0}^{n} \binom{n}{t} \Delta^n_{\frac{1}{2}h} f(y_{2r+t}) \right)^+ = (\Delta^n_h f(y_{2r}))^+ = \lambda_r,$$

so that $\sum_{t=0}^{n} \binom{n}{t} \mu_{2r+t} \geqq \lambda_r$, and hence, by (3), (1),

$$q_{D'} \geqq \sum_{r=0}^{m-n} \binom{m-r-1}{n-1} \lambda_r = p_D$$

Together with (4) this shows that $p_{D'} \geqq p_D$, as required.

Lemma 4. (i) $t(x) = p(x) + n(x)$.

(ii) $p(x) - n(x) = f(x) - \sum_{r=0}^{n-1} \left(\frac{x^r}{r!}\right) \lim_{h \to 0+} (\Delta^r_h f(0)/h^r)$.

Proof. The case $x = 0$ is trivial. Accordingly we suppose that $0 < x \leqq 1$ By Lemma 2,

$$p_D = n_D + f(x_m) - \sum_{r=0}^{n-1} \binom{m}{r} \Delta^r_{h_D} f(0),$$

so that

$$p(x) \geqq n_D + f(x_m) - \sum_{r=0}^{n-1} \binom{m}{r} \Delta^r_{h_D} f(0).$$

In view of Lemma 3 we may choose h arbitrarily small. We note that

$$mh = x_m > x - h$$

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so that $x_m \to x$ and $m \to \infty$ as $h \to 0_+$. Hence

$$p(x) \geqq n(x)+f(x)-\sum_{r=0}^{n-1} \varliminf_{h\to 0+} \binom{m}{r} \Delta_h^r f(0),$$

but

$$\varliminf_{h\to 0+} \binom{m}{r} \Delta_h^r f(0) = \varliminf_{h\to 0+} \left(\frac{x_m^r}{r!}\right)\left(\frac{m!}{m^r(m-r)!}\right)\left(\frac{\Delta_h^r f(0)}{h^r}\right)$$

$$= \frac{x^r}{r!} \varliminf_{h\to 0+} \left(\frac{\Delta_h^r f(0)}{h^r}\right), \quad 0 \leqq r \leqq n-1,$$

so that

$$p(x)-n(x) \geqq f(x)-\sum_{r=0}^{n-1} \left(\frac{x^r}{r!}\right) \varliminf_{h\to 0+} (\Delta_h^r f(0)/h^r). \tag{1}$$

Similarly, the inequality

$$p_D - f(x_m)+\sum_{r=0}^{n-1} \binom{m}{r} \Delta_{h_D}^r f(0) \leqq n(x)$$

leads to

$$p(x)-n(x) \leqq f(x)-\sum_{r=0}^{n-1} \left(\frac{x^r}{r!}\right) \varlimsup_{h\to 0+} (\Delta_h^r f(0)/h^r). \tag{2}$$

From (1) and (2),

$$\varliminf_{h\to 0+} (\Delta_h^r f(0)/h^r) = \varlimsup_{h\to 0+} (\Delta_h^r f(0)/h^r), \quad 0 \leqq r \leqq n-1,$$

so that

$$p(x)-n(x) = f(x)-\sum_{r=0}^{n-1} \left(\frac{x^r}{r!}\right) \lim_{h\to 0+} (\Delta_h^r f(0)/h^r). \tag{3}$$

Since $t_D = p_D + n_D$, $t(x) \leqq p(x)+n(x)$. But $t_D = (p_D - n_D)+2n_D$, so that

$$2n_D + f(x_m)-\sum_{r=0}^{n-1} \binom{m}{r} \Delta_{h_D}^r f(0) \leqq t(x),$$

and hence

$$2n(x)+f(x)-\sum_{r=0}^{n-1} \left(\frac{x^r}{r!}\right) \lim_{h\to 0+} (\Delta_h^r f(0)/h^r) \leqq t(x),$$

i.e. $p(x)+n(x) \leqq t(x)$, using (3). Thus $t(x) = p(x)+n(x)$, which completes the proof.

Remark. The continuity of f figures essentially in the above. For example there are discontinuous solutions (unbounded in any finite interval) of the functional equation $f(x+y) = f(x)+f(y)$. (Use a Hamel basis for the reals over the rationals.) For such an f, $t(x) = p(x) = n(x) = 0$ but the full statement of Lemma 4 is not meaningful (except in the case $n = 1$).

It remains to prove that, e.g. $p(x)$ belongs to A_n. It seems convenient to travel by a somewhat indirect route.

Lemma 5. *If* $1 \leqq s \leqq n$, $0 \leqq x < 1$, *then for any* $k > \varepsilon > 0$ *such that*

$$x + sk + \varepsilon \leqq 1,$$

$$p(x) + \sum_{r=1}^{\frac{1}{2}s} \binom{s}{2r} p(x+2rk+\varepsilon) \geqq \sum_{r=1}^{\frac{1}{2}s-1} \binom{s}{2r+1} p(x+(2r+1)k-\varepsilon) + sp(x+k), \qquad s \text{ even}.$$

$$sp(x+k) + \sum_{r=1}^{\frac{1}{2}(s-1)} \binom{s}{2r+1} p(x+(2r+1)k+\varepsilon) \geqq \sum_{r=1}^{\frac{1}{2}(s-1)} \binom{s}{2r} p(x+2rk-\varepsilon) + p(x), \qquad s \text{ odd}.$$

Proof. Let D be an n-admissible subdivision of $[0, x+sk]$ with mesh length $h < \varepsilon/s$. For definiteness let $D = \{x_r\}_{r=0}^{j}$. Let t be the largest integer such that $x_t \leqq x$. (t may be zero i.e. if $x = 0$). Let m be the largest integer such that $x_{t+m} \leqq x+k$. Take h sufficiently small to ensure $t \geqq n$ if $x > 0$, and to ensure $m \geqq n$ if $x = 0$.

We set $x_{t+r} = x_t + rh$, $r = 0, 1, \ldots, sm$ (this may or may not be a proper extension of D) and write

$$p_w = \sum_{q=0}^{t+mw-n} \binom{t+mw-q-1}{n-1} (\Delta_h^n f(x_q))^+, \quad w = 1, \ldots, s,$$

$$p_0 = \sum_{q=0}^{t-n} \binom{t-q-1}{n-1} (\Delta_h^n f(x_q))^+ \quad \text{if } x > 0,$$

and $p_0 = 0$ if $x = 0$.

For $w = 2, \ldots, s$,

$$\begin{aligned} |x_{wm+t} - (x+wk)| &= |x_t + wmh - x - wk| \\ &\leqq |x - x_t + k - mh| + (w-1)|k - mh| \\ &= |x_{t+m} - (x+k)| + (w-1)|k-mh| < sh < \varepsilon. \end{aligned}$$

It follows that $x + wk - \varepsilon \leqq x_{wm+t} \leqq x + wk + \varepsilon$ for $w = 2, \ldots, s$.

We now have

$$\left.\begin{aligned} &p(x+wk-\varepsilon) \leqq \sup_D p_w \leqq p(x+wk+\varepsilon),\ w = 2, \ldots, s, \\ &\sup_D p_1 = p(x+k), \text{ and } \sup_D p_0 = p(x), \end{aligned}\right\} \tag{1}$$

where the sup is over all subdivisions of the type described above. However

$$\sum_{w=0}^{s} (-1)^{s-w} p_w = \sum_{q=0}^{t+ms-n} \beta_q (\Delta_h^n f(x_q))^+,$$

where

$$\beta_q = \sum_r (-1)^{s-r} \binom{s}{r} \binom{t+mr-q-1}{n-1},$$

and the integer r satisfies $\max((q-t+n)/m, 0) \leqq r \leqq s$. It follows that β_q

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is the coefficient of $z^{ms+t-q-n}$ in the expression $(1-z^m)^s(1-z)^{-n}$, and hence that $\beta_q \geqq 0$. This gives

$$\sum_{r=0}^{s}(-1)^{s-r}\binom{s}{r}p_r \geqq 0. \tag{2}$$

(1) and (2) combine to give the required result.

The proof of the next Lemma follows a standard pattern used to prove the continuity of a convex function. Recall our standing assumption that f is a continuous function with bounded nth variation on $[0, 1]$.

Lemma 6. *If $n \geqq 2$ then p is continuous in* $[0, 1[$.

Proof. By the previous Lemma p is non-negative, non-decreasing and satisfies

$$p(x+2k+\varepsilon)-p(x+k) \geqq p(x+k)-p(x) \tag{1}$$

whenever $k>\varepsilon>0$, $x \geqq 0$, $x+2k+\varepsilon \leqq 1$.

Suppose that there exists $\delta>0$ such that $p(x+h)-p(x) \geqq \delta$ whenever $h>0$. Choose $h = \frac{1}{2}m(m+1)$, $\varepsilon = 1/m^2(m+1)$ where $m>2$ is a positive integer satisfying $y+1/m \leqq 1$. By repeated application of (1), for different choices of x, k, we obtain, for $r = 2, \ldots, m+1$,

$$p(y+rh+\tfrac{1}{2}r(r-1)\varepsilon)-p(y+(r-1)h+\tfrac{1}{2}(r-1)(r-2)\varepsilon) \geqq p(y+h)-p(y) \geqq \delta.$$

Summation gives

$$\sum_{r=2}^{m+1}[p(y+rh+\tfrac{1}{2}(r-1)r\varepsilon)-p(y+(r-1)h+\tfrac{1}{2}(r-1)(r-2)\varepsilon)] \geqq m\delta.$$

Therefore $p(y+1/m) = p(y+(m+1)h+\frac{1}{2}m(m+1)\varepsilon) \geqq m\delta \to \infty$, as $m \to \infty$. This contradicts the boundedness of p, and shows that p is right hand continuous on $[0, 1[$. A similar argument shows that p is left hand continuous on $]0, 1[$.

Definition. Suppose f belongs to $C[0, 1]$ and has bounded nth variation on $[0, 1]$. We define

$$f_1(x) = p(x)+\sum_{r=0}^{n-1}\left(\frac{x^r}{r!}\right)\lim_{h\to 0+}(\Delta_h^r f(0)/h^r)^+, \quad 0 \leqq x \leqq 1$$

$$f_2(x) = n(x)+\sum_{r=0}^{n-1}\left(\frac{x^r}{r!}\right)\lim_{h\to 0+}(\Delta_h^r f(0)/h^r)^-, \quad 0 \leqq x \leqq 1.$$

Note that $f(x) = f_1(x)-f_2(x)$, $0 \leqq x \leqq 1$, by Lemma 4 (ii).

Lemma 7. *If $0 \leqq x \leqq x+rh \leqq 1$, $0 \leqq r \leqq n$ then*

$$\Delta_h^r f_1(x) \geqq 0, \quad \Delta_h^r f_2(x) \geqq 0.$$

Proof. Since p is non-negative non-decreasing on $[0, 1]$ in all cases, it is not hard to see that f_1, f_2 are non-negative, non-decreasing. This gives the full result in the case $n = 1$.

If $n \geqq 2$, it follows from Lemmas 5, 6 that p satisfies $\Delta_h^r p(x) \geqq 0$ for

$$0 \leqq x \leqq x+rh<1,$$

$0 \leqq r \leqq n$. It follows easily that a similar result holds for f_1, f_2. Since f_1, f_2 are non-negative, non-decreasing this may be strengthened to read

$$\Delta_h^r f_1(x) \geqq 0,\ \Delta_h^r f_2(x) \geqq 0 \text{ for } 0 \leqq x \leqq x+rh \leqq 1,\ 0 \leqq r \leqq n.$$

Lemma 8. *If* $f(x) = f_3(x) - f_4(x)$, $0 \leqq x \leqq 1$, *where* f_3, f_4 *satisfy the conditions*

$$\Delta_h^r f_3(x) \geqq 0,\ \Delta_h^r f_4(x) \geqq 0 \textit{ for } 0 \leqq x \leqq x+rh \leqq 1,\ 0 \leqq r \leqq n$$

then

$$f_1(x) \leqq f_3(x), f_2(x) \leqq f_4(x) \quad 0 < x \leqq 1.$$

Proof.

$$\Delta_h^r (f_3 - f)(x) = \Delta_h^r f_4(x) \geqq 0,\ 0 \leqq x \leqq x+rh \leqq 1,\ 0 \leqq r \leqq n,$$

hence

$$\Delta_h^r f_3(x) \geqq (\Delta_h^r f(x))^+,\ 0 \leqq x \leqq x+rh \leqq 1,\ 0 \leqq r \leqq n.$$

If D is an n-admissible subdivision, say $\{x_r\}_{r=0}^m$, of $[0, x]$, where $0 < x \leqq 1$, then $f_3(x_m) - \sum_{r=0}^{n-1} \binom{m}{r} \Delta_{h_D}^r f_3(0) \geqq p_D$ (cf. Lemma 2). *A fortiori*,

$$f_3(x) \geqq p_D + \sum_{r=0}^{n-1} \binom{m}{r} (\Delta_{h_D}^r f(0))^+$$

and, reasoning as in the proof of Lemma 4, we deduce that $f_3(x) \geqq f_1(x)$. Similarly $f_4(x) \geqq f_2(x)$. This completes the proof.

Theorem 1. *Let f belong to $C[0, 1]$ and suppose that n is a positive integer. Then f has bounded nth variation on $[0, 1]$ if and only if f is of the form $f = f_1 - f_2$, with $f_1, f_2 \in A_n$.*

Proof. If $f \in A_n$ then $V_n(f) \leqq f(1)$ (see Lemma 2 (ii)). Therefore if $f = f_1 - f_2$, with $f_1, f_2 \in A_n$,

$$V_n(f) \leqq V_n(f_1) + V_n(f_2) \leqq f_1(1) + f_2(1) < \infty.$$

Suppose now that f belongs to $C[0, 1]$ and has bounded nth variation. Suppose first that $n \geqq 2$. In view of Lemmas 6, 7 it remains only to prove that the functions f_1, f_2 appearing in the statement of Lemma 7 are continuous at 1.

Let
$$f_3(x) = f_1(x),\ 0 \leqq x < 1,\ f_3(1) = \lim_{x \to 1-} f_1(x),$$
$$f_4(x) = f_2(x),\ 0 \leqq x < 1,\ f_4(1) = \lim_{x \to 1-} f_2(x).$$

It is easily seen that $f_3(1) - f_4(1) = f(1)$. Moreover f_3, f_4 have their first n differences non-negative on $[0, 1]$ so that, by Lemma 8, $f_3(1) \geqq f_1(1)$ and $f_4(1) \geqq f_2(1)$. However the reverse inequalities hold since f_1, f_2 are non-decreasing on $[0, 1]$. It follows that f_1, f_2 are indeed continuous at 1. In the case $n = 1$ it is possible to prove f_1, f_2 right hand continuous in $[0, 1[$ and left hand continuous in $]0, 1]$ by making an appropriate choice of f_3, f_4 in an argument similar to the above. Since such proofs are already well known in this situation we omit the details.

4. Cone norm on S_n

Bonsall proves in (1) that if A is a closed semi-algebra in a Banach algebra then the difference algebra $S = A - A$ is itself a Banach algebra with respect to the cone norm $\| \ \|_s$. ($\| \ \|_s$ is the Minkowski functional of the absolutely convex hull of the unit ball of A). In (2) we study the semi-algebras A_n (which are closed in the uniform norm topology of the Banach algebra $C[0, 1]$) obtaining an integral representation for the cone norm and relating the order structure of S_n to that norm. We shall now show that the results of Section 3 enable us to interpret the cone norm of S_n as the nth variation norm.

Definition. For $f \in C[0, 1]$ and any positive integer n,

$$\| f \|_n = V_n(f) + \sum_{r=0}^{n-1} \left| \overline{\lim_{h \to 0+}} \, (\Delta_h^r f(0)/h^r) \right| / r!$$

With slight abuse of language we call $\| \ \|_n$ the nth *variation norm*. (In fact $\| \ \|_n$ is a sub-additive extended real valued functional on $C[0, 1]$).

Theorem 2. *Let n be a positive integer.*

(i) $\{f \in C[0, 1] : \| f \|_n < \infty\} = S_n = A_n - A_n$.

(ii) S_n, *under* $\| \ \|_n$, *is a Banach algebra with maximal ideal space* $[0, 1]$.

(iii) $\{f \in S_n : \| f^{n+1} \|_n = (\| f \|_n)^{n+1}\} = \pm A_n$.

(iv) S_n *with norm* $\| \ \|_n$ *and the order induced by* A_n *is an abstract* (L)*-space.*

Proof. (i) If $f \in S_n$ then $\| f \|_n = f_1(1) + f_2(1) < \infty$, where $f = f_1 - f_2$ is the canonical decomposition appearing in Lemma 7. The rest follows from Theorem 1.

(ii) The definition of the cone norm, $\| \ \|_{S_n}$, shows that

$$\| f \|_{S_n} = \inf (\| f_3 \| + \| f_4 \|),$$

where $\| \ \|$ denotes the uniform norm and the inf is over all $f_3, f_4 \in A_n$ such that $f = f_3 - f_4$ (cf. (2)). By Lemma 8, $f_1(1) \leqq f_3(1)$ and $f_2(1) \leqq f_4(1)$. It follows that

$$\| f \|_{S_n} = \| f_1 \| + \| f_2 \| = f_1(1) + f_2(1) = \| f \|_n,$$

since the uniform norm coincides with evaluation at 1 on A_n.

We have now proved that $\| \ \|_n$ coincides with the cone norm on S_n, and, as a result, parts (ii), (iii), (iv) of the theorem coincide with Theorems 2, 3, 4 of (2).

5. The case of S_∞

Definition. For $f \in C[0, 1]$, let $\| f \|_\infty = \sup_n \| f \|_n$.

Lemma 9. $S_\infty = \{f \in C[0, 1] : \|f\|_\infty < \infty\}$ *and* $\| \ \|_\infty$ *coincides with the cone norm on* S_∞.

Proof. Suppose that $f \in S_\infty$ and that $f = f_1^{(n)} - f_2^{(n)}$ is the canonical decomposition of f regarded as an element of S_n, $n = 1, 2, \ldots$. Then

$$\|f\|_n = f_1^{(n)}(1) + f_2^{(n)}(1), \quad n = 1, 2, \ldots.$$

If $f = f_3 - f_4$, where $f_3, f_4 \in S_\infty$, then by repeated application of Lemma 8

$$f_3(1) - f_4(1) \geqq \|f\|_{n+1} \geqq \|f\|_n, \quad n = 1, 2, \ldots.$$

Hence $\sup_n \|f\|_n \leqq f_3(1) + f_4(1) < \infty$. Let $f_1 = \sup_n f_1^{(n)}$, $f_2 = \sup_n f_2^{(n)}$. In fact,

$$f_1(x) = \lim_{n\to\infty} f_1^{(n)}, \quad f_2(x) = \lim_{n\to\infty} f_2^{(n)}(x),$$

so that f_1, f_2 have non-negative nth differences for each $n = 1, 2, \ldots$, and we can show that $f_1, f_2 \in A_n$ (cf. the proof of Theorem 1).

Arguing as in the proof of Theorem 2 we deduce that $f_1(1) + f_2(1)$ equals the cone norm of f, and hence that $\|f\|_\infty$ equals the cone norm of f.

Finally, if we are given $f \in C[0, 1]$ with $\|f\|_\infty < \infty$ then we can construct f_1, f_2 as above.

Remarks. (i) The Lemma shows that a function in $C[0, 1]$ has an absolutely convergent Taylor series if and only if its nth variation norms are uniformly bounded. Note however that

$$C^{(\infty)}[0, 1] = \{f \in C[0, 1] : V_n(f) < \infty, \quad n = 1, 2, \ldots\} = \bigcap_{n=1}^{\infty} S_n,$$

where $C^{(\infty)}[0, 1]$ denotes the infinitely differentiable functions on $[0, 1]$. There is no norm under which $C^{(\infty)}[0, 1]$ is a Banach algebra, see (3).

(ii) It is not difficult to establish an isometric isomorphism between S_∞, with the cone norm, and the sequence space l^1. This means we have exact analogues of parts (i) and (iv) of Theorem 2. The maximal ideal space of S_∞ is easily seen to be the closed unit disc, but the analogue of part (iii), which amounts to describing those elements of S_∞ on which spectral radius coincides with norm, is rather tedious to state. The details appear in (2).

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The University, Liverpool