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The determination of turbulence-model statistics from the velocity–acceleration correlation

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For inhomogeneous turbulent flows at high Reynolds number, it is shown that the redistribution term in Reynolds-stress turbulence models can be determined from the velocity–acceleration correlation. It is further shown that the drift coefficient in the generalized Langevin model (which is used in probability density function (PDF) methods) can be determined from the Reynolds stresses and the velocity–acceleration correlation. These observations are valuable, since the second moments of velocity and acceleration can be measured in experiments, in direct numerical simulations and in well-resolved large-eddy simulations (LES), and hence these turbulence-model quantities can be determined. The redistribution is closely related to the pressure–rate-of-strain, and the unknown in the PDF equation is closely related to the conditional mean pressure gradient (conditional on velocity). In contrast to the velocity–acceleration moments, these pressure statistics are much more difficult to obtain, and our knowledge of them is quite limited. It is also shown that the generalized Langevin model can be re-expressed to provide a direct connection between the drift term and the fluid acceleration. All of these results are first obtained using the constant-property Navier–Stokes equations, but it is then shown that the results are simply extended to variable-density flows.

Key words: turbulence modelling, turbulent flows

1. Introduction

Turbulence models are usually derived from exact equations for selected statistics, and these equations are in turn derived from the Navier–Stokes equations (see, e.g., Pope 2000). As a consequence of the inevitable closure problem, these exact equations contain some 'unknown' statistics that must be modelled as functions of the 'known' quantities represented by the models. For example, in Reynolds-stress models

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(Pope 2000, chap. 11), a principal term to be modelled is the pressure–rate-of-strain

$$\mathscr{R}_{ij} \equiv \left\langle \frac{p'}{\rho} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle, \tag{1.1}$$

where p' is the pressure fluctuation, ρ is the density and $\boldsymbol{u}(\boldsymbol{x}, t)$ is the Eulerian velocity fluctuation. As a second example, in models based on the transport equation for the joint probability density function (PDF) of velocity (Pope 2000, 2011), a principal term to be modelled is the conditional pressure gradient

$$\mathscr{G}_i(\boldsymbol{v}) \equiv \frac{1}{\rho} \left\langle \frac{\partial p'}{\partial x_i} \middle| \, \boldsymbol{u} = \boldsymbol{v} \right\rangle, \tag{1.2}$$

where $\boldsymbol{v}$ is a sample-space variable for the velocity fluctuation.

In the development and testing of turbulence models, it is highly desirable to have a knowledge of the behaviour of the quantities to be modelled, such as $\mathscr{R}_{ij}$ and $\mathscr{G}_i(\boldsymbol{v})$. This knowledge can, in principle, be obtained either from experiments or from direct numerical simulations (DNS). Historically, accurate resolved measurements of the fluctuating pressure in turbulent flows have proved difficult. Recently, miniaturized velocity–pressure probes have been developed (Terashima *et al.* 2014) that are capable of measuring one-point statistics – but not gradients. Hence, in spite of this progress, $\mathscr{R}_{ij}$ and $\mathscr{G}_i(\boldsymbol{v})$ remain inaccessible to measurements in laboratory flows. There are, however, some measurements of pressure–rate-of-strain statistics in the atmospheric boundary layer (Nguyen *et al.* 2013) in which the scales to be resolved are much larger than in the laboratory. Extremely valuable information has been obtained from DNS in simple canonical flows such as homogeneous shear flow (Sawford & Yeung 2000), channel flow (Kim, Moin & Moser 1987; Moser, Kim & Mansour 1999) and the flat-plate boundary layer (Spalart 1988). In DNS, in principle, all information about the flow is available with excellent resolution and accuracy. However, in some of the numerical methods used (e.g. spectral projection methods, Rogallo & Moin 1985) an accurate representation of the pressure is not readily available. For these and similar reasons, the information about quantities such as $\mathscr{R}_{ij}$ and $\mathscr{G}_i(\boldsymbol{v})$ available from experiments and DNS is quite limited.

In the last decade, great advances have been made in experimental methods capable of simultaneously measuring velocity and acceleration in turbulent flows. These include particle-tracking techniques (La Porta *et al.* 2001; Gerashchenko *et al.* 2008) and tomographic particle image velocimetry (T-PIV, Elsinga *et al.* 2006; Coriton, Steinberg & Frank 2014). Moreover, in DNS, the velocity $\boldsymbol{U}(\boldsymbol{x}, t)$ is obviously known, and the acceleration $\boldsymbol{A}(\boldsymbol{x}, t)$ can be obtained quite easily by either Lagrangian (Yeung & Pope 1989; Sawford & Yeung 2000) or Eulerian (Vedula & Yeung 1999) approaches.

In this paper we show that information almost equivalent to $\mathscr{R}_{ij}$ and $\mathscr{G}_i(\boldsymbol{v})$ can be extracted from the first and second moments of the velocity and acceleration.

For simplicity, we consider the turbulent flow of a constant-property Newtonian fluid at high Reynolds number (Re), but (as shown in § 4) the approach presented can be extended simply to the general variable-property case. Hence, the governing Navier–Stokes equations initially considered are $\nabla \cdot \boldsymbol{U} = 0$ and

$$\frac{\mathrm{D}U_i}{\mathrm{D}t} = \nu \nabla^2 U_i - \frac{1}{\rho} \frac{\partial p}{\partial x_i}, \tag{1.3}$$

where ν is the kinematic viscosity.

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In all of the equations derived from (1.3) given below, all terms originating from $\mathrm{D}U_i/\mathrm{D}t$ are kept on the left-hand side, and all other terms are kept on the right-hand side. This is contrary to normal practice, but it is helpful to see more clearly the origin of the various contributions.

2. Reynolds-stress models

With $\boldsymbol{U} = \langle \boldsymbol{U} \rangle + \boldsymbol{u}$ being the Reynolds decomposition of the Eulerian velocity $\boldsymbol{U}(\boldsymbol{x}, t)$, the transport equation for the Reynolds-stress tensor $\langle u_i u_j \rangle$ can be derived from (1.3) by evaluating the quantity

$$N_{ij} \equiv \left\langle u_i \frac{\mathrm{D}U_j}{\mathrm{D}t} + u_j \frac{\mathrm{D}U_i}{\mathrm{D}t} \right\rangle . \tag{2.1}$$

After manipulation, the result is

$$\begin{aligned} &\frac{\partial \langle u_i u_j \rangle}{\partial t} + \langle U_k \rangle \frac{\partial \langle u_i u_j \rangle}{\partial x_k} + \frac{\partial \langle u_i u_j u_k \rangle}{\partial x_k} + \langle u_i u_k \rangle \frac{\partial \langle U_j \rangle}{\partial x_k} + \langle u_j u_k \rangle \frac{\partial \langle U_i \rangle}{\partial x_k} \\ &= \left\langle \frac{p'}{\rho} \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right\rangle - 2\nu \left\langle \frac{\partial u_i}{\partial x_k} \frac{\partial u_j}{\partial x_k} \right\rangle - \frac{1}{\rho} \left(\frac{\partial \langle u_i p' \rangle}{\partial x_j} + \frac{\partial \langle u_j p' \rangle}{\partial x_i} \right) + \nu \nabla^2 \langle u_i u_j \rangle \\ &= N_{ij} = \widehat{R}_{ij} - \frac{2}{3} \widehat{\epsilon} \delta_{ij}, \end{aligned} \tag{2.2}$$

where the last part is simply the unique decomposition of N_{ij} (i.e. the left-hand side on the first line) into its deviatoric component $\widehat{R}_{ij}$ (with $\widehat{R}_{ii} = 0$) and its isotropic component $(2/3)\widehat{\epsilon}\,\delta_{ij}$, i.e.

$$\widehat{R}_{ij} \equiv N_{ij} - \tfrac{1}{3} N_{kk} \delta_{ij} \quad \text{and} \quad \widehat{\epsilon} \equiv \tfrac{1}{2} N_{kk}. \tag{2.3a,b}$$

On the left-hand side of (2.2), the first two terms represent the rate of change following the mean flow, the middle term is convective transport by the fluctuating velocity and the last two terms are production. On the right-hand side, the first term is the pressure–rate-of-strain $\mathscr{R}_{ij}$, the second is (minus) the dissipation tensor ε_{ij}, the third is pressure transport and the final term is viscous transport. The dissipation of kinetic energy (or strictly the pseudo-dissipation) is $\varepsilon \equiv \varepsilon_{ii}/2$. (One should note the distinction in the notation between $\widehat{\epsilon}$ and ε, and between $\widehat{R}_{ij}$ and $\mathscr{R}_{ij}$.)

We now make several observations about the terms in the Reynolds-stress equation.

(*a*) The viscous transport term is of relative order Re^{-1} and is henceforth neglected.

(*b*) It is well established that at high Reynolds number the dissipation is isotropic (to an excellent approximation), and so henceforth we take $\varepsilon_{ij} = (2/3)\varepsilon\delta_{ij}$.

(*c*) In homogeneous turbulence (with mean velocity gradients), all of the transport terms are zero. It then follows that (in this case) we have $\widehat{R}_{ij} = \mathscr{R}_{ij}$ and $\widehat{\epsilon} = \varepsilon$.

(*d*) For inhomogeneous flows, to the extent that the pressure transport has been studied, it is generally found to make a minor contribution to the Reynolds-stress budget (see, e.g., Spalart 1988). For example, in the plane-jet experiment of Terashima *et al.* (2014), the peak pressure transport is just 15 % of the peak production in the cross-stream profile of the kinetic energy budget. If the pressure transport is zero, we again have $\widehat{R}_{ij} = \mathscr{R}_{ij}$ and $\widehat{\epsilon} = \varepsilon$.

(*e*) In the construction of Reynolds-stress models, there is usually no explicit model for the pressure transport, and hence the appropriate 'redistribution' term should be a model for $\widehat{R}_{ij}$ rather than for $\mathscr{R}_{ij}$ (although, as argued above, the difference is likely to be minor).

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The upshot of these observations is that, while ideally we would like to have knowledge of $\mathscr{R}_{ij}$ and ε_{ij}, almost the same information is provided by $\widehat{R}_{ij}$ and $\widehat{\epsilon}$.

A caveat is that $\widehat{\epsilon}$ is not guaranteed to be non-negative. An exact result, due to Mann, Søren & Andersen (1999), is $\widehat{\epsilon} = \varepsilon + \boldsymbol{\nabla} \cdot \langle \boldsymbol{u} p' \rangle / \rho$, and so it is theoretically possible for the pressure transport term to cause $\widehat{\epsilon}$ to be negative. Also, while the pressure transport is of minor importance in the relatively simple flows mentioned above, without further evidence, this cannot be safely assumed in more complex flows.

We now make the simple observation that $\widehat{R}_{ij}$ and $\widehat{\epsilon}$ are determined exactly by the velocity–acceleration correlation. For, from (2.1) and (2.2), we obtain

$$\left\langle u_i \frac{\mathrm{D}U_j}{\mathrm{D}t} + u_j \frac{\mathrm{D}U_i}{\mathrm{D}t} \right\rangle = \langle u_i A_j + u_j A_i \rangle = \langle u_i a_j \rangle + \langle u_j a_i \rangle = \widehat{R}_{ij} - \frac{2}{3} \widehat{\epsilon} \delta_{ij}, \tag{2.4}$$

where the Reynolds decomposition of acceleration is $\boldsymbol{A} = \langle \boldsymbol{A} \rangle + \boldsymbol{a}$. From the trace of this equation we obtain

$$\widehat{\epsilon} = -\langle u_i a_i \rangle, \tag{2.5}$$

and its deviatoric part yields

$$\widehat{R}_{ij} = \langle u_i a_j \rangle + \langle u_j a_i \rangle - \tfrac{2}{3} \langle u_k a_k \rangle \delta_{ij}. \tag{2.6}$$

In summary, from a measurement of the velocity–acceleration correlation $\langle u_i a_j \rangle$ the above equations exactly determine $\widehat{R}_{ij}$ and $\widehat{\epsilon}$, which are the primary quantities to be modelled in a Reynolds-stress model. These quantities provide good approximations to the pressure–rate-of-strain $\mathscr{R}_{ij}$ and dissipation ε.

3. Probability density function models

The one-point, one-time joint PDF of the Eulerian velocity is denoted by $f(\boldsymbol{V}; \boldsymbol{x}, t)$, where $\boldsymbol{V}$ is a sample-space variable corresponding to velocity. It follows from (1.3) that this PDF evolves by

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = -\frac{\partial}{\partial V_i} \left[f \langle \nu \nabla^2 U_i \,|\, \boldsymbol{V} \rangle - f \left\langle \frac{1}{\rho} \frac{\partial p}{\partial x_i} \right| \boldsymbol{V} \right\rangle \bigg], \tag{3.1}$$

where the means on the right-hand side are conditional on $\boldsymbol{U}(\boldsymbol{x}, t) = \boldsymbol{V}$. These terms can be manipulated to be re-expressed in terms of the conditional dissipation tensor and the conditional pressure gradient (1.2) or the conditional pressure–rate-of-strain (see, e.g., Pope 2000, chap. 12). There are very few data on these quantities from experiment and DNS.

This PDF equation can also be written concisely in terms of the conditional acceleration, i.e.

$$\frac{\partial f}{\partial t} + V_i \frac{\partial f}{\partial x_i} = -\frac{\partial}{\partial V_i} \left[f \langle A_i \,|\, \boldsymbol{V} \rangle \right]. \tag{3.2}$$

The predominant approach to modelling the velocity PDF equation is via a generalized Langevin model (GLM, Haworth & Pope 1986), which is a stochastic differential equation (SDE) for $\boldsymbol{U}^*(t)$ – which is a model for the velocity of a fluid particle with position $\boldsymbol{X}^*(t)$. The GLM can be written as

$$\mathrm{d}U_i^* = M_i \,\mathrm{d}t + G_{ij} u_j^* \,\mathrm{d}t + (C_0 \varepsilon)^{1/2} \,\mathrm{d}W_i, \tag{3.3}$$

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where M_i, G_{ij} and C_0 are model coefficients, $\boldsymbol{u}^* \equiv \boldsymbol{U}^* - \langle \boldsymbol{U}(\boldsymbol{X}^*, t)\rangle$ is the fluctuating component of the particle velocity and $\boldsymbol{W}(t)$ is an isotropic Wiener process. We denote by $f^*(\boldsymbol{V}; \boldsymbol{x}, t)$ the Eulerian PDF of velocity obtained from the GLM. This evolves by the equation

$$\frac{\partial f^*}{\partial t} + V_i \frac{\partial f^*}{\partial x_i} = -\frac{\partial}{\partial V_i}[f^*(M_i + G_{ij}v_j)] + \frac{1}{2} C_0 \varepsilon \frac{\partial^2 f^*}{\partial V_i \partial V_i}, \tag{3.4}$$

with the definition $\boldsymbol{v} \equiv \boldsymbol{V} - \langle \boldsymbol{U}(\boldsymbol{x}, t)\rangle$.

It is natural to attempt to relate the coefficients in the GLM to the fluid acceleration. However, this cannot be done directly, since $\boldsymbol{U}^*(t)$ (given by (3.3)) is nowhere differentiable; or, put less rigorously, $\mathrm{d}\boldsymbol{U}^*/\mathrm{d}t$ is white noise.

Instead, using the PDF equations (3.2) and (3.4) we can relate the coefficients in the GLM to the conditional acceleration. To do so, following Sawford & Yeung (2000), we define

$$G_{ij}^{(p)} \equiv G_{ij} + \tfrac{1}{2} C_0 \varepsilon \lambda_{ij}, \tag{3.5}$$

where λ_{ij} is the i–j component of the inverse of the Reynolds-stress tensor, and then we rewrite (3.4) as

$$\begin{aligned} \frac{\partial f^*}{\partial t} + V_i \frac{\partial f^*}{\partial x_i} &= -\frac{\partial}{\partial V_i}\left[f^*(M_i + G_{ij}^{(p)} v_j) - \frac{1}{2} C_0 \varepsilon \left(\frac{\partial f^*}{\partial V_i} + f^* \lambda_{ij} v_j\right)\right] \\ &= -\frac{\partial}{\partial V_i}\left[f^*(M_i + G_{ij}^{(p)} v_j) - \frac{1}{2} C_0 f^* \varepsilon \frac{\partial (\ln f^* - \ln f_{JN})}{\partial V_i}\right], \end{aligned} \tag{3.6}$$

where f_{JN} is the joint-normal PDF with the same first and second moments as f^* (which has the property $\partial f_{JN}/\partial V_i = -f_{JN}\lambda_{ij}v_j$). We now make the following observations about the term in C_0:

(*a*) the term makes no contribution to the mean momentum equation (obtained by multiplying (3.6) by V_k and integrating over all $\boldsymbol{V}$);

(*b*) the term makes no contribution to the Reynolds-stress equation (obtained by multiplying (3.6) by $v_k v_\ell$ and integrating over all $\boldsymbol{V}$);

(*c*) the term vanishes for the case of f^* being joint normal (as it is in homogeneous turbulence, to an excellent approximation).

(It should be noted that, if f^* is joint normal, then (*a*) and (*b*) follow from (*c*); however, more generally, (*a*) and (*b*) hold regardless of the nature of f^*.)

Based on these observations, for the present purposes we neglect the term in C_0 in (3.6). Then, comparing this equation with (3.2), we conclude that the PDFs f and f^* satisfy the same evolution equation provided that

$$M_i + G_{ij}^{(p)} v_j = \langle A_i \,|\, \boldsymbol{V}\rangle. \tag{3.7}$$

Taking the (unconditional) mean of this equation, we obtain the well-known result

$$M_i = \langle A_i \rangle = \nu \nabla^2 \langle U_i \rangle - \frac{1}{\rho}\frac{\partial \langle p \rangle}{\partial x_i}, \tag{3.8}$$

with the viscous term being negligible. Then, subtracting (3.8) from (3.7), we obtain the condition for the GLM model to be correct:

$$G_{ij}^{(p)} v_j = \langle a_i \,|\, \boldsymbol{v}\rangle. \tag{3.9}$$

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To be precise, provided that (*a*) the term in C_0 in (3.6) can be neglected, (*b*) the conditional acceleration is linear in the velocity and (*c*) the coefficient $G_{ij}^{(p)}$ satisfies (3.9), then the evolution of the PDF f^* is identical to that of f.

The DNS of Sawford & Yeung (2000) provides some evidence that the conditional acceleration is linear in velocity (to an approximation) in homogeneous shear flow, but there is very little evidence to test more generally this assumption made in the GLM. If the conditional acceleration is indeed linear, then by multiplying (3.9) by v_k and taking the unconditional mean, we obtain

$$G_{ij}^{(p)} \langle u_j u_k \rangle = \langle a_i u_k \rangle, \tag{3.10}$$

and then post-multiplying by $\lambda_{k\ell}$ we obtain an explicit expression for $G_{ij}^{(p)}$:

$$G_{ij}^{(p)} = \langle a_i u_k \rangle \lambda_{kj}. \tag{3.11}$$

In general, whether or not the conditional acceleration is linear in velocity, (3.11) provides the best approximation to $G_{ij}^{(p)}$ in the sense that it yields the correct rate of change of the Reynolds stresses.

4. Discussion

It is instructive to consider the simplest case of decaying homogeneous isotropic turbulence, for which we have the following exact results:

$$\langle a_i u_i \rangle = \frac{\mathrm{d}k}{\mathrm{d}t} = -\varepsilon, \quad \langle a_i u_j \rangle = -\frac{1}{3}\varepsilon\delta_{ij}, \quad \lambda_{ij} = \frac{3}{2k}\delta_{ij}. \tag{4.1a–c}$$

Equations (3.5), (3.11) and (4.1) then yield

$$G_{ij}^{(p)} = -\frac{1}{2}\frac{\varepsilon}{k}\delta_{ij}, \quad G_{ij} = -\left(\frac{1}{2} + \frac{3}{4}C_0\right)\frac{\varepsilon}{k}\delta_{ij}, \tag{4.2a,b}$$

the latter being the well-known result for the GLM, which reduces to the simplified Langevin model (SLM) for the isotropic case considered.

According to the Kolmogorov hypotheses (K41, Kolmogorov 1941), for high-Reynolds-number turbulence, small-scale quantities (such as acceleration) are independent of velocity. At first sight, the result in (4.1), $\langle a_i u_i \rangle = -\varepsilon$, may appear to be at odds with K41. It is not, however, since the correlation coefficient between u_i and a_i decreases with the turbulent Reynolds number $Re_T \equiv k^2/(\varepsilon\nu)$, approximately as $Re_T^{-3/8}$ according to the DNS data of Vedula & Yeung (1999). This implies that the correlation between velocity and acceleration arises from the larger-scale motions; this observation is important for the present considerations, as it suggests that accurate measurements of $\langle a_i u_i \rangle$ can be made without full resolution of the smallest scales, both in experiments and in simulations. Hence, in addition to experiments and DNS, well-resolved large-eddy simulations (LES) may also be used to determine the velocity–acceleration correlation, and hence $\widehat{R}_{ij}$ and $G_{ij}^{(p)}$.

Based on (3.5) and (3.6), and following Sawford & Yeung (2000), we rewrite the generalized Langevin model in the revealing form

$$\mathrm{d}\boldsymbol{U}^* = \boldsymbol{A}^{(p)}\,\mathrm{d}t + \mathrm{d}\boldsymbol{U}^{(n)}, \tag{4.3}$$

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with the definitions

$$A_i^{(p)} \equiv M_i + G_{ij}^{(p)} u_j^*, \quad \mathrm{d}U_i^{(n)} \equiv -\tfrac{1}{2} C_0 \varepsilon \lambda_{ij} u_j^* \, \mathrm{d}t + (C_0 \varepsilon)^{1/2} \, \mathrm{d}W_i. \tag{4.4a,b}$$

Then, the condition (3.7) for the GLM to yield the correct evolution of the velocity PDF is, simply,

$$\langle \boldsymbol{A}^{(p)} \mid \boldsymbol{V} \rangle = \langle \boldsymbol{A} \mid \boldsymbol{V} \rangle. \tag{4.5}$$

The quantity $\boldsymbol{A}^{(p)}$ can be regarded as the 'physical' component of acceleration, whereas $\mathrm{d}\boldsymbol{U}^{(n)}$ can be viewed as the 'non-physical' component of the velocity increment, in which the drift term in λ_{ij} 'cancels' the diffusion term in $\boldsymbol{W}$, in the sense that $\mathrm{d}\boldsymbol{U}^{(n)}$ makes no contribution to the evolution equations for the first and second moments of velocity, nor to the PDF equation when the PDF is joint normal. The coefficient C_0 appears only in $\mathrm{d}\boldsymbol{U}^{(n)}$, and its value does not affect $\boldsymbol{A}^{(p)}$, nor the evolution of the Reynolds stresses. The value of C_0 primarily affects two-time velocity correlations.

To every generalized Langevin model (defined by C_0 and G_{ij}), there is a corresponding Reynolds-stress model, i.e. a corresponding redistribution term $\widehat{R}_{ij}$ (Pope 1994). In terms of $G_{ij}^{(p)}$, from (2.6) and (3.10), we determine the corresponding redistribution term to be

$$\widehat{R}_{ij} = G_{ik}^{(p)} \langle u_k u_j \rangle + G_{jk}^{(p)} \langle u_k u_i \rangle - \tfrac{2}{3} \widehat{\epsilon} \delta_{ij}, \tag{4.6}$$

the trace of which gives the constraint on any model for $G_{ij}^{(p)}$:

$$G_{ij}^{(p)} \langle u_i u_j \rangle = -\widehat{\epsilon}. \tag{4.7}$$

It should be noted that, in contrast to the corresponding relations for G_{ij}, the above two equations do not contain C_0.

While Sawford & Yeung (2000) used the velocity–acceleration correlation obtained from DNS of homogeneous shear flow to test existing forms of the Langevin model, it is believed that the explicit expression for the drift term (3.11) is original to the present work. In considering the different problem of relative dispersion, Sawford & Yeung (2010) use similar reasoning to construct a stochastic model for the velocity difference between two fluid particles, and to relate the drift term to the conditional rate of change of the velocity difference.

A different method of determining the coefficients in the generalized Langevin model (including an anisotropic diffusion tensor) is described by Pope (2002) for the case of homogeneous shear flow. This requires as an input the tensor of Lagrangian integral time scales, and the resulting model yields good agreement with DNS data on two-time statistics. In contrast, the present method requires only one-time information, and the prediction of two-time statistics (which is controlled by C_0) is not addressed.

Since PDF methods are used extensively in turbulent combustion (see, e.g., Haworth 2010; Pope 2013), it is important to consider variable-density flows, and this is very straightforward. Density-weighted (Favre) means are used, e.g. $\widetilde{\boldsymbol{U}} \equiv \langle \rho \boldsymbol{U} \rangle / \langle \rho \rangle$, and the density-weighted second moments are, for example, $\widetilde{u_i'' a_j''}$, where u_i'' denotes the fluctuation, $u_i'' = U_i - \widetilde{U}_i$. Essentially, all of the above results apply in the variable-density case, but with the (volume-weighted) means replaced by their density-weighted counterparts. The condition on the GLM (4.5) becomes

$$\langle \rho \boldsymbol{A}^{(p)} \mid \boldsymbol{V} \rangle = \langle \rho \boldsymbol{A} \mid \boldsymbol{V} \rangle, \tag{4.8}$$

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from which (in place of (3.8)) we obtain

$$M_i = \widetilde{A}_i = -\frac{1}{\langle\rho\rangle}\frac{\partial\langle p\rangle}{\partial x_i}, \tag{4.9}$$

and (in place of (3.11))

$$G_{ij}^{(p)} = \widetilde{a_i'' u_k''}\widetilde{\lambda}_{kj}, \tag{4.10}$$

where $\widetilde{\lambda}_{ij}$ denotes the i–j component of the inverse of the density-weighted Reynolds-stress tensor $\widetilde{u_i'' u_j''}$.

In this paper, we have focused on velocity and acceleration. However, the same ideas can be applied to other quantities such as a conserved scalar $\xi(\boldsymbol{x}, t) = \langle\xi\rangle + \xi'$, which evolves by

$$\frac{\mathrm{D}\xi}{\mathrm{D}t} = \nabla\cdot(\Gamma\nabla\xi), \tag{4.11}$$

where Γ is the molecular diffusivity. (For simplicity of notation, we revert to considering constant-density flow.) From a measurement of ξ and its rate of change $\mathrm{D}\xi/\mathrm{D}t$, the scalar dissipation $\chi \equiv 2\Gamma\langle\nabla\xi'\cdot\nabla\xi'\rangle$ can be obtained, since we have

$$\begin{aligned}\left\langle -2\xi\frac{\mathrm{D}\xi}{\mathrm{D}t}\right\rangle &= 2\Gamma\langle\nabla\xi\cdot\nabla\xi\rangle - \Gamma\nabla^2\langle\xi^2\rangle \\ &= \chi + 2\Gamma\nabla\langle\xi\rangle\cdot\nabla\langle\xi\rangle - \Gamma\nabla^2\langle\xi^2\rangle \approx \chi,\end{aligned} \tag{4.12}$$

where the error in this approximation is of relative order Re^{-1}. Similarly, the conditional mean of the diffusion term $\nabla\cdot(\Gamma\nabla\xi)$, which is a major unknown in PDF methods, can be obtained (without approximation) as the conditional mean of $\mathrm{D}\xi/\mathrm{D}t$.

5. Conclusions

It has been shown that the redistribution term $\widehat{R}_{ij}$ in Reynolds-stress models is twice the symmetric deviatoric part of the velocity–acceleration correlation (2.6). The generalized Langevin model has been re-expressed (4.3) so that the drift term $\boldsymbol{A}^{(p)}$ (4.4) corresponds directly to the fluid acceleration (in conditional expectation, (4.5)). Subject to the assumption that the conditional acceleration is linear in velocity, the drift coefficient $G_{ij}^{(p)}$ in the GLM is explicitly given by the velocity–acceleration correlation, post-multiplied by the inverse of the Reynolds-stress tensor (3.11).

The velocity–acceleration correlation $\langle u_i a_j\rangle$ arises from the larger-scale turbulent motions, and hence can be obtained from measurements (without Kolmogorov-scale resolution), from DNS and from well-resolved LES. In particular, $\langle u_i a_j\rangle$ can be measured using particle-tracking velocimetry and tomographic PIV. This route provides a simpler and potentially more productive way to 'measure' turbulence-model statistics (compared with considering fluctuating pressure statistics), and consequently may be useful in the development and testing of turbulence models.

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