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Elliptic isles in families of area-preserving maps

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Abstract. We prove that every one-parameter family of area-preserving maps unfolding a homoclinic tangency has a sequence of parameter intervals, approaching the bifurcation parameter, where the dynamics exhibits wild hyperbolic sets accumulated by elliptic isles. This is a parametric conservative analogue of a famous theorem of Newhouse on the abundance of wild hyperbolic sets.

1. *Introduction*

This paper is about the dynamics of area-preserving surface diffeomorphisms. We assume the reader to be familiar with hyperbolic theory concepts such as 'hyperbolic periodic orbit' and 'hyperbolic basic set' of a diffeomorphism, as well as the bifurcation theory concepts of 'homoclinic' and 'heteroclinic tangencies'. A tangency between stable and unstable leaves of a hyperbolic set Λ is said to be a homoclinic tangency of Λ. A hyperbolic basic set is said to be wild if it has homoclinic tangencies which are persistent in the sense that they cannot be avoided by small perturbations of the underlying diffeomorphism. The concept of wild hyperbolic set was introduced by Newhouse [**13**] to disprove the density of Ω-stable diffeomorphisms on the sphere S^2. Later, in [**14**], he showed that for dissipative dynamics, this concept implies the co-existence of infinitely many sinks. Finally, in [**15**], he established the abundance of infinitely many sinks around a wild hyperbolic set. For surface dissipative diffeomorphisms, this phenomenon appears at the unfolding of every homoclinic tangency. A parametric version of the theorem appeared a couple of years later in the work of Robinson [**17**].

The techniques used in [**15**] and [**17**] do not apply to the conservative case. In [**4**] we proved a conservative analogue of the Newhouse theorem on the abundance of wild hyperbolic sets. Here we prove a parametric version of that theorem, which is also a conservative analogue of Robinson's theorem. Our result depends crucially on an asymptotic formula for the splitting angle of the Hénon mapping separatrices, which was obtained by Gelfreich in [**7**].

This article was essentially written some six years ago as part of a larger work on the Newhouse phenomenon for higher-dimensional symplectic dynamics. This broader project did not, however, come through; so with the consent of the co-authors I have decided to come forward with this contribution on the two-dimensional dynamics. The arguments here rely heavily on a previous paper [**3**], which the reader may find helpful to read in parallel.

2. *Statement of results*

Let M^2 denote a two-dimensional symplectic manifold, i.e. an orientable surface together with some area form ω. A symplectic, or area-preserving, map is any diffeomorphism $f : M^2 \to M^2$ which preserves the area form ω. We denote by $\mathrm{Diff}^r(M^2)$ and $\mathrm{Diff}^r_\omega(M^2)$ the group of class-C^r diffeomorphisms and the group of class-C^r symplectic maps $f : M^2 \to M^2$, respectively.

We assume the reader to be familiar with basic concepts from hyperbolic theory, namely those of hyperbolic periodic orbit, homoclinic orbit, hyperbolic invariant set, hyperbolic basic set, stable and unstable manifolds. As usual, $W^s(P) = W^s(P, f)$ and $W^u(P) = W^u(P, f)$ will denote, respectively, the stable and unstable manifolds of a point P for a map f. A similar notation $W^s(\Lambda) = W^s(\Lambda, f)$ and $W^u(\Lambda) = W^u(\Lambda, f)$ is used to denote, respectively, the stable and unstable sets of a given hyperbolic set Λ for a map f. See [**18**] for a good introduction to hyperbolic theory.

Given a hyperbolic f-invariant set Λ and two points $x, y \in \Lambda$, an intersection point in $W^s(x, f) \cap W^u(y, f) - \Lambda$ is called a homoclinic point of Λ. This homoclinic point is called a homoclinic tangency point if the corresponding intersection is not transverse.

Let Λ be a basic set for a map f. Recall that the analytic continuation of Λ is the maximal invariant set in a neighbourhood U of Λ which is known to be another hyperbolic basic set, conjugated to Λ, for all maps in some neighbourhood $\mathcal{U}$ of f in $\mathrm{Diff}^r(M^2)$. Following Newhouse, we say that Λ is a wild basic set over an open set $\mathcal{U} \subseteq \mathrm{Diff}^r(M^2)$, containing the map f, if for all maps $g \in \mathcal{U}$:

(1) the analytic continuation Λ_g is a hyperbolic basic set conjugated to Λ; and
(2) there is at least one orbit of homoclinic tangencies of Λ_g.

We shall refer to the open set $\mathcal{U}$ as a Newhouse region for the wild hyperbolic set Λ. The proof of the following proposition is quite standard; see [**16**], or see [**2**] for a conservative argument.

PROPOSITION 1. *Let Λ be a wild hyperbolic set over an open set of maps $\mathcal{U} \subseteq \mathrm{Diff}^r_\omega(M^2)$ with $r \geq 4$. Then:*

(1) *given any periodic point $P \in \Lambda$, there is a dense subset $\mathcal{D} \subseteq \mathcal{U}$ such that for every $g \in \mathcal{D}$, the periodic point P_g has an orbit of homoclinic tangencies;*
(2) *there is a residual subset $\mathcal{R} \subseteq \mathcal{U}$, i.e. a countable intersection of open subsets dense in $\mathcal{U}$, such that for every $g \in \mathcal{R}$, the basic set Λ_g is contained in the closure of all generic elliptic periodic points of g.*

A periodic point P with period n of $f \in \mathrm{Diff}^r_\omega(M^2)$ $(r \geq 4)$ is said to be a generic elliptic point if both eigenvalues λ and λ^{-1} of Df^n_P sit on the unit circle without resonances of order less than or equal to 3, i.e. $|\lambda| = 1$ with $\lambda^2 \neq 1$ and $\lambda^3 \neq 1$, and the first coefficient of the Birkhoff normal form of f^n at point P is non-zero. Under these non-resonance conditions, the Birkhoff normal form theorem says that after some smooth symplectic change of coordinates mapping point P to origin, the diffeomorphism f^n takes the form

$$f^n(r\cos\theta, r\sin\theta) = (r\cos(\theta+\alpha+\beta r^2),\ r\sin(\theta+\alpha+\beta r^2)) + O(r^4),$$

where $\lambda = e^{i\alpha}$, and β is a symplectic invariant of f^n at the fixed point P, the so-called Birkhoff normal form first coefficient. If $\beta \neq 0$ and $r \geq 5$, then Moser's theorem applies, giving the existence of an invariant set Σ, with full Lebesgue density at P, which is a union of invariant curves. In each of these curves the map f^n is conjugated to an irrational rotation of the circle. This structure around P is usually described in the literature as an 'elliptic isle'.

A class-C^r function $(\mu, x) \mapsto f_\mu(x)$ defined on $I \times M^2$, where I is an interval of real numbers, with values in M^2 and such that $f_\mu \in \mathrm{Diff}^r(M^2)$ for all $\mu \in I$ is called a class-C^r one-parameter family of diffeomorphisms. If $f_\mu \in \mathrm{Diff}^r_\omega(M^2)$ for all $\mu \in I$, we say that $\{f_\mu\}_\mu$ is one-parameter family of symplectic maps.

We say that a family f_μ unfolds generically an orbit of homoclinic quadratic tangencies at $(\mu_0, Q_0) \in I \times M^2$ associated with some hyperbolic periodic point P if, denoting by P_μ the analytic continuation of P for the map f_μ, the following hold.

(1) $W^s(P, f_{\mu_0})$ and $W^u(P, f_{\mu_0})$ have a quadratic tangency at Q_0.

(2) If ℓ is any smooth curve that is transverse to $W^s(P, f_{\mu_0})$ and $W^u(P, f_{\mu_0})$ at Q_0, then the local intersections of $W^s(P_\mu, f_\mu)$ and $W^u(P_\mu, f_\mu)$ with ℓ cross each other with relative non-zero velocity at (μ_0, Q_0).

Let f_μ be a one-parameter family of maps in $\mathrm{Diff}^r(M^2)$. Take a parameter interval $\Delta \subseteq \mathbb{R}$, and let $\{\Lambda_\mu\}_{\mu\in\Delta}$ be a continuous family of basic sets. This means that for each $\mu \in \Delta$, Λ_μ is a hyperbolic basic set of f_μ and, furthermore, the correspondence $\mu \mapsto \Lambda_\mu$ is continuous with respect to Hausdorff distance. It follows that all basic sets Λ_μ are conjugated to each other. We say that the Λ_μ are wild basic sets over Δ if for all $\mu \in \Delta$, there is at least one orbit of homoclinic quadratic tangencies of Λ_μ which unfolds generically with μ; we shall also say that Δ is a Newhouse interval for the basic sets Λ_μ. More strongly, we will say that the basic sets Λ_μ are C^r-stably-wild basic sets over Δ if they are wild basic sets over Δ for all class-C^r one-parameter families uniformly close to f_μ. Uniform proximity of one-parameter families refers to the following distance. The topology of the group $\mathrm{Diff}^r(M^2)$ is clearly metrizable. Taking any metric d_{C^r} inducing the topology of $\mathrm{Diff}^r(M^2)$, we define the distance between one-parameter families of maps in $\mathrm{Diff}^r(M^2)$ by

$$d(\{f_\mu\}_\mu, \{g_\mu\}_\mu) = \sup_{\mu\in I} d_{C^r}(f_\mu, g_\mu).$$

The parametric version of Proposition 1 is obtained in a similar way.

PROPOSITION 2. *Let Λ be a wild hyperbolic set over an interval Δ for a one-parameter family of maps $f_\mu \in \mathrm{Diff}^r_\omega(M^2)$, where $r \geq 4$. Then the following holds.*

(1) *Given any periodic point $P \in \Lambda$, there is a dense subset $D \subseteq \Delta$ such that for every $\mu \in D$, the periodic point P_μ has an orbit of homoclinic tangencies.*

(2) *There is a residual subset $R \subseteq \Delta$, i.e. a countable intersection of open subsets dense in Δ, such that for every $\mu \in R$, the basic set Λ_μ is contained in the closure of all generic elliptic periodic points of f_μ.*

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The *conservative Hénon family* is the family of area-preserving maps $H_a : \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$H_a(x, y) = (y, -x + a - y^2). \tag{1}$$

Note that when $a = -1$ the Hénon map has a parabolic fixed point at $x = y = -1$ which splits into two fixed points, a saddle $O_s = (-1 - \sqrt{1+a}, -1 - \sqrt{1+a})$ and an elliptic point $O_e = (-1 + \sqrt{1+a}, -1 + \sqrt{1+a})$, for $a > -1$. It has been proved in [**7**] that for all $a > -1$ sufficiently close to the bifurcation value $a = -1$, the saddle fixed point O_s has a transverse homoclinic orbit. This transversality implies the existence of a hyperbolic set for all values of a just after the bifurcation point. We examine the structure of this set, following the arguments in [**3**] and using asymptotics for the homoclinic angle from [**7**]. Specifically, we prove that for all $a > -1$ close to the bifurcation point, the Hénon map has a C^2-stably-wild binary horseshoe which includes the saddle point O_s. This theorem is the main result of the paper.

THEOREM A. *The Hénon map family (1) has a sequence of Newhouse intervals* Δ_k *associated with* C^2*-stably-wild horseshoes containing the saddle fixed point* O_s*. The sequence* Δ_k *converges to the bifurcation value* $a = -1$ *as* $k \to +\infty$*.*

THEOREM B. *Let* f_μ *be a class-*C^r *one-parameter family of symplectic maps in* $\mathrm{Diff}^r_\omega(M^2)$ $(r \geq 6)$*. Let* O *be a periodic hyperbolic orbit and* Γ *an orbit of quadratic homoclinic tangencies of* f_0 *which unfolds generically at* $\mu = 0$*. Take any small neighborhood* U *of* $O \cup \Gamma$*. Then there is a sequence of Newhouse intervals* Δ_k *converging to* $\mu = 0$*. Each Newhouse interval* Δ_k *is associated with a* C^2*-stably-wild hyperbolic basic set* Λ_k *such that* $O \subseteq \Lambda_k \subseteq U$*.*

Theorem B follows from Theorem A. As explained in [**4**, §4], the argument uses a standard technique for renormalizing the dynamics at the unfolding of a homoclinic tangency, with the Hénon map showing up in the limit process. The conservative two-dimensional case of this renormalization process is treated in [**9**], based on Shil'nikov coordinates. For simplicity, in [**4**] we have assumed all maps to be of class C^∞, but class C^r with $r \geq 6$ is actually enough. If f_μ is a class-C^r one-parameter family of maps unfolding a homoclinic tangency, the renormalized maps converge to the Hénon map family in the C^{r-4} topology, as shown in [**9**]. Letting $r \geq 6$, this guarantees at least C^2 convergence to the Hénon family, which ensures that the C^2-stably-wild basic sets of the Hénon map still persist in the renormalized dynamics at the homoclinic tangency unfolding.

COROLLARY. *Under the same assumptions there is a non-meager set of parameters* R*, i.e. a set which is not a countable union of nowhere dense subsets of* $\mathbb{R}$*, having the homoclinic bifurcation parameter value* $\mu = 0$ *as an accumulation point, such that for every* $\mu \in R$*, the closure of generic elliptic periodic points of* f_μ *contains a wild basic set* Λ_k *including the periodic orbit* O*.*

This corollary follows from Proposition 2 and Theorem B.

Let us now make precise the construction of the wild set for the Hénon map. Stable-wildness comes from a large 'thickness' condition. The notion of thickness of a hyperbolic basic set Λ of a two-dimensional C^2-diffeomorphism, denoted by $\tau(\Lambda)$, was introduced by Newhouse who proved the following.

THEOREM. (Newhouse) *Let Λ be a hyperbolic basic set of a diffeomorphism $f \in \mathrm{Diff}^2(M^2)$. Assume $\tau(\Lambda) = \tau^s(\Lambda)\tau^u(\Lambda) > 1$ and that some periodic point $P \in \Lambda$ has an orbit Γ of quadratic homoclinic tangencies. Finally, let f_μ be a one-parameter family of maps in $\mathrm{Diff}^2(M^2)$, with $f_0 = f$, that unfolds generically the orbit of homoclinic tangencies Γ. Then there are parameter intervals over which Λ is a C^2-stably-wild basic set.*

Next we describe the mechanism introduced in [**13**] to prove the existence of C^2-stably-wild hyperbolic sets. Let Λ be a hyperbolic basic set such that at some point H there is a tangency between stable and unstable leaves of Λ. Consider the Cantor-like foliations $\mathcal{F}^s = W^s_{\mathrm{loc}}(\Lambda)$ and $\mathcal{F}^u = W^u_{\mathrm{loc}}(\Lambda)$ and iterate them backward and forward, respectively, until they meet at H. Extend these iterated foliations C^1-smoothly to a neighborhood of H; then there is a C^1 curve ℓ through H that consists of tangencies between these extended foliations. Consider the Cantor sets K^s and K^u formed by the points where, respectively, the first backward iteration of $\mathcal{F}^s$ and the first forward iteration of $\mathcal{F}^u$ intersect the curve ℓ. By definition of ℓ, Λ has a 'homoclinic' tangency at each point in $K^s \cap K^u$. In this construction, persistent homoclinic tangencies of Λ are equivalent to persistent intersections between the Cantor sets K^u and K^s. The device used to guarantee the 'persistent intersections' is the concept of *thickness* $\tau(K)$ of a one-dimensional Cantor set K, lying inside some curve ℓ, which we shall define below. Let us call each connected component of the complement $I - K$, where I is the interval spanned by K (i.e. the smallest closed connected subset of ℓ containing K), a *gap* of K. Roughly, the thickness of a Cantor set measures the relative size of its gaps, with large thickness corresponding to small gaps. The following intersection criterion holds.

GAP LEMMA. *Let K^s, K^u be two Cantor sets in the same open curve ℓ such that the intervals spanned by K^s and K^u intersect, but K^s is not contained inside a gap of K^u and neither is K^u contained inside a gap of K^s. If*

$$\tau(K^s)\tau(K^u) > 1, \tag{2}$$

then the Cantor sets intersect, i.e. $K^s \cap K^u \neq \emptyset$.

Of course, (2) is a stable condition only if we have continuity of thickness; in fact, it was proved in [**15**] that for dynamically defined Cantor sets, such as K^s and K^u in the previous context, thickness does depend continuously on the map, for the C^2 topology. Later, in [**10**], the new concepts of *left thickness* $\tau_L(K)$ and *right thickness* $\tau_R(K)$ of a Cantor set K were introduced, together with a remark that the hypothesis (2) in the gap lemma can be replaced by the weaker pair of conditions

$$\tau_L(K^s)\tau_R(K^u) > 1 \quad \text{and} \quad \tau_R(K^s)\tau_L(K^u) > 1. \tag{3}$$

The usual definition of thickness, or lateral thicknesses, is strictly geometric and can be applied to any compact set lying on a curve. Of course, to have continuity we must restrict to dynamically defined Cantor sets. Here, as in [**3**], we will adopt a more dynamical definition of thickness which only applies to dynamically defined Cantor sets. This slightly different definition is not equivalent to the usual geometric one. Nevertheless, the same results, namely the continuity of thickness and the gap lemma, still hold. Finally, and because this will suffice for our purposes, we shall restrict the scope of our definitions to *binary* Cantor sets and horseshoes, although these definitions can easily be generalized to arbitrary combinatorics.

By a *binary Cantor set* we mean any pair (K, ψ), where K is a Cantor subset of an open curve I and $\psi : I_0 \cup I_1 \to I$ is a C^1 expanding map defined on the union $I_0 \cup I_1$ of two subintervals of I, such that the restriction of ψ to $K = \bigcap_{n\geq 0} \psi^{-n}(I_0 \cup I_1)$ is topologically conjugated to the Bernoulli shift $\sigma : \{0, 1\}^{\mathbb{N}} \to \{0, 1\}^{\mathbb{N}}$. We may assume that I is the interval spanned by K and that for each $i = 0, 1$, I_i is the interval spanned by $K \cap I_i$; then $\{I_0, I_1\}$ is a Markov partition for (K, ψ). The gaps of (K, ψ) are ordered in the following way. Let us call the intervals spanned by the Cantor set components

$$K(a_0, \ldots, a_n) = \bigcap_{i=0}^{n} \psi^{-i}(K \cap I_{a_i}),$$

where $(a_0, \ldots, a_n) \in \{0, 1\}^{n+1}$, the *covering intervals of order* n. Then I_0 and I_1 are the covering intervals of order zero. $U_0 = I - (I_0 \cup I_1)$ is said to be the *gap of order* 0. In general, the components of the complement in I of the union of all covering intervals of order less than or equal to n which are not gaps of order less than or equal to $n - 1$ are called *gaps of order* n. It is easy to check that every gap is obtained by this procedure and therefore has some definite order.

The definitions below of left thickness and right thickness require the curve I to be oriented. Given a gap U of K, we denote by L_U, respectively R_U, the unique covering interval with the same order as U that is left-, respectively right-, adjacent to U. The greatest lower bounds

$$\tau_L(K, \psi) = \inf\left\{ \frac{|L_U|}{|U|} : U \text{ is a gap of } K \right\},$$
$$\tau_R(K, \psi) = \inf\left\{ \frac{|R_U|}{|U|} : U \text{ is a gap of } K \right\},$$
$$\tau(K, \psi) = \min\{\tau_L(K, \psi), \tau_R(K, \psi)\},$$

are called the *left thickness*, the *right thickness* and the *thickness* of (K, ψ), respectively. $|U|$ denotes the length of an interval $U \subseteq I$. These three thicknesses are continuous functions of (K, ψ) over the space of all $C^{1+\alpha}$ binary Cantor sets ($\alpha > 0$) with its natural $C^{1+\alpha}$ topology. It was remarked in [**10**] that the lateral thicknesses may be discontinuous for non-binary Cantor sets. However, with our 'dynamical' definition the lateral thicknesses are always continuous. The same argument as for usual thickness applies; see, for instance, [**16**]. To prove that the gap lemma, with condition (3) replacing (2), still holds, just follow the proof in [**10**] but argue that one could obtain pairs of linked gaps of ever higher order, rather than ever smaller lengths. The conclusion is then the same because as we consider gaps with strictly increasing order, their lengths converge to zero.

Let us say that the binary Cantor set (K, ψ) is *positive* when the restriction of ψ to each interval I_i $(i = 0, 1)$ preserves orientation. In order to estimate lateral thicknesses notice that, by the (orientation-preserving) self-similarity property of positive binary

Cantor sets, the ratio $|L_U|/|U|$, respectively $|R_U|/|U|$, is, up to a distortion factor, equal to $\tilde{\tau}_L(K, \psi) := |I_0|/|U_0|$, respectively $\tilde{\tau}_R(K, \psi) := |I_1|/|U_0|$. We shall refer to $\tilde{\tau}_L(K, \psi)$ and $\tilde{\tau}_R(K, \psi)$ as *top scale thicknesses* of (K, ψ). For affine Cantor sets, where the distortion factor is one, we have $\tilde{\tau}_L = \tau_L$ and $\tilde{\tau}_R = \tau_R$. In general, if distortion is small, then the top scale thicknesses $\tilde{\tau}_L$ and $\tilde{\tau}_R$ are good approximations of τ_L and τ_R, respectively. Lateral thicknesses are useless for *non-positive* binary Cantor sets, because in this case both left and right thicknesses equal the usual thickness.

A *binary horseshoe* is any pair (Λ, T) such that $T : S_0 \cup S_1 \to \mathbb{R}^2$ is a one-to-one local diffeomorphism of class C^2, where S_0 and S_1 are disjoint compact rectangles (up to diffeomorphism), and $\Lambda = \bigcap_{n\in\mathbb{Z}} T^{-n}(S_0 \cup S_1)$ is a hyperbolic basic set conjugated to the Bernoulli shift $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$. For each $i = 0, 1$ there is a unique fixed point $P_i \in S_i$, and we assume $\{S_0, S_1\}$ to form a Markov partition bounded by pieces of stable and unstable manifolds of the fixed points P_0 and P_1. When both fixed points have positive eigenvalues we say that (Λ, T) is a *positive binary horseshoe*.

Let $\mathcal{F}^s = W^s_{\text{loc}}(\Lambda) \cap (S_0 \cup S_1)$ and $\mathcal{F}^u = W^u_{\text{loc}}(\Lambda) \cap (T(S_0) \cup T(S_1))$. These sets may be seen as Cantor-like foliations where the leaves are just the connected components of the sets $\mathcal{F}^s$ and $\mathcal{F}^u$; they extend to C^1-foliations over $S_0 \cup S_1$ and $T(S_0) \cup T(S_1)$, respectively. The two foliations are transverse to each other. Pick the leaves I^s_* in $\mathcal{F}^s$ and I^u_* in $\mathcal{F}^u$ containing the fixed point P_0. Then the Cantor sets $\Lambda^s = \Lambda \cap I^u_*$ and $\Lambda^u = \Lambda \cap I^s_*$ can be identified, respectively, with the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ via projections $\pi_s : \mathcal{F}^s \to \Lambda^s$ and $\pi_u : \mathcal{F}^u \to \Lambda^u$ whose fibers are precisely the leaves of the respective Cantor-like foliations. The map $\psi^s : \Lambda^s \to \Lambda^s$, $\psi^s = \pi_s \circ T$ describes the action of T on the foliation $\mathcal{F}^u$; and the map $\psi^u : \Lambda^u \to \Lambda^u$, $\psi^u = \pi_u \circ T^{-1}$ describes the action of T^{-1} on the foliation $\mathcal{F}^s$. Moreover, ψ^u and ψ^s extend as C^1 expanding maps to $I^s_* \cap T(S_0) \cup I^s_* \cap T(S_1)$ and $I^u_* \cap S_0 \cup I^u_* \cap S_1$, respectively.

Given a positive binary horseshoe (Λ, T), we orient the invariant local separatrices of P_0, I^s_* and I^u_* so that orbits flow in the positive direction. These orientations in I^s_* and I^u_* induce orientations in all leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$, and also induce transverse orientations to these foliations. We remark that if we had chosen the other fixed point P_1, then all these orientations would be reversed. Finally, notice that (Λ^s, ψ^s) and (Λ^u, ψ^u) are positive binary Cantor sets. We define the *left–right thickness* of (Λ, T) as

$$\tau_{LR}(\Lambda, T) = \min\{\tau_L(\Lambda^s, \psi^s)\tau_R(\Lambda^u, \psi^u),\ \tau_L(\Lambda^u, \psi^u)\tau_R(\Lambda^s, \psi^s)\}.$$

Once again, if we fix orientations with respect to the second fixed point P_1, then the left and right thicknesses of Cantor sets (Λ^s, ψ^s) and (Λ^u, ψ^u) would be exchanged but the left–right thickness of (Λ, T) would stay unchanged. We define the *top scale left–right thickness* of (Λ, T) to be

$$\tilde{\tau}_{LR}(\Lambda, T) = \min\{\tilde{\tau}_L(\Lambda^s, \psi^s)\tilde{\tau}_R(\Lambda^u, \psi^u),\ \tilde{\tau}_L(\Lambda^u, \psi^u)\tilde{\tau}_R(\Lambda^s, \psi^s)\}.$$

As before, when the stable and unstable distortions of (Λ, T) are both small, $\tilde{\tau}_{LR}(\Lambda, T)$ approximates $\tau_{LR}(\Lambda, T)$ well.

From standard distortion estimates, see [**15, 16**], it can proved that this thickness depends continuously on (Λ, T) in the C^2 topology. We can now prove the following key result.

PROPOSITION 3. *Let $f_\mu : M^2 \to M^2$ be a one-parameter family of class-C^2 symplectic maps, and let (Λ_μ, T_μ) be a family of positive binary horseshoe maps defined on the union of two smooth rectangles $S_0(\mu) \cup S_1(\mu) \subseteq M^2$ as*

$$T_\mu(x) = \begin{cases} f_\mu(x) & \text{if } x \in S_0(\mu), \\ (f_\mu)^N(x) & \text{if } x \in S_1(\mu), \end{cases} \qquad N \geq 1.$$

Suppose that at $\mu = 0$, $\tau_{LR}(\Lambda_0, T_0) > 1$ and the invariant manifolds of a fixed point $O = f_\mu(O) \in \Lambda_\mu$ unfold generically an orbit of quadratic homoclinic tangencies. Then there is a sequence of parameter intervals Δ_k, accumulating at $\mu = 0$, such that Λ_μ is C^2-stably-wild over each Δ_k.

Proof. We orient the stable and unstable branches of $W^s(O) - O$ and $W^u(O) - O$ so that orbits flow in the positive direction. Let us say that a homoclinic tangency of O is *positive* if the orientations on the stable and unstable branches agree at the point of tangency.

Assume first that the homoclinic tangency of O, which by hypothesis unfolds generically at $\mu = 0$, is a positive one. As before, let H denote one homoclinic point in this orbit of tangencies, and let ℓ be the curve through H of tangencies between the C^1-extensions of the backward and forward iterations of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively. Again, let K^s and K^u be the Cantor sets formed by the points where the first backward and first forward iteration of $\mathcal{F}^s$ and $\mathcal{F}^u$, respectively, intersect the curve ℓ. By hypothesis, the condition (3) is fulfilled for the Cantor sets Λ^s and Λ^u. But locally K^s and K^u are the images, by the holonomies along the stable and unstable foliations, of the Cantor sets Λ^s and Λ^u, respectively. Both holonomies take the point O to H. Consider now the components $\Lambda^s_n = \Lambda^s(0, \dots, 0)$ and $\Lambda^u_n = \Lambda^u(0, \dots, 0)$, of order n, in the binary Cantor sets Λ^s and Λ^u, respectively. These are small neighbourhoods of O in Λ^s and Λ^u. They are both binary Cantor sets which obviously satisfy $\tau_L(\Lambda^t_n) \geq \tau_L(\Lambda^t)$ and $\tau_R(\Lambda^t_n) \geq \tau_R(\Lambda^t)$ for $t = s, u$. Thus these small Cantor sets also satisfy condition (3).

For each n, let K^s_n and K^u_n be the full images, by the holonomies, of the Cantor sets Λ^s_n and Λ^u_n. These images are also binary Cantor sets. To estimate their thicknesses, observe that these holonomies are maps of class C^1, and therefore they are almost linear, with very small distortion, near O. Since the map φ_μ preserves orientation, the iterations of the foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ around H inherit the orientations from $\mathcal{F}^s$ and $\mathcal{F}^u$; moreover, since the tangency at H is positive, their transversal orientations agree along the curve ℓ. But this means that the holonomy maps transforming Λ^s_n onto K^s_n and Λ^u_n onto K^u_n preserve orientation. Therefore, taking n large enough, the open condition (3) will still be satisfied by the Cantor sets K^s_n and K^u_n. This means that for some parameter interval Δ with $0 \in \partial\Delta$, for all $\mu \in \Delta$ the two intervals spanned by K^s_n and K^u_n each have a boundary point interior to the other and, furthermore, the Cantor sets K^s_n and K^u_n fulfil condition (3). Since these are open conditions, for all maps that are C^2-close to some φ_μ with $\mu \in \Delta$, we may apply the gap lemma to show that the horseshoe corresponding to Λ_μ has some 'homoclinic' tangency due to an intersection in $K^s \cap K^u$. Therefore Λ_μ is C^2-stably-wild over Δ.

This completes the proof in the case where the homoclinic tangency H is positive. If it is not positive, then arguing as in [**16**, Theorem 1 of §3.1], one can easily prove that the bifurcation parameter $\mu = 0$ is accumulated by two alternating sequences μ^+_k and μ^-_k where positive and negative quadratic homoclinic tangencies are unfolded near H.

Applying the previous case to parameters μ_k^+, there is a sequence of small intervals Δ_k, with $\mu_k^+ \in \partial\Delta_k$, such that Λ_μ is C^2-stably-wild over Δ_k. □

Theorem A follows by applying this abstract proposition to families of basic sets whose existence is stated in the next lemma.

LEMMA A. *For each $n \geq 4$, there is a continuous family of hyperbolic basic sets $\Lambda_n = \Lambda_n(a)$ for the Hénon map (1), defined in a small parameter interval Δ_n, such that:*

(1) *the sequence of intervals Δ_n converges to $a = -1$;*

(2) *the union of all intervals Δ_n covers $]-1, -1/2]$;*

(3) *each basic set Λ_n contains the fixed point O_s;*

(4) $\lim_{n\to+\infty} \tau_{LR}(\Lambda_n) = +\infty$.

Next, we provide a rough sketch of the construction of basic sets and give the heuristics behind the thickness asymptotics.

The Hénon map is reversible with respect to the canonical involution $I(x, y) = (y, x)$. Recall that a symplectic diffeomorphism $f : M^2 \to M^2$ is called *reversible* if there is a smooth map $I : M^2 \to M^2$ such that $I \circ I = \mathrm{Id}_{M^2}$ and $I^*\omega = -\omega$ (where ω denotes the area form in M^2) which conjugates f with its inverse, i.e. $f \circ I = I \circ f^{-1}$. The map I is called an *involution*. A set which is invariant with respect to f and I is called a *symmetric f-invariant set*. A periodic orbit is called symmetric if, viewed as a set, it is a symmetric invariant set. In particular, symmetric fixed points are common fixed points of f and I.

At $a = -1$ the Hénon family goes through a 'saddle-centre' bifurcation where a pair of symmetric fixed points is created: a saddle O_s and an elliptic point O_e. It was proved in [**1**] that for $a > -1$, the unstable manifold of the saddle O_s has a transverse intersection with the symmetry line $\mathrm{Fix}(I) = \{(x, y) : x = y\}$ at some point Ω. By reversibility, the stable manifold also intersects this symmetry line at Ω. The transversality of these intersections with $\mathrm{Fix}(I)$ implies that the symmetric homoclinic point Ω depends analytically on the parameter $a > -1$; but this is not enough to guarantee the transversality of the intersection between the invariant manifolds at Ω. It follows from [**6**] that the splitting angle at this intersection must be an exponentially small function of $a + 1$ as $a \to -1$. The transversality was established in [**7**] where the authors give an asymptotic expression for the Lazutkin splitting invariant at Ω; see §7 for the definition of Lazutkin invariant.

From this transverse homoclinic intersection at Ω we can argue, as in the classical Birkhoff's theorem, that for each $a > -1$ the saddle O_s is accumulated by two sequences of symmetric periodic points Q_n and Q'_n with even period $2n$. The points Q_n and Q'_n, as well as their nth iterates, sit in the symmetry line $x = y$, close to O_s and Ω, respectively. Both periodic points are hyperbolic. The eigenvalues of Q_n are both positive, while those of Q'_n are negative. For n large enough, it is clear that the stable manifold of Q_n intersects transversely the unstable manifold of O_s, and the unstable manifold of Q'_n intersects transversely the stable manifold of O_s. Let S^n be the square bounded by the local invariant manifolds of O_s and Q_n; let S_0^n be the rectangle formed by points in S^n whose first iteration stays inside S^n; and finally let S_1^n be the rectangle of points in S^n which return to S^n after

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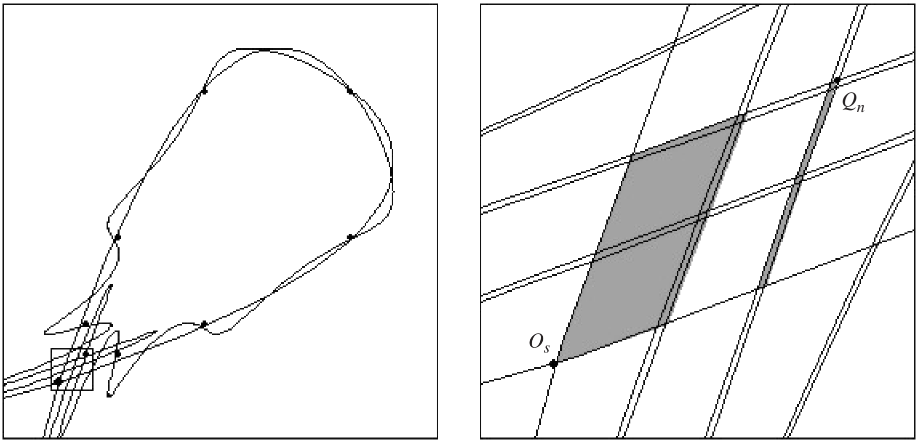

FIGURE 1. The binary horseshoe $T_4 : \Lambda_4 \to \Lambda_4$ for $\delta \approx 1.09$.

$2n$ iterations. For each $(x_0, y_0) \in S_0^n \cup S_1^n$, denote by $\{(x_i, y_i)\}$ the forward orbit of the Hénon map with this initial state, and define the map $T_n : S_0^n \cup S_1^n \to S^n$ by setting

$$T_n(x_0, y_0) = \begin{cases} (x_1, y_1) & \text{if } (x_0, y_0) \in S_0^n, \\ (x_{2n}, y_{2n}) & \text{if } (x_0, y_0) \in S_1^n. \end{cases}$$

Observe that S_0^n contains the fixed point O_s while S_1^n contains the periodic point Q_n. These two rectangles are bounded by the invariant manifolds of O_s and Q_n, and together they form a Markov partition for the binary horseshoe

$$\Lambda_n = \bigcap_{k \in \mathbb{Z}} (T_n)^{-k} (S_0^n \cup S_1^n);$$

see Figure 1. Since the eigenvalues of O_s and Q_n are positive, (Λ_n, T_n) is a positive binary horseshoe.

We now want to estimate the left–right thickness of Λ_n. Notice that as n tends to infinity each branch of the map T_n becomes more 'linear' while its distortion tends to zero. Consider the *vertical* rectangles S^n, S_0^n and S_1^n, and let w^n, w_0^n and w_1^n be their respective widths measured along the unstable direction. Let δ be the logarithm of the (larger) eigenvalue of the saddle O_s, and denote by λ_n the larger eigenvalue of Q_n. If n is large enough, then $\lambda_n^{-1} = o(\delta)$, and since T_n becomes almost linear in each branch, we have $w_0^n \sim w^n \, e^{-\delta}$ and $w_1^n \sim w^n \, \lambda_n^{-1}$. Therefore we obtain the following asymptotics for the left and right stable thicknesses of Λ_n:

$$\tau_L(\Lambda_n^s) \sim \frac{w_0^n}{w^n - w_0^n - w_1^n} = \frac{1}{e^\delta - 1 - \lambda_n^{-1}} = \mathcal{O}(\delta^{-1}),$$

$$\tau_R(\Lambda_n^s) \sim \frac{w_1^n}{w^n - w_0^n - w_1^n} = \frac{\lambda_n^{-1}}{1 - e^{-\delta} - \lambda_n^{-1}} = \mathcal{O}(\delta^{-1}\lambda_n^{-1}).$$

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By reversibility, the unstable thicknesses have the same values, hence we also get an asymptotic expression for the left–right thickness of Λ_n:

$$\tau_{LR}(\Lambda_n, T_n) \sim \mathcal{O}(\delta^{-2}\lambda_n^{-1}). \tag{4}$$

Thus, as $n \to \infty$, the map T_n becomes more linear with smaller distortion, but, along with this, the left–right thickness decreases to zero. So we need to compromise by choosing carefully the number of iterations n, which has to be large if we want small distortion, but not too large if we also want to keep the left–right thickness large. Denoting by $\theta = \theta_\delta$ the splitting angle at the symmetric homoclinic point Ω, we choose n so that $e^{2\,\delta\,n}\,\theta \sim \delta^{-3/2}$, which one can easily check to be the asymptotic value of λ_n. Thus, replacing λ_n by $\delta^{-3/2}$ in (4), we obtain $\tau_{LR}(\Lambda_n, T_n) \sim \delta^{-1/2}$ which tends to infinity as $a \to -1$, or as $\delta \to 0$. Of course, now we have to prove that for this particular value of n (depending on a or δ), the symmetric periodic saddle Q_n and the corresponding horseshoe Λ_n already exist. Moreover, we need to show that the distortion of (Λ_n, T_n) tends to zero when $a \to -1$.

Let us now outline the proof of Lemma A. The construction of Λ_n is carried out in Birkhoff coordinates. We rescale the Hénon maps, for $a > -1$, in order to make the distance between the fixed points O_s and O_e constant. This is done in the first part of §4. Define δ to be the logarithm of the eigenvalue at O_s. Parametrizing the rescaled mappings in δ, we obtain a family of maps close to the identity, $F_\delta = \mathrm{Id}_{\mathbb{R}^2} + \delta F_0 + \mathcal{O}(\delta^2)$, where F_0 is a quadratic Hamiltonian vector field with two fixed points—a saddle O_s and an elliptic point O_e. For this part we follow closely the construction in [**3**] for maps near the identity. The main assumption of [**3**, Theorem 1] is a bounding condition on the intersection geometry between the stable and unstable separatrices of the saddle point O_s. In our setting, this condition essentially comes from the asymptotics in [**7**] for the Lazutkin invariant at Ω, but the analytic dependence of Birkhoff coordinates on the parameter is also needed. For the latter, we show in §3 that for analytic families of symplectic maps near the identity (such as the ones above), coordinates exist that depend analytically on $\delta > 0$ and which reduce each map F_δ to its Birkhoff normal form over a fixed-size neighbourhood of O_s. Then, in the second part of §4, by working in Birkhoff coordinates we translate the asymptotics on the splitting angle in [**7**] into a condition on the C^2 geometry of the unfolding of the separatrices at $\delta = 0$. As mentioned above, this condition is the main assumption of [**3**, Theorem 1]; the rest of our construction closely follows the work in [**3**]. Distortion estimates, which are needed to estimate thickness, follow from [**3**, Theorem 2]. Unfortunately, the construction of Λ_n is not a logical consequence of results in [**3**]; some adaptations are needed, and in §5 we give some technical details of these adjustments. To finish this section, we provide a short description, a kind of road map, to help the reader navigate through §5 and [**3**].

In §5 we associate an integer $n = n(\delta)$, called the *half return time*, to each $\delta > 0$, such that $e^{2\delta n}\theta \sim \delta^{-3/2}$. For this value of n, the periodic point Q_n will be exponentially close to $O_s = (0, 0)$. More precisely, $Q_n \sim (\delta^{3/4}\sqrt{\theta_\delta}, \delta^{3/4}\sqrt{\theta_\delta})$. Since the scale of $\Lambda_{n(\delta)}$ shrinks as $\delta \to 0$, we perform one last rescaling which brings Q_n close to $(1, 1)$ and S^n close to a unit size square $[0, 1]^2$. Because the second derivatives of T_n, which are needed to estimate distortion, are scale-dependent, this 'normalization' is required in [**3**, Theorem 2]. Then, in these final coordinates we compute estimates for the derivatives of T_n.

(1) In the first branch S_0,

$$DT_n = \begin{pmatrix} e^{\delta} & 0 \\ 0 & e^{-\delta} \end{pmatrix} + \mathcal{O}(\delta^{3/2}\,\theta_\delta),$$

and all the second derivatives of T_n on this branch are of exponentially small order $\mathcal{O}(\delta^{3/2}\,\theta_\delta)$; Similar bounds hold for T_n^{-1} on $T_n(S_0)$.

(2) In the second branch S_1,

$$DT_n = \begin{pmatrix} -\delta^{-3/2} & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\delta^{-1}) & o(1) \\ o(1) & o(1) \end{pmatrix},$$

and the second derivatives of T_n on this branch are all uniformly bounded, except for the second derivative in the variable x of the first component of T_n, which is unbounded of order $\mathcal{O}(\delta^{-5/2})$; again, similar bounds hold for T_n^{-1} on $T_n(S_1)$.

From these asymptotics the construction of Λ_n follows easily. It is clear that the left–right thickness of Λ_n has, up to a distortion factor, order $\delta^{-1/2}$. Applying [**3**, Theorem 2], we obtain from (1) and (2) above that distortion is small, of order $\mathcal{O}(\delta^{1/2})$. This shows that distortion factors are close to 1. The half return time $n(\delta)$ is asymptotically equivalent to $-(\log(\delta^{3/2}\theta))/(2\delta)$, and tends to $+\infty$ as $\delta \to 0$. Thus, when $a \to -1$, we have $\delta \to 0$ and so $\tau_{LR}(\Lambda_n, T_n) \to +\infty$.

Finally, the lemma below is proved in §6.

LEMMA B. *There is a sequence of parameter values accumulating at $a = (-1)^+$, for which the saddle point O_s of the Hénon map has an orbit of quadratic homoclinic tangencies that unfolds generically with parameter a.*

Lemmas A and B, in view of Proposition 3, imply Theorem A.

3. *Analytic Birkhoff coordinates*

For area-preserving maps we have Siegel and Moser's theorem on the convergence of the Birkhoff normal form around a hyperbolic fixed point; see [**19**]. This theorem says that given an analytic area-preserving map $F : U \subseteq \mathbb{R}^2 \to \mathbb{R}^2$ defined in a neighbourhood U of $(0, 0)$ with a hyperbolic fixed saddle sitting at the origin such that $F(x, y) = (\lambda\, x + \cdots, \lambda^{-1}\, y + \cdots)$ where $|\lambda| \neq 1$, there exist an analytic change of coordinates $\zeta_F(x, y) = (x + \cdots, y + \cdots)$ defined in a neighbourhood of $(0, 0)$ and an analytic function $\alpha_F(\omega) = \log(\lambda) + \cdots$ of one variable ω defined in some neighbourhood of 0, such that for all (x, y) close to $(0, 0)$,

$$(F \circ \zeta_F)\,(x, y) = \zeta_F(e^{\alpha_F(xy)}\, x, e^{-\alpha_F(xy)}\, y). \tag{5}$$

The map $L_F : \mathbb{R}^2 \to \mathbb{R}^2$, $L_F(x, y) = (e^{\alpha_F(xy)}\, x, e^{-\alpha_F(xy)}\, y)$, is called 'a Birkhoff normal form' for F.

Following Birkhoff, the maps $\zeta_F(x, y)$ and $\alpha_F(\omega)$ with $\omega = xy$ in this theorem are found as formal power series

$$\zeta_F(x, y) = \left(x + \sum_{n+m\geq 2} a_{nm}(F) x^n y^m,\ y + \sum_{n+m\geq 2} b_{nm}(F) x^n y^m \right), \tag{6}$$

$$\exp(\alpha_F(\omega)) = \lambda + \sum_{n=1}^{\infty} c_{2n}(F)\omega^n \tag{7}$$

which solve the conjugacy relation (5). The uniqueness of the formal solutions (6) and (7) of equation (5) is obtained by adding the following 'normalizing' condition for the formal solution (6):

$$a_{n+1\,n}(F) = b_{n\,n+1}(F) = 0 \quad \text{for all } n \geq 1. \tag{8}$$

These coefficients are obtained by recursive relations which involve the coefficients of the Taylor series of F. Therefore each $a_{km}(F)$, $b_{km}(F)$ or $c_{2m}(F)$ is a polynomial in the Taylor coefficients of F. By taking the weak topology (of pointwise convergence of coefficients) in the space of formal series, this proves that the formal solutions $\zeta_F(x, y)$ and $\alpha_F(\omega)$ depend continuously on F.

Unfortunately, the formal transformation $\zeta_F(x, y)$ does not, formally speaking, preserve area; equation (7) is not the appropriate normalizing condition. The reason we consider this condition is that it makes the convergence proof much easier. In any case, the normal form thus obtained (although not unique) is, at least formally, a reversible area-preserving map.

We now outline Siegel and Moser's convergence proof in order to justify why the maps ζ_F and α_F depend continuously, even analytically, on F.

Let U be an open neighbourhood of the origin in $\mathbb{C}^2$. Consider the set of systems $\mathcal{S} = \mathcal{S}(U)$ formed by all holomorphic maps $F \in \mathcal{H}(U, \mathbb{C}^2)$ of the form $F(x, y) = (\lambda x + \ldots, \lambda^{-1} y + \cdots)$ for some $\lambda \in \mathbb{C}$ with $|\lambda| \neq 1$, and endow $\mathcal{S}$ with the topology of $\mathcal{H}(U, \mathbb{C}^2)$†.

Some definitions are required. Let $t(x, y) = \sum_{n=0}^{\infty} \alpha_{nk}\, x^n y^k$ and $s(x, y) = \sum_{n=0}^{\infty} \beta_{nk}\, x^n y^k$ be formal power series in the variables x and y. We say that $s(x, y)$ dominates $t(x, y)$, and write $t(x, y) \prec s(x, y)$, if for all $n, k \in \mathbb{N}$, $|\alpha_{nk}| \leq \beta_{nk}$. The relation $\prec$ partially orders the set $\mathbb{R}^+(x, y)$ of all formal series with non-negative coefficients, which we shall call *positive* formal series. A similar definition is given for formal power series in a single variable.

Let $F_0 \in \mathcal{S}$ be given. Choose $c > 0$ sufficiently large so that the closed polydisc $\Delta_{c^{-1}} = \overline{D}(0, c^{-1}) \times \overline{D}(0, c^{-1})$ is contained inside the domain U. Take a neighbourhood $\mathcal{U}$ of F_0 in $\mathcal{S}$ such that all maps $F \in \mathcal{U}$ are uniformly bounded on $\Delta_{c^{-1}}$ by some constant $M > 0$. Then the Taylor coefficients of all $F \in \mathcal{U}$ are bounded by the sequence $\{Mc^n\}$ and we may assume, by taking a larger c if necessary, that $M = 1$.

Consider the positive, convergent, formal series

$$G_c(x, y) = \frac{c(x+y)^2}{1 - c(x+y)} = \sum_{n=2}^{\infty} c^{n-1}(x+y)^n,$$

and note that both components of the second-order Taylor remainder of any $F \in \mathcal{U}$ are dominated by $G_c(x, y)$.

Write $\zeta_F(x, y) = (\varphi_F(x, y), \psi_F(x, y))$ and $\lambda_F(\omega) = \exp\{\alpha_F(\omega)\}$. Then a positive formal series $W_F(\omega) = \sum_{n=1}^{\infty} w_n(F)\omega^n$, with zero constant term, is constructed from the data $\varphi_F(x, y)$, $\psi_F(x, y)$ and $\lambda_F(\omega)$ which satisfy the dominance relations

† This is the space of holomorphic functions on the open set U with the natural topology of uniform convergence on compact subsets of U.

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$$\varphi_F(x, y) - x \prec (x + y) W_F(x + y),$$
$$\psi_F(x, y) - y \prec (x + y) W_F(x + y),$$
$$\lambda_F(x\, y)^{-1} - \lambda^{-1} \prec W_F(x + y)$$

and the following 'dominance equation'

$$\omega\, W_F(\omega) \prec \frac{c_1}{1 - c_2\, W_F(\omega)}\, G_c(\omega(W_F(\omega) + 1), \omega(W_F(\omega) + 1)). \tag{9}$$

Proving this fact is the main step in Siegel and Moser's theorem, but we skip its proof here. Moving on, equation (9) easily implies

$$W_F(\omega) \prec \frac{c_3\, \omega(1 + W_F)^2}{1 - c_2\, W_F - 2\, c\, \omega(1 + W_F)}, \tag{10}$$

where $W_F = W_F(\omega)$ and c_1, c_2 and c_3 are positive constants defined by open conditions that depend only on λ and c. Consider now the *quadratic* fixed point equation in U,

$$U(\omega) = \frac{c_3\, \omega(1 + U)^2}{1 - c_2\, U - 2\, c\, \omega(1 + U)}, \tag{11}$$

which has two solutions that satisfy $U(0) = 1$ and $U(0) = 0$, respectively, with both being analytic in a neighbourhood of the origin. Let $U(\omega)$ be the second solution, which defines a formal series around the origin with zero constant term, and has a positive radius of convergence $r > 0$. This radius r may be computed explicitly from the values c, c_1, c_2 and c_3.

The coefficients u_n of $U(\omega) = \sum_{n=1} u_n\, \omega^n$ may be recursively computed from (11). Similarly, the coefficients $w_n(F)$ of $W_F(\omega) = \sum_{n=1} w_n(F)\omega^n$ can be recursively estimated from (10). Comparing the two sequences of coefficients, starting at $w_0(F) = 0 = u_0$, it can be shown inductively that $w_n(F) \le u_n$ for all $n \in \mathbb{N}$. The reason for this is that the right-hand side of (11), which is the same as in (10), expands as a positive power series in the variables ω and U.

Therefore $W_F(\omega) \prec U(\omega)$ for all $F \in \mathcal{U}$, and this proves that the family of formal series $\{\zeta_F(x, y)\}_{F \in \mathcal{U}}$ converges uniformly in $(F, x, y) \in \mathcal{U} \times \overline{D}(0, r/2)^2$. Analogously, the family $\{\lambda_F(\omega)^{-1}\}_{F \in \mathcal{U}}$ converges uniformly in $(F, \omega) \in \mathcal{U} \times \overline{D}(0, r)$.

Given a family $\{F_h : h \in \Omega\}$ of maps in $\mathcal{U}$, holomorphic in $(h, x, y) \in \Omega \times U$, the coefficients $a_{k\,m}(h) := a_{k\,m}(F_h)$, $b_{k\,m}(h) := b_{k\,m}(F_h)$ and $c_{2\,m}(h) := c_{2\,m}(F_h)$ are polynomials in the Taylor coefficients of F_h at $(x, y) = (0, 0)$, and so depend analytically on h. Therefore, with $F = F_h$, the partial sums of the formal power series (6) and (7) are holomorphic functions in $(h, x, y) \in \Omega \times \mathbb{C}^2$ and $(h, \omega) \in \Omega \times \mathbb{C}$, respectively. Thus the mappings $(h, x, y) \mapsto \zeta_{F_h}(x, y)$ and $(h, \omega) \mapsto \alpha_{F_h}(\omega)$, which are uniform limits of their partial sums over the domains $\Omega \times \overline{D}(0, r/2)^2$ and $\Omega \times \overline{D}(0, r)$, respectively, are also holomorphic in these domains.

The proof in [**19**] therefore shows the following.

THEOREM 1. (Siegel–Moser) *Given $F_0 \in \mathcal{S}(U)$, there is some $r > 0$ and an open set $\mathcal{U} \subseteq \mathcal{S}(U)$ containing F_0 such that:*

(1) *for every $F \in \mathcal{U}$, the formal series $\zeta_F(x, y)$ and $\alpha_F(\omega)$ given in (6) and (7), which are uniquely determined by (5) and the normalizing condition (8), converge uniformly to holomorphic functions defined on $D(0, r/2)^2$ and $D(0, r)$, respectively;*

(2) *the transformations $\zeta : \mathcal{U} \to \mathcal{H}(D(0, r/2)^2, \mathbb{C}^2)$, $F \mapsto \zeta_F$, and $\alpha : \mathcal{U} \to \mathcal{H}(D(0, r), \mathbb{C})$, $F \mapsto \alpha_F$ are both continuous;*

(3) *given a holomorphic function $F : \Omega \times U \to \mathbb{C}^2$, $F(h, x, y) = F_h(x, y)$ that defines a family of mappings $F_h \in \mathcal{U} \subseteq \mathcal{S}(U)$ in the complex parameter $h \in \Omega \subseteq \mathbb{C}$, the functions $\zeta(h;\ x, y) := \zeta_{F_h}(x, y)$ and $\alpha(h;\ \omega) := \alpha_{F_h}(\omega)$ are holomorphic in the domains $\Omega \times D(0, r/2)^2$ and $\Omega \times D(0, r)$, respectively.*

Remark 1. Although $\zeta_F(x, y)$ is not area-preserving, for all $|x|, |y| < r/2$ we have

$$\det D\zeta_F(x, 0) = 1 = \det D\zeta_F(0, y).$$

To simplify the notation, we will omit the subscript F in ζ_F. Since there is a 'symmetry' in the two relations above, it is enough to prove the first one. Differentiating (5) with respect to x and y at the point $(x, 0)$, one obtains the relations

$$DF_{\zeta(x,0)} \cdot \zeta_x(x, 0) = \lambda\, \zeta_x(\lambda x, 0),$$
$$DF_{\zeta(x,0)} \cdot \zeta_y(x, 0) = \lambda x^2\, \alpha'(0)\zeta_x(\lambda x, 0) \;+\; \lambda^{-1}\, \zeta_y(\lambda x, 0).$$

Therefore

$$\omega(\zeta_x(\lambda x, 0), \zeta_y(\lambda x, 0)) = \omega(\zeta_x(x, 0), \zeta_y(x, 0)),$$

and this relation holds for any power of λ. Taking negative powers λ^{-n} with $n \to +\infty$, we obtain

$$\det D\zeta_{(x,0)} = \omega(\zeta_x(x, 0), \zeta_y(x, 0)) = \omega(\zeta_x(0, 0), \zeta_y(0, 0)) = 1.$$

Remark 2. Given $F \in \mathcal{S}$, for all $n \in \mathbb{Z} - \{0\}$ we have

$$\zeta_{F^n}(x, y) = \zeta_F(x, y) \quad \text{and} \quad \alpha_{F^n}(\omega) = n\, \alpha_F(\omega).$$

In particular, all these formal series converge on the same domain.

Remark 3. Given a Hamiltonian vector field X with flow $\phi^t \in \mathcal{S}$ $(t \neq 0)$, define $\zeta_X := \zeta_{\phi^1}$ and $\alpha_X := \alpha_{\phi^1}$. Then for all $t \in \mathbb{R} - \{0\}$,

$$\zeta_{\phi^t}(x, y) = \zeta_X(x, y) \quad \text{and} \quad \alpha_{\phi^t}(\omega) = t\, \alpha_X(\omega).$$

Remark 4. Given $F \in \mathcal{S}$ that is reversible with respect to the canonical involution $I(x, y) = (y, x)$, i.e. $F \circ I = I \circ F^{-1}$, we have $\zeta_F \circ I = I \circ \zeta_F$. In particular, the time-reversing symmetries of F and L_F are conjugated by ζ_F.

Remarks 2 and 3 are quite obvious. Let us prove Remark 4 which follows from the uniqueness condition (8). Since

$$F^{-1} \circ (I \circ \zeta_F \circ I) = F^{-1} \circ I \circ \zeta_F \circ I = I \circ F \circ \zeta_F \circ I = I \circ \zeta_F \circ L_F \circ I$$
$$= I \circ \zeta_F \circ I \circ (L_F)^{-1} = (I \circ \zeta_F \circ I) \circ L_{F^{-1}}$$

and $I \circ \zeta_F \circ I$ trivially fulfils condition (8), it follows from Remark 2 that $I \circ \zeta_F \circ I = \zeta_{F^{-1}} = \zeta_F$; therefore $\zeta_F \circ I = I \circ \zeta_F$.

Remark 5. Given any $\alpha \in \mathcal{H}(D(0, r), \mathbb{C})$, and defining $W_r = \{(x, y) \in \mathbb{C}^2 : |xy| < r\}$, the normal form

$$L(x, y) = (e^{\alpha(xy)}\, x,\ e^{-\alpha(xy)}\, y)$$

defines a global holomorphic diffeomorphism of W_r onto W_r, with inverse

$$L^{-1}(x, y) = (e^{-\alpha(xy)}\, x,\ e^{\alpha(xy)}\, y).$$

Thus, if $\mathcal{U}$ and $r > 0$ are as in Theorem 1, then the maps $L^n : \mathcal{U} \to \mathcal{H}(W_r, \mathbb{C}^2)$, $F \mapsto L_{F^n} = (L_F)^n$, $n \in \mathbb{Z}$, are continuous.

Remark 6. Let α and L be as in Remark 5. The region

$$e^{-\alpha(xy)} < \left|\frac{x}{y}\right| < e^{\alpha(xy)}$$

is a *fundamental domain* for the restriction of L to the invariant set $W_r^* = \{(x, y) \in \mathbb{C}^2 : 0 < |xy| < r\}$. Assuming that $|\alpha(\omega)| < A$ for all $|\omega| < r$, it follows that every L orbit in W_r goes through the polydisc $D(0, \sqrt{re^A})^2$.

Remark 7. Given $r > 0$ and $\mathcal{U}$ as in Theorem 1, there is some r_1 such that for each $F \in \mathcal{U}$ we can extend ζ_F holomorphically to the domain W_{r_1}. Let $r_1 < r^2 e^{-A}/4$ where $|\alpha_F(\omega)| < A$ for all $|\omega| < r$. Then, by Remark 6, all L_F orbits in W_{r_1} must go through the domain $D(0, r/2)^2$ where $\zeta_F(x, y)$ is defined. Thus, given $(x, y) \in W_{r_1}$, we can take some $n \in \mathbb{Z}$ such that $(L_F)^{-n}(x, y) \in D(0, r/2)^2$, and define

$$\tilde{\zeta}_F(x, y) := (F^n \circ \zeta_F \circ (L_F)^{-n})\,(x, y).$$

By virtue of (5), $\tilde{\zeta}_F(x, y)$ is well-defined and holomorphic in W_{r_1}. The transformation $\tilde{\zeta} : \mathcal{U} \to \mathcal{H}(W_{r_1}, \mathbb{C}^2)$, $F \mapsto \tilde{\zeta}_F$ is easily seen to be continuous.

Consider an analytic family F_δ of area-preserving maps. Suppose all maps F_δ have holomorphic extensions to some open set $U \subseteq \mathbb{C}^2$ containing the origin, and the family F_δ is a perturbation of the identity, i.e.

$$F_\delta = \mathrm{Id}_{\mathbb{R}^2} + \epsilon F_1 + \mathcal{O}(\epsilon^2), \tag{12}$$

where $\epsilon = \epsilon(\delta)$ satisfies

$$\lim_{\delta \to 0} \epsilon(\delta)/\delta = c > 0. \tag{13}$$

The variation direction is that of the Hamiltonian vector field F_1, which also extends holomorphically to the same open set $U \subseteq \mathbb{C}^2$. Assume the origin is a diagonalized hyperbolic fixed point, so that

$$F_\delta(x, y) = (\lambda_\delta\, x + \cdots\ ,\ \lambda_\delta^{-1}\, y + \cdots), \tag{14}$$

where $\lambda_\delta = 1 + a\,\epsilon + \mathcal{O}(\epsilon^2)$ with $a \neq 0$, and

$$F_1(x, y) = (ax + \cdots\ ,\ -ay + \cdots), \tag{15}$$

the dots meaning terms in $x^i y^j$ of order $i + j \geq 2$. Each map F_δ is therefore in the class $\mathcal{S}(U)$. Finally, assume that the flow ϕ^t of the vector field F_1 extends to holomorphic maps $\phi^t : U \to \mathbb{C}^2$ for any real time t.

For each $\delta \neq 0$, we define the formal series

$$\begin{aligned} \zeta(\delta, x, y) = \zeta_\delta(x, y) &:= \zeta_{F_\delta}(x, y), \\ \alpha(\delta, \omega) = \alpha_\delta(\omega) &:= \alpha_{F_\delta}(\omega). \end{aligned} \tag{16}$$

For $\delta = 0$, let us set

$$\begin{aligned} \zeta(0, x, y) = \zeta_0(x, y) &:= \zeta_{F_1}(x, y), \\ \alpha(0, \omega) = \alpha_0(\omega) &:= 0, \end{aligned} \tag{17}$$

where $F_1 = (\partial F_\delta / \partial \delta)_{\delta=0}$.

In [**6**] Fontich and Simó proved that for some small δ_0 the series $\zeta_\delta(x, y)$ and $\alpha_\delta(\omega)$ converge, uniformly in $\delta \in [-\delta_0, \delta_0] - \{0\}$, over some fixed open domain around the origin. They went through the argument in [**19**] and checked that the coefficients $a_{nm}(\delta)$, $b_{nm}(\delta)$ and $c_n(\delta)$ can be uniformly bounded in δ. The next proposition is a direct corollary of Theorem 1 which generalizes [**6**, Proposition 3.1].

PROPOSITION 4. *Given a family F_δ as above, there are constants $r > 0$ and $\delta_0 > 0$ such that:*

(1) *for all $|\delta| < \delta_0$, the formal series $\zeta_\delta(x, y)$ and $\alpha_\delta(\omega)$ converge uniformly to holomorphic functions defined on $D(0, r/2)^2$ and $D(0, r)$, respectively;*
(2) *the maps $\zeta : [-\delta_0, \delta_0] \to \mathcal{H}(D(0, r/2)^2, \mathbb{C}^2)$, $\delta \mapsto \zeta_\delta$ and $\alpha : [-\delta_0, \delta_0] \to \mathcal{H}(D(0, r), \mathbb{C})$, $\delta \mapsto \alpha_\delta$ are both continuous;*
(3) *$\zeta(\delta, x, y) = \zeta_\delta(x, y)$ and $\alpha(\delta, \omega) = \alpha_\delta(\omega)$ are real analytic functions in the real domains $\{(\delta, x, y) : 0 < |\delta| < \delta_0,\ |x|, |y| < r/2\}$ and $\{(\delta, \omega) : 0 < |\delta| < \delta_0,\ |\omega| < r\}$, respectively.*
(4) *uniformly in $\omega \in D(0, r)$ we have*

$$\alpha_{F_1}(\omega) = \lim_{\delta \to 0} \frac{\alpha_\delta(\omega)}{\epsilon(\delta)}.$$

To prove this proposition we need a simple lemma. Given $t > 0$, denote by $[t/\epsilon]$ the integer part of $t/\epsilon(\delta)$. As before, let $\phi^t : U \to \mathbb{C}^2$ be the (real time) flow of the Hamiltonian vector field $F_1(x, y)$. For each compact set $K \subseteq U$, we consider the following seminorm

$$\|f\|_K = \max\{\|f(x, y)\| \mid (x, y) \in K\}.$$

LEMMA 1. *The following limit holds in $\mathcal{H}(U, \mathbb{C}^2)$ for any $t > 0$:*

$$\lim_{\delta \to 0} (F_\delta)^{[t/\epsilon(\delta)]} = \phi^t.$$

Proof. Since

$$\begin{aligned} F_\delta &= \mathrm{Id}_{\mathbb{C}^2} + \epsilon\, F_1 + \epsilon^2\, F_2 \quad \text{and} \\ \phi^\epsilon &= \mathrm{Id}_{\mathbb{C}^2} + \epsilon\, F_1 + \mathcal{O}(\epsilon^2), \end{aligned}$$

there is some constant $C > 0$, depending on K, such that $\|DF_\delta\|_K \leq 1 + C\epsilon$ and $\|F_\delta - \phi^\epsilon\|_K \leq C\epsilon^2$. Thus

$$\begin{aligned} \|F_\delta^n - \phi^{n\epsilon}\|_K &\leq \|F_\delta\, F_\delta^{n-1} - F_\delta\, \phi^{(n-1)\epsilon}\|_K + \|F_\delta\, \phi^{(n-1)\epsilon} - \phi^\epsilon\, \phi^{(n-1)\epsilon}\|_K \\ &\leq (1 + C\epsilon)\|F_\delta^{n-1} - \phi^{(n-1)\epsilon}\|_K + C\epsilon^2, \end{aligned}$$

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so, by induction,

$$\|F_\delta^n - \phi^{n\epsilon}\|_K \le C\epsilon^2 \,\frac{(1+C\epsilon)^n - 1}{C\epsilon} = \epsilon((1+C\epsilon)^n - 1).$$

Finally, by taking $n = [t/\epsilon]$ we get

$$\begin{aligned}\|F_\delta^n - \phi^t\|_K &\le \|F_\delta^n - \phi^{n\epsilon}\|_K + \|\phi^{n\epsilon} - \phi^t\|_K \\ &\le \epsilon e^{Ct} + \|\mathrm{Id}_{\mathbb{C}^2} - \phi^{t-n\epsilon}\|_K,\end{aligned}$$

and this converges to zero as $\delta \to 0$, since $\epsilon = \epsilon(\delta) \to 0$ and $|t - n\epsilon| \le \epsilon$. □

Proof of Proposition 4. The time-one map ϕ^1 of F_1 is in the class $\mathcal{S}(U)$. For this map ϕ^1, take $r > 0$ and $\mathcal{U}$ according to Theorem 1. By Lemma 1, there is $\delta_0 > 0$ such that for all $\delta \in [-\delta_0, \delta_0]$, $(F_\delta)^{[1/\epsilon]} \in \mathcal{U}$. Given $|\delta'| < \delta_0$, let us prove the continuity of ζ and α at $\delta = \delta'$. Suppose first that $\delta' \ne 0$, and fix $n = [1/\epsilon(\delta')]$. Of course, for all δ near δ', $(F_\delta)^n$ is in $\mathcal{U}$. By item (2) of Theorem 1, ζ and α are continuous on $\mathcal{U}$, and from Remark 2 we have $\zeta_\delta = \zeta_{(F_\delta)^n}$ and $\alpha_\delta = n^{-1}\,\alpha_{(F_\delta)^n}$; thus the continuity of $\delta \mapsto \zeta_\delta$ and $\delta \mapsto \alpha_\delta$ at $\delta = \delta'$ follows. Notice that we also obtain the analyticity of $\zeta(\delta, x, y)$ and $\alpha(\delta, \omega)$ at $\delta = \delta'$ from item (3) of Theorem 1.

On the other hand, again by Remark 2,

$$\begin{aligned}\zeta_\delta(x, y) &:= \zeta_{F_\delta}(x, y) = \zeta_{(F_\delta)^{[1/\epsilon]}}(x, y) \to \zeta_{\phi^1} =: \zeta_0,\\ \alpha_\delta(x, y) &:= \alpha_{F_\delta}(x, y) = \frac{\alpha_{(F_\delta)^{[1/\epsilon]}}(x, y)}{[1/\epsilon]} \to 0,\end{aligned}$$

which proves continuity at $\delta = 0$.

By Lemma 1 and the continuity of $\alpha : \mathcal{U} \to \mathcal{H}(D(0, r), \mathbb{C})$, we have

$$[1/\epsilon]\,\alpha_\delta := \alpha_{(F_\delta)^{[1/\epsilon]}} \to \alpha_{\phi^1} =: \alpha_{F_1}$$

as $\delta \to 0$, and therefore

$$\lim_{\delta\to 0} \frac{\alpha_\delta}{\epsilon(\delta)} = \lim_{\delta\to 0} \frac{1}{[1/\epsilon]\,\epsilon(\delta)}\; [1/\epsilon]\,\alpha_\delta = \alpha_{F_1}.$$ □

4. *Splitting of separatrices*

In [7] the authors call the family of maps $\tilde{F}_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\tilde{F}_\epsilon(x, y) = (x + \epsilon\,(y + \epsilon(x - x^2)),\; y + \epsilon(x - x^2)) \tag{18}$$

a conservative Hénon family. The relationship with our model (1) is given by the following rescaling affine mappings $\tau_\epsilon : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\tau_\epsilon(x, y) = \left(\frac{1}{2} + \frac{y+1}{\epsilon^2},\; \frac{y-x}{\epsilon^3}\right).$$

One can easily verify that for each $\epsilon \in \mathbb{R}$, by setting $a = -1 - \epsilon^4/4$ we have

$$\tau_\epsilon(y, a - x - y^2) = (\tilde{F}_\epsilon \circ \tau_\epsilon)\,(x, y).$$

The mappings $\tilde{F}_\epsilon$ are of course reversible for the corresponding involution $I_\epsilon = \tau_\epsilon \circ I \circ \tau_\epsilon^{-1}$, which is computed to be $I_\epsilon(x, y) = (x - \epsilon y, -y)$. We can write (18) in the form $\tilde{F}_\epsilon = \mathrm{Id}_{\mathbb{R}^2} + \epsilon \tilde{F}_1 + \epsilon^2 \tilde{F}_2$, where $\tilde{F}_1(x, y) = (y, x - x^2)$ and $\tilde{F}_2(x, y) = (x - x^2, 0)$; this shows that $\tilde{F}_\epsilon$ is a perturbation of the identity in the direction of the Hamiltonian vector field $\tilde{F}_1$. Notice that the origin corresponds to the saddle O_s for all maps $\tilde{F}_\epsilon$, with

$$\tilde{F}_\epsilon(x, y) = ((1 + \epsilon^2)x + \epsilon y + \cdots , \epsilon x + y + \cdots).$$

Denoting by $e^{\pm\delta}$ the eigenvalues of this saddle, we have

$$2 + \epsilon^2 = \mathrm{trace}\, D(\tilde{F}_\epsilon)_{(0,0)} = e^\delta + e^{-\delta} \tag{19}$$

with the following asymptotic relation at $\delta = 0$:

$$\epsilon = \delta - \frac{\delta^3}{24} + O(\delta^5). \tag{20}$$

Upon computing the eigenvectors of the linear part of $\tilde{F}_\epsilon$, we construct the 'linearizing matrix'

$$M_\epsilon = \frac{1}{2\sqrt[4]{4 + \epsilon^2}} \begin{pmatrix} \epsilon + \sqrt{4 + \epsilon^2} & -\epsilon + \sqrt{4 + \epsilon^2} \\ 2 & -2 \end{pmatrix}, \tag{21}$$

normalized to have determinant equal to -1, which diagonalizes $D(\tilde{F}_\epsilon)_{(0,0)}$ and transforms, via conjugacy, the family of involutions I_δ back to the canonical involution $I(x, y) = (y, x)$. Letting $F_\delta = M_\epsilon^{-1} \circ \tilde{F}_\epsilon \circ M_\epsilon$, where $\epsilon = \epsilon(\delta)$ is implicitly defined by (19), we have $F_\delta = \mathrm{Id}_{\mathbb{R}^2} + \epsilon F_1 + \mathcal{O}(\epsilon^2)$, with

$$F_1(x, y) = \left(x - \frac{\sqrt{2}}{4}(x + y)^2, -y + \frac{\sqrt{2}}{4}(x + y)^2 \right) \tag{22}$$

and

$$F_\delta(x, y) = (e^\delta x + \cdots , e^{-\delta} y + \cdots).$$

The map $I(x, y) = (y, x)$ is now the time-reversing involution for all F_δ. The family F_δ is again a perturbation of the identity, now in the direction of the Hamiltonian vector field F_1.

The vector fields $\tilde{F}_1$ and F_1 have homoclinic loops associated with the saddles sitting at the origin, which are described by the critical level equations of the Hamiltonians

$$\tilde{H}_1 = \frac{y^2}{2} - \frac{x^2}{2} + \frac{x^3}{3} = 0 \quad \text{and}$$
$$H_1 = \frac{(x - y)^2}{4} - \frac{(x + y)^2}{4} + \frac{(x + y)^3}{6\sqrt{2}} = 0.$$

These curves intersect the symmetry lines of the involutions I_0 and I at the symmetric homoclinic points $\tilde{\Omega} = (3/2, 0)$ and $\Omega = (3\sqrt{2}/4, 3\sqrt{2}/4)$, respectively. Let us now compute the corresponding homoclinic lengths, $\lambda(\tilde{\Omega})$ and $\lambda(\Omega)$; see §7 for the definition of *homoclinic length*. It is straightforward to check that the function $\tilde{\gamma} : \mathbb{R} \to \mathbb{R}^2$, $\tilde{\gamma}(t) = (x(t), y(t))$, where

$$x(t) = \frac{3}{2}\left(1 - \frac{(1 - t)^2}{(1 + t)^2}\right) \quad \text{and} \quad y(t) = x(t)\frac{(1 - t)^2}{(1 + t)^2}, \tag{23}$$

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linearizes the unstable curve of the saddle at the origin of $\tilde{F}_1$. By symmetry, $I_0 \circ \tilde{\gamma}$ also linearizes the stable curve. Thus, since $\tilde{\gamma}(1) = I_0(\tilde{\gamma}(1)) = (3/2, 0)$, the homoclinic length of the loop of $\tilde{F}_1$ is $\lambda(\tilde{\Omega}) = 6\sqrt{2}$, which is the square root of the area of the parallelogram spanned by the vectors $\tilde{\gamma}'(0)$ and $(I_0 \circ \tilde{\gamma})'(0)$; see 39 and the remarks that follow it. The loop of F_1 has exactly the same homoclinic length, $\lambda(\Omega) = 6\sqrt{2}$, since $\tilde{F}_1$ and F_1 are conjugated by an area-preserving linear map with determinant -1.

Finally, we analyse the rescaled Hénon family F_δ at the symmetric homoclinic point Ω_δ. Let $(x, y) = \zeta_\delta(\overline{x}, \overline{y})$ be the Birkhoff coordinates as given by Proposition 4 applied to the family F_δ. These coordinate transformations ζ_δ take each mapping F_δ to the Birkhoff normal form

$$L_\delta(\overline{x}, \overline{y}) = (e^{\alpha_\delta(\overline{x}\,\overline{y})}\, \overline{x},\, e^{-\alpha_\delta(\overline{x}\,\overline{y})}\, \overline{y}). \tag{24}$$

By Remark 7, there is then some $r > 0$ such that the coordinates $(x, y) = \zeta_\delta(\overline{x}, \overline{y})$ extend analytically to the domain W_r where L_δ acts as a global diffeomorphism; see Remark 5. The mappings $\zeta_\delta(\overline{x}, \overline{y})$, as well as all their derivatives, are equicontinuous on the domain W_r and depend continuously on $\delta \in [-\delta_0, \delta_0]$ (or analytically on $\delta \in [-\delta_0, \delta_0] - \{0\}$).

Since all the maps F_δ are reversible with respect to the canonical involution I, we have, by Remark 4, that $\zeta_\delta \circ I = I \circ \zeta_\delta$ for all small δ. Thus, if $(w(\delta), 0) \in W_r$ is such that $\zeta_\delta(w(\delta), 0) = \Omega_\delta$, then

$$\zeta_\delta(0, w(\delta)) = \zeta_\delta \circ I(w(\delta), 0) = I \circ \zeta_\delta(w(\delta), 0) = I(\Omega_\delta) = \Omega_\delta.$$

Observing that $\gamma_u(\overline{x}) = \zeta_\delta(\overline{x}, 0)$ and $\gamma_s(\overline{y}) = \zeta_\delta(0, \overline{y})$ linearize the invariant manifolds of the saddle at the origin of F_δ, we see that $w(\delta)$ is precisely the homoclinic length of Ω_δ. Thus $w(0) = \lambda(\Omega_0) = 6\sqrt{2}$, and $w(\delta)$ converges to this number as $\delta \to 0$.

We are now going to define, using Birkhoff coordinates $(\overline{x}, \overline{y})$, a new system of coordinates (t, E) in a neighbourhood of the homoclinic points Ω_δ, according to which the mapping F_δ can be described by the shift translation

$$\sigma^\delta : (t, E) \mapsto (t + \delta, E), \tag{25}$$

where the unstable curve is represented by the axis $\{E = 0\}$ and the stable curve is the graph of a periodic function $E = g_\delta(t)$.

We begin by taking a small (but fixed-size) neighbourhood U of the point Ω_0 which is covered by all the images $\zeta_\delta(W_r)$ and on which we have well-defined inverse branches $(\overline{x}, \overline{y}) = (\zeta_\delta)^{-1}(x, y)$ that, along with all their derivatives, depend continuously on δ. Define $(t, E) = \eta_\delta(x, y)$ in such a neighbourhood by setting

$$E = E_\delta(x, y) = \overline{x}\,\overline{y},$$
$$t = t_\delta(x, y) = (\delta/\alpha_\delta(\overline{x}\,\overline{y}))\ \log(\overline{x}/w(\delta)),$$

with $(\overline{x}, \overline{y}) = (\zeta_\delta)^{-1}(x, y)$. Clearly, these new coordinates are defined for all small δ in a fixed neighbourhood U of Ω_0 and, along with their derivatives, depend continuously on δ. One can trivially verify that these coordinates conjugate F_δ with the shift σ^δ in (25). In other words, given any small δ and $(x, y) \in U$ such that $F_\delta(x, y) \in U$,

$$\eta_\delta \circ F_\delta(x, y) = \eta_\delta(x, y) + (\delta, 0).$$

From the definition, one has $\eta_\delta(\Omega_\delta) = (0, 0)$. By Remark 1, one computes easily that

$$\det D\eta_\delta(\Omega_\delta) = 1. \tag{26}$$

It is also clear that the line $\{E = 0\}$ corresponds to the $\overline{x}$-axis, which in turn represents the unstable manifold. Since the stable manifold must be invariant under the shift σ^δ, in (t, E) coordinates it has to be the graph of a periodic function $E = g_\delta(t)$ with period δ. Of course $g_\delta(0) = 0$ because $(t, E) = (0, 0)$ are the coordinates of the homoclinic point Ω_δ. When $\delta = 0$, both invariant manifolds merge in the homoclinic loop of F_1, and we have $g_0(t) \equiv 0$.

Consider the function $\gamma : \mathbb{R} \to \mathbb{R}^2$ defined by

$$\gamma(t) = M_0 \cdot \tilde{\gamma}(e^t),$$

where M_0 is the matrix (21) with $\epsilon = 0$, and $\tilde{\gamma}$ is as defined in (23). Then $\gamma(t)$ is the homoclinic solution of the vector field F_1 which satisfies the initial condition $\gamma(0) = \Omega_0$. This homoclinic solution $\gamma(t)$ extends to complex time with poles at the numbers $t = i\pi + 2n\pi i$, $n \in \mathbb{Z}$. Therefore $\gamma(t)$ is holomorphic in the horizontal complex strip $|\mathrm{Im}\, t| \le r = 3\pi/4 < \pi$. It then follows from [**6**] that the Fourier coefficients of $g_\delta(t)$,

$$c_n(\delta) = \frac{1}{\delta} \int_0^\delta g_\delta(t) e^{i\, 2\pi n t/\delta}\, dt \quad (n \in \mathbb{Z}),$$

have the uniform upper bounds

$$|c_n(\delta)| \le C \exp\left\{-\frac{2\pi r|n|}{\delta}\right\} = C \exp\left\{-\frac{3\pi^2|n|}{2\delta}\right\}, \tag{27}$$

where the constant $C > 0$ is independent of δ and $n \in \mathbb{Z}$.

In [**7**] the authors proved that there is some constant $\theta_0 > 0$ (which numerically is known to be large: $\theta_0 \approx 2.474 \cdot 10^6$) such that the Lazutkin invariant of F_δ at Ω_δ has the asymptotic behaviour

$$\theta(\Omega_\delta) = \frac{64\pi\; e^{-2\pi^2/\epsilon}}{9\,\epsilon^7}\,(\theta_0 + \mathcal{O}(\epsilon)). \tag{28}$$

Using (20), it can easily be checked that

$$\frac{e^{-2\pi^2/\epsilon}}{\epsilon^7} = \frac{e^{-2\pi^2/\delta}}{\delta^7}\,(1 + \mathcal{O}(\delta)),$$

and therefore we can replace ϵ by δ in (28).

In the coordinates $(t, E) = \eta_\delta(x, y)$, the Lazutkin invariant is, up to some factor of order $1 + \mathcal{O}(\delta)$, the negative of the derivative $(g_\delta)'(0)$. By (26), the coordinate transformations $(t, E) = \eta_\delta(x, y)$ preserve the Lazutkin invariant at Ω_δ. Thus, for all small $\delta > 0$,

$$(g_\delta)'(0) = \frac{64\pi\; e^{-2\pi^2/\delta}}{9\,\delta^7}\,(-1 + \mathcal{O}(\delta)). \tag{29}$$

But from the Fourier series of $g_\delta(t)$ and the upper bounds (27),

$$\begin{aligned}(g_\delta)'(0) &= \frac{2\pi i}{\delta}\,(c_1(\delta) - c_{-1}(\delta)) + \frac{2\pi i}{\delta} \sum_{|n|\ge 2} n\, c_n(\delta)\\ &= \frac{2\pi i}{\delta}\,(c_1(\delta) - c_{-1}(\delta)) + \mathcal{O}(e^{-3\pi^2/\delta}),\end{aligned}$$

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and since the remainder $\mathcal{O}(e^{-3\pi^2/\delta})$ is exponentially small compared with the right-hand side of (29), we obtain that $c_1(\delta) = \overline{c_{-1}(\delta)} = i\, b_1(\delta) \in i\mathbb{R}$ and, for all sufficiently small $\delta > 0$,

$$b_1(\delta) = \frac{16\, e^{-2\pi^2/\delta}}{9\, \delta^6}\, (-1 + \mathcal{O}(\delta)). \tag{30}$$

Thus

$$g_\delta(t) = b_1(\delta)\left(\sin\left(\frac{2\pi t}{\delta}\right) + \mathcal{O}(e^{-\pi^2/\delta})\right), \tag{31}$$

where the remainder is the sum of a Fourier series with all coefficients exponentially small.

Clearly, the family of periodic functions $g_\delta(t)$ has *bounded C^2-geometry* in the sense of the following definition; this concept synthesizes everything that will be used in the next section.

Given a family of periodic smooth functions $g_\delta(t)$ depending on a small parameter $\delta > 0$ and satisfying the periodicity condition $g_\delta(t + \delta) = g_\delta(t)$, define, for $i = 0, 1, 2$,

$$M_i(\delta) = \max\left\{\delta^i \left|\frac{d^i g_\delta}{dt^i}(t)\right| : t \in \mathbb{R}\right\}, \tag{32}$$

$$m_1(\delta) = \min\left\{\delta \left|\frac{dg_\delta}{dt}(t)\right| : g_\delta(t) = 0\right\} \quad \text{and}$$

$$m_2(\delta) = \min\left\{\delta^2 \left|\frac{d^2 g_\delta}{dt^2}(t)\right| : \frac{dg_\delta}{dt}(t) = 0\right\}.$$

Definition 1. Let us say that $g_\delta(t)$ has bounded C^2-geometry if and only if there is some constant $C > 0$ such that for all sufficiently small $\delta > 0$, $C\, m_1(\delta) > M_2(\delta)$ and $C\, m_2(\delta) > M_0(\delta)$.

Remark 8. It follows from Definition 1 that all the functions of δ—M_0, M_1, M_2, m_1 and m_2—are asymptotically equivalent, in the sense that the quotient of any pair is bounded away from 0 and from ∞. In particular, all of them are of exponentially small order.

The family of periodic functions $g_\delta(t)$ will play the role of the Melnikov function $M_\delta(t)$ in [**3**]. The quantities (32) correspond exactly to the quantities defined in [**3**, (1)]. Note that in [**3**] the 'flow time' t was scaled so that all Melnikov functions $M_\delta(t)$ have period one. This accounts for the factors δ^i that appear in (32) but which are missing from [**3**, (1)].

5. *Thick horseshoes*

The basic set construction is based on an abstract bounding distortion result in [**3**]. In [**3**] $\mathcal{F}$ was defined to be the class of all (positive binary horseshoe) maps $f : S_0 \cup S_1 \to \mathbb{R}^2$ where: (1) S_0 and S_1 are compact subsets, diffeomorphic to rectangles, with non-empty interior; (2) f is a map of class C^2 in a neighborhood of $S_0 \cup S_1$, mapping this compact set diffeomorphically onto its image $f(S_0) \cup f(S_1)$; (3) the maximal invariant set $\Lambda(f) = \bigcap_{n\in\mathbb{Z}} f^{-n}(S_0 \cup S_1)$ is a hyperbolic basic set conjugated to the Bernoulli shift $\sigma : \{0, 1\}^{\mathbb{Z}} \to \{0, 1\}^{\mathbb{Z}}$; (4) $\mathcal{P} = \{S_0, S_1\}$ is a Markov partition for $f : \Lambda(f) \to \Lambda(f)$ and, in particular, f has two fixed points $P_0 \in S_0$ and $P_1 \in S_1$ whose stable and unstable manifolds contain the boundaries of S_0 and S_1; and finally, (5) the fixed points P_0 and P_1 both have positive eigenvalues.

Given two positive small numbers $\epsilon > 0$ and $\gamma > 0$, we define $\mathcal{F}(\epsilon, \gamma)$ to be the class of all maps $f : S_0 \cup S_1 \to \mathbb{R}^2$, $f \in \mathcal{F}$, such that:

(1) f preserves area;

(2) $\mathrm{diam}(S_0 \cup S_1) = \mathrm{diam}(f(S_0) \cup f(S_1)) = 1$;

(3) upon writing $f(x, y) = (f_1(x, y), f_2(x, y))$ and $f^{-1}(x, y) = (\tilde{f}_1(x, y), \tilde{f}_2(x, y))$, we have
  (a) $|\partial f_2/\partial y| < 1 < |\partial f_1/\partial x| \leq 2/\epsilon$,
  (b) $|\partial f_1/\partial y|, |\partial f_2/\partial x| \leq \epsilon(|\partial f_1/\partial x| - 1)$;

(4) the second derivatives satisfy
  (a) $|\partial^2 \tilde{f}_1/\partial x \partial y|, |\partial^2 \tilde{f}_1/\partial y^2|, |\partial^2 \tilde{f}_2/\partial x^2|, |\partial^2 \tilde{f}_2/\partial x \partial y| \leq \gamma\, (|\partial \tilde{f}_2/\partial y| - 1)$,
  (b) $|\partial^2 f_1/\partial x \partial y|, |\partial^2 f_1/\partial y^2|, |\partial^2 f_2/\partial x^2|, |\partial^2 f_2/\partial x \partial y| \leq \gamma\, (|\partial f_1/\partial x| - 1)$,
  (c) $|\partial^2 \tilde{f}_2/\partial y^2|, |\partial^2 \tilde{f}_1/\partial x^2| \leq \gamma\, |\partial \tilde{f}_2/\partial y|(|\partial \tilde{f}_2/\partial y| - 1)$,
  (d) $|\partial^2 f_1/\partial x^2|, |\partial^2 f_2/\partial y^2| \leq \gamma\, |\partial f_1/\partial x|(|\partial f_1/\partial x| - 1)$;

(5) the variation of $\log |(\partial f_1/\partial x)(x, y)|$ in each rectangle S_i is less than or equal to $\gamma\, (1 - \alpha_i^{-1})$, where $\alpha_i = \max_{(x,y)\in S_i} |(\partial f_1/\partial x)(x, y)|$;

(6) the gap sizes satisfy

$$\mathrm{dist}(S_0, S_1) \geq \frac{\epsilon}{\gamma} \quad \text{and} \quad \mathrm{dist}(f(S_0), f(S_1)) \geq \frac{\epsilon}{\gamma}.$$

The normalizing condition (2) avoids having all subsequent items refer to the scale of the basic set. With the above notation, the following result was proved in [**3**, cf. Theorem 2].

THEOREM 2. *For all small enough $\epsilon > 0$ and $\gamma > 0$, given $f \in \mathcal{F}(\epsilon, \gamma)$, the basic set $\Lambda(f)$ has dynamically defined Cantor sets (Λ^u, ψ^u) and (Λ^s, ψ^s) with small distortion, bounded by $D(\epsilon, \gamma) = 20\gamma + 2\epsilon$. In particular,*

$$e^{-2\, D(\epsilon,\gamma)}\, \tilde{\tau}_{LR}(\Lambda(f)) \leq \tau_{LR}(\Lambda(f)) \leq e^{2\, D(\epsilon,\gamma)}\, \tilde{\tau}_{LR}(\Lambda(f)).$$

In the rest of this section we shall outline the basic set construction in the class $\mathcal{F}(\epsilon, \gamma)$, and estimate the corresponding top scale $\tilde{\tau}_{LR}$-thickness, in order to apply the above theorem. All proofs will refer to [**3**], with the following adaptations in notation. In all formulas of [**3**], one should either drop the variable μ, when it appears as an argument, or take $\mu = 1$, when it appears as a factor in some expression. Here the eigenvalue is $\lambda = e^{\delta}$, so $\log \lambda$ should be replaced by δ in all expressions of [**3**]. Of course $\lambda_{\delta,\mu}(t)$ is to be understood as $\lambda_\delta(t) = e^{\alpha_\delta(t)}$. The filter $\mathcal{N}$ in our setting will just be the filter of all neighbourhoods of $\delta = 0$ in the parameter half-line $[0, +\infty)$. Many computations here will be reduced by half owing to the reversible character of the Hénon map, which was not assumed of maps $f_{\delta,\mu}$ in [**3**]. A crucial quantity in [**3**] is $\theta = \theta_\delta$ defined in [**3**, formula 4], which appears in 'almost all' derivative bounds given thereafter. Here the same role is played by

$$\theta = \theta_\delta = -(g_\delta)'(0) > 0, \tag{33}$$

for which one has the upper and lower bounds given in (27) and (29).

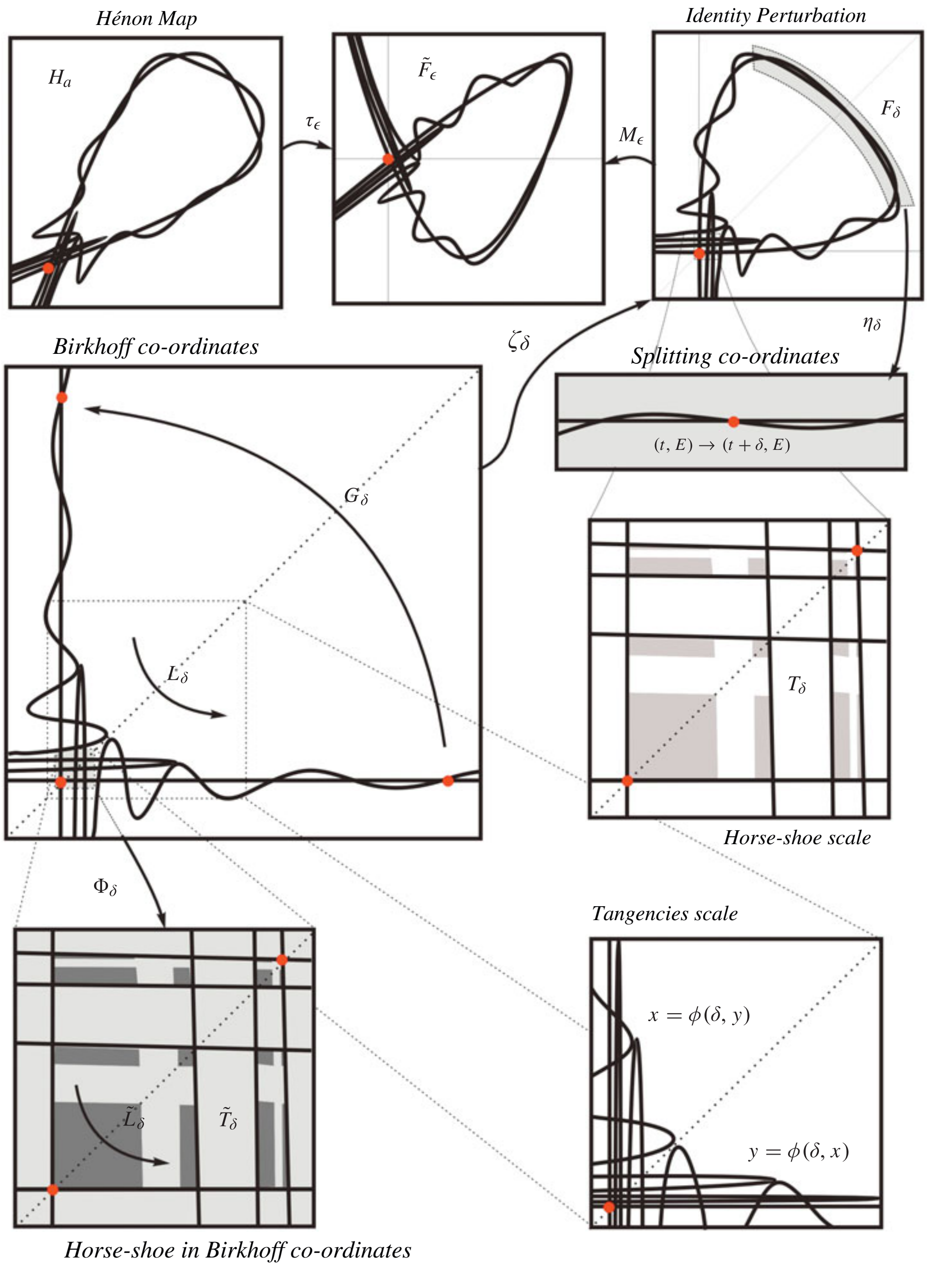

FIGURE 2. Construction coordinates.

Definition 2. (The half return time) We define the *half return time* as

$$n(\delta) = \text{ the integer part of } \frac{-\log(\theta_\delta\, \delta^{3/2})}{2\,\delta}.$$

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Clearly $\lim_{\delta\to 0} n(\delta) = +\infty$. From item (4) in Proposition 4 we can deduce the following.

LEMMA 2. *There are constants $C > 0$ and $\delta_0 > 0$ such that the following inequalities hold for all $0 < \delta < \delta_0$ and all $|t| < 2$:*

(1) $C^{-1}\delta \le \alpha_\delta(t) \le C\,\delta$;

(2) $|(\alpha_\delta)'(t)| \le C\,\delta$;

(3) $|(\alpha_\delta)''(t)| \le C\,\delta$.

Using these facts we prove as in [**3**, Lemma 6.4] that the following holds.

LEMMA 3. *Writing $n = n(\delta)$ for all small enough $\delta > 0$, we have:*

(1) $e^{-2n\delta} = \delta^{3/2}\,\theta_\delta(1 + \mathcal{O}(\delta))$;

(2) $n\,\theta_\delta = o(\sqrt{\theta_\delta})$;

(3) $e^{2n(\alpha_\delta(t)-\delta)} = 1 + \mathcal{O}(\sqrt{\theta_\delta})$ *for* $0 \le t \le e^{-2n\alpha_\delta(t)}$.

Consider the normal form L_δ associated with the map F_δ, with the notation in (24). Conformally rescaling F_δ, we can take $w(\delta) = 1$; in other words, we can make the homoclinic length of Ω_δ constant and equal to 1. Therefore $\zeta_\delta(1, 0) = \Omega_\delta = \zeta_\delta(0, 1)$, and for some small $r > 0$ both restriction maps $\zeta_\delta^- = \zeta_\delta|_{B_r(1,0)}$ and $\zeta_\delta^+ = \zeta_\delta|_{B_r(0,1)}$ are one-to-one onto a neighbourhood of Ω_0.

Definition 3. (The transition map) For each $\delta > 0$, we define the *transition map* $G_\delta = (\zeta_\delta^+)^{-1} \circ \zeta_\delta^-$ in a small but fixed neighbourhood of $(1, 0)$.

Clearly these maps satisfy $G_\delta(1, 0) = (0, 1)$, the compatibility relation $L_\delta \circ G_\delta = G_\delta \circ L_\delta$, and also the reversibility equation

$$G_\delta \circ I = I \circ (G_\delta)^{-1}. \tag{34}$$

Denote the components of $G_\delta(x, y)$ by

$$G_\delta(x, y) = (g_1(\delta, x, y),\ g_2(\delta, x, y)).$$

Then from (34) one has

$$G_\delta^{-1}(x, y) = (g_2(\delta, y, x), g_1(\delta, y, x)),$$

and so all bounds on the derivatives of G_δ automatically give bounds on the derivatives of G_δ^{-1}. One can easily prove that at $\delta = 0$, $g_1(0, x, 0) = 0$ and $g_2(0, x, 0) = x^{-1}$. Thus, using the symplectic character of G_0 along the homoclinic loop (see Remark 1 and also [**3**, Lemma 6.2]), we obtain the next result.

LEMMA 4. *For all $x \in [1/2, 3/2]$, we have $(g_2)_x(0, x, 0) = -x^{-2} = -1 + O(x - 1)$ and $(g_1)_y(0, x, 0) = x^{-2} = 1 + O(x - 1)$.*

LEMMA 5. *For some constant $C > 0$ and all small enough $\delta > 0$, the function $g_1(\delta, x, 0)$ and its first and second derivatives with respect to x are bounded by $C\delta\,\theta_\delta$, $C\theta_\delta$ and $C\theta_\delta/\delta$, respectively.*

Moreover, by relating the transition map G_δ with the periodic function $g_\delta(t)$ through the coordinate transformation η_δ, one easily proves the following.

LEMMA 6. $(g_1)_x(\delta, 1, 0) = -\theta_\delta$.

5.1. *Rescaling the basic set.* The map T_δ will be defined by two different branches where it coincides with the maps L_δ and $(L_\delta)^n \circ G_\delta \circ (L_\delta)^n$. These two branches are defined, respectively, on the two very small rectangles

$$S_0 = \{(x, y) : |x| < e^{-(n+1/2)\delta},\ |y| < e^{-(n-1/2)\delta}\} \quad \text{and}$$
$$S_1 = \{(x, y) : |x - e^{-n\,\delta}| < 2\delta^{3/2} e^{-n\,\delta},\ |y| < e^{-(n-1/2)\delta}\}.$$

The domain $S_0 \cup S_1$ has diameter of order $e^{-n\delta} \sim \sqrt{\delta^{3/2}\,\theta_\delta}$. To scale this domain up to the unit square, we introduce the scaling maps $\Phi_\delta : \mathbb{R}^2 \to \mathbb{R}^2$,

$$\Phi_\delta(x, y) = (e^{n\,\alpha_\delta(xy)}\,x,\ e^{n\,\alpha_\delta(xy)}\,y),$$

where $n = n(\delta)$ is the half return time. The product of the components of Φ_δ, namely $e^{2n\alpha_\delta(xy)}xy$, is a function of the product xy. Therefore the inverse map $\Phi_\delta^{-1} : \mathbb{R}^2 \to \mathbb{R}^2$ is given by

$$\Phi_\delta^{-1}(x, y) = (e^{-n\,\alpha_\delta(t_\delta(xy))}x,\ e^{-n\,\alpha_\delta(t_\delta(xy))}y),$$

where $t_\delta(s)$ is defined implicitly by $t_\delta(0) = 0$ and the relation

$$e^{2n\,\alpha_\delta(t_\delta(s))}\,t_\delta(s) = s \quad \text{for all } |s| < 2.$$

We scale T_δ, setting $\tilde{T}_\delta = \Phi_\delta \circ T_\delta \circ \Phi_\delta^{-1}$. This map has the two branches $\tilde{L}_\delta = \Phi_\delta \circ L_\delta \circ \Phi_\delta^{-1}$ and $\tilde{G}_\delta = \Phi_\delta \circ L_\delta^n \circ G_\delta \circ L_\delta^n \circ \Phi_\delta^{-1}$. To compute the first, define

$$\tilde{\alpha}(s) = \tilde{\alpha}_\delta(s) := \alpha_\delta \circ t_\delta\,(s).$$

Notice that if we write $(x_1, y_1) = \Phi_\delta^{-1}(x, y)$ and $(x_2, y_2) = L_\delta(x_1, y_1)$, then $x_2 y_2 = x_1 y_1 = t_\delta(xy)$. Therefore

$$\begin{aligned}
\tilde{L}_\delta(x, y) &= \Phi_\delta \circ L_\delta(e^{-n\,\alpha_\delta(t_\delta(xy))}x,\ e^{-n\,\alpha_\delta(t_\delta(xy))}y) \\
&= \Phi_\delta(e^{\alpha_\delta(t_\delta(xy)) - n\,\alpha_\delta(t_\delta(xy))}x,\ e^{-\alpha_\delta(t_\delta(xy)) - n\,\alpha_\delta(t_\delta(xy))}y) \\
&= (e^{\alpha_\delta(t_\delta(xy))}x,\ e^{-\alpha_\delta(t_\delta(xy))}y) \\
&= (e^{\tilde{\alpha}_\delta(xy)}\,x,\ e^{-\tilde{\alpha}_\delta(xy)}\,y).
\end{aligned}$$

A simple computation gives $\tilde{G}_\delta(1, 0) = (0, 1)$ for all δ. It is obvious that the scaling map Φ_δ commutes with the involution I. Therefore, using (34), we obtain the reversibility of $\tilde{G}_\delta$, i.e. $\tilde{G}_\delta \circ I = I \circ \tilde{G}_\delta^{-1}$. To make $\tilde{G}_\delta$ explicit, we introduce an auxiliary function

$$p(x, y) = p_\delta(x, y) := g_1(x, e^{-2n\,\tilde{\alpha}(xy)}y) \cdot g_2(x, e^{-2n\,\tilde{\alpha}(xy)}y),$$

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where $\tilde{\alpha} = \tilde{\alpha}_\delta$, $g_1(\cdot, \cdot) = g_1(\delta, \cdot, \cdot)$, $g_2(\cdot, \cdot) = g_2(\delta, \cdot, \cdot)$ and $n = n(\delta)$. Then, from [**3**, formula (11)] we get

$$\tilde{G}_\delta(x, y) = (e^{2n\alpha\circ p(x,y)} g_1(x, e^{-2n\,\tilde{\alpha}(xy)} y),\ g_2(x, e^{-2n\,\tilde{\alpha}(xy)} y)).$$

We will consider the domains of the rescaled maps G_δ and G_δ^{-1} to be, respectively, the rectangles

$$\tilde{S}_1(\delta) = \{(x, y) : |x - 1| \leq 2\,\delta^{3/2} \text{ and } 0 \leq y \leq 1 + \delta/2\},$$
$$\tilde{S}_1'(\delta) = \{(x, y) : 0 \leq x \leq 1 + \delta/2 \text{ and } |y - 1| \leq 2\,\delta^{3/2}\}.$$

To estimate the derivatives of $\tilde{G}_\delta$, first observe the following.

LEMMA 7. *There are constants $C > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$ and $s \in [0, 2]$, we have $|t_\delta(s)|, |t_\delta'(s)|, |t_\delta''(s)| \leq C\,\delta^{3/2}\,\theta_\delta$.*

Proof. This proof depends on Lemma 3; see [**3**, Lemma 7.1]. □

From the bounds in Lemma 7 together with Lemmas 2, 4 and 5, we can show the following estimates.

LEMMA 8. *There are constants $C > 0$ and $\delta_0 > 0$ such that for all $(x, y) \in \tilde{S}_1(\delta)$:*

(1) $|p(x, y)| \leq C\,\delta^{3/2}\,\theta_\delta$;

(2) $\left|\frac{\partial p}{\partial x}(x, y)\right| \leq C\,\theta_\delta$;

(3) $\left|\frac{\partial p}{\partial y}(x, y)\right| \leq C\,\delta^{3/2}\,\theta_\delta$;

(4) $\left|\frac{\partial^2 p}{\partial x^2}(x, y)\right| \leq C\,\delta^{-1}\,\theta_\delta$.

Proof. See [**3**, Lemma 7.2]. □

5.2. *Bounds on the derivatives of $\tilde{T}_\delta$.* Consider the first branch $\tilde{L}_\delta$ of $\tilde{T}_\delta$ to be defined on the rectangle $\tilde{S}_0(\delta) = [0, 1 - \delta/2] \times [1 + \delta/2]$, while $\tilde{L}_\delta^{-1}$ is defined on $\tilde{S}_0'(\delta) = [0, 1 + \delta/2] \times [1 - \delta/2]$.

LEMMA 9. *There is $\delta_0 > 0$ such that for $0 < \delta < \delta_0$, the Jacobian matrix of $\tilde{L}_\delta$ satisfies*

$$D\tilde{L}_\delta = \begin{pmatrix} e^\delta & 0 \\ 0 & e^{-\delta} \end{pmatrix} + \mathcal{O}(\delta^{3/2}\,\theta_\delta)$$

on $\tilde{S}_0(\delta)$, and all the second derivatives of $\tilde{L}_\delta$ are of exponentially small order $\mathcal{O}(\delta^{3/2}\,\theta_\delta)$. Similar bounds hold for $\tilde{L}_\delta^{-1}$ on $\tilde{S}_0'$.

Proof. See [**3**, Lemmas 8.1 and 8.2]. □

On the other branch the following holds.

LEMMA 10. *There exists $\delta_0 > 0$ such that, for $0 < \delta < \delta_0$, the Jacobian matrix of $\tilde{G}_\delta$ satisfies*

$$D\tilde{G}_\delta = \begin{pmatrix} -\delta^{-3/2} & -1 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} \mathcal{O}(\delta^{-1}) & o(1) \\ o(1) & o(1) \end{pmatrix}$$

on the domain $\tilde{S}_1$, and the second derivatives of $\tilde{G}_\delta$ are all uniformly bounded except for $\partial^2 g_1/\partial x^2 = \mathcal{O}(\delta^{-5/2})$. Again, similar bounds hold for $\tilde{G}_\delta^{-1}$ on S_1'.

Proof. This result follows from Lemma 8; see the arguments in [**3**, Lemmas 8.3 and 8.4]. In the proof of [**3**, Lemmas 8.3] the estimate

$$\frac{\partial g_1}{\partial x}(\delta, \mu, 1, 0) = -\mu\, \theta_\delta + \mathcal{O}(\mu^2)$$

should be replaced by the explicit value given in Lemma 6. □

LEMMA 11. *There are constants $C > 2$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, the maps $\tilde{L}_\delta$ and $\tilde{L}_\delta^{-1}$ over $\tilde{S}_0$, the map $\tilde{G}_\delta$ over $\tilde{S}_1$, and the map $\tilde{G}_\delta^{-1}$ over $\tilde{S}'_1$ satisfy conditions (3), (4) and (5) of the class $\mathcal{F}(3\delta^{3/2}/2,\ 3C\delta^{1/2})$ definition.*

Proof. This lemma follows from the previous ones, Lemmas 9 and 10. See the proof of [**3**, Lemma 8.5]. □

5.3. *Thick horseshoes.*

LEMMA 12. *Besides $(0, 0) \in \tilde{S}_0$, the map $\tilde{T}_\delta$ has a second symmetric fixed point with coordinates $(x_1, x_1) \in \tilde{S}_1 \cap \tilde{S}'_1$, where $x_1 = 1 + \mathcal{O}(\delta^{5/2})$. Moreover, there is a family of smooth functions $\gamma_\delta : [0, 1 + \delta/2] \to \mathbb{R}$ such that $\gamma_\delta(x_1) = x_1$ and, for all $t \in [0, 1 + \delta/2]$:*
(1) $-\frac{3}{2}\,\delta^{3/2} \le (\gamma_\delta)'(t) \le -\frac{2}{3}\,\delta^{3/2}$;
(2) $|\gamma_\delta(t) - 1| \le \frac{7}{4}\,\delta^{3/2}$.
The graphs $\{(t, \gamma_\delta(t)) : t \in [0, 1 + \delta/2]\}$ and $\{(\gamma_\delta(t), t) : t \in [0, 1 + \delta/2]\}$ are, respectively, the local invariant unstable and stable manifolds of the fixed point (x_1, x_1). In particular, these pieces of invariant manifolds are contained in $\tilde{S}'_1$ and $\tilde{S}_1$, respectively.

Proof. See [**3**, Lemmas 8.6 and 8.7]. Note that the remainder of the expression for x_1 is $\mathcal{O}(\delta^{5/2})$ here, instead of the $\mathcal{O}(\mu \log^{3/2} \lambda)$ in [**3**]; this can be obtained by replacing $\frac{\partial g_1}{\partial y}(\delta, \mu, 1, 0) = 1 + \mathcal{O}(\mu)$ with $\frac{\partial g_1}{\partial y}(\delta, 1, 0) = 1 + \mathcal{O}(\delta)$, which in turn follows from $\frac{\partial g_1}{\partial y}(0, 1, 0) = 1$; see Lemma 4. □

LEMMA 13. *For all small enough $\delta > 0$, the local invariant manifolds of the fixed points $(0, 0)$ and (x_1, x_1) are the boundary curves of a Markov partition $\mathcal{P}_\delta = \{S_0(\delta), S_1(\delta)\}$ such that $S_0 \subseteq \tilde{S}_0$, $\tilde{T}_\delta(S_0) \subseteq \tilde{S}'_0$, $S_1 \subseteq \tilde{S}_1$ and $\tilde{T}_\delta(S_1) \subseteq \tilde{S}'_1$.*

Proof. See the proof of [**3**, Lemma 8.8]. □

Since both fixed points have positive eigenvalues, we obtain the following.

LEMMA 14. *For all small enough $\delta > 0$, the restriction $\tilde{T}_\delta|_{S_0 \cup S_1} : S_0 \cup S_1 \to \mathbb{R}^2$ is a positive binary horseshoe map, i.e. $\tilde{T}_\delta \in \mathcal{F}$.*

LEMMA 15. *For all small enough $\delta > 0$, $dist(S_0, S_1) = dist(\tilde{T}(S_0), \tilde{T}(S_1)) = \mathcal{O}(\delta)$.*

Proof. This follows essentially from Lemma 12; See the proof of [**3**, Lemma 8.9]. □

Thus, from Lemmas 14 and 15, the following can be deduced.

COROLLARY 5. *There are constants $C > 0$ and $\delta_0 > 0$ such that for all $0 < \delta < \delta_0$, $\tilde{T}_\delta \in \mathcal{F}(3\delta^{3/2}/2,\ 3C\delta^{1/2})$.*

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In particular, by Theorem 2, the $\tilde{T}_\delta$-invariant horseshoe $\Lambda_\delta = \bigcap_{j=-\infty}^{\infty} (\tilde{T}_\delta)^{-j}(\tilde{S}_0 \cup \tilde{S}_1)$ has stable and unstable distortion of order $\mathcal{O}(\delta)$. It is also easy to compute that the width of the rectangles $S_0(\delta)$, $S_1(\delta)$ and the gap between them are of orders 1, $\delta^{3/2}$ and δ, respectively. Therefore, the top scale left–right thickness of $(\Lambda_\delta, \tilde{T}_\delta)$ is of order $\mathcal{O}(\delta^{-1/2})$.

LEMMA 16. *There is some* $\delta_0 > 0$ *such that for all* $0 < \delta < \delta_0$, *we have:*
(1) $\tilde{\tau}_L(\Lambda_\delta^s), \tilde{\tau}_L(\Lambda_\delta^u) \geq 1/\delta$;
(2) $\tilde{\tau}_R(\Lambda_\delta^s), \tilde{\tau}_R(\Lambda_\delta^u) \geq \sqrt{\delta}/4$.
In particular, $\tilde{\tau}_{LR}(\Lambda_\delta) \geq 1/(4\sqrt{\delta})$.

Proof. See the proof of [**3**, Lemma 8.10]. □

Finally, combining Theorem 2 with the lemma above allows us to conclude that

$$\lim_{\delta\to 0} \tau_{LR}(\Lambda_\delta, \tilde{T}_\delta) = +\infty.$$

6. *Tangencies in the Hénon map*

In this section we will prove Lemma B stated in §2. The graph $E = g_\delta(t)$ describes a piece of the stable manifold in the coordinates $(t, E) = \eta_\delta(x, y)$. Translating this into Birkhoff coordinates $(\overline{x}, \overline{y}) = \zeta_\delta^{-1}(x, y)$, the same arc of stable manifold is characterized as the graph $\overline{y} = \phi(\delta, \overline{x})$ of a function $\phi(\delta, x)$ implicitly defined by

$$\begin{aligned}\phi(\delta, x) &= g_\delta\bigg(\frac{\delta}{\alpha(\delta, x\phi)} \log x\bigg)\Big/x \\ &= g_\delta(\log x - \tau)/x, \end{aligned} \tag{35}$$

where ϕ stands for $\phi(\delta, x)$. From $\alpha(\delta, 0) = \log\lambda_\delta = \delta$ we can derive the following expression for $\tau = \tau(\delta, x)$:

$$\tau(\delta, x) = x \log x\, \phi(\delta, x)\, \alpha(\delta, x\phi)^{-1} \int_0^1 \frac{d\alpha}{d\omega}(\delta, sx\phi)\, ds. \tag{36}$$

We are going to consider the following domain for $\phi(\delta, x)$:

$$\Xi = \{(\delta, x) : |\delta| < \delta_0 \text{ and } \sqrt{|b_1(\delta)|}/3 < x < 3/\sqrt{|b_1(\delta)|}\}.$$

A simple application of the implicit function theorem, and computing the derivatives of the implicitly defined functions $\phi(\delta, x)$ and $\tau(\delta, x)$, gives the next lemma.

LEMMA 17. *The unique functions* $\phi(\delta, x)$ *and* $\tau(\delta, x)$ *which are defined implicitly on* Ξ *by equations (35) and (36) have the following asymptotics over* Ξ*:*

$$\begin{aligned} &\phi(\delta, x) = \mathcal{O}(\sqrt{|b_1(\delta)|}), && \tau(\delta, x) = \mathcal{O}(|b_1(\delta)|), \\ &\phi_x(\delta, x) = \mathcal{O}(\delta^{-1}), && \tau_x(\delta, x) = \mathcal{O}(\delta^{-1}\sqrt{|b_1(\delta)|}), \\ &\phi_{xx}(\delta, x) = \mathcal{O}(\delta^{-2}\sqrt{|b_1(\delta)|}^{-1}), && \tau_{xx}(\delta, x) = \mathcal{O}(\delta^{-2}). \end{aligned}$$

Observe that for $(\delta, x) \in \Xi$, both of the points $(x, \phi(\delta, x))$ and $(\phi(\delta, x), x)$ belong to the open set W_r where Birkhoff coordinates can be extended. The graphs $\{(x, \phi(\delta, x)) : (\delta, x) \in \Xi\}$ and $\{(\phi(\delta, x), x) : (\delta, x) \in \Xi\}$ represent, respectively in these coordinates, arcs of stable and unstable manifolds of the origin for the map F_ϵ $(\epsilon = \epsilon(\delta))$. See Figure 3. The following proposition is a statement about the existence of quadratic homoclinic tangencies between these symmetric invariant manifolds.

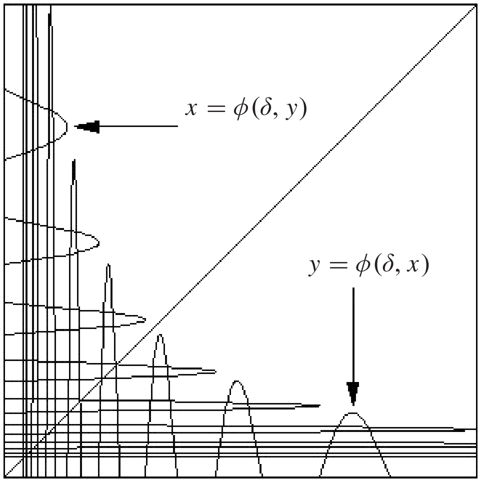

FIGURE 3. Symmetric homoclinic tangencies.

PROPOSITION 6. *There are sequences δ_n and x_n converging to zero such that for all $n \in \mathbb{N}$, $(\delta_n, x_n) \in \Xi$ and:*

(1) $\lim_{n\to\infty} e^{2n\delta} |b_1(\delta)| = 1, \ \lim_{n\to\infty} e^{n\,\delta}\, x_n = 1$;
(2) $\phi(\delta_n, x_n) = x_n$;
(3) $\phi_x(\delta_n, x_n) = 1$;
(4) $\phi_{xx}(\delta_n, x_n) < 0$;
(5) *for some $\delta_n' > \delta_n$ and all $\delta_n < \delta < \delta_n'$, $\phi_\delta(\delta, x_n) > 0$.*

Proof of Lemma B. Take δ_n and x_n as in Proposition 6. By item (2) of this proposition, $\zeta_{\delta_n}(x_n, x_n)$ are homoclinic points, and non-transversality follows from item (3). By item (4) these homoclinic tangencies are quadratic. If we could show that $\phi_\delta(\delta_n, x_n) > 0$, then these tangencies would unfold generically. But since there are transverse homoclinic orbits, the stable and unstable local manifolds of the origin are accumulated on the right by other arcs of the same manifolds. Because the tangency at (x_n, x_n) is negative, δ_n is the limit of a decreasing sequence of parameters where other homoclinic tangencies unfold near (x_n, x_n). The statement in item (5) then implies that these new tangencies unfold generically. □

Proof of Proposition 6. The graph $\overline{y} = \phi(\delta, \overline{x})$ oscillates from exponentially small amplitudes at $x = 1$ to very large almpitudes near $x = 0$. We rescale each concave part in this graph, scaling its width measured along the $\overline{x}$-axis to an interval of size δ, and scaling its height measured along the $\overline{y}$-axis to an interval of size one. Define

$$\Delta_n = \{\delta \ : 1/2 < e^{2n\delta}|b_1(\delta)| < 2\}$$

and

$$\Xi_n = \{(\delta, x) : \delta_n \in \Delta_n \text{ and } x \in [1 - 3\delta/8,\ 1 - \delta/8]\}.$$

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It is straightforward to check that for each $n \geq 1$,

$$(\delta, x) \in \Xi_n \Rightarrow (\delta, e^{-n\delta} x) \in \Xi.$$

Therefore we may define, for $(\delta, x) \in \Xi_n$,

$$\begin{aligned} \phi_n(\delta, x) &= e^{n\delta}\, \phi(\delta, e^{-n\delta}\, x) \\ &= e^{2n\delta}\, g_\delta(\log x - \tau_n(\delta, x)), \end{aligned} \tag{37}$$

where $\tau_n(\delta, x) = \tau(\delta, e^{-n\delta}\, x)$. From Lemma 17 we obtain that:

(1) $\tau_n(\delta, x) = \mathcal{O}(|b_1(\delta)|)$;

(2) $(\tau_n)_x(\delta, x) = \mathcal{O}(\delta^{-1}\, |b_1(\delta)|)$; and

(3) $(\tau_n)_{xx}(\delta, x) = \mathcal{O}(\delta^{-2}\, |b_1(\delta)|)$.

Combining (31) with (37), and noting that (37) is strictly positive and bounded away from zero in its domain Ξ_n, one easily obtains, for $(\delta, x) \in \Xi_n$,

$$\phi_n(\delta, x) = \gamma_n(1 + \sigma_n)\, \sin\left(\frac{2\pi(1-x)}{\delta}\right),$$

where $\gamma_n = -e^{2n\delta}\, b_1(\delta)$, and $\sigma_n = \sigma_n(\delta, x)$ together with its derivatives $(\sigma_n)_x$ and $(\sigma_n)_{xx}$ are small functions of order $\mathcal{O}(\delta)$. By (30) the factor γ_n is positive. Notice also that

$$(\delta, x) \in \Xi_n \quad \Leftrightarrow \quad \frac{\pi}{4} < \frac{2\pi(1-x)}{\delta} < \frac{3\pi}{4},$$

and the sine function is strictly positive and concave in the interval $[\pi/4, 3\pi/4]$. By the definition of Δ_n, as δ goes through Δ_n the factor γ_n runs across the interval $[1/2, 2]$. Therefore there exists some parameter value $\delta = \delta_n$ for which γ_n is close to 1 and the graph $y = \phi_n(\delta_n, x)$ is tangent to the diagonal $y = x$ at some point (x_n^*, x_n^*) near $(1, 1)$. Then, since ϕ_n is just a rescaling of ϕ, the sequences δ_n and $x_n = e^{-n\delta} x_n^*$ satisfy items (1), (2), (3) and (4) of Proposition 6. But if we choose δ_n to be the last value of $\delta \in \Delta_n$ such that $y = \phi_n(\delta_n, x)$ has some tangency with $y = x$, then clearly $(\phi_n)_\delta(\delta, x_n) \geq 0$ for all $\delta \geq \delta_n$ which are sufficiently close to δ_n. If $(\phi_n)_\delta(\delta_n, x_n) > 0$, item (5) is obvious. If not, then by analyticity of $(\phi_n)_\delta$ its zeros are isolated and one must have $(\phi_n)_\delta(\delta, x_n) > 0$ for all $\delta > \delta_n$ which are sufficiently close to δ_n. Thus item (5) follows anyway. □

7. *Symplectic invariants*

In this section we recall two symplectic invariants associated with homoclinic orbits. Given a fixed point P of a symplectic map $\varphi : M^2 \to M^2$, with multipliers $0 < \lambda^{-1} < 1 < \lambda$, assume that two separatrices $\gamma^s(P)$ of $W^s(P)$ and $\gamma^u(P)$ of $W^u(P)$ intersect at some homoclinic point Q. Take vectors v_s and v_u tangent to $\gamma^s(P)$ and $\gamma^u(P)$, respectively, at point P. Consider the (unique) 'linearizing' maps $\gamma_s : \mathbb{R} \to W^s(P) \subseteq M^2$ and $\gamma_u : \mathbb{R} \to W^u(P) \subseteq M^2$ such that:

(1) $\gamma_s(0) = P$ and $\gamma_u(0) = P$;

(2) $(\gamma_s)'(0) = v_s$ and $(\gamma_u)'(0) = v_u$;

(3) $\varphi(\gamma_s(t)) = \gamma_s(\lambda^{-1}\, t)$ and $\varphi(\gamma_u(t)) = \gamma_u(\lambda\, t)$.

There are positive real numbers $T_s, T_u \in \mathbb{R}^+$ such that $Q = \gamma_s(T_s) = \gamma_u(T_u)$.

The first invariant, the *Lazutkin invariant*, is the ratio

$$\theta(Q) = \frac{\omega_p((\gamma_s)'(T_s), (\gamma_u)'(T_u))}{\omega_p(v_s, v_u)}. \tag{38}$$

The second invariant might be known by another name in the literature, but we call it the *homoclinic length* of Q,

$$\lambda(Q) = \sqrt{T_s \cdot T_u \cdot |\omega_P(v_u, I(v_u))|}. \tag{39}$$

Both of these numbers do not depend on the choice of the vectors v_s and v_u; they both remain constant along the orbits, and they are easily seen to be invariant under area-preserving changes of coordinates.

The Lazutkin invariant is used as a symplectic invariant measure of the splitting angle at some transverse homoclinic point. $\theta(Q) = 0$ means that the two separatrices are tangent or coincide.

The homoclinic length is always positive. Suppose we choose (area-preserving) Birkhoff coordinates taking the map φ to its normal form around P, so that the coordinates of Q are $(c, 0)$ and $(0, c)$. Then c is precisely the homoclinic length of Q.

In the case of Hamiltonian vector fields with a homoclinic loop, the length is defined with respect to the induced flow maps ϕ^t, $t \neq 0$. Is not difficult to see that all points in the loop have the same length, which is also the same for all flow maps ϕ^t. Assume now that X is a Hamiltonian vector field in the surface M^2 which is reversible with respect to some involution I, meaning that $DI_x \cdot X(x) = -X(I(x))$. Suppose we are given a symmetric fixed saddle $P = I(P)$, $X(P) = 0$, with multipliers $-\lambda < 0 < \lambda$, together with a homoclinic loop cutting the fixed point set of I in some symmetric homoclinic point Q. In order to compute the length of the homoclinic connection, we take a non-zero vector $v_u \in T_P(M^2)$ tangent to the unstable manifold at P, and then find a solution $\gamma_u : \mathbb{R}^+ \to M^2$ to the following problem. For all $t > 0$:

(1) $\lambda t \, (d\gamma_u/dt) = X(\gamma_u(t))$;

(2) $\lim_{t\to 0} \gamma_u(t) = P$;

(3) $\lim_{t\to 0} t^{-1} \, (d\gamma_u/dt) = v_u$.

The curve γ_u linearizes the unstable manifold of P and, by symmetry, $\gamma_s = I \circ \gamma_u$ linearizes the stable manifold of P. Let $T \in \mathbb{R}^+$ be the time corresponding to the symmetric homoclinic point, i.e. $\gamma_u(T) = Q = (I \circ \gamma_u)\,(T)$. Then the homoclinic length of the connection is given by $\sqrt{T^2 \cdot |\omega(v_u, I(v_u))|}$.

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