

MEAGER-ADDITIVE SETS IN TOPOLOGICAL GROUPS

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Abstract. By the Galvin–Mycielski–Solovay theorem, a subset X of the line has Borel’s strong measure zero if and only if $M + X \neq \mathbb{R}$ for each meager set M .

A set $X \subseteq \mathbb{R}$ is *meager-additive* if $M + X$ is meager for each meager set M . Recently a theorem on meager-additive sets that perfectly parallels the Galvin–Mycielski–Solovay theorem was proven: A set $X \subseteq \mathbb{R}$ is meager-additive if and only if it has sharp measure zero, a notion akin to strong measure zero.

We investigate the validity of this result in Polish groups. We prove, e.g., that a set in a locally compact Polish group admitting an invariant metric is meager-additive if and only if it has sharp measure zero. We derive some consequences and calculate some cardinal invariants.

§1. Introduction. More than 100 years ago Borel [4] defined the notion of strong measure zero: a metric space X has *strong measure zero* (hereafter Smz) if for any sequence $\langle \varepsilon_n \rangle$ of positive numbers there is a cover $\{U_n\}$ of X such that $\text{diam } U_n \leq \varepsilon_n$ for all n . In the same paper, Borel conjectured that every Smz set of reals is countable. This statement, known as the *Borel Conjecture*, attracted a lot of attention. It is well-known that the Borel Conjecture is independent of ZFC, the usual axioms of set theory. The proof of consistency of its failure was settled by 1948 by Sierpiński [24], who proved in 1928 that the Luzin set (that exists under the Continuum Hypothesis) is a counterexample, and Gödel [8], who proved in 1948 the consistency of the Continuum Hypothesis. The consistency of the Borel Conjecture was proved in 1976 in Laver’s ground-breaking paper [13].

Over time numerous characterizations of Smz were discovered. Maybe the most interesting is the Galvin–Mycielski–Solovay Theorem: Confirming Prikry’s conjecture, Galvin et al. [6] proved the following:

THEOREM [6]. *A set $X \subseteq \mathbb{R}$ is Smz if and only if $M + X \neq 2^\omega$ for each meager set $M \subseteq 2^\omega$.*

This theorem led to the study of small sets defined in a similar manner. We focus on the meager-additive sets. By definition, a set $X \subseteq \mathbb{R}$ is *meager-additive* if $M + X$ is meager for each meager set M . The definition easily extends to any topological group. Meager-additive sets in 2^ω have received a lot of attention. They were investigated by many, most notably by Bartoszyński and Judah [1], Pawlikowski [20] and Shelah [23].

In a recent paper [28], the author of the present paper used combinatorial properties of meager-additive sets described by Shelah [23] and Pawlikowski [20]

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to characterize meager-additive sets in 2^ω in a way that nicely parallels Borel's definition of strong measure zero, thus obtaining a theorem that is very similar to the Galvin–Mycielski–Solovay Theorem.

In more detail, a set X has *sharp measure zero* (thereinafter Smz^\sharp) if for any sequence (ε_n) of positive numbers there is a γ -groupable cover $\{U_n\}$ of X such that $\text{diam } U_n \leq \varepsilon_n$ for all n . Note that the only difference between strong measure zero and sharp measure zero is that the cover is required to be γ -groupable (the notion is defined below). With these definitions, the theorem reads

THEOREM [28, Theorem 5.1]. *A set $X \subseteq 2^\omega$ has sharp measure zero if and only if it is meager-additive.*

With some effort, the theorem was shown to hold also in \mathbb{R} and Euclidean spaces. In the present paper we attempt to extend it to a wider class of Polish groups.

The inspiration comes from recent results on strong measure zero: Kysiak [12] and Fremlin [5] showed that the Galvin–Mycielski–Solovay Theorem holds in all locally compact Polish groups. Hrušák et al. [9] and Hrušák and Zapletal [10] (see also [11]) found that among Polish groups admitting an invariant metric, under the Continuum Hypothesis the Galvin–Mycielski–Solovay Theorem fails for groups that are not locally compact.

This of course raises questions about the scope of the above theorem. Does it hold for locally compact metric groups? Does it consistently fail for other Polish groups? We give partial answers to the former questions; to date, the latter remains a mystery.

It turns out that in addition to sharp measure zero and meager-additive sets it is handy to consider a subfamily of meager-additive sets, the so called *sharply meager-additive* sets (see Section 3 for the definition).

The following are the main results of the paper.

THEOREM 1.1. *Let \mathbb{G} be a Polish group equipped with a right-invariant metric. Then every sharply \mathcal{M} -additive set in \mathbb{G} is Smz^\sharp .*

THEOREM 1.2. *Let \mathbb{G} be a locally compact Polish group and $S \subseteq \mathbb{G}$. Then S is Smz^\sharp if and only if S is sharply \mathcal{M} -additive.*

THEOREM 1.3. *Let \mathbb{G} be a Polish group admitting an invariant metric. If $X \subseteq \mathbb{G}$ is \mathcal{M} -additive, then X is Smz^\sharp (in any metric on \mathbb{G}).*

As a corollary to these theorems, we get an equivalence of the three properties in locally compact groups admitting an invariant metric. This class of groups includes all compact or locally compact abelian Polish groups.

THEOREM 1.4. *Let \mathbb{G} be a locally compact group admitting an invariant metric. Let $X \subseteq \mathbb{G}$. The following are equivalent:*

- (i) X is \mathcal{M} -additive,
- (ii) X is sharply \mathcal{M} -additive,
- (iii) X is Smz^\sharp .

Consequences include, for instance:

- Meager-additive sets are preserved by continuous mappings between groups (Theorem 6.1).

- A product of two meager-additive sets is meager-additive (Theorem 6.2).
- Meager-additive sets are universally meager and transitively meager (Propositions 6.5 and 6.6).
- All γ -sets are meager-additive (Proposition 6.8).

We also calculate the uniformity number of the ideal of meager-additive sets in any Polish group admitting an invariant metric (Theorem 7.1).

The paper is organized as follows. In Section 2 we recall elementary material on sharp measure zero and a few technical facts. In Section 3 we introduce sharply meager-additive sets and prove Theorem 1.1. In Section 4 we show that in locally compact groups the classes of sharp measure zero and sharply meager-additive sets coincide, i.e., we prove Theorem 1.2. In Section 5 Theorems 1.3 and 1.4 are proved. In Section 6 we derive the aforementioned consequences and in Section 7 we calculate the uniformity number of meager-additive sets. The last section lists some open problems that we consider interesting.

Some common notation used throughout the paper includes $|A|$ for the cardinality of a set A , ω for the set of natural numbers, ω^ω for the family of all sequences of natural numbers. We also write $E_n \nearrow E$ to denote that $\langle E_n \rangle$ is an increasing sequence of sets with union E . All metric spaces and topological groups we consider are separable. The diameter of a set E in a metric space is denoted $\text{diam } E$. A closed ball of radius r centered at x is denoted by $B(x, r)$.

§2. Sharp measure zero: review. In this section we review a few facts from the theory of sharp measure zero, as developed in [28]. There are many equivalent definitions listed there. We will make use of two of them.

Let's recall once again that by Borel's definition [4] a metric space X has *strong measure zero* (abbreviated as Smz) if for every sequence $\langle \varepsilon_n \rangle$ of positive reals there is a cover $\langle U_n \rangle$ of X such that $\text{diam } U_n \leq \varepsilon_n$ for all n . Our first goal is to describe sharp measure zero in a way that parallels this definition.

We need to recall a few notions regarding covers. Let $\langle U_n \rangle$ be a sequence of subsets of a set X .

- A sequence of sets $\langle U_n \rangle$ is called a γ -cover if each $x \in X$ belongs to all but finitely many U_n .
- A sequence of sets $\langle U_n \rangle$ is called a γ -groupable cover if there is a partition $\omega = I_0 \cup I_1 \cup I_2 \cup \dots$ into consecutive finite intervals (i.e., I_{j+1} is on the right of I_j for all j) such that the sequence $\langle \bigcup_{n \in I_j} U_n : j \in \omega \rangle$ is a γ -cover. The partition $\langle I_j \rangle$ will be occasionally called *witnessing* and the finite families $\{U_n : n \in I_j\}$ will be occasionally called *witnessing families*.

DEFINITION 2.1. A metric space X has *sharp measure zero* (abbreviated as Smz^\sharp) if for every sequence $\langle \varepsilon_n \rangle$ of positive reals there is a γ -groupable cover $\langle U_n \rangle$ of X such that $\text{diam } U_n \leq \varepsilon_n$ for all n .

For a metric space X , the family of all Smz^\sharp subsets of X will be denoted $\text{Smz}^\sharp(X)$.

Sharp measure zero is a σ -additive property and it is preserved by uniformly continuous maps:

PROPOSITION 2.2 [28, 1.2]. (i) For a metric space X , the family $\text{Smz}^\sharp(X)$ is a σ -ideal.

- (ii) If $f : X \rightarrow Y$ is a uniformly continuous mapping and $S \subseteq X$ is Smz^\sharp , then $f(S)$ is Smz^\sharp .

The latter yields another property of Smz^\sharp that is worth mentioning: Smz^\sharp is a *uniform* property: it depends only on the uniformity induced by a metric, not the metric itself. Also, it is not a topological property; we prove that in Remark 7.2.

- LEMMA 2.3 [28, 3.4].
- (i) Every Smz^\sharp set admits a countable cover by totally bounded sets.
 - (ii) In particular, every Smz^\sharp set S in a complete metric space is contained in a σ -compact set.

This lets us relax the uniform continuity condition in Proposition 2.2(ii):

PROPOSITION 2.4. Let X be a complete or σ -compact metric space and $f : X \rightarrow Y$ a continuous mapping. If $S \subseteq X$ is Smz^\sharp , then so is $f(S)$.

PROOF. Lemma 2.3 yields, in either case, a countable cover $\{K_n\}$ of S by compact sets. The maps $f \upharpoonright K_n$ are thus uniformly continuous for all $n \in \omega$. Hence $f(S \cap K_n)$ are Smz^\sharp by 2.2(ii) and $f(S)$ is Smz^\sharp by 2.2(i). \dashv

2.1. Hausdorff measures. The other definition of Smz^\sharp that we will utilize is set up in terms of Hausdorff measure and its modification. We recall the definitions and basic facts.

A non-decreasing, right-continuous function $h : [0, \infty) \rightarrow [0, \infty)$ such that $h(0) = 0$ and $h(r) > 0$ if $r > 0$ is called a *gauge*.

If $\delta > 0$, a cover \mathcal{A} of a set $E \subseteq X$ is termed δ -fine if $\text{diam } A \leq \delta$ for all $A \in \mathcal{A}$. Let h be a gauge. For each $\delta > 0$ set

$$\mathcal{H}_\delta^h(E) = \inf \left\{ \sum_{n \in \omega} h(\text{diam } E_n) : \{E_n\} \text{ is a countable } \delta\text{-fine cover of } E \right\}$$

and define

$$\mathcal{H}^h(E) = \sup_{\delta > 0} \mathcal{H}_\delta^h(E).$$

The set function \mathcal{H}^h is called the *h -dimensional Hausdorff measure*. Properties of Hausdorff measures are well-known. The following, including the two lemmas, can be found e.g., in [22]. The restriction of \mathcal{H}^h to Borel sets is a G_δ -regular Borel measure.

Sharp measure zero requires a slight modification of Hausdorff measure, as it was introduced in [28]. Let h be a gauge. For each $\delta > 0$ set

$$\overline{\mathcal{H}}_\delta^h(E) = \inf \left\{ \sum_{n=0}^N h(\text{diam } E_n) : \{E_n\} \text{ is a finite } \delta\text{-fine cover of } E \right\}.$$

Note that the only difference is that only finite covers are taken in account, as opposed to countable covers in the definition of $\mathcal{H}_\delta^h(E)$. Then let

$$\overline{\mathcal{H}}_0^h(E) = \sup_{\delta > 0} \overline{\mathcal{H}}_\delta^h(E).$$

It is easy to check that $\overline{\mathcal{H}}_0^h$ is a finitely subadditive set function. However, it is not a measure, since it need not be σ -additive. That is why another step is required: we need to apply the operation known as Munroe’s *Method I construction* (cf. [16] or [22]):

$$\overline{\mathcal{H}}^h(E) = \inf \left\{ \sum_{n \in \omega} \overline{\mathcal{H}}_0^h(E_n) : E \subseteq \bigcup_{n \in \omega} E_n \right\}.$$

Thus defined set function is indeed an outer measure whose restriction to Borel sets is a Borel measure. It is called the *h-dimensional upper Hausdorff measure*.

Detailed information on the upper Hausdorff measure is provided in [28], see also [27]. Here we only recall two properties that we will make use of.

LEMMA 2.5 [28, Lemma 3.4]. *Let h be a gauge and E a set in a metric space. If E is σ -compact, then $\overline{\mathcal{H}}^h(E) = \mathcal{H}^h(E)$.*

LEMMA 2.6 [28, Lemma 3.9]. *Let h be a gauge and E a set in a metric space. If E has a γ -groupable cover $\langle U_n \rangle$ such that $\sum_{n \in \omega} h(\text{diam } U_n) < \infty$, then $\overline{\mathcal{H}}^h(E) = 0$.*

By a classical result of Besicovitch [2, 3] (cf. [28] for detailed proofs), a metric space X is Smz if and only if $\mathcal{H}^h(X) = 0$ for all gauges h . It is nice that Smz $^\sharp$ sets are characterized in exactly the same way, only that the upper Hausdorff measures are used in place of Hausdorff measures:

THEOREM 2.7 [28, Theorem 3.7]. *A metric space X is Smz $^\sharp$ if and only if $\overline{\mathcal{H}}^h(X) = 0$ for each gauge h .*

§3. Sharply \mathcal{M} -additive sets. Given a metric space X , we write $\mathcal{M}(X)$ to denote the family of all meager sets in X , and likewise for a topological group. In most cases there is no danger of confusion, so we write only \mathcal{M} instead.

Let us clarify our terminology regarding Polish groups first. A *Polish group* is a topological group that is homeomorphic to a Polish space, i.e., a completely metrizable separable topological space. (A *metric on a group* is a metric whose induced topology is the group topology.) Trivially, every Polish group admits a complete metric. It also admits a right-invariant metric. The two metrics, though, do not have to coincide: there are Polish groups that do not have a metric that is simultaneously right-invariant and complete. Also, not every Polish group has a (two-sided) invariant metric. However, if the group is abelian or compact, then it has an invariant metric. Any invariant metric on a Polish group is complete.

Recall that Prikry [21] proved that if \mathbb{G} is a separable group with a right-invariant metric and $X \subseteq \mathbb{G}$ is a set satisfying $\forall M \in \mathcal{M} \ MX \neq \mathbb{G}$, then X has strong measure zero. This is the easy part of the Galvin–Mycielski–Solovay Theorem discussed above. In this section we prove a counterpart of Prikry’s result for sharp measure zero. We first provide a formal definition of an \mathcal{M} -additive set.

DEFINITION 3.1. Let \mathbb{G} be a Polish group and $X \subseteq \mathbb{G}$. The set X is called *\mathcal{M} -additive* (or *meager-additive*) if for every meager set $M \subseteq \mathbb{G}$, the set MX is meager.

We will write $\mathcal{M}^*(\mathbb{G})$ for the σ -ideal of \mathcal{M} -additive sets in \mathbb{G} .

\mathcal{M} -additive sets are difficult to deal with in a general Polish group. The following property, though appearing more complex, is much easier, as we shall see.

PROPOSITION 3.2. *Let \mathbb{G} be a Polish group and $X \subseteq \mathbb{G}$. The following are equivalent.*

- (i) $\forall M \in \mathcal{M} \exists K \supseteq X \sigma$ -compact $MK \neq \mathbb{G}$ and
- (ii) $\forall M \in \mathcal{M} \exists K \supseteq X \sigma$ -compact $MK \in \mathcal{M}$.

PROOF. Let X satisfy (i) and $M \in \mathcal{M}$. We may suppose M is F_σ . Let $Q \subseteq \mathbb{G}$ be countable and dense. Then QM is meager, hence there is a σ -compact set $K \supseteq X$ and $z \notin QMK$. The latter yields $Q^{-1}z \cap MK = \emptyset$. Since Q is dense, so is $Q^{-1}z$, thus MK is disjoint with a dense set. If we prove that MK is F_σ , we are done. To that end it is enough show that if C is compact and F is closed, then FC is closed. Suppose $y \notin FC$. Let d be a left-invariant metric on \mathbb{G} . Define a function on C by $f(x) = d(Fx, y) = d(x, F^{-1}y)$. Since Fx is closed for each $x \in C$, f positive and it is clearly continuous. Therefore there is $\delta > 0$ such that $\inf_{x \in C} f(x) > \delta$. Hence $d(Fx, y) > \delta$ for all $x \in C$ and $d(FC, y) \geq \delta > 0$ follows. We proved that $d(FC, y) > 0$ for each $y \notin FC$, i.e., FC is closed, which concludes the proof. \dashv

It is clear that the property described by this proposition is stronger than \mathcal{M} -additivity. We capture it in the following definition.

DEFINITION 3.3. Let \mathbb{G} be a Polish group and $X \subseteq \mathbb{G}$. The set X is called *sharply \mathcal{M} -additive* if for every meager set $M \subseteq \mathbb{G}$ there is a σ -compact set $K \supseteq X$ such that MK is meager.

We will write $\mathcal{M}^\sharp(\mathbb{G})$ for the σ -ideal of sharply \mathcal{M} -additive sets in \mathbb{G} .

As proved in [28], in 2^ω and \mathbb{R} and its finite powers, every \mathcal{M} -additive set is sharply \mathcal{M} -additive. We shall see later that the same remains true in a wide class of locally compact Polish groups.

Our first theorem shows that sharply \mathcal{M} -additive sets are of sharp measure zero.

THEOREM 3.4. *Let \mathbb{G} be a Polish group equipped with a right-invariant metric. Then every sharply \mathcal{M} -additive set in \mathbb{G} is Smz^\sharp .*

PROOF. Let d be a right-invariant metric on \mathbb{G} and let $X \subseteq \mathbb{G}$ be a sharply \mathcal{M} -additive set. Let g be an arbitrary gauge. We want to show that $\overline{\mathcal{H}}^g(X) = 0$. Let $Q \subseteq \mathbb{G}$ be a countable dense symmetric set. Since Hausdorff measures are G_δ -regular, there is a G_δ -set $H \supseteq Q$ such that $\mathcal{H}^g(H) = 0$. Since Q is symmetric, we may, replacing H with $H \cap H^{-1}$ if necessary, suppose that H is symmetric as well. The set $M = \mathbb{G} \setminus H$ is obviously meager. Since X is sharply \mathcal{M} -additive, there is a σ -compact set $K \supseteq X$ and $z \notin MK$. Routine manipulation leads to $K \subseteq H^{-1}z = Hz$. Since the underlying metric is right-invariant, with the aid of Lemma 2.5 we get

$$\overline{\mathcal{H}}^g(X) \leq \overline{\mathcal{H}}^g(K) = \mathcal{H}^g(K) \leq \mathcal{H}^g(Hz) = \mathcal{H}^g(H) = 0,$$

as required. \dashv

§4. Sharply \mathcal{M} -additive sets in locally compact groups. In this section we prove that in locally compact Polish groups, sharp meager-additivity and sharp measure zero are equivalent notions. The following theorem may be regarded a “Galvin–Mycielski–Solovay theorem for \mathcal{M} -additive sets.”

THEOREM 4.1. *Let \mathbb{G} be a locally compact Polish group and $S \subseteq \mathbb{G}$. Then S is Smz^\sharp if and only if S is sharply \mathcal{M} -additive.*

It turns out that this theorem is a particular case of a more general result. Its proof is postponed until the end of the section. Let us first note that it follows from Proposition 2.4 that the notion of sharp measure zero in a locally compact metric group is independent of the choice of metric.

PROPOSITION 4.2. *Let d_1 and d_2 be two metrics on a locally compact metrizable group \mathbb{G} . If $X \subseteq \mathbb{G}$ is Smz^\sharp with respect to d_1 , then so it is with respect to d_2 .*

Thus we may refer to a Smz^\sharp set in a locally compact group without referring to a metric on \mathbb{G} .

The following theorem is a group-free version of Theorem 4.1.

THEOREM 4.3. *Let Y be a Polish metric space and X a separable locally compact metric space. Let $\phi : Y \times X \rightarrow X$ be a continuous mapping such that for each $y \in Y$ and every compact nowhere dense set $P \subseteq X$ the image $\phi(\{y\} \times P)$ is nowhere dense. If $S \subseteq Y$ is Smz^\sharp and $M \subseteq X$ is meager, then there is a σ -compact set $\widehat{S} \supseteq S$ such that $\phi(\widehat{S} \times M)$ is meager.*

PROOF. We need the following combinatorial lemma.

LEMMA 4.4 [10, Lemma 6.2]. *Let K be a compact metric space and X a separable locally compact metric space. Let $U \subseteq X$ be an open set with compact closure $C = \overline{U}$ and $P \subseteq X$ be compact nowhere dense. Let $\phi : K \times X \rightarrow X$ be a continuous mapping such that for each $y \in K$ the image $\phi(\{y\} \times P)$ is nowhere dense. Then*

$$\forall \varepsilon > 0 \exists \delta > 0 \forall x \in C \forall y \in K \exists z \in C \quad B(z, \delta) \subseteq B(x, \varepsilon) \setminus \phi(B(y, \delta) \times P).$$

Since S is Smz^\sharp and Y is complete, S is contained in a σ -compact subset of Y , so we may suppose Y is σ -compact. Let K_n be compact sets in Y with $K_n \nearrow Y$ and let P_n be compact nowhere dense sets with $P_n \nearrow M$. Let $\{U_k\}$ be a countable base of X . For each k choose $x_0^k \in X$ and $\varepsilon_0^k > 0$ such that $B(x_0^k, \varepsilon_0^k) \subseteq U_k$ is compact, and let $C_k = B(x_0^k, \varepsilon_0^k)$. Use Lemma 4.4 to recursively construct a sequence $\langle \varepsilon_n \rangle \in (0, \infty)^\omega$ such that

$$\forall n \forall i \leq n \forall x \in C_i \forall y \in K_n \exists z \in C_i \quad B(z, \varepsilon_n) \subseteq B(x, \varepsilon_{n-1}) \setminus \phi((B(y, \varepsilon_n) \cap K_n) \times P_n). \quad (1)$$

Since S is Smz^\sharp , there is an γ -groupable cover $\{E_n\}$ of S such that $\text{diam } E_n < \varepsilon_n$ for all n . Hence for each n there is y such that $E_n \subseteq B(y, \varepsilon_n)$. Therefore we may use (1) to construct sequences $\langle x_n^k : n \in \omega \rangle$ such that for all k

$$B(x_{n+1}^k, \varepsilon_{n+1}) \subseteq B(x_n^k, \varepsilon_n) \setminus \phi((E_{n+1} \cap K_{n+1}) \times P_{n+1}). \quad (2)$$

It is easy to check that since $\{E_n\}$ is a γ -groupable cover of S and $K_n \nearrow Y$, the family $\{E_n \cap K_n\}$ is also a γ -groupable cover of S . Thus we might have supposed that $E_n \subseteq K_n$, and also that all E_n 's are closed. Therefore (2) simplifies to

$$B(x_{n+1}^k, \varepsilon_{n+1}) \subseteq B(x_n^k, \varepsilon_n) \setminus \phi(E_{n+1} \times P_{n+1}). \quad (3)$$

In particular, $B(x_n^k, \varepsilon_n)$ is a decreasing sequence of compact balls for all k and thus there is a point $x^k \in U_k$ such that

$$x^k \notin \bigcup_{n \in \omega} \phi(E_n \times P_n). \quad (4)$$

The set \widehat{S} is constructed as follows: Let \mathcal{G}_j be the families of E_n 's witnessing the γ -groupability of $\{E_n\}$. Put $G_n = \bigcup_{n \in \mathcal{G}_j} E_n$ and let $F_n = \bigcap_{i < n} G_i$ and $\widehat{S} = \bigcup_{n \in \omega} F_n$. It is clear that since E_n 's are closed, the set \widehat{S} is F_σ , and clearly $S \subseteq \widehat{S}$. Moreover, routine calculation shows that $\widehat{S} \times M \subseteq \bigcup_n E_n \times P_n$. Therefore (4) yields $x^k \notin \phi(\widehat{S} \times M)$ for all k . So letting $D = \{x^k : k \in \omega\}$, the set D is disjoint with $\phi(\widehat{S} \times M)$ and it is dense in X . Since \widehat{S} and M are σ -compact, so is $\phi(\widehat{S} \times M)$. Therefore $\phi(\widehat{S} \times M)$ is an F_σ set disjoint with a dense set, and is thus meager. \dashv

The following is obviously a particular case of the theorem.

COROLLARY 4.5. *Let $\phi : \mathbb{G} \curvearrowright X$ be an action of a Polish group \mathbb{G} on a σ -compact space X . If $S \subseteq \mathbb{G}$ is Smz^\sharp , then there is σ -compact set $\widehat{S} \supseteq S$ such that $\phi(\widehat{S} \times M)$ is meager.*

PROOF OF THEOREM 4.1. Let $X = \mathbb{G}$, define $\phi : \mathbb{G} \curvearrowright \mathbb{G}$ by $\phi(x, y) = yx$ and apply the corollary to get the forward implication. The reverse implication follows from Theorem 3.4. \dashv

§5. Meager-additive sets. In the previous sections we linked sharp measure zero to sharp meager-additivity. In this section we investigate \mathcal{M} -additive sets in the hope that we can link them to sharp measure zero sets as well.

We first note that if a set in a Polish group is Smz^\sharp in some complete metric, then it is Smz^\sharp in any other metric as well. This follows at once from Proposition 2.4. Thus in the following theorem the particular metric does not matter.

THEOREM 5.1. *Let \mathbb{G} be a Polish group admitting an invariant metric. If $X \subseteq \mathbb{G}$ is \mathcal{M} -additive, then X is Smz^\sharp (in any metric on \mathbb{G}).*

Let us recall once again that not every Polish group admits an invariant metric, but if it is compact or abelian, then it does. Also, any invariant metric on a Polish group is complete. Polish groups that admit an invariant metric are referred to as TSI groups.

The following easily follows from Theorems 5.1 and 4.1.

THEOREM 5.2. *Let \mathbb{G} be a locally compact TSI group. Let $X \subseteq \mathbb{G}$. The following are equivalent:*

- (i) X is \mathcal{M} -additive,
- (ii) X is sharply \mathcal{M} -additive,
- (iii) X is Smz^\sharp .

The rest of this section is devoted to the proof of Theorem 5.1.

The symbol \mathbb{G} stands for the Polish group. We suppose it is equipped with an invariant (and thus complete) metric throughout this section. The metric will be denoted by d . We will use additive notation for the group operation, though \mathbb{G} is not a priori assumed to be abelian.

We shall deal with series in \mathbb{G} . Write $\|x\| = d(0, x)$. Suppose that $\langle x_n \rangle$ is a sequence in \mathbb{G} such that $\sum_{n \in \omega} \|x_n\| < \infty$. Then the sequence of partial sums is Cauchy and therefore converges. We will thus write $\sum_{n \in \omega} x_n$ for $\lim_{n \rightarrow \infty} (x_0 + x_1 + \dots + x_n)$.

Given $\varepsilon > 0$, a set $B \subseteq \mathbb{G}$ is ε -separated if $d(p, q) \geq \varepsilon$ for any two distinct points in B .

The grid. For each $n \in \omega$ set $\varepsilon_n = 2^{-n}$. We use this notation throughout. Define a system of grids in \mathbb{G} as follows.

- Let Q_0 be a set that is maximal among all ε_0 -separated sets in \mathbb{G} containing 0.
- For $n \geq 1$ let Q_n be a set that is maximal among all ε_n -separated subsets of the closed ball $B(0, \varepsilon_{n-1})$ containing 0.

The maximality ensures that any point in $B(0, \varepsilon_{n-1})$ is within distance ε_n from some point of Q_n .

Each Q_n is finite or countable. Provide it with a well-ordering so that 0 is the first point in the order and

- let $Q_{n,m}$ be the set of the first m points in Q_n .

Then set

- $Q_n^* = Q_{0,n} + Q_{1,n} + \dots + Q_{n,n}$ and
- $Q = \bigcup_{n \in \omega} Q_n^*$.

It is easy to check that Q is a dense set.

Base for meager sets. Next we define canonical meager sets in \mathbb{G} . We will use the following notation: ω^ω is the family of all sequences of natural numbers; $\omega^{\uparrow\omega} \subseteq \omega^\omega$ is the family of nondecreasing unbounded sequences of natural numbers. The quantifiers \forall^∞ and \exists^∞ have the usual meaning: “for all except finitely many” and “for infinitely many.” The symbol \underline{d} denotes the lower distance.

DEFINITION 5.3. Let $f \in \omega^{\uparrow\omega}$, $x \in \mathbb{G}$ and $c > 0$. Let

$$H(f, x, c) = \{z \in \mathbb{G} : \forall^\infty n \underline{d}(z, Q_{f(n)}^* + x) \geq c \varepsilon_{f(n+1)}\}.$$

LEMMA 5.4. The set $H(f, x, c)$ is meager for any $f \in \omega^{\uparrow\omega}$, $x \in \mathbb{G}$ and $c > 0$.

PROOF. $H(f, x, c)$ is disjoint with the set

$$\{z \in \mathbb{G} : \exists^\infty n z \in Q_{f(n)}^* + x\} = \bigcap_{n \in \omega} \bigcup_{m \geq n} (Q_{f(m)}^* + x) = Q + x.$$

Since Q is dense in \mathbb{G} , so is $Q + x$. On the other hand, $H(f, x, c)$ is obviously F_σ . Hence it is meager. ⊥

Not only that the sets $H(f, x, c)$ are meager, they actually form a base for meager sets in \mathbb{G} . We need yet a little more:

LEMMA 5.5. $\forall M \in \mathcal{M} \forall f \in \omega^{\uparrow\omega} \exists g \in \omega^{\uparrow\omega} \exists x \in \mathbb{G} M \subseteq H(f \circ g, x, 1)$

PROOF. Suppose F_n are closed nowhere dense sets such that $F_n \nearrow M$. Write $G_n = \mathbb{G} \setminus F_n$.

We recursively define a sequence $\langle x_n \rangle$ in \mathbb{G} and an increasing sequence $\langle k_n \rangle$ of integers subject to the following conditions. Write $y_n = x_0 + x_1 + \dots + x_n$.

$$x_{n+1} \in Q_{f(k_n)+1} + Q_{f(k_n)+2} + \dots + Q_{f(k_{n+1})}, \tag{5}$$

$$\underline{d}(F_n, Q_{f(k_n)}^* + y_{n+1}) > 3\varepsilon_{f(k_{n+1})}. \tag{6}$$

Let $k_0 = 0$ and $x_0 = 0$ and proceed by induction. Suppose that x_n and k_n are defined. The sets $-x_n - q - G_n$ are open dense for each $q \in \mathbb{G}$. Since $Q_{f(k_n)}^*$ is finite, there is $z \in \mathbb{G}$ and $\delta > 0$ such that

$$B(z, \delta) \subseteq \bigcap_{q \in Q_{f(k_n)}^*} (-y_n - q + G_n) \cap B(0, \varepsilon_{f(k_n)}). \tag{7}$$

Choose $k_{n+1} > k_n$ large enough to satisfy $4\varepsilon_{f(k_{n+1})} < \delta$.

Denote for the moment $m = f(k_n)$ and $j = f(k_{n+1}) - m$.

- Since $\|z\| \leq \varepsilon_m$, there is $t_1 \in Q_{m+1}$ such that $d(t_1, z) \leq \varepsilon_{m+1}$.
- \vdots
- Since $\| -t_1 + z \| \leq \varepsilon_{m+1}$, there is $t_2 \in Q_{m+2}$ such that $d(t_2, -t_1 + z) \leq \varepsilon_{m+2}$.
- Proceed by induction up to $m + j$ to get, for each $i \in [1, j]$, $t_i \in Q_{m+i}$ such that

$$\| -t_j - t_{j-1} - \dots - t_1 + z \| \leq \varepsilon_{m+j}. \tag{8}$$

Put $x_{n+1} = t_1 + t_2 + \dots + t_j$. Condition (5) is clearly satisfied.

Using left invariance of the metric, inequality (8) reads $d(z, x_{n+1}) \leq \varepsilon_{f(k_{n+1})}$.

Therefore $4\varepsilon_{f(k_{n+1})} < \delta$ and (7) yield

$$B(x_{n+1}, 3\varepsilon_{f(k_{n+1})}) \subseteq B(z, 4\varepsilon_{f(k_{n+1})}) \subseteq B(z, \delta) \subseteq -y_n - q + G_n$$

for all $q \in Q_{f(k_n)}^*$. Using left invariance of the metric again, we get $B(q + y_{n+1}, 3\varepsilon_{f(k_{n+1})}) \cap F_n = \emptyset$ and (6) follows. Since $Q_{j+1} \subseteq B(0, \varepsilon_j)$ for all j , (5) yields $\|x_{n+1}\| \leq \sum_{j=f(k_n)}^\infty \varepsilon_j = 2\varepsilon_{f(k_n)}$. In particular, $\sum \|x_n\| < \infty$, whence the series $\sum x_n$ converges. Set $x = \sum x_n = \lim y_n$ and $g(n) = k_n$.

(5) also yields $d(x, y_{n+1}) \leq 2\varepsilon_{f(k_{n+1})}$. Combine this inequality with (6) and the triangle inequality to get, for each $q \in Q_{f(k_n)}^*$,

$$\begin{aligned} \underline{d}(F_n, q + x) &\geq \underline{d}(F_n, q + y_{n+1}) - d(q + y_{n+1}, q + x) \\ &\geq 3\varepsilon_{f(k_{n+1})} - d(y_{n+1}, x)\varepsilon_{f(k_{n+1})} = \varepsilon_{fg(n+1)}. \end{aligned}$$

It follows that $\underline{d}(F_n, Q_{f(k_n)}^* + x) \geq \varepsilon_{fg(n+1)}$ for all n , which is enough. -†

Key lemma. We now prove an important combinatorial lemma on the canonical meager sets.

LEMMA 5.6. *Let $x, y \in \mathbb{G}$ and $f, g \in \omega^{\uparrow\omega}$. Suppose that for each $k \in \omega$*

$$|Q_{f(k)}| > 1, \tag{9}$$

$$Q_{f(k)}^* \text{ is } 3\varepsilon_{f(k+1)}\text{-separated,} \tag{10}$$

$$f(k + 1) \geq f(k) + 4. \tag{11}$$

If $H(f, x, \frac{1}{4}) \subseteq H(f \circ g, y, 1)$, then

$$\forall^\infty n \in \omega \exists k \in [g(n), g(n + 1)) \quad \underline{d}(Q_{f(k)}^* + x, Q_{f(k)}^* + y) \leq \varepsilon_{f(k+1)}.$$

PROOF. Using right invariance of the metric we may assume $y = 0$. Set

$$S = \{n \in \omega : \forall k \in [g(n), g(n + 1)) \quad \underline{d}(\mathcal{Q}_{f(k)}^* + x, \mathcal{Q}_{f(k)}^*) > \varepsilon_{f(k+1)}\}.$$

We need to prove that S is finite. So aiming towards contradiction suppose that $|S| = \omega$. Our goal is to construct $\tau \in \mathbb{G}$ such that

$$\forall n \in S \quad \underline{d}(\tau, \mathcal{Q}_{fg(n)}^*) < \varepsilon_{fg(n+1)}, \tag{12}$$

$$\forall^\infty k \in \omega \quad \underline{d}(\tau, \mathcal{Q}_{f(k)}^* + x) \geq \frac{1}{4}\varepsilon_{f(k+1)}. \tag{13}$$

Once we have such a τ , since $|S| = \omega$, condition (12) will ensure that $\tau \notin H(f \circ g, 0, 1)$. On the other hand, condition (13) will ensure that $\tau \in H(f, x, \frac{1}{4})$: the desired contradiction. ⊥

τ will be defined as a sum of a series. For each $k \in \omega$ we shall choose $\tau_k \in \mathcal{Q}_{f(k+1)}$ according to the following rules, and then we let $\tau = \sum_{k \in \omega} \tau_k$.

For any k put $T_k = \sum_{i < k} \tau_i$ and $t_k = \sum_{i \geq k} \tau_i$.

The rules for recursive construction of τ_k 's are as follows:

- (a) If there is $n \in S$ such that $k \in [g(n), g(n + 1))$, put $\tau_k = 0$.
- (b) If there is $n \notin S$ such that $k \in [g(n), g(n + 1))$ and $\underline{d}(T_k, \mathcal{Q}_{f(k)}^* + x) \geq \frac{1}{2}\varepsilon_{f(k+1)}$, put $\tau_k = 0$.
- (c) If there is $n \notin S$ such that $k \in [g(n), g(n + 1))$ and $\underline{d}(T_k, \mathcal{Q}_{f(k)}^* + x) < \frac{1}{2}\varepsilon_{f(k+1)}$, let τ_k be the first nonzero element of $\mathcal{Q}_{f(k+1)}$. Condition (9) ensures that such a choice is possible.

The rules ensure that

$$T_k \in \mathcal{Q}_{f(1),2} + \mathcal{Q}_{f(2),2} + \dots + \mathcal{Q}_{f(k),2} \subseteq \mathcal{Q}_{f(k)}^*. \tag{14}$$

Notice also that since $\tau_i \in \mathcal{Q}_{f(i+1)}$, we have $\|\tau_i\| \leq 2\varepsilon_{f(i+1)}$ and thus condition (11) yields

$$\|t_k\| \leq \frac{2}{15}\varepsilon_{f(k)}. \tag{15}$$

We prove (12) first. Suppose $n \in S$. Since $T_{g(n)} \in \mathcal{Q}_{fg(n)}^*$ by (14), we have

$$\underline{d}(\tau, \mathcal{Q}_{fg(n)}^*) \leq \underline{d}(T_{g(n)} + t_{g(n)}, T_{g(n)}) = \|t_{g(n)}\|.$$

Since $n \in S$, rule (a) yields $t_{g(n)} = t_{g(n+1)}$ and thus (15) gives

$$\underline{d}(\tau, \mathcal{Q}_{fg(n)}^*) \leq \|t_{g(n+1)}\| \leq \frac{2}{15}\varepsilon_{fg(n+1)} < \varepsilon_{fg(n+1)}.$$

Thus (12) is proved. The proof of (13) is split into three cases corresponding to the three rules (a)–(c).

CLAIM 1. If $n \in S$ and $k \in [g(n), g(n + 1))$, then $\underline{d}(\tau, \mathcal{Q}_{f(k)}^* + x) \geq \frac{1}{4}\varepsilon_{f(k+1)}$.

PROOF. Since $n \in S$, (14) implies $\underline{d}(\mathcal{Q}_{f(k)}^* + x, T_k) > \varepsilon_{f(k+1)}$. Rule (a) yields $t_k = t_{g(n+1)}$. Hence $\|t_k\| = \|t_{g(n+1)}\| \leq \frac{2}{15}\varepsilon_{fg(n+1)} \leq \frac{2}{15}\varepsilon_{f(k+1)}$. Combine and use triangle inequality and left invariance of the metric to get

$$\begin{aligned} \underline{d}(\mathcal{Q}_{f(k)}^* + x, \tau) &\geq \underline{d}(\mathcal{Q}_{f(k)}^* + x, T_k) - \underline{d}(T_k, \tau) \\ &> \varepsilon_{f(k+1)} - \|t_k\| \geq \varepsilon_{f(k+1)} - \frac{2}{15}\varepsilon_{f(k+1)} > \frac{1}{4}\varepsilon_{f(k+1)}. \quad \dashv \end{aligned}$$

CLAIM 2. If $n \notin S$, $k \in [g(n), g(n + 1))$ and $\underline{d}(T_k, \mathcal{Q}_{f(k)}^* + x) \geq \frac{1}{2}\varepsilon_{f(k+1)}$, then $\underline{d}(\tau, \mathcal{Q}_{f(k)}^* + x) \geq \frac{1}{4}\varepsilon_{f(k+1)}$

PROOF. This time rule (b) applies. Since $\tau_k = 0$, we have $t_k = t_{k+1}$. Use the same arguments as before and the assumption.

$$\begin{aligned} \underline{d}(\mathcal{Q}_{f(k)}^* + x, \tau) &\geq \underline{d}(\mathcal{Q}_{f(k)}^* + x, T_k) - \underline{d}(T_k, \tau) \\ &\geq \frac{1}{2}\varepsilon_{f(k+1)} - \|t_k\| = \frac{1}{2}\varepsilon_{f(k+1)} - \|t_{k+1}\| \\ &\geq \frac{1}{2}\varepsilon_{f(k+1)} - \frac{2}{15}\varepsilon_{f(k+1)} > \frac{1}{4}\varepsilon_{f(k+1)}. \quad \dashv \end{aligned}$$

CLAIM 3. If $n \notin S$, $k \in [g(n), g(n + 1))$ and $\underline{d}(T_k, \mathcal{Q}_{f(k)}^* + x) < \frac{1}{2}\varepsilon_{f(k+1)}$, then $\underline{d}(\tau, \mathcal{Q}_{f(k)}^* + x) \geq \frac{1}{4}\varepsilon_{f(k+1)}$.

PROOF. This is the only nontrivial case. Geometrically, an approximation T_k of τ is close to some point of $\mathcal{Q}_{f(k)}^* + x$, so that the next term τ_k in the sequence is chosen large enough to move the new approximation T_{k+1} of τ away from that point. We however have to make sure that T_{k+1} does not get close to another point of $\mathcal{Q}_{f(k)}^* + x$. That is why condition (10) is required.

Write $\tau = T_k + \tau_k + t_{k+1}$. We know that $\|t_{k+1}\| \leq \frac{2}{15}\varepsilon_{f(k+1)}$. Since $\tau_k \neq 0$, we also have $\varepsilon_{f(k+1)} \leq \|\tau_k\| \leq 2\varepsilon_{f(k+1)}$.

By assumption, there is $q_0 \in \mathcal{Q}_{f(k)}^*$ such that $d(q_0 + x, T_k) < \frac{1}{2}\varepsilon_{f(k+1)}$. First estimate $d(q_0 + x, \tau)$ from above:

$$\begin{aligned} d(q_0 + x, \tau) &\leq d(q_0 + x, T_k) + \|\tau_k\| + \|t_{k+1}\| \\ &\leq \left(\frac{1}{2} + 2 + \frac{2}{15}\right)\varepsilon_{f(k+1)} < \frac{8}{3}\varepsilon_{f(k+1)}. \end{aligned}$$

Use this inequality to estimate $d(p + x, \tau)$ from below for any $p \in \mathcal{Q}_{f(k)}^*$, $p \neq q_0$. Here condition (10) gets in play: it yields $d(p + x, q_0 + x) = d(p, q_0) \geq 3\varepsilon_{f(k+1)}$ and thus

$$\begin{aligned} d(p + x, \tau) &\geq d(p + x, q_0 + x) - d(q_0 + x, \tau) \\ &\geq 3\varepsilon_{f(k+1)} - \frac{8}{3}\varepsilon_{f(k+1)} > \frac{1}{4}\varepsilon_{f(k+1)}. \end{aligned}$$

It remains to show that $d(q_0 + x, \tau) > \frac{1}{4}\varepsilon_{f(k+1)}$:

$$\begin{aligned} d(q_0 + x, \tau) &\geq \|\tau_k\| - d(q_0 + x, T_k) - \|t_{k+1}\| \\ &\geq \varepsilon_{f(k+1)} - \frac{1}{2}\varepsilon_{f(k+1)} - \frac{2}{15}\varepsilon_{f(k+1)} > \frac{1}{4}\varepsilon_{f(k+1)}. \quad \dashv \end{aligned}$$

We can now finish the proof of Lemma 5.6: Claims 1–3 yield (13). Therefore $\tau \in H(f, x, \frac{1}{4}) \setminus H(f \circ g, 0, 1)$: the required contradiction.

PROOF OF THEOREM 5.1. Let $X \subseteq \mathbb{G}$ be \mathcal{M} -additive. We will show that $\overline{\mathcal{H}}^h(X) = 0$ for every gauge h .

Begin by recursively building $f \in \omega^{\uparrow\omega}$ subject to conditions (9)–(11) and

$$\forall k \in \omega \quad h(2\varepsilon_{f(k+1)}) < \frac{2^{-k}}{|\mathcal{Q}_{f(k)}^*|^2}. \tag{16}$$

Consider the set $H(f, 0, \frac{1}{4})$. It is meager by Lemma 5.4. Therefore $H(f, 0, \frac{1}{4}) + X$ is meager as well. By virtue of Lemma 5.5 there thus exists $g \in \omega^{\uparrow\omega}$ and $y \in \mathbb{G}$ such that $H(f, 0, \frac{1}{4}) + x \subseteq H(f \circ g, y, 1)$ for all $x \in X$. Right invariance of the metric gives $H(f, 0, \frac{1}{4}) + x = H(f, x, \frac{1}{4})$. It follows that $H(f, x, \frac{1}{4}) \subseteq H(f \circ g, y, 1)$ for all $x \in X$. Lemma 5.6 thus yields

$$\forall x \in X \forall^\infty n \in \omega \exists k \in [g(n), g(n+1)) \underline{d}(\mathcal{Q}_{f(k)}^* + x, \mathcal{Q}_{f(k)}^* + y) \leq \varepsilon_{f(k+1)}. \tag{17}$$

The latter inequality means that there are $p, q \in \mathcal{Q}_{f(k)}^*$ such that $d(p + x, q + y) \leq \varepsilon_{f(k+1)}$ which is, via left invariance of the metric, equivalent to

$$x \in B(-p + q + y, \varepsilon_{f(k+1)}).$$

Therefore (17) can be phrased as follows:

$$\forall x \in X \forall^\infty n \in \omega \exists k \in [g(n), g(n+1)) \exists p, q \in \mathcal{Q}_{f(k)}^* \quad x \in B(-p + q + y, \varepsilon_{f(k+1)}). \tag{18}$$

Set

$$\begin{aligned} 2\mathcal{B}_k &= \{B(-p + q + y, \varepsilon_{f(k+1)}) : p, q \in \mathcal{Q}_{f(k)}^*\}, & k \in \omega, \\ \mathcal{G}_n &= \bigcup \{\mathcal{B}_k : g(n) \leq k < g(n+1)\}, & n \in \omega, \\ \mathcal{G} &= \bigcup_{k \in \omega} \mathcal{B}_k = \bigcup_{n \in \omega} \mathcal{G}_n. \end{aligned}$$

With this setting, (18) reads

$$\forall x \in X \forall^\infty n \in \omega \exists G \in \mathcal{G}_n \quad x \in G.$$

For all $k \in \omega$ we have $|\mathcal{B}_k| \leq |\mathcal{Q}_{f(k)}^*|^2$. In particular, the families \mathcal{G}_n are finite, witnessing that \mathcal{G} is a γ -groupable cover of X .

By Lemma 2.6, in order to prove that $\overline{\mathcal{H}}^h(X) = 0$ it is enough to show that $\sum_{G \in \mathcal{G}} h(\text{diam } G)$ is finite. Clearly $\text{diam } B(-p + q + y, \varepsilon_{f(k+1)}) \leq 2\varepsilon_{f(k+1)}$. Therefore (16) yields

$$\begin{aligned} \sum_{G \in \mathcal{G}} h(\text{diam } G) &= \sum_{k \in \omega} \sum_{G \in \mathcal{B}_k} h(\text{diam } G) \\ &= \sum_{k \in \omega} \sum_{p, q \in \mathcal{Q}_{f(k)}^*} h(\text{diam } B(-p + q + y, \varepsilon_{f(k+1)})) \end{aligned}$$

$$\leq \sum_{k \in \omega} |Q_{f(k)}^*|^2 h(2\varepsilon_{f(k+1)}) \leq \sum_{k \in \omega} 2^{-k} < \infty.$$

The proof of Theorem 5.1 is complete. \dashv

§6. Some consequences.

Continuous images and cartesian products. Meager-additive sets are preserved by continuous mappings between Polish groups and by cartesian products as follows:

THEOREM 6.1. *Let \mathbb{G}_1 be a TSI Polish group and \mathbb{G}_2 a locally compact Polish group. Let $f : \mathbb{G}_1 \rightarrow \mathbb{G}_2$ a continuous mapping. If $X \subseteq \mathbb{G}_1$ is \mathcal{M} -additive, then so is $f(X)$.*

PROOF. Since X is \mathcal{M} -additive it is $\text{Smz}^\#$ by Theorem 5.1. By Proposition 2.4, $f(X)$ is $\text{Smz}^\#$ as well. By Theorem 4.1, $f(X)$ is \mathcal{M} -additive. \dashv

THEOREM 6.2. *Let $\mathbb{G}_1, \mathbb{G}_2$ be TSI locally compact groups. Let $X_1 \subseteq \mathbb{G}_1, X_2 \subseteq \mathbb{G}_2$.*

- (i) *If X_1 is Smz and X_2 is \mathcal{M} -additive, then $X_1 \times X_2$ is Smz .*
- (ii) *If X_1 and X_2 are \mathcal{M} -additive, then so is $X_1 \times X_2$.*

PROOF. (i) By Theorem 5.2, X_2 is $\text{Smz}^\#$. By [28, Theorem 3.14], a product of a Smz and $\text{Smz}^\#$ sets is Smz .

(ii) By Theorem 5.2, both X_1 and X_2 are $\text{Smz}^\#$. By [28, Theorem 3.14], a product of two $\text{Smz}^\#$ sets is $\text{Smz}^\#$. Now apply Theorem 5.2 to conclude that $X_1 \times X_2$ is \mathcal{M} -additive. \dashv

COROLLARY 6.3. *Let $\mathbb{G}_1, \mathbb{G}_2$ be TSI locally compact groups. Let $X \subseteq \mathbb{G}_1 \times \mathbb{G}_2$. The following are equivalent.*

- (i) *X is \mathcal{M} -additive,*
- (ii) *$\text{proj}_1 X$ and $\text{proj}_2 X$ are \mathcal{M} -additive, and*
- (iii) *$\text{proj}_1 X \times \text{proj}_2 X$ is \mathcal{M} -additive.*

PROOF. Write $X_1 = \text{proj}_1 X, X_2 = \text{proj}_2 X$. (iii) \Rightarrow (i) is trivial.

(i) \Rightarrow (ii) Suppose X is \mathcal{M} additive. Then by Theorem 6.1 both X_1 and X_2 are \mathcal{M} additive.

(ii) \Rightarrow (iii) Suppose X_1 and X_2 are \mathcal{M} additive. Then by Theorem 6.2 $X_1 \times X_2$ is \mathcal{M} additive. \dashv

Binary operations. Theorems 4.1 and 5.2 reveal a surprising fact about \mathcal{M} -additive sets in TSI locally compact groups:

COROLLARY 6.4. *Let \mathbb{G} be a TSI locally compact group. Let \star be a continuous binary operation on \mathbb{G} such that $N \star x$ is nowhere dense for each nowhere dense set N and $x \in \mathbb{G}$. If $X \subseteq \mathbb{G}$ is \mathcal{M} -additive, then $M \star X$ is meager for each meager set M .*

In particular, this holds for any action $\phi : \mathbb{G} \curvearrowright \mathbb{G}$.

Universally meager sets. Recall that a separable metric space E is termed *universally meager* [25, 26] if for any perfect Polish spaces Y, X such that $E \subseteq X$ and every continuous one-to-one mapping $f : Y \rightarrow X$ the set $f^{-1}(E)$ is meager in Y . By [28, Proposition 6.11] every $\text{Smz}^\#$ set is universally meager. Thus Theorem 5.2 yields

PROPOSITION 6.5. *Every \mathcal{M} -additive set in a TSI Polish group is universally meager.*

Transitively meager sets. Nowik et al. [17] studied the notion of transitively meager sets on the line. By the definition, a set $S \subseteq \mathbb{G}$ in a Polish group is *transitively meager* if

$$\forall P \subseteq \mathbb{G} \text{ perfect } \exists F \supseteq S \text{ } \sigma\text{-compact } \forall x \in \mathbb{G} P \cap xF \text{ is meager in } P.$$

PROPOSITION 6.6. *Every \mathcal{M} -additive set in a TSI Polish group is transitively meager.*

PROOF. This is an easy consequence of a technical [28, Lemma 6.10]:

LEMMA 6.7 [28, Lemma 6.10]. *Let X, Y, Z be perfect Polish spaces and $\phi : Y \rightarrow X$ a continuous one-to-one mapping. Let \mathcal{F} be an equicontinuous family of uniformly continuous mappings of Z into X . If $E \subseteq Z$ is Smz^\sharp , then there is a σ -compact set $F \supseteq E$ such that $\phi^{-1}f(F)$ is meager in Y for all $f \in \mathcal{F}$.*

Let S be an \mathcal{M} -additive set in \mathbb{G} . By Theorem 5.1 it is Smz^\sharp . Now apply the lemma with $Y = P, X = Z = \mathbb{G}, \phi : P \rightarrow \mathbb{G}$ the identity inclusion, and \mathcal{F} the family of all left translates. ⊣

γ -sets. Recall the notion of γ -set, as introduced by Gerlits and Nagy [7]. A family \mathcal{U} of open sets in a separable metric space X is called an ω -cover of X if every finite subset of X is contained in some $U \in \mathcal{U}$. A metric space X is a γ -set if every ω -cover of X contains a γ -cover. Generalizing a result of Nowik and Weiss [18, Proposition 3.7], [28, Proposition 6.6] proved that every γ -set is Smz^\sharp . This, together with Theorem 4.1, yields

PROPOSITION 6.8. *Every γ -set in a locally compact Polish group is sharply \mathcal{M} -additive.*

§7. Uniformity number of meager-additive sets. Let \mathbb{G} be a Polish group. We have considered the following σ -ideals of small sets:

- $\mathcal{M}^*(\mathbb{G})$, the ideal of \mathcal{M} -additive sets in \mathbb{G} ,
- $\mathcal{M}^\sharp(\mathbb{G})$, the ideal of sharply \mathcal{M} -additive sets in \mathbb{G} , and
- $\text{Smz}^\sharp(\mathbb{G})$, the ideal of sharp measure zero sets in \mathbb{G} . This notion may depend, unlike the other two, on the metric.

We will calculate uniformity numbers of $\mathcal{M}^*(\mathbb{G})$ and $\text{Smz}^\sharp(\mathbb{G})$ for TSI Polish groups. Recall that if \mathcal{J} is an ideal of subsets of a set X , then

- $\text{non } \mathcal{J} = \min\{|A| : A \notin \mathcal{J}\}$ (uniformity of \mathcal{J}),
- $\text{add } \mathcal{J} = \min\{|A| : A \subseteq \mathcal{J}, \bigcup A \notin \mathcal{J}\}$ (additivity of \mathcal{J}),
- $\text{cov } \mathcal{J} = \min\{|A| : A \subseteq \mathcal{J}, \bigcup A = \mathbb{G}\}$ (covering of \mathcal{J}).

The invariants $\text{non } \mathcal{M}(2^\omega)$, $\text{add } \mathcal{M}(2^\omega)$ and $\text{cov } \mathcal{M}(2^\omega)$ are commonly denoted just $\text{non } \mathcal{M}$, $\text{add } \mathcal{M}$ and $\text{cov } \mathcal{M}$.

The cardinal invariant $\text{non } \mathcal{M}^*(2^\omega)$ is termed *transitive additivity of \mathcal{M}* and denoted $\text{add}^* \mathcal{M}$ in [1, 19], where a combinatorial characterization of $\text{add}^* \mathcal{M}$ is also provided [1, Theorem 2.7.14].

THEOREM 7.1. *Let \mathbb{G} be a TSI Polish group.*

- (i) *If \mathbb{G} is locally compact, then $\text{non Smz}^\sharp(\mathbb{G}) = \text{non } \mathcal{M}^*(\mathbb{G}) = \text{add}^* \mathcal{M}$.*
- (ii) *If \mathbb{G} is not locally compact, then $\text{non Smz}^\sharp(\mathbb{G}) = \text{non } \mathcal{M}^*(\mathbb{G}) = \text{add } \mathcal{M}$.*

PROOF. (i) Let $K_n \nearrow \mathbb{G}$ be a sequence of compact sets. Let $X \subseteq \mathbb{G}$ be a not Smz^\sharp set such that $|X| = \text{non Smz}^\sharp(\mathbb{G})$. There is n such that $X \cap K_n$ is not Smz^\sharp , therefore $|X \cap K_n| = \text{non Smz}^\sharp(\mathbb{G})$. Since K_n is compact, it is *dyadic*: there is a continuous mapping $f : 2^\omega \rightarrow K_n$ onto K_n . For each $x \in X \cap K_n$ pick $\hat{x} \in f^{-1}(x)$ and set $\hat{X} = \{\hat{x} : x \in X \cap K_n\}$. Then $f(\hat{X}) = X \cap K_n$, hence \hat{X} is by Proposition 2.4 not Smz^\sharp , and clearly $|\hat{X}| = |X \cap K_n| = \text{non Smz}^\sharp(\mathbb{G})$. It follows that $\text{non Smz}^\sharp(2^\omega) \leq \text{non Smz}^\sharp(\mathbb{G})$.

On the other hand, by the Perfect set theorem, \mathbb{G} contains a copy of 2^ω . It follows that $\text{non Smz}^\sharp(2^\omega) \geq \text{non Smz}^\sharp(\mathbb{G})$.

Combine the two inequalities to get $\text{non Smz}^\sharp(2^\omega) = \text{non Smz}^\sharp(\mathbb{G})$. By Theorem 5.2 we have $\text{Smz}^\sharp(\mathbb{G}) = \mathcal{M}^*(\mathbb{G})$ and likewise for 2^ω . Thus $\text{non Smz}^\sharp(2^\omega) = \text{non } \mathcal{M}^*(2^\omega) = \text{add}^* \mathcal{M}$ and $\text{non Smz}^\sharp(\mathbb{G}) = \text{non } \mathcal{M}^*(\mathbb{G})$ and the result follows.

(ii) We will consider Smz^\sharp sets in ω^ω . The metric on ω^ω is the usual least difference metric: for distinct $f, g \in \omega^\omega$ we let $d(f, g) = 2^{-n}$, where n is the first number for which $f(n) \neq g(n)$.

Recall that a set $X \subseteq \omega^\omega$ is *bounded* if $\exists g \in \omega^\omega \forall f \in X \forall^\infty n \in \omega f(n) \leq g(n)$. Let $\mathfrak{b} = \min\{|X| : X \subseteq \omega^\omega \text{ is not bounded}\}$ denote the *bounding number*.

Let $X \subseteq \omega^\omega$ be an unbounded set such that $|X| = \mathfrak{b}$. Since X is not bounded, it is not contained in a σ -compact set and in particular (by Lemma 2.3) it is not Smz^\sharp . It follows that $\text{non Smz}^\sharp(\omega^\omega) \leq \mathfrak{b}$.

Miller and Fremlin [15] constructed a set $X \subseteq \omega^\omega$ that is not Smz and $|X| = \text{cov } \mathcal{M}$. This set is clearly not Smz^\sharp and thus $\text{non Smz}^\sharp(\omega^\omega) \leq \text{cov } \mathcal{M}$.

It follows that $\text{non Smz}^\sharp(\omega^\omega) \leq \min(\text{cov } \mathcal{M}, \mathfrak{b})$. Miller [14] proved that $\min(\text{cov } \mathcal{M}, \mathfrak{b}) = \text{add } \mathcal{M}$ (see also [1, 2.2.9]). Overall we have

$$\text{non Smz}^\sharp(\omega^\omega) \leq \text{add } \mathcal{M}. \tag{19}$$

Our group \mathbb{G} is not locally compact and is equipped with an invariant metric. By [9, Lemma 5.4], such a group contains a uniform copy of ω^ω and thus $\text{non Smz}^\sharp(\mathbb{G}) \leq \text{non Smz}^\sharp(\omega^\omega)$. Combine with (19) to get

$$\text{non Smz}^\sharp(\mathbb{G}) \leq \text{add } \mathcal{M}. \tag{20}$$

It is well-known that $\text{add } \mathcal{M}(X) = \text{add } \mathcal{M}$ for every uncountable Polish space. Thus if X is a set in \mathbb{G} such that $|X| < \text{add } \mathcal{M}$, then for each meager set $M \subseteq \mathbb{G}$ the set MX is a union of less than $\text{add } \mathcal{M}$ many meager sets and is thus meager. It follows that

$$\text{non } \mathcal{M}^*(\mathbb{G}) \geq \text{add } \mathcal{M}. \tag{21}$$

Finally, by Theorem 5.1, $\mathcal{M}^*(\mathbb{G}) \subseteq \text{Smz}^\sharp(\mathbb{G})$ and hence

$$\text{non } \mathcal{M}^*(\mathbb{G}) \leq \text{non Smz}^\sharp(\mathbb{G}). \tag{22}$$

Now combine (21), (22), and (20):

$$\text{add } \mathcal{M} \leq \text{non } \mathcal{M}^*(\mathbb{G}) \leq \text{non Smz}^\sharp(\mathbb{G}) \leq \text{add } \mathcal{M}$$

and the result follows. \dashv

REMARK 7.2. Theorem 7.1 provides a simple argument proving that Smz^\sharp is consistently not a topological property. (Remind that under Borel Conjecture Smz^\sharp is a topological property for a trivial reason.)

The Baer–Specker group \mathbb{Z}^ω is a TSI Polish group. On the other hand, it is homeomorphic to the set of irrational numbers, so regard it as a subset of \mathbb{R} . By Theorem 7.1(ii) there is a set $X \subseteq \mathbb{Z}^\omega$ such that $|X| = \text{add } \mathcal{M}$ and that is not Smz^\sharp in the invariant metric. On the other hand, if $\text{add } \mathcal{M} < \text{add}^* \mathcal{M}$, then X is by Theorem 7.1(i) Smz^\sharp in the metric of the real line. Since $\text{add } \mathcal{M} < \text{add}^* \mathcal{M}$ is relatively consistent with ZFC (as proved by Pawlikowski [19]), X is consistently a set that is Smz^\sharp in one metric on \mathbb{Z}^ω and not Smz^\sharp in another homeomorphic metric.

§8. Questions. We conclude with a few open problems.

\mathcal{M} -additive vs. Smz^\sharp under the Continuum Hypothesis. As to Smz sets in groups, let us recall that Galvin–Mycielski–Solovay Theorem has been extended to all locally compact Polish groups by Kysiak [12] and Fremlin [5]. Further extension of the theorem to Polish groups has been proven to be impossible: Hrušák et al. [9] and Hrušák and Zapletal [10] found that under the Continuum Hypothesis the Galvin–Mycielski–Solovay Theorem fails for all TSI Polish groups that are not locally compact.

We ask if an analogous situation occurs for Smz^\sharp sets. We know from Theorem 5.1 that for a TSI Polish group we have $\mathcal{M}^*(\mathbb{G}) \subseteq \text{Smz}^\sharp(\mathbb{G})$. Does the converse inclusion hold?

QUESTION 8.1. *Assume the Continuum Hypothesis. Is there a TSI, not locally compact Polish group \mathbb{G} such that $\mathcal{M}^*(\mathbb{G}) \neq \text{Smz}^\sharp(\mathbb{G})$?*

We conjecture the answer to be negative. More specifically, we may ask about the Baer–Specker group:

QUESTION 8.2. *Assume the Continuum Hypothesis. Is there a Smz^\sharp set in \mathbb{Z}^ω that is not \mathcal{M} -additive?*

TSI condition. The proof of Theorem 5.1 relies on the two-sided invariance of the metric. Can we do better?

QUESTION 8.3. *Do Theorems 5.1 and 5.2 extend to a wider class of Polish groups? In particular, can the assumption that the group is TSI be dropped or relaxed to the existence of a complete left-invariant metric?*

\mathcal{E} -additive sets. Let \mathbb{G} be locally compact Polish group. Denote by \mathcal{E} the σ -ideal consisting of sets that are contained in an F_σ -set of Haar measure zero. A set $X \subseteq \mathbb{G}$ is \mathcal{E} -additive if $EX \in \mathcal{E}$ for every $E \in \mathcal{E}$. The notion of *sharply \mathcal{E} -additive* sets can be defined, too, in the obvious manner.

In [28, Theorem 5.2] it is shown that a set $X \subseteq 2^\omega$ is \mathcal{M} -additive if and only if it is \mathcal{E} -additive if and only if it is sharply \mathcal{E} -additive. The proofs heavily depend on the combinatorics of sets in \mathcal{E} . We have no idea if this theorem or at least some inclusions extend to other locally compact groups.

- QUESTION 8.4. (i) *In what locally compact Polish groups is every \mathcal{M} -additive set \mathcal{E} -additive (or sharply \mathcal{E} -additive)?*
 (ii) *In what locally compact Polish groups is every \mathcal{E} -additive (or sharply \mathcal{E} -additive) set \mathcal{M} -additive?*
 (iii) *In what locally compact Polish groups is every \mathcal{E} -additive sharply \mathcal{E} -additive?*

The two new classes extend the family of ideals that we are interested in to five: Smz^\sharp , \mathcal{M}^* , \mathcal{M}^\sharp , \mathcal{E}^* and \mathcal{E}^\sharp , leading to a number of questions about possible inclusions. For TSI locally compact groups, the twelve questions reduce, due to Theorem 5.2, to the five listed in Question 8.4.

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