Bull. Aust. Math. Soc. **91** (2015), 167–174
doi:10.1017/S0004972714000689

CARDINALITY OF INVERSE LIMITS WITH UPPER SEMICONTINUOUS BONDING FUNCTIONS

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(Received 12 June 2014; accepted 13 August 2014; first published online 23 September 2014)

Abstract

We explore the cardinality of generalised inverse limits. Among other things, we show that, for any $n \in \{\aleph_0, c, 1, 2, 3, \ldots\}$, there is an upper semicontinuous function with the inverse limit having exactly n points. We also prove that if f is an upper semicontinuous function whose graph is a continuum, then the cardinality of the corresponding inverse limit is either 1, $\aleph_0$ or c. This generalises the recent result of I. Banič and J. Kennedy, which claims that the same is true in the case where the graph is an arc.

2010 *Mathematics subject classification*: primary 54C60; secondary 54B10, 54D80.

Keywords and phrases: continua, limits, inverse limits, cardinality, upper semicontinuous set-valued functions.

1. Introduction

In the present paper, we explore the cardinality of generalised inverse limits on intervals induced by an upper semicontinuous set-valued bonding function $f : I \to 2^I$. It is a well-known fact that in the case where the bonding function $f : I \to I$ is a continuous single-valued function, the corresponding inverse limit is either an arc-like continuum or a singleton. In the first case the inverse limit consists of uncountably many points and in the other of a single point. Therefore the cardinality of such inverse limit can either be 1 or c. It has been shown by I. Banič and J. Kennedy in [2] that there are examples of upper semicontinuous set-valued functions that produce inverse limits with cardinality $\aleph_0$. The main results proved in this paper are the following (see Theorems 3.9 and 3.11):

(1) Let n be a positive integer. Then there is an upper semicontinuous function $f : I \to 2^I$ such that $\varprojlim f$ contains exactly n points.
(2) Let $f : I \to 2^I$ be an upper semicontinuous function whose graph $G(f)$ is a continuum. Then the cardinality of $\varprojlim f$ is either 1 or infinity.

We proceed as follows. In Section 2 we give basic definitions and introduce notation that will be used in the paper. In Section 3 we prove the main results of the paper.

2. Definitions and notation

Our definitions and notation mostly follow [4] and [6].

A *continuum* is a nonempty, compact and connected metric space.

If (X, d) is a compact metric space, then 2^X denotes the set of all nonempty closed subsets of X.

Let X and Y be compact metric spaces. A function $f : X \to 2^Y$ is *upper semicontinuous* if, for each open set $V \subseteq Y$, the set $\{x \in X \mid f(x) \subseteq V\}$ is an open set in X.

The *graph* $G(f)$ of a function $f : X \to 2^Y$ is the set of all points $(x, y) \in X \times Y$ such that $y \in f(x)$.

The following characterisation of upper semicontinuous functions can be found in [4, page 120]:

THEOREM 2.1. *Let X and Y be compact metric spaces and $f : X \to 2^Y$ a function. Then f is an upper semicontinuous function if and only if its graph $G(f)$ is closed in $X \times Y$.*

The *inverse limit* of an inverse sequence $\{X_n, f_n\}_{n=0}^{\infty}$ is defined to be the subspace of the product space $\prod_{n=0}^{\infty} X_n$ of all $\mathbf{x} = (x_0, x_1, x_2, \ldots) \in \prod_{n=0}^{\infty} X_n$, such that $x_n \in f_n(x_{n+1})$ for each n. The inverse limit is denoted by $\varprojlim \{X_n, f_n\}_{n=1}^{\infty}$.

These inverse limits have been studied intensively in the last decade. One reason for such intense research is the fact that inverse sequences with very simple spaces and simple bonding functions can produce very complicated continua as their inverse limits. This may happen even in the case where all the spaces are closed unit intervals and all the bonding functions are the same. They are a generalisation of standard inverse limits and were introduced in [4, 5] by Ingram and Mahavier. The concept of these generalised inverse limits has become very popular since their introduction and has been studied by many authors; see [1, 3] where more references can be found.

In this paper we deal only with the case where, for each n, X_n is the closed unit interval $I = [0, 1]$, and all f_n are the same function $f : I \to 2^I$. In this case we denote the inverse limit $\varprojlim \{X_n, f_n\}_{n=1}^{\infty}$ simply by $\varprojlim f$.

For any upper semicontinuous function f, the set $\mathcal{P}(f)$ was first introduced in [2] to show some properties of inverse limits regarding their cardinality. We will also use the set $\mathcal{P}(f)$, which is defined as follows.

DEFINITION 2.2. For an upper semicontinuous function f on $[0, 1]$, and a positive integer n, define

$$\mathcal{P}_n(f) = \{x \in [0, 1] : \text{there is } x_n \in [0, 1] \text{ such that } (x_n, x) \in G(f^n)\},$$

and let

$$\mathcal{P}(f) = \bigcap_{n=1}^{\infty} \mathcal{P}_n(f).$$

We will also use the following result about $\mathcal{P}(f)$ from [2].

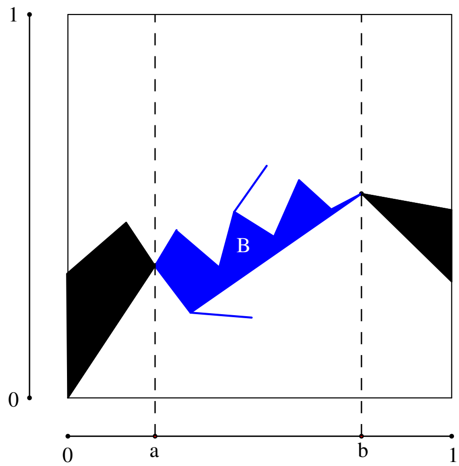

FIGURE 1. An example of the function from Lemma 3.1.

PROPOSITION 2.3. *Suppose that $f : I \to 2^I$ is upper semicontinuous. Then*

$$\varprojlim f = \varprojlim(\mathcal{P}(f), f|_{\mathcal{P}(f)}).$$

For every set A, we denote the cardinality of A by $|A|$.

3. The cardinality of inverse limits

In this section we prove Theorems 3.9 and 3.11, the main results of the paper. We start with several lemmas, which we use to prove Theorem 3.9.

LEMMA 3.1. *Suppose that $f : I \to 2^I$ is an upper semicontinuous function whose graph $G(f)$ is a continuum. Let $a, b \in I$, $a < b$. If $|f(a)| = |f(b)| = 1$, then $B = ([a, b] \times I) \cap G(f)$ is a continuum (see Figure 1).*

PROOF. B is obviously nonempty and compact. Suppose that the sets U', V' are a separation of B. One can easily see that U' and V' are compact.

(1) If $\{a\} \times f(a) \subseteq U'$, $\{b\} \times f(b) \subseteq V'$, we take $U = U' \cup ([0, a] \times I) \cap G(f)$ and $V = V' \cup ([b, 1] \times I) \cap G(f)$. It is obvious that U and V are not empty, $U \cup V = G(f)$ and $U \cap V = \emptyset$. Since U and V are compact, they are also closed in $G(f)$. Therefore, U and V are a separation of $G(f)$.
(2) If $\{a\} \times f(a) \subseteq V'$, $\{b\} \times f(b) \subseteq U'$, one can easily obtain the same result as above.
(3) If $\{a\} \times f(a), \{b\} \times f(b) \subseteq U'$, we take $U = U' \cup (([0, a] \times I) \cup ([b, 1] \times I)) \cap G(f)$ and $V = V'$. Similarly as above, U and V are a separation of $G(f)$.
(4) If $\{a\} \times f(a), \{b\} \times f(b) \subseteq V'$, the proof is similar to the previous case.

In all cases we get a separation of $G(f)$, which is a continuum—a contradiction. Therefore B is a continuum. □

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Lemma 3.2. *Suppose that $f : I \to 2^I$ is an upper semicontinuous function whose graph $G(f)$ is a continuum. Then there exists $a \in I$ such that $(a, a) \in G(f)$.*

Proof. Let D be the diagonal of $I \times I$, $D = \{(x, x) \mid x \in I\}$. Suppose that $G(f) \cap D$ is empty. Then there exist $(a, b), (c, d) \in G(f)$ such that $a < b$ and $c > d$ (such points exist since $\{0\} \times I \neq \emptyset$ and $\{1\} \times I \neq \emptyset$). We define $U' = \{(x, y) \in I \times I \mid y < x\}$ and $V' = \{(x, y) \in I \times I \mid y > x\}$, which are mutually disjoint open sets in $I \times I$. Taking $U = G(f) \cap U'$ and $V = G(f) \cap V'$, we get a separation of the continuum $G(f)$, which is a contradiction. Therefore there exists $a \in I$ such that $(a, a) \in G(f)$. □

Lemma 3.3. *Let $f : I \to 2^I$ be an upper semicontinuous function such that the cardinality of $\varprojlim f$ is finite. If $(a, a) \in G(f)$, then $|f(a)| = 1$.*

Proof. Suppose that there is $b \neq a$ such that $(a, b) \in G(f)$, i.e. $b \in f(a)$. Let $b_0 = b$, $b_{-1} \in f(b_0), b_{-2} \in f(b_{-1}), \ldots, b_{-n-1} \in f(b_{-n}), \ldots$. Then, for each $m \geq 1$, $(b_{-m}, b_{-m+1}, \ldots, b_{-2}, b_{-1}, b_0, a, a, a, \ldots) \in \varprojlim f$, which is a contradiction, since $\varprojlim f$ is finite. Therefore $\{a\} \times I$ intersects $G(f)$ only at the point (a, a). □

Lemma 3.4. *Suppose that $f : I \to 2^I$ is an upper semicontinuous function whose graph $G(f)$ is a continuum such that $\varprojlim f$ is finite. Then there exists exactly one $a \in I$ such that $(a, a) \in G(f)$.*

Proof. Suppose that there is $b \in I$, $b \neq a$, such that $(b, b) \in G(f)$. Without loss of generality let $a < b$. We define $B = ([a, b] \times I) \cap G(f)$. It follows from Lemmas 3.3 and 3.1 that B is a continuum.

First we prove that, for all $x \in [a, b]$, there exists $y \in [a, b]$ such that $x \in f(y)$. Suppose that this is not true. Then there exists $x \in (a, b)$ such that, for all $y \in [a, b]$, $x \notin f(y)$. We define $U' = [a, b] \times [0, x)$ and $V' = [a, b] \times (x, 1]$ which are open sets in $[a, b] \times I$. Then $U = U' \cap B$ and $V = V' \cap B$ are a separation of B, which is in contradiction with B being a continuum.

Therefore, for every $x \in [a, b]$, there exists a point in $\varprojlim f$ with x in the first coordinate. This means that $\varprojlim f$ is not finite, which is a contradiction. Therefore, there is exactly one point of $G(f)$ on the diagonal D. □

Lemma 3.5. *Suppose that $f : I \to 2^I$ is an upper semicontinuous function whose graph $G(f)$ is a continuum such that $\varprojlim f$ is finite but contains more than one point. If $(a, a) \in G(f)$ then $a \neq 0$ and $a \neq 1$.*

Proof. Suppose that $a = 0$. We show that $G(f)$ is entirely below the diagonal. Suppose that there exists $(c, d) \in G(f)$ such that $c < d$. Then, for every $t \in (0, c)$, there exists $z \in (0, c)$ such that $(z, t) \in G(f) \cap \{(x, y) \mid y \geq x\}$. For every $t \in (0, c)$ there is $z_1 \in (0, c)$ such that $t \in f(z_1)$, there is $z_2 \in (0, c)$ such that $z_2 \in f(z_1)$, and so on. Therefore $(t, z_1, z_2, \ldots) \in \varprojlim f$. That means we get infinitely many points in the inverse limit—a contradiction. We have proved that $G(f)$ is entirely below the diagonal. It follows that there is $(x_1, x_2, \ldots) \in \varprojlim f$ such that $x_1 \neq 0$ and $x_1 < x_2 < x_3 < \cdots$. Since this sequence is increasing and bounded, it converges to some $0 < s \leq 1$. The sequence

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$(x_n, x_{n+1}) \in G(f)$ converges to (s, s) and therefore $(s, s) \in G(f)$. But from Lemma 3.4, $(0, 0)$ is the only point of $G(f)$ on the diagonal. By a similar argument we show that $a \neq 1$. □

The following three lemmas are obtained as parts of the proof of [2, Theorem 3.14]. They have also turned out to be very useful when proving other results, so we state them as separate lemmas.

Lemma 3.6. *Suppose that $f : I \to 2^I$ is an upper semicontinuous function and $\varprojlim f$ is finite. Then every point of $\varprojlim f$ is periodic (if $\mathbf{z} \in \varprojlim f$, then there is a finite sequence $(e_0, e_1, \ldots, e_l)$ such that $\mathbf{z} = (e_0, e_1, \ldots e_l, e_0, e_1, \ldots e_l, \ldots)$).*

Proof. See [2, proof of Theorem 3.14, items 6 and 9]. □

Lemma 3.7. *Let $f : I \to 2^I$ be an upper semicontinuous function. For every $x \in \mathcal{P}(f)$ there is some point $\mathbf{z} \in \varprojlim f$ with x in the first coordinate. Therefore, if $\varprojlim f$ is finite, $\mathcal{P}(f)$ must be finite.*

Proof. We take any $x \in \mathcal{P}(f)$ and define the sequence $(\mathbf{z}_n)$ as follows:

$$\begin{aligned}
\mathbf{z}_1 &= (x, y_1^1, 1, 1, 1, \ldots), \quad x \in f(y_1^1);\\
\mathbf{z}_2 &= (x, y_1^2, y_2^2, 1, 1, \ldots), \quad x \in f(y_1^2),\ y_1^2 \in f(y_2^2);\\
\mathbf{z}_3 &= (x, y_1^3, y_2^3, y_3^3, 1, \ldots), \quad x \in f(y_1^3),\ y_1^3 \in f(y_2^3),\ y_2^3 \in f(y_3^3);\\
&\vdots
\end{aligned}$$

It follows from the definition of $\mathcal{P}(f)$ that such a sequence exists. Let $\mathbf{s} = (x, s_1, s_2, s_3, \ldots)$ be a limit of some convergent subsequence of $\mathbf{z}_1, \mathbf{z}_2, \mathbf{z}_3, \ldots$. One can easily see that $\mathbf{s} \in \varprojlim f$, since $(s_1, x), (s_2, s_1), (s_3, s_2), \ldots$ are in $G(f)$. □

Lemma 3.8. *Let $f : I \to 2^I$ be an upper semicontinuous function such that $\varprojlim f$ is finite. Then the function $f|_{\mathcal{P}(f)} : \mathcal{P}(f) \longrightarrow 2^I$ can be interpreted as a function $\mathcal{P}(f) \longrightarrow \mathcal{P}(f)$ which is a single-valued bijection on $\mathcal{P}(f)$.*

Proof. See [2, proof of Theorem 3.14, items 10 and 11]. □

Theorem 3.9. *Suppose that $f : I \to 2^I$ is an upper semicontinuous function whose graph $G(f)$ is a continuum. Then the cardinality of $\varprojlim f$ is either one or infinity.*

Proof. Suppose that the cardinality of $\varprojlim f$ is finite, but contains more then one point. We take the uniquely defined a from Lemma 3.4. We denote $L = \varprojlim f$.

(1) Let $\mathrm{Orb}(x) = \{x, f(x), f^2(x), \ldots\}$ be the orbit of x. Recall that the function $f|_{\mathcal{P}(f)} : \mathcal{P}(f) \to \mathcal{P}(f)$ is bijective. Then, under $f|_{\mathcal{P}(f)}$, the orbit of each point of $\mathcal{P}(f)$ is finite: since $P(f)$ is finite there exist $i, j \in \mathbb{N}, i < j$, such that $f^i(x) = f^j(x)$ and therefore $x = f^{j-i}(x)$. This means that $\mathrm{Orb}(x)$ is finite. Furthermore, different orbits of elements in $\mathcal{P}(f)$ are disjoint and $\mathrm{Orb}(a) = \{a\}$.

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(2) We take $p \in \mathcal{P}(f)$, $p \neq a$ (we know that $|\mathcal{P}(f)| > 1$ since $\varprojlim f$ contains more than one point). Let J be the smallest closed interval that contains the orbit of p. Then $J = [p_1, p_2]$ where $p_1, p_2 \in \mathrm{Orb}(p)$ and of course $p_1 \neq p_2$.

(3) We claim that $J \subseteq f(J)$. Since $(J \times I) \cap G(f)$ is a continuum (this follows from Lemma 3.1), $f(J)$ is connected. It is enough to show that $p_1 \in f(J)$ and $p_2 \in f(J)$. We know that $p_1 \in \mathrm{Orb}(p)$ and therefore $p_1 = f^i(p)$ for some integer $i > 1$. This means that $p_1 = f(f^{i-1}(p))$, where $f^{i-1}(p) \in J$. We have proved that $p_1 \in f(J)$. By a similar argument we show that $p_2 \in f(J)$.

(4) If we take arbitrary $x \in J$, there exist $x_1 \in J$ such that $x \in f(x_1)$, $x_2 \in J$ such that $x_1 \in f(x_2)$, and so on. For each $x \in J$, there is a point $(x, x_1, x_2, \ldots) \in L$ and this is a contradiction, since L is finite.

So if $\varprojlim f$ contains more than one point, it is infinite. □

Inverse limits can be many things. One can get from a one-point continuum to any arc-like continuum using single-valued bonding functions. So one can easily find functions g and h such that $|\varprojlim g| = 1$ and $|\varprojlim h| = c$. The next example gives an upper semicontinuous function f such that $|\varprojlim h| = \aleph_0$.

Example 3.10. Let $f : I \to 2^I$ be an upper semicontinuous function whose graph $G(f)$ is the union of four line segments (see Figure 2):

- the first connects $(0, \frac{1}{3})$ to $(\frac{1}{3}, \frac{1}{3})$;
- the second connects $(\frac{1}{3}, \frac{1}{3})$ to $(0, \frac{1}{2})$;
- the third connects $(0, \frac{1}{2})$ to $(\frac{1}{3}, \frac{2}{3})$;
- the fourth connects $(\frac{1}{3}, \frac{2}{3})$ to $(1, \frac{2}{3})$.

It is easy to see that

$$\varprojlim f = \left\{\left(\frac{2}{3}, \frac{2}{3}, \frac{2}{3}, \ldots\right), \left(\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right), \left(\frac{2}{3}, \frac{1}{3}, \frac{1}{3}, \ldots\right), \left(\frac{2}{3}, \frac{2}{3}, \frac{1}{3}, \ldots\right), \ldots\right\}.$$

Hence, $|\varprojlim f| = \aleph_0$.

We have seen in Theorem 3.9 and Example 3.10 that $|\varprojlim f| \in \{1, \aleph_0, c\}$ if $G(f)$ is connected. If $G(f)$ is not connected, other possibilities may occur. In Theorem 3.11 we show that for any positive integer n there is an upper semicontinuous function such that the corresponding inverse limit contains exactly n points. If $n \geq 2$ than the graph of each such function is disconnected.

Theorem 3.11. *Let n be a positive integer. Then there is an upper semicontinuous function $f : I \to 2^I$ such that $\varprojlim f$ contains exactly n points.*

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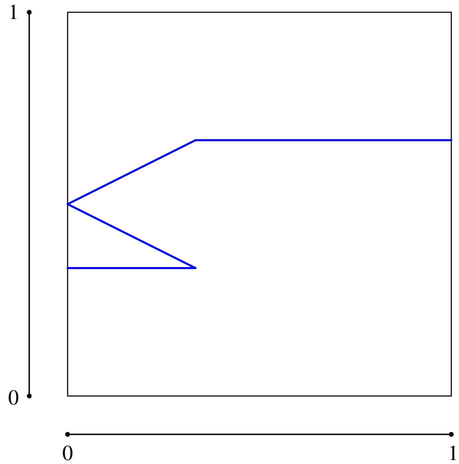

Figure 2. The graph of the function from Example 3.10.

Proof. For $n = 1$, we define the function f with $G(f) = I \times \{0\}$. It is obvious that $\varprojlim f = \{(0, 0, 0, \ldots)\}$.

For every $n \geq 2$, we define the function $f_n : I \to 2^I$ by

$$f_n(x) = \begin{cases} \dfrac{0}{n-1} & \text{if } 0 \leq x \leq \dfrac{1}{n}, \\ \dfrac{1}{n-1} & \text{if } \dfrac{1}{n} \leq x \leq \dfrac{2}{n}, \\ \dfrac{2}{n-1} & \text{if } \dfrac{2}{n} \leq x \leq \dfrac{3}{n}, \\ \vdots & \\ \dfrac{n-1}{n-1} & \text{if } \dfrac{n-1}{n} \leq x \leq 1. \end{cases}$$

The function f_4 is shown in Figure 3.

It is easy to see that, for every $k \in \{0, 1, 2, \ldots, n-1\}$, we have $k/(n-1) \in [k/n, (k+1)/n]$. Therefore $(0, 0), ((1/n-1), (1/n-1)), \ldots, ((n-2)/(n-1), (n-2)/(n-1)), (1, 1) \in G(f_n)$. It follows that $(k/(n-1), k/(n-1), k/(n-1), \ldots) \in \varprojlim f_n$ for all $k \in \{0, 1, 2, \ldots, n-1\}$. Also, for every $t \in I \setminus \{0, 1/(n-1), \ldots, (n-2)/(n-1), 1\}$, there is no $u \in I$ such that $t \in f_n(u)$. Hence $\varprojlim f_n = \{(k/(n-1), k/(n-1), k/(n-1), \ldots) \mid k \in \{0, 1, \ldots, n-1\}\}$. Therefore the cardinality of $\varprojlim f$ is n. □

To summarise, we have shown in Theorems 3.9 and 3.11 and Example 3.10 that for any $n \in \{\aleph_0, c, 1, 2, 3, \ldots\}$ there is an upper semicontinuous function $f : I \to 2^I$ such that the inverse limit $\varprojlim f$ has exactly n points. In the special case, if the graph of f is a continuum, the cardinality of the corresponding inverse limit is either 1, $\aleph_0$

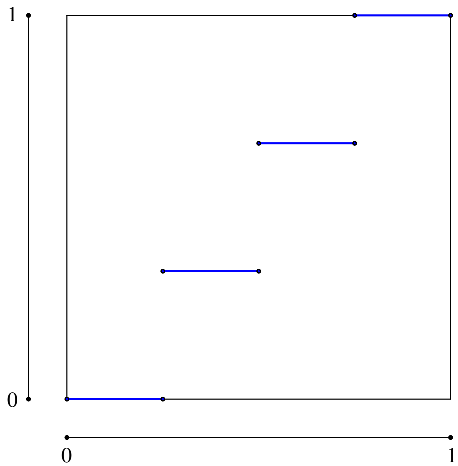

FIGURE 3. The function f_4.

or c. We conclude the present paper with the following open problem about possible generalisations of these results.

PROBLEM 3.12. Let X be any continuum and let $f : X \to 2^X$ be an upper semicontinuous function whose graph is a continuum. Is it true that $|\varprojlim f| \in \{1, \aleph_0, c\}$?

Acknowledgement

We want to thank Iztok Banič for many useful suggestions.

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