Canadian Mathematical Bulletin
doi:10.4153/CMB-2011-076-x

Similarity and Coincidence Isometries for Modules

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Abstract. The groups of (linear) similarity and coincidence isometries of certain modules Γ in d-dimensional Euclidean space, which naturally occur in quasicrystallography, are considered. It is shown that the structure of the factor group of similarity modulo coincidence isometries is the direct sum of cyclic groups of prime power orders that divide d. In particular, if the dimension d is a prime number p, the factor group is an elementary abelian p-group. This generalizes previous results obtained for lattices to situations relevant in quasicrystallography.

1 Introduction

The classification of colour symmetries and that of grain boundaries in crystals and quasicrystals are closely related to the existence of similar and coincidence sublattices of the underlying lattice of periods or the corresponding translation module; *cf.* [2] and [4]. It is thus of interest to understand the corresponding groups of isometries from a more mathematical perspective. For a free $\mathbb{Z}$-module $M \subset \mathbb{R}^d$ of finite rank that spans $\mathbb{R}^d$, an element $R \in \mathrm{O}(d, \mathbb{R})$ is called a *coincidence isometry of* M if RM and M are commensurate, written $RM \sim M$, which means that their intersection has finite index both in M and in RM. We let $\mathrm{OC}(M)$ denote the set of all coincidence isometries of M. More generally, a *similarity isometry of* M is an element $T \in \mathrm{O}(d, \mathbb{R})$ with $\alpha TM \sim M$ for some positive real number α. This definition was first introduced for lattices in [6]. The set $\mathrm{OS}(M)$ of all similarity isometries of M obviously contains $\mathrm{OC}(M)$ as a subset.

For subrings $\mathcal{S}$ of the rings of integers of real algebraic number fields, we consider the similarity and coincidence isometries of free $\mathcal{S}$-modules $\Gamma \subset \mathbb{R}^d$ of rank d that span $\mathbb{R}^d$. For a separate treatment of the crystallographic case $\mathcal{S} = \mathbb{Z}$, where Γ is a lattice, the reader is referred to [14]. We show that the similarity isometries of Γ form a group that contains the coincidence isometries as a normal subgroup. The corresponding factor group of similarity modulo coincidence isometries is the direct sum of cyclic groups of prime power orders that divide d (Theorem 3.14). In the case of $\mathcal{S}$-*modules over* K *in* $\mathbb{R}^d$ (*cf.* Definition 3.17), where K is the quotient field of $\mathcal{S}$, the factor group is either trivial or an elementary abelian 2-group, depending on the parity of d. This includes the standard icosahedral modules and the rings of cyclotomic integers in complex cyclotomic fields.

Received by the editors November 10, 2008; revised February 27, 2009.
Published electronically April 25, 2011.
AMS subject classification: **20H15**, 82D25, 52C23.

2 $\mathbb{Z}$-modules

Let us begin by recalling some well-known facts on abelian groups. If M is an abelian group and $N \subset M$ a subgroup of finite index $[M : N] = k$, then a direct consequence of Lagrange's Theorem is that kM is a subgroup of N.

Fact 2.1 Let M be a free $\mathbb{Z}$-module of finite rank.
(1) If N is a submodule of M, then N is also a free $\mathbb{Z}$-module with $\operatorname{rank}(N) \leqslant \operatorname{rank}(M)$.
(2) If N is a submodule of M of finite index, then N has the same rank as M.

Proof *Cf.* [1, Theorem 6.2] for (1). If $[M : N] = k \in \mathbb{N}$, then $kM \subset N$. If $\{m_1, \dots, m_r\}$ is a $\mathbb{Z}$-basis of M, then $\{km_1, \dots, km_r\}$ forms a $\mathbb{Z}$-basis for the free module kM. Using (1) yields $\operatorname{rank}(M) = \operatorname{rank}(kM) \leqslant \operatorname{rank}(N)$. Now (2) follows from (1). ■

Fact 2.2 ([12, Chapter 2, Lemma 6.1.1]) If M is a torsion-free abelian group of rank r, and N is a subgroup which is also of rank r, then the index $[M : N]$ is finite and equals the absolute value of the determinant of the transition matrix from any basis of M to any basis of N.

Together with Fact 2.1(2), this gives the following equivalence.

Lemma 2.3 *Let $M \subset \mathbb{R}^d$ be a free $\mathbb{Z}$-module of finite rank and let $N \subset M$ be a submodule. Then the index $[M : N]$ is finite if and only if* $\operatorname{rank}(N) = \operatorname{rank}(M)$.

Fact 2.4 Two free $\mathbb{Z}$-modules $M, M' \subset \mathbb{R}^d$ of finite rank are called *commensurate* if their intersection has finite (subgroup) index both in M and in M'. In this case we write $M \sim M'$. Furthermore, if $M \sim M'$, then $M \cap M'$ is a free $\mathbb{Z}$-module with $\operatorname{rank}(M) = \operatorname{rank}(M') = \operatorname{rank}(M \cap M')$ by Facts 2.1(1) and 2.1(2).

Lemma 2.5 *Commensurability of free $\mathbb{Z}$-modules of finite rank r contained in $\mathbb{R}^d$ is an equivalence relation.*

Proof Reflexivity and symmetry are clear by definition. For the transitivity, let $M_1 \sim M_2$ and $M_2 \sim M_3$. In particular, the indices $s_{12} = [M_2 : (M_1 \cap M_2)]$ and $s_{23} = [M_2 : (M_2 \cap M_3)]$ are finite. One obtains $s_{12}M_2 \subset (M_1 \cap M_2)$ and $s_{23}M_2 \subset (M_2 \cap M_3)$. Since M_1, M_3 and $s_{12}s_{23}M_2$ are free $\mathbb{Z}$-modules of rank r in $\mathbb{R}^d$, Lemma 2.3 together with the relation

$$s_{12}s_{23}M_2 \subset (M_1 \cap M_2 \cap M_3)$$

now implies that $M_1 \cap M_2 \cap M_3$ is of finite index both in M_1 and M_3. As a consequence, one obtains $M_1 \sim M_3$. ■

Alternatively to the definition of $\operatorname{OS}(M)$ above, one easily verifies that

$$\operatorname{OS}(M) = \{R \in \mathrm{O}(d, \mathbb{R}) \mid \beta RM \subset M \text{ for some } \beta \in \mathbb{R}_+\}, \tag{1}$$

where $\mathbb{R}_+$ denotes the set of positive real numbers.

Therefore, $\mathrm{OS}(M)$ consists of all linear isometries that arise from similarity mappings of M into itself. We call a submodule of M of the form βRM a *similar submodule of M*. (For similar submodules in four dimensions, see [8].)

Lemma 2.6 *Let $M \subset \mathbb{R}^d$ be a free $\mathbb{Z}$-module of finite rank r that spans $\mathbb{R}^d$. The sets* $\mathrm{OS}(M)$ *and* $\mathrm{OC}(M)$ *are subgroups of* $\mathrm{O}(d, \mathbb{R})$.

Proof Let $R, S \in \mathrm{OS}(M)$. Due to equation (1), there exist positive real numbers α, β with $\alpha RM \subset M$ and $\beta SM \subset M$. Hence $RS \in \mathrm{OS}(M)$, because

$$\alpha\beta RSM = \alpha R(\beta SM) \subset \alpha RM \subset M.$$

One also has $M \subset \alpha^{-1}R^{-1}M$, which implies that the group index $[\alpha^{-1}R^{-1}M : M] = s$ is finite by Lemma 2.3. Thus $s\alpha^{-1}R^{-1}M \subset M$ is also of finite index. This shows $R^{-1} \in \mathrm{OS}(M)$. For the group property of $\mathrm{OC}(M)$ let $R_1, R_2 \in \mathrm{OC}(M)$. $R_2M \sim M$ yields $M \sim R_2^{-1}M$, and hence $R_1M \sim R_1R_2^{-1}M$. On the other hand, $M \sim R_1M \sim R_1R_2^{-1}M$ implies $M \sim R_1R_2^{-1}M$, because commensurability is transitive by Lemma 2.5. ■

Let us briefly turn to the subgroup of orientation preserving similarity isometries $\mathrm{SOS}(M) \subset \mathrm{OS}(M)$, which are by definition those similarity isometries R with $\det(R) = 1$. For planar lattices, these similarity rotations are rather well understood; *cf.* [11]. A $\mathbb{Z}$-module in an algebraic number field K of degree n is called *full*, if it contains n linearly independent elements over $\mathbb{Q}$. If a full $\mathbb{Z}$-module in K is a ring and contains the number 1, it is called an *order of K*. Any order of K is contained in the ring of algebraic integers $\mathcal{O}_K$ of K, which is itself an order. Hence $\mathcal{O}_K$ is also called the *maximal order of K*; *cf.* [12]. In the following results on orders of imaginary algebraic number fields, we parametrise the Euclidean plane by the complex numbers $\mathbb{C}$ and furthermore, we use $\mathrm{SO}(2, \mathbb{R}) \simeq \mathbb{S}^1$.

Lemma 2.7 *Let K be an imaginary algebraic number field and let $\mathcal{O}$ be an order of K. Then*

$$\mathrm{SOS}(\mathcal{O}) = \left\{ a/|a| \;\middle|\; a \in \mathcal{O} \setminus \{0\} \right\}.$$

Proof For $0 \neq a \in \mathcal{O}$, one has $|a| \cdot a/|a|\mathcal{O} \subset \mathcal{O}$, because $\mathcal{O}$ is a ring. Hence $a/|a| \in \mathrm{SOS}(\mathcal{O})$. Conversely, let $r \in \mathrm{SOS}(\mathcal{O})$, meaning $r \in \mathbb{S}^1$ with $\lambda r\mathcal{O} \subset \mathcal{O}$ for some $\lambda \in \mathbb{R}_+$. Since $1 \in \mathcal{O}$, this yields $\lambda r \in \mathcal{O}$, say $\lambda r = \beta$. Thus $|\lambda| = |\beta|$, because $r \in \mathbb{S}^1$. This shows that $r = \pm\beta/|\beta|$ is an $\mathcal{O}$-direction. ■

There is a close connection between similar submodules of orders $\mathcal{O}$ of algebraic number fields K that arise from rotations and the principal ideals of these orders. The special cases where K is an n-th cyclotomic field of class number 1 and $\mathcal{O} = \mathbb{Z}[\zeta_n]$ (with ζ_n an n-th primitive root of unity), or where $\mathcal{O}$ is a planar lattice with non-generic multiplier ring, can be found in [4] and [11].

Theorem 2.8 *Let K be an imaginary algebraic number field and let $\mathcal{O}$ be an order of K. Then the similar submodules of $\mathcal{O}$ of the form $\alpha R\mathcal{O}$ with $R \in \mathrm{SOS}(\mathcal{O})$ are precisely the principal ideals of $\mathcal{O}$,* i.e., *the ideals of the form $\kappa\mathcal{O}$ with $\kappa \in \mathcal{O}$. Moreover, one has*

$$[\mathcal{O} : \kappa\mathcal{O}] = |N(\kappa)|,$$

where N denotes the field norm of K.

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Proof Let $R \in \mathrm{SOS}(\mathcal{O})$ and $\alpha \in \mathbb{R}_+$ with $\alpha R\mathcal{O} \subset \mathcal{O}$. Due to Lemma 2.7, there exists a nonzero $\delta \in \mathcal{O}$ such that $R = \delta/|\delta|$. Then $1 \in \mathcal{O}$ implies $\alpha\delta/|\delta| \in \mathcal{O}$. Hence $\alpha R\mathcal{O}$ is a principal ideal of $\mathcal{O}$. Conversely, for any nonzero $\kappa' \in \mathcal{O}$ one has $\kappa'\mathcal{O} \subset \mathcal{O}$, because $\mathcal{O}$ is a ring. Setting $R' = \kappa'/|\kappa'|$, one has $R' \in \mathrm{SOS}(\mathcal{O})$ by Lemma 2.7, and $|\kappa'|R'\mathcal{O} \subset \mathcal{O}$. The second claim follows by a standard argument in Minkowski theory; *cf.* [12, Section 3, Chapter 2]. Considering a Minkowski representation $x(\mathcal{O})$ of $\mathcal{O}$, one finds

$$[\mathcal{O} : \kappa\mathcal{O}] = [x(\mathcal{O}) : x(\kappa\mathcal{O})] = |N(\kappa)|. \qquad \blacksquare$$

3 $\mathcal{S}$-Modules

Let $\mathcal{S} \subset \mathbb{R}$ be a ring with unity that is also a finitely generated $\mathbb{Z}$-module, hence a free $\mathbb{Z}$-module of finite rank r. Furthermore, let K be the field of fractions of $\mathcal{S}$. Throughout this section, let $\Gamma \subset \mathbb{R}^d$ be a free $\mathcal{S}$-module of rank d that spans $\mathbb{R}^d$, meaning that it is the $\mathcal{S}$-span of an $\mathbb{R}$-basis of $\mathbb{R}^d$. Besides the case $\mathcal{S} = \mathbb{Z}$, where Γ is a lattice in $\mathbb{R}^d$, this also covers many important examples relevant in quasicrystallography; *cf.* Example 3.18.

Remark 3.1

(1) Every element of $\mathcal{S}$ is an algebraic integer and K is a real algebraic number field.
(2) $\mathcal{S}$ is integrally closed if and only if $\mathcal{S}$ is the ring of integers $\mathcal{O}_K$ in K.
(3) Γ is a free $\mathbb{Z}$-module of rank rd.
(4) In fact, by standard results from algebra, rings $\mathcal{S}$ as above are precisely the subrings of rings of integers in real algebraic number fields; *cf.* [15] for more on this.

Remark 3.2 Let Γ_1, Γ_2 be free $\mathcal{S}$-modules of rank d that span $\mathbb{R}^d$. If Γ_1 and Γ_2 are commensurate, then $\Gamma_1 \cap \Gamma_2$ is a free $\mathbb{Z}$-module of rank rd (*cf.* Fact 2.4) and it spans $\mathbb{R}^d$. Namely, if $\Gamma_1 \sim \Gamma_2$, then one has $m = [\Gamma_1 : (\Gamma_1 \cap \Gamma_2)] < \infty$. Hence $m\Gamma_1 \subset (\Gamma_1 \cap \Gamma_2)$, which implies that $\Gamma_1 \cap \Gamma_2$ contains an $\mathbb{R}$-basis of $\mathbb{R}^d$.

The following result is of fundamental importance; compare [20, Theorem 2.1] for the special case $\mathcal{S} = \mathbb{Z}$ $(K = \mathbb{Q})$.

Theorem 3.3 *Let $\Gamma_1, \Gamma_2 \subset \mathbb{R}^d$ be free $\mathcal{S}$-modules of rank d that span $\mathbb{R}^d$. Further, let $B_1, B_2 \in \mathrm{GL}(d, \mathbb{R})$ be basis matrices of the $\mathcal{S}$-modules Γ_1 and Γ_2, respectively. Then, one has*

$$\Gamma_1 \sim \Gamma_2 \iff B_2^{-1}B_1 \in \mathrm{GL}(d, K).$$

Proof Let firstly $\Gamma_1 \sim \Gamma_2$. By Remark 3.2, the intersection $\Gamma' = \Gamma_1 \cap \Gamma_2$ contains an $\mathbb{R}$-basis $\mathcal{B}$ of $\mathbb{R}^d$. Let $B \in \mathrm{GL}(d, \mathbb{R})$ be the associated matrix. Then there exist non-singular matrices $Z_1, Z_2 \in \mathrm{Mat}(d, \mathcal{S})$ such that

$$B_1 Z_1 = B = B_2 Z_2,$$

whence $B_2^{-1}B_1 = Z_2 Z_1^{-1} \in \mathrm{GL}(d, K)$ by the standard formula for the inverse of a matrix. Conversely, if $B_2^{-1}B_1 \in \mathrm{GL}(d, K)$, then there is a non-zero number $s \in \mathcal{S}$

such that $B = sB_2^{-1}B_1 \in \mathrm{Mat}(d, \mathcal{S})$. Setting $\Gamma' = \Gamma_1 \cap \Gamma_2$, the identity $sB_1 = B_2B$ implies that $s\Gamma_1 \subset \Gamma' \subset \Gamma_1$. Since $s\Gamma_1$ and Γ_1 are both free $\mathbb{Z}$-modules of rank rd, one obtains $[\Gamma_1 : \Gamma'] < \infty$. By symmetry, one also has $[\Gamma_2 : \Gamma'] < \infty$. Hence, $\Gamma_1 \sim \Gamma_2$. ■

Definition 3.4 For an arbitrary element $R \in \mathrm{O}(d, \mathbb{R})$, define

$$\mathrm{scal}_\Gamma(R) = \{\alpha \in \mathbb{R} \mid \Gamma \sim \alpha R\Gamma\}.$$

Note that $\mathrm{OS}(\Gamma) = \{R \in \mathrm{O}(d, \mathbb{R}) \mid \mathrm{scal}_\Gamma(R) \neq \varnothing\}$.

Remark 3.5 If $\beta \in \mathrm{scal}_\Gamma(R)$, then there exists a nonzero element $t \in \mathbb{Z}$ such that $t\beta R\Gamma \subset \Gamma$. For if $\beta \in \mathrm{scal}_\Gamma(R)$, then the index $[\beta R\Gamma : (\Gamma \cap \beta R\Gamma)] = t$ is finite and one has $t\beta R\Gamma \subset (\Gamma \cap \beta R\Gamma) \subset \Gamma$.

Lemma 3.6 *For all elements $\alpha \in \mathrm{scal}_\Gamma(R)$ one has $\alpha^d \in K$. Thus α is an algebraic number.*

Proof One has $\alpha R\Gamma \sim \Gamma$ by assumption. Let B be a basis matrix for Γ. Then αRB is a basis matrix for $\alpha R\Gamma$. By Theorem 3.3, one has $B^{-1}\alpha RB \in \mathrm{GL}(d, K)$, which immediately yields

$$\alpha^d = \pm \det(\alpha R) \in K.$$

Hence α is algebraic over K, which implies that $K(\alpha)$ is a finite field extension of K, and thus also of $\mathbb{Q}$. Therefore α is algebraic over $\mathbb{Q}$. ■

Let $\mathbb{R}^\bullet$ denote the multiplicative group formed by the nonzero real numbers. Denoting by $\mathbb{R}^\bullet\,\mathrm{GL}(d, K)$ the group consisting of all elements of the form tH with $t \in \mathbb{R}^\bullet$ and $H \in \mathrm{GL}(d, K)$, there is the following consequence of Theorem 3.3.

Corollary 3.7 *For any basis matrix B_Γ of Γ, one has*

$$\mathrm{OS}(\Gamma) = \Big(B_\Gamma\big(\mathbb{R}^\bullet\,\mathrm{GL}(d, K)\big)B_\Gamma^{-1}\Big) \cap \mathrm{O}(d, \mathbb{R}) \tag{2}$$

and

$$\mathrm{OC}(\Gamma) = \big(B_\Gamma\,\mathrm{GL}(d, K)B_\Gamma^{-1}\big) \cap \mathrm{O}(d, \mathbb{R}).$$

Proof Let $\{\gamma_1, \dots, \gamma_d\}$ be an $\mathcal{S}$-basis of Γ and denote by B_Γ the associated matrix. For $R \in \mathrm{OS}(\Gamma)$ there exists a positive real number α with $\alpha R\Gamma \sim \Gamma$. The set $\{\alpha R\gamma_1, \dots, \alpha R\gamma_d\}$ is an $\mathcal{S}$-basis of $\alpha R\Gamma$ with associated matrix $B_{\alpha R\Gamma} = \alpha RB_\Gamma$. Theorem 3.3 then implies that there exists an $H \in \mathrm{GL}(d, K)$ with $H = B_\Gamma^{-1}\alpha RB_\Gamma$. Thus, one has $R = B_\Gamma\alpha^{-1}HB_\Gamma^{-1}$. If on the other hand $S \in \mathrm{O}(d, \mathbb{R})$ and $S = B_\Gamma\beta JB_\Gamma^{-1}$ for some $\beta \in \mathbb{R}^\bullet$ and $J \in \mathrm{GL}(d, K)$, then one has

$$B_\Gamma^{-1}B_{\beta^{-1}S\Gamma} = B_\Gamma^{-1}\beta^{-1}SB_\Gamma \in \mathrm{GL}(d, K).$$

Theorem 3.3 therefore implies $\beta^{-1}S\Gamma \sim \Gamma$, which shows $S \in \mathrm{OS}(\Gamma)$. ■

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Remark 3.8 By Corollary 3.7, every element $R \in \mathrm{OS}(\Gamma)$ can be written as $R = B_\Gamma \beta H B_\Gamma^{-1}$ with $\beta \in \mathbb{R}^\bullet$ and $H \in \mathrm{GL}(d,K)$. Theorem 3.3 implies $\beta R\Gamma \sim \Gamma$ and hence $\beta \in \mathrm{scal}_\Gamma(R)$, which shows that β is an algebraic number and that $\beta^d \in K$ by Lemma 3.6. But the set of all algebraic numbers is countable and so is $\mathrm{GL}(d,K)$. Therefore the group $\mathrm{OS}(\Gamma)$ is countable and in particular, the subgroup $\mathrm{OC}(\Gamma)$ is countable as well. The explanations above imply that, in Corollaries 3.7 and 3.9, $\mathbb{R}^\bullet$ can be replaced by the set of all nonzero real numbers δ with $\delta^d \in K$.

Corollary 3.9 *Let $\Gamma \subset K^d$. One has*

$$\mathrm{OS}(\Gamma) = \big(\mathbb{R}^\bullet\, \mathrm{GL}(d,K)\big) \cap \mathrm{O}(d,\mathbb{R})$$

and

$$\mathrm{OC}(\Gamma) = \mathrm{O}(d,K).$$

Proof The assumption $\Gamma \subset K^d$ yields $B_\Gamma \in \mathrm{GL}(d,K)$. Therefore, one has $B_\Gamma\, \mathrm{GL}(d,K) B_\Gamma^{-1} = \mathrm{GL}(d,K)$, and the claim follows from Corollary 3.7. ■

Lemma 3.10 *For $R \in \mathrm{OS}(\Gamma)$, the following assertions hold:*
(1) $b \cdot \mathrm{scal}_\Gamma(R) = \mathrm{scal}_\Gamma(R)$ for all $b \in K \setminus \{0\}$,
(2) $r\Gamma \sim \Gamma$ with $r \in \mathbb{R}$ implies $r \in K$,
(3) $\alpha\beta^{-1} \in K$ for all $\alpha, \beta \in \mathrm{scal}_\Gamma(R)$.

Proof Let $\alpha \in \mathrm{scal}_\Gamma(R)$. Since K is the field of fractions of $\mathcal{S}$, every nonzero element $b \in K$ can be written as $b = b_1/b_2$ with $b_1, b_2 \in \mathcal{S} \setminus \{0\}$. Then, one finds

$$\frac{b_1}{b_2}\alpha R\Gamma \sim \frac{1}{b_2}\alpha R\Gamma \sim \frac{1}{b_2}\Gamma.$$

One easily observes that $\frac{1}{b_2}\Gamma \sim \Gamma$. Hence $b\alpha R\Gamma \sim \Gamma$, yielding $b \cdot \mathrm{scal}_\Gamma(R) \subset \mathrm{scal}_\Gamma(R)$. Thus, one also has $b^{-1} \cdot \mathrm{scal}_\Gamma(R) \subset \mathrm{scal}_\Gamma(R)$, which proves (1). In order to show (2), let $u \in \mathbb{R}$ with $u\Gamma \sim \Gamma$. Due to Remark 3.5, there exists a nonzero integer k such that $ku\Gamma \subset \Gamma$. Let $\gamma \in \Gamma$ be represented in terms of an $\mathcal{S}$-basis $\gamma_1, \dots, \gamma_d$ of Γ as $\gamma = \sum_{i=1}^d c_i\gamma_i$ with $c_i \in \mathcal{S}$. On the other hand, $ku\gamma$ can be represented as $ku\gamma = \sum_{i=1}^d a_i\gamma_i$, where $a_i \in \mathcal{S}$. Thus

$$\sum_{i=1}^d kuc_i\gamma_i = \sum_{i=1}^d a_i\gamma_i.$$

By assumption, Γ spans the $\mathbb{R}^d$. Hence $\{\gamma_1, \dots, \gamma_d\}$ forms an $\mathbb{R}$-basis of $\mathbb{R}^d$. Therefore, one has $kuc_i = a_i$, yielding $u = a_i c_i^{-1} k^{-1} \in K$. Finally, (3) is obtained from (2) as follows. By assumption, one has

$$\beta R\Gamma \sim \Gamma \sim \alpha R\Gamma.$$

Multiplying with $1/\beta$ gives $R\Gamma \sim \frac{\alpha}{\beta} R\Gamma$, which completes the proof. ■

In accordance with the previous notation, denote by $K^\bullet$ the multiplicative group formed by the nonzero elements of K.

Remark 3.11 As a direct consequence of Lemma 3.10, one has $\mathrm{scal}_\Gamma(R) = \alpha K^\bullet$ for any $\alpha \in \mathrm{scal}_\Gamma(R)$.

Define the map

$$\begin{aligned} \eta\colon\ \mathrm{OS}(\Gamma) &\longrightarrow \mathbb{R}^\bullet/K^\bullet, \\ R &\longmapsto \mathrm{scal}_\Gamma(R). \end{aligned}$$

This map is well-defined due to the fact that $\mathrm{scal}_\Gamma(R)$ is non-empty for $R \in \mathrm{OS}(\Gamma)$ and by Remark 3.11.

Lemma 3.12 *The map* η *is a group homomorphism with* $\mathrm{Ker}(\eta) = \mathrm{OC}(\Gamma)$.

Proof Let $R, S \in \mathrm{OS}(\Gamma)$ and $\alpha \in \mathrm{scal}_\Gamma(R)$, $\beta \in \mathrm{scal}_\Gamma(S)$. We need to show that $\alpha\beta \in \mathrm{scal}_\Gamma(RS)$. By assumption, one has

$$\Gamma \sim \alpha R\Gamma \sim \alpha R(\beta S\Gamma) = \alpha\beta RS\Gamma.$$

Thus $\alpha\beta \in \mathrm{scal}_\Gamma(RS)$, hence η is a group homomorphism. It remains to show that $\mathrm{Ker}(\eta) = \mathrm{OC}(\Gamma)$. For $R \in \mathrm{OC}(\Gamma)$ the set $\mathrm{scal}_\Gamma(R)$ contains 1, thus $\eta(R) = \mathrm{scal}_\Gamma(R) = K^\bullet$ by Remark 3.11. Conversely, if $S \in \mathrm{Ker}(\eta)$, one has $\mathrm{scal}_\Gamma(S) = K^\bullet$, which implies $S \in \mathrm{OC}(\Gamma)$. ∎

As the kernel of a group homomorphism, $\mathrm{OC}(\Gamma)$ is a normal subgroup of $\mathrm{OS}(\Gamma)$. The factor group $\mathrm{OS}(\Gamma)/\mathrm{OC}(\Gamma)$ is isomorphic to the image of η, which is a subgroup of $\mathbb{R}^\bullet/K^\bullet$ and thus abelian. Furthermore, $\mathrm{OS}(\Gamma)/\mathrm{OC}(\Gamma)$ is countable by Remark 3.8. The corresponding result holds for the special case of orientation-preserving isometries by considering the restriction of η to SOS. To unfold the structure of the factor group $\mathrm{OS}(\Gamma)/\mathrm{OC}(\Gamma)$, we need the following result from the theory of abelian groups.

Proposition 3.13 ([18, Theorems 5.1.9 and 5.1.12]) *Let G be an abelian group.*

(1) If a prime number p exists such that $x^p = 1$ *for all* $x \in G$*, then G is the direct sum of subgroups of order p.*

(2) If a positive integer n exists such that $x^n = 1$ *for all* $x \in G$*, then G is the direct sum of cyclic groups of prime power orders.*

Theorem 3.14 *The group* $\mathrm{OS}(\Gamma)/\mathrm{OC}(\Gamma)$ *is the direct sum of cyclic groups of prime power orders that divide d.*

Proof We consider again the group homomorphism $\eta\colon \mathrm{OS}(\Gamma) \longrightarrow \mathbb{R}^\bullet/K^\bullet$. Let $R \in \mathrm{OS}(\Gamma)$. According to Lemma 3.6, one has $\alpha^d \in K^\bullet$ for any nonzero $\alpha \in \mathrm{scal}_\Gamma(R)$, which yields

$$\eta(R)^d = \mathrm{scal}_\Gamma(R)^d = (\alpha K^\bullet)^d = \alpha^d K^\bullet = K^\bullet \tag{3}$$

in $\mathbb{R}^\bullet/K^\bullet$. Using the group isomorphism $\eta\big(\mathrm{OS}(\Gamma)\big) \simeq \mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)$, this shows that the order of each element of $\mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)$ divides d. Proposition 3.13(2) then implies that the group $\mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)$ is the direct sum of cyclic groups of prime power orders. Consequently, the prime power order of each cyclic group divides d. ■

Example 3.15 Denote by $\{e_1, \dots, e_d\}$ the canonical basis of $\mathbb{R}^d$. Let $n \geqslant 1$ be a natural number with $\xi = \sqrt[d]{n} \notin \mathbb{Q}$. The $\mathbb{Z}$-span Γ of $\{\xi^i e_i \mid 1 \leqslant i \leqslant d\}$ is a lattice in $\mathbb{R}^d$. Consider the cyclic permutation $\sigma = (12\cdots d)$ of the symmetric group S_d. Then σ induces a linear isomorphism R of $\mathbb{R}^d$ by permuting the canonical basis vectors, *i.e.*, $Re_i = e_{\sigma(i)}$. Since $\xi R\Gamma \subset \Gamma$, R is a similarity isometry of Γ. But R is not a coincidence isometry of Γ (because $\sqrt[d]{n} \notin \mathbb{Q}$). Setting $m = \min_i \{\xi^i \mid \xi^i \in \mathbb{Q}\}$, one easily verifies that the factor group $\mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)$ contains the cyclic group C_m of order m generated by the equivalence class of R. If m is not a prime power, then the fundamental theorem of finitely generated abelian groups states that C_m is the direct sum of cyclic groups of prime power orders. Examples of the module case can be constructed similarly.

Corollary 3.16 *If $d = p$ is a prime number, then* $\mathrm{OS}(\Gamma)/\,\mathrm{OC}(\Gamma)$ *is an elementary abelian p-group,* i.e., *it is the direct sum of cyclic groups of order p.*

Definition 3.17 Denote by $\langle\,\cdot\,,\,\cdot\,\rangle$ the standard scalar product of $\mathbb{R}^d$. We call Γ an $\mathcal{S}$*-module over K in* $\mathbb{R}^d$ if it satisfies $\langle\gamma,\gamma\rangle \in K$ for all $\gamma \in \Gamma$.

Example 3.18

(1) Let $\mathcal{S} = \mathbb{Z}$. Then, $K = \mathbb{Q}$ and the $\mathcal{S}$-modules Γ over K in $\mathbb{R}^d$ are precisely the *rational lattices* in $\mathbb{R}^d$; *cf.* [13] for examples.

(2) For $n \in \mathbb{N}$, $\mathcal{S}^n$ is an $\mathcal{S}$-module over K in $\mathbb{R}^n$.

(3) Consider the quadratic number field $L := \mathbb{Q}(\tau)$, where τ is the golden ratio, *i.e.*, $\tau = (1+\sqrt{5})/2$. Then, one has $\mathcal{O}_L = \mathbb{Z}[\tau]$. The icosian ring

$$\mathbb{I} = \big\langle (1,0,0,0), (0,1,0,0), \tfrac{1}{2}(1,1,1,1), \tfrac{1}{2}(1-\tau,\tau,0,1)\big\rangle_{\mathcal{O}_L} \subset L^4$$

is an $\mathcal{O}_L$-module over L in $\mathbb{R}^4$ (see [8]). Further, both the standard body centred icosahedral module $\mathcal{M}_{\mathrm{B}}$ and the standard face centred icosahedral module $\mathcal{M}_{\mathrm{F}}$ of quasicrystallography are $\mathcal{O}_L$-modules over L in $\mathbb{R}^3$; *cf.* [2], [9] and references therein.

(4) Consider the quadratic number field $L := \mathbb{Q}(\sqrt{2})$. Then, $\mathcal{O}_L = \mathbb{Z}[\sqrt{2}]$ and further, the octahedral (or cubian) ring

$$\mathbb{K} = \langle (1,0,0,0), \tfrac{1}{\sqrt{2}}(1,1,0,0), \tfrac{1}{\sqrt{2}}(1,0,1,0), \tfrac{1}{2}(1,1,1,1)\rangle_{\mathcal{O}_L} \subset L^4$$

is an $\mathcal{O}_L$-module over L in $\mathbb{R}^4$; *cf.* [8], [9] and references therein.

(5) Consider the complex cyclotomic field $\mathbb{Q}(\zeta_m)$, where $m \geq 3$ and ζ_m is a primitive m-th root of unity in $\mathbb{C}$ (*e.g.*, $\zeta_m = e^{2\pi i/m}$). Recall that $\mathbb{Q}(\zeta_m)$ is a finite Galois extension of $\mathbb{Q}$ with maximal real subfield $L := \mathbb{Q}(\zeta_m) \cap \mathbb{R} = \mathbb{Q}(\zeta_m + \bar{\zeta}_m)$;

https://doi.org/10.4153/CMB-2011-076-x Published online by Cambridge University Press

cf. [19, Theorem 2.5]. Further, it is well known that $\mathcal{O}_{\mathbb{Q}(\zeta_m)} = \mathbb{Z}[\zeta_m]$ and $\mathcal{O}_L = \mathbb{Z}[\zeta_m + \bar{\zeta}_m]$; *cf.* [19, Theorem 2.6 and Proposition 2.16]. Moreover, since $\zeta_m^2 = (\zeta_m + \bar{\zeta}_m)\zeta_m - 1$, the ring $\mathbb{Z}[\zeta_m]$ is the $\mathbb{Z}[\zeta_m + \bar{\zeta}_m]$-span of the $\mathbb{R}$-basis $\{1, \zeta_m\}$ of $\mathbb{C}$. Identifying the complex numbers $\mathbb{C}$ with $\mathbb{R}^2$, one can now verify that $\mathbb{Z}[\zeta_m]$ is an $\mathcal{O}_L$-module over L in $\mathbb{R}^2$. In particular, rings of integers in complex cyclotomic fields can be used to construct planar mathematical quasicrystals such as the vertex sets of Penrose, Ammann–Beenker or shield tilings; *cf.* [7], [3], [5], [17].

Theorem 3.19 *For an $\mathcal{S}$-module Γ over K in $\mathbb{R}^d$ one has*

$$\operatorname{OS}(\Gamma)^2 \subset \operatorname{OC}(\Gamma),$$

where $\operatorname{OS}(\Gamma)^2 = \{R^2 \mid R \in \operatorname{OS}(\Gamma)\}$. Thus, the factor group $\operatorname{OS}(\Gamma)/\operatorname{OC}(\Gamma)$ is an elementary abelian 2-group when d is even. If d is odd, one has $\operatorname{OS}(\Gamma) = \operatorname{OC}(\Gamma)$.

Proof Let $R \in \operatorname{OS}(\Gamma)$. Then there exists an element $\alpha \in \mathbb{R}_+$ with $\alpha R\Gamma \subset \Gamma$. By assumption, one has $\langle \alpha R\gamma, \alpha R\gamma \rangle \in K$ for all $\gamma \in \Gamma$. Hence $\alpha^2 \in K^\bullet$, say $\alpha^2 = s_1/s_2$, where $s_1, s_2 \in \mathcal{S} \setminus \{0\}$. Since $s_2\alpha^2 = s_1 \in \mathcal{S}$ and $\alpha R\Gamma \subset \Gamma$, this yields

$$\Gamma \supset s_2\alpha R(\alpha R\Gamma) = s_2\alpha^2 R^2\Gamma \subset R^2\Gamma,$$

whence $s_1R^2\Gamma \subset (\Gamma \cap R^2\Gamma)$. Since Γ, $R^2\Gamma$ and $s_1R^2\Gamma$ are $\mathbb{Z}$-modules of the same finite rank, one obtains that both $[\Gamma : s_1R^2\Gamma]$ and $[R^2\Gamma : s_1R^2\Gamma]$ are finite. It follows that $\Gamma \sim R^2\Gamma$, meaning that R^2 is a coincidence isometry of Γ. Consequently, $\operatorname{OS}(\Gamma)^2 \subset \operatorname{OC}(\Gamma)$. Thus, every element of the factor group $\operatorname{OS}(\Gamma)/\operatorname{OC}(\Gamma)$ is of order 1 or 2. By Proposition 3.13(1), the factor group is an elementary abelian 2-group. If d is odd, set $d = 2m + 1$ with $m \in \mathbb{N}$. Then $\alpha(\alpha^2)^m = \alpha^d \in K^\bullet$ yields $\alpha \in K^\bullet$, because $\alpha^2 \in K^\bullet$. Thus $\eta(R) = \operatorname{scal}_\Gamma(R) = \alpha K^\bullet = K^\bullet$ in $\mathbb{R}^\bullet/K^\bullet$ for all $R \in \operatorname{OS}(\Gamma)$, whence $\operatorname{OS}(\Gamma)/\operatorname{OC}(\Gamma)$ is the trivial group. In other words, one has $\operatorname{OS}(\Gamma) = \operatorname{OC}(\Gamma)$ for d odd. ∎

Example 3.20 Using the notation of Example 3.18(5), consider a cyclotomic field $\mathbb{Q}(\zeta_m)$ of class number one, meaning that its ring of integers $\mathbb{Z}[\zeta_m]$ is a unique factorization domain; see [10] for more on this. $\mathbb{Z}[\zeta_m]$ is an $\mathcal{O}_L$-module over L in $\mathbb{R}^2$. Since we are working in 2-space, Theorem 3.19 implies that the factor group of similarity modulo coincidence isometries is an elementary abelian 2-group. Combining the results of [10] and [16, Proposition 1.89], one immediately obtains

$$\operatorname{SOS}(\mathbb{Z}[\zeta_m])/\operatorname{SOC}(\mathbb{Z}[\zeta_m]) \simeq C_2 \times C_2^{(\aleph_0)},$$

where SOS and SOC indicate the restriction to orientation-preserving isometries, and $C_2^{(\aleph_0)}$ stands for the direct sum of countably many cyclic groups of order 2.

Acknowledgments The author is thankful to M. Baake, C. Huck, R. V. Moody and P. Zeiner for valuable discussions and comments on the manuscript. This work was supported by the German Research Council (DFG), within the CRC 701.

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