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A GENERALIZED DAVENPORT EXPANSION

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Abstract We prove a new generalization of Davenport's Fourier expansion of the infinite series involving the fractional part function over arithmetic functions. A new Mellin transform related to the Riemann zeta function is also established.

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1. Introduction and main result

In 1937, Davenport presented infinite series over arithmetic functions, using the Fourier series of the fractional part function, $\{x\} = x - [x]$, where $[x]$ denotes the integer part of x. The integer part function is defined as

$$[x] := \begin{cases} \lfloor x \rfloor, & \text{if } x \geq 0, \\ \lceil x \rceil, & \text{if } x < 0. \end{cases}$$

Here $\lfloor x \rfloor$ returns the greatest integer that is $\leq x$, and $\lceil x \rceil$ returns the smallest integer that is $\geq x$. The main result [4, Equation (2)] is the explicit formula,

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} \left(\{nx\} - \frac{1}{2} \right) = -\frac{1}{\pi} \sum_{n=1}^{\infty} \frac{A(n)}{n} \sin(2\pi nx). \tag{1.1}$$

Here $a(n)$ is an arithmetic function and $A(n) = \sum_{d|n} a(d)$. To date, many authors have researched (1.1) and associated identities, including its convergence [1,2,5,6,9,10]. The principal idea of proving (1.1) through Mellin transforms can be found in Segal [10]. A number of authors have generalized (1.1) through the use of periodic Bernoulli polynomials and Mellin inversion [2,6].

The purpose of this article is to offer a new generalization of (1.1), and in accomplishing this, we obtain a new Mellin transform. Our main theorem provides a Fourier series with coefficients for both sine and cosine, giving (1.1) as the special case $N = 1$. The Mellin transform given in our Theorem 1.2 is a more general form of an integral that has been used to obtain many interesting results, including the functional equation for the Riemann zeta function. Recall that the Riemann zeta function is $\zeta(s) = \sum_{n\geq 1} n^{-s}$, for $\Re(s) > 1$.

Theorem 1.1. *Let $a(n)$ be chosen so that $L(s) = \sum_{n\geq 1} a(n)n^{-s}$ is analytic for $\Re(s) > 1$. For $N \geq 1$, and $F_k(n) := \sum_{d|n} d^{-k}a(n/d)$, we have for real $x > 0$,*

$$\sum_{n=1}^{\infty} \frac{a(n)}{n} \left(\{nx\}^N + N! \sum_{k=0}^{N-1} \frac{\zeta(-k)}{(N-k)!k!} \right)$$
$$= -N! \sum_{k=0}^{N-1} \frac{(-1)^k}{(N-k)!} \left(\frac{\cos(\frac{\pi}{2}k)}{\pi} \sum_{n=1}^{\infty} \frac{F_k(n)}{n} \sin(2\pi nx) - \frac{\sin(\frac{\pi}{2}k)}{\pi} \sum_{n=1}^{\infty} \frac{F_k(n)}{n} \cos(2\pi nx) \right).$$

Our proof ensures that if the infinite series on one side of the theorem converges then the infinite series on the other side converges as well. The Mellin transform we need to establish Theorem 1.1 is given in the following and fits in neatly with the family of integrals given in [3]. Let $(s)_k = \Gamma(s+k)/\Gamma(s)$ denote the Pochhammer symbol.

Theorem 1.2. *For $0 < \Re(s) < N$, we have,*

$$\int_0^{\infty} \{y\}^N y^{-s-1} dy = N! \sum_{k=0}^{N-1} \frac{(-1)^k \zeta(s-k)}{(N-k)!(-s)_{k+1}}.$$

Proof. A direct computation gives

$$\begin{aligned} \int_1^{\infty} \{y\}^N y^{-s-1} \, \mathrm{d}y &= \sum_{k=1}^{\infty} \int_k^{k+1} \{y\}^N y^{-s-1} \, \mathrm{d}y \\ &= \sum_{k=1}^{\infty} \int_0^1 \frac{y^N}{(y+k)^{s+1}} \, \mathrm{d}y \\ &= \int_0^1 y^N \zeta(s+1, y+1) \, \mathrm{d}y. \end{aligned} \tag{1.2}$$

From [7, p. 184, Equation (12.2)], we have

$$\int_0^1 y^N \zeta(s, y) \, \mathrm{d}y = N! \sum_{k=0}^{N-1} (-1)^k \frac{\zeta(s-k-1)}{(N-k)!(1-s)_{k+1}}, \tag{1.3}$$

where $\zeta(s, y)$ is the Hurwitz zeta function. The left side may be written

$$\int_0^1 y^N \zeta(s, y+1) \, \mathrm{d}y + \frac{1}{N-s+1}. \tag{1.4}$$

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Note that, for $0 < \Re(s) < N$,

$$\int_0^1 \frac{\{y\}^N}{y^{s+1}}\,\mathrm{d}y = \int_0^1 y^{N-s-1}\,\mathrm{d}y = \frac{1}{N-s}. \tag{1.5}$$

Combining (1.2–1.5) gives the theorem. □

2. Proof of main theorem

To prove Theorem 1.1, we first obtain a different form of our main integral result contained in Theorem 1.2. After this is accomplished, we generalize the proof of Segal [10].

Proof of Theorem 1.1**.** By Theorem 1.2,

$$\{y\}^N = \frac{N!}{2\pi i}\int_{(c)}\left(\sum_{k=0}^{N-1}\frac{(-1)^k\zeta(s-k)}{(N-k)!(-s)_{k+1}}\right)y^s\,\mathrm{d}s, \tag{2.1}$$

if $0 < \Re(s) = c < N$. The integrand has a simple pole at $s = 0$. Note that

$$\lim_{s\to 0}\left(s\frac{\zeta(s-k)}{(-s)_{k+1}}y^s\right) = -\frac{\zeta(-k)}{\Gamma(k+1)}. \tag{2.2}$$

Therefore, computing the residue at the pole $s = 0$, and moving the line of integration to $-1 < \Re(s) = d < 0$ in (2.1),

$$\begin{aligned}\frac{1}{2\pi i}&\int_{(c)}\left(N!\sum_{k=0}^{N-1}\frac{(-1)^k\zeta(s-k)}{(N-k)!(-s)_{k+1}}\right)y^s\,\mathrm{d}s\\ &= N!\sum_{k=0}^{N-1}\frac{\zeta(-k)}{(N-k)!k!} + \frac{1}{2\pi i}\int_{(d)}\left(N!\sum_{k=0}^{N-1}\frac{(-1)^k\zeta(s-k)}{(N-k)!(-s)_{k+1}}\right)y^s\,\mathrm{d}s.\end{aligned} \tag{2.3}$$

Collectively, we have for $y > 0$

$$\{y\}^N - N!\sum_{k=0}^{N-1}\frac{\zeta(-k)}{(N-k)!k!} = \frac{1}{2\pi i}\int_{(d)}\left(N!\sum_{k=0}^{N-1}\frac{(-1)^k\zeta(s-k)}{(N-k)!(-s)_{k+1}}\right)y^s\,\mathrm{d}s. \tag{2.4}$$

By absolute convergence of $L(1-s)$ in the region $-1 < \Re(s) = d < 0$, we may invert the desired series over the coefficients $a(n)$ in (2.4) after replacing y by ny. This gives us

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{a(n)}{n}&\left(\{ny\}^N - N!\sum_{k=0}^{N-1}\frac{\zeta(-k)}{(N-k)!k!}\right)\\ &= N!\sum_{k=0}^{N-1}\frac{(-1)^k}{(N-k)!}\frac{1}{2\pi i}\int_{(d)}\frac{\zeta(s-k)}{(-s)_{k+1}}L(1-s)y^s\,\mathrm{d}s.\end{aligned} \tag{2.5}$$

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Notice that, by the functional equation for the Riemann zeta function [11, p. 13, Theorem 2.1],

$$\begin{aligned}\frac{\zeta(s-k)L(1-s)y^s}{(-s)_{k+1}} &= \frac{\sin(\frac{\pi}{2}(s-k))\Gamma(1+k-s)\zeta(1+k-s)L(1-s)\Gamma(-s)y^s}{\Gamma(k+1-s)}\\ &= \Gamma(-s)\sin\left(\frac{\pi}{2}(s-k)\right)\zeta(1+k-s)L(1-s)y^s,\end{aligned}$$

and $\sin((\pi/2)(s-k)) = \sin((\pi/2)s)\cos((\pi/2)k) + \cos((\pi/2)s)\sin((\pi/2)k)$. Note that $\zeta(k+s)L(s) = \sum_{n\geq 1} F_k(n)n^{-s}$, for $\Re(s) > 1$. Therefore, replacing s by $-s$ in our integral in (2.5), and employing [8, p. 406]

$$\int_0^\infty y^{s-1}\cos(2\pi y)\,\mathrm{d}y = (2\pi)^{-s}\Gamma(s)\cos\left(\frac{\pi}{2}s\right),$$
$$\int_0^\infty y^{s-1}\sin(2\pi y)\,\mathrm{d}y = (2\pi)^{-s}\Gamma(s)\sin\left(\frac{\pi}{2}s\right),$$

both valid for $0 < \Re(s) < 1$, we obtain the Fourier series in the right-hand side in Theorem 1.1. To see this, note that $\zeta(1+k-s)$ is analytic for $-1 < \Re(s) < 0$ since $k \geq 0$. The same is true for $L(1-s)$ by hypothesis. Therefore, replacing s by $-s$ moves the line of integration into the region $0 < \Re(s) < 1$, and we may interchange the series produced from the product $\zeta(1+k+s)L(1+s)$ with the integral by absolute convergence. The resulting formula is valid for real $x > 0$, by Mellin inversion [8, p. 80]. □

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