The primitive ideal space of the C*-algebra of the affine semigroup of algebraic integers

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Abstract

The purpose of this paper is to give a complete description of the primitive ideal space of the C*-algebra $\mathfrak{T}[R]$ associated to the ring of integers R in a number field K in the recent paper [**5**]. As explained in [**5**], $\mathfrak{T}[R]$ can be realized as the Toeplitz C*-algebra of the affine semigroup $R \rtimes R^\times$ over R and as a full corner of a crossed product $C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*$, where $\Omega_{\mathbb{A}_f}$ is a certain adelic space. Therefore $\mathrm{Prim}(\mathfrak{T}[R])$ is homeomorphic to the primitive ideal space of this crossed product. Using a recent result of Sierakowski together with the fact that every quasi-orbit for the action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$ contains at least one point with trivial stabilizer we show that $\mathrm{Prim}(\mathfrak{T}[R])$ is homeomorphic to the quasi-orbit space for the action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$, which in turn may be identified with the power set $2^{\mathcal{P}}$ of the set of prime ideals $\mathcal{P}$ of R equipped with the power-cofinite topology.

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1. *Introduction*

Let R be the ring of integers in a number field K. In the recent paper [**5**] Cuntz, Deninger and Laca introduce an algebra $\mathfrak{T}[R]$ attached to R and they show that this algebra has a very interesting KMS-structure relative to a certain natural one parameter group of automorphisms. The algebra $\mathfrak{T}[R]$ is an extension of the C*-algebra $A[R]$ studied previously in [**2**, **3**, **4**] and, in contrast to $A[R]$, it is functorial under homomorphisms of rings. While $A[R]$ is always simple, the new algebra $\mathfrak{T}[R]$ has a fairly rich ideal structure, and it is the aim of this paper to give a detailed description of the primitive ideal space $\mathrm{Prim}(\mathfrak{T}[R])$ as a topological space. Since the closed ideals in a C*-algebra A correspond bijectively and inclusion-preserving to the open subsets of $\mathrm{Prim}(A)$, we obtain a complete picture of the ideal structure of $\mathfrak{T}[R]$. A related extension $\mathcal{T}(\mathbb{N} \rtimes \mathbb{N}^\times)$ of the C*-algebra $\mathcal{Q}_{\mathbb{N}}$ from [**2**] had been introduced in [**12**], and although strictly speaking it is not equal to $\mathfrak{T}[\mathbb{Z}]$, its structure is similar enough for our methods to apply there as well.

It is shown in [**5**] that $\mathfrak{T}[R]$ is a full corner in a crossed product $C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*$, where $\Omega_{\mathbb{A}_f}$ is a certain quotient space of the product $\mathbb{A}_f \times \mathbb{A}_f$ of the finite adele space $\mathbb{A}_f$ over K by itself. Since Morita equivalent C*-algebras have homeomorphic primitive ideal spaces, it

therefore suffices to describe the primitive ideal space of this crossed product. The situation is somewhat similar to the computation of the primitive ideal space of the original Bost–Connes algebra as performed by Laca and Raeburn in [11]. Since the Bost–Connes algebra is Morita equivalent to the crossed product $C_0(\mathbb{A}_f) \rtimes \mathbb{Q}_+^*$ and since $\mathbb{Q}_+^*$ is an abelian group one can use a theorem of Williams based on the Mackey machine (e.g. see [11, theorem 1·1]) to compute the primitive ideal space of the crossed product as a certain quotient of $\mathbb{A}_f \times \widehat{\mathbb{Q}_+^*}$, where $\widehat{\mathbb{Q}_+^*}$ denotes the Pontrjagin dual of $\mathbb{Q}_+^*$.

The situation becomes a bit more complicated in the present case since the $ax + b$-group $K \rtimes K^*$ is not abelian and the action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$ has wildly varying stabilizer groups. However, we shall see in Section 3 below that every quasi-orbit for the action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$ contains at least one orbit with trivial stabilizers. It turns out that this suffices to prove that $\mathrm{Prim}(C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*)$ is homeomorphic to the quasi-orbit space $\Omega_{\mathbb{A}_f}/ \sim$ via an induction procedure (recall that if G is a group acting on a topological space X, then two elements $x, y \in X$ lie in the same *quasi-orbit* if $\overline{Gx} = \overline{Gy}$). We deduce this fact in §2 in a much more general setting from a recent result of Sierakowski [14] which extends earlier results of Renault [13] and of Archbold and Spielberg [1].

We give a precise description of the quasi-orbit space $\Omega_{\mathbb{A}_f}/ \sim$ in Section 3, showing that it is homeomorphic to the power set $2^{\mathcal{P}}$ of the set of prime ideals $\mathcal{P}$ of R equipped with the power-cofinite topology.

2. *The primitive ideal space of crossed products by essentially free actions*

Let G be a countable discrete group acting by automorphisms on a separable C*-algebra A. We will give a detailed description of the primitive ideal space of the reduced crossed product $A \rtimes_r G$ in the case where the action of G on A is exact and the associated action of G on the space $\widehat{A}$ of equivalence classes of irreducible representations of A is *essentially free* in the sense that every closed invariant subset C of $\widehat{A}$ contains a dense invariant subset D such that G acts freely on D. Our method builds up on recent work of Sierakowski [14], which relies on a central result of the paper [1] by Archbold and Spielberg and extends earlier results by Renault (see [13, corollary 4·9]),

Recall that an action of the group G on A is *exact*, if for any G-invariant ideal $I \subseteq A$ the corresponding sequence of reduced crossed products

$$0 \longrightarrow I \rtimes_r G \longrightarrow A \rtimes_r G \longrightarrow (A/I) \rtimes_r G \longrightarrow 0$$

is exact. The group G is called *exact* if every action of G on any C*-algebra A is exact. The class of exact groups is quite large. It contains all amenable groups and all countable groups which can be embedded (as abstract groups) in a linear group over some field K (in particular all free groups, see [8]). We refer to the discussion in [14] for further details.

For a C*-algebra A we denote by $\mathcal{I}(A)$ the set of closed ideals in A equipped with the *Fell topology* for which a sub-base of open sets is given by the sets of the form

$$U_I := \{J \in \mathcal{I}(A) : I \nsubseteq J\}, \quad I \in \mathcal{I}(A).$$

Restricted to $\mathrm{Prim}(A)$, this is the usual Jacobson topology. Note that if A and B are two C*-algebras and $\varphi : \mathcal{I}(A) \to \mathcal{I}(B)$ is any map which preserves inclusion of ideals, then φ is continuous. If (A, G, α) is a system, we denote by $\mathcal{I}^G(A) \subseteq \mathcal{I}(A)$ the set of G-invariant closed ideals in A equipped with the topology restricted from $\mathcal{I}(A)$. We will need the following formulation of Sierakowski's results.

THEOREM 2·1 (Sierakowski). *Suppose that (A, G, α) is a dynamical system with G discrete such that the action of G on A is exact and the action of G on $\widehat{A}$ is essentially free. Then the map*

$$\text{res} : \mathcal{I}(A \rtimes_r G) \longrightarrow \mathcal{I}^G(A); \quad J \longmapsto J \cap A$$

is a homeomorphism with inverse map given by

$$\text{ind} : \mathcal{I}^G(A) \longrightarrow \mathcal{I}(A \rtimes_r G); \quad \text{ind}(I) = I \rtimes_r G.$$

Proof. This follows from combining [**14**, proposition 1·3] with [**14**, theorem 1·20] and observing that since both maps obviously preserve inclusion of ideals, they are homeomorphisms with respect to the Fell-topologies.

Remark 2·2. (1) The assumption that the action of G on $\widehat{A}$ is essentially free is certainly weaker than the condition that the action of G on $\text{Prim}(A)$ is essentially free. This follows directly from the definitions of essential freeness and of the Jacobson topologies on $\widehat{A}$ and $\text{Prim}(A)$ (e.g. see [**6**, chapter 3]).

(2) It is shown in [**14**, section 2] that the assumption that the action of G on $\widehat{A}$ is essentially free can be replaced by a somewhat weaker assumption which Sierakowski calls the *residual Rokhlin* property*. Thus all results discussed below will remain true if the assumption that the action of G on $\widehat{A}$ is essentially free is replaced by the assumption that the action satisfies the residual Rokhlin* property.

Suppose that (A, G, α) is a C*-dynamical system. A G-invariant ideal $I \subseteq A$ is called *G-prime* if for any pair of closed G-invariant ideals $J_1, J_2 \subseteq A$ with $J_1 \cap J_2 \subseteq I$ we have $J_i \subseteq I$ for $i = 1$ or $i = 2$. We denote by $\text{Prime}^G(A)$ the set of G-prime ideals in A and we denote by $\text{Prime}(A)$ the set of prime ideals in A (the case where G is the trivial group). The Fell-topology on $\mathcal{I}(A)$ induces topologies on $\text{Prime}^G(A)$ and $\text{Prime}(A)$.

PROPOSITION 2·3. *Suppose that (A, G, α) satisfies the assumptions of Theorem* 2·1. *Then the map* res : $\mathcal{I}(A \rtimes_r G) \to \mathcal{I}^G(A)$ *restricts to a homeomorphism* $\text{Prime}(A \rtimes_r G) \cong \text{Prime}^G(A)$.

Proof. Suppose that $I \in \text{Prime}(A \rtimes_r G)$ and let $J_1, J_2 \in \mathcal{I}^G(A)$ such that $J_1 \cap J_2 \subseteq I \cap A$. It follows then from Theorem 2·1 that $J_1 \rtimes_r G \cap J_2 \rtimes_r G \subseteq (I \cap A) \rtimes_r G = I$ and hence that $J_i \rtimes_r G \subseteq I$ for some $i \in \{1, 2\}$. But then we also have $J_i = (J_i \rtimes_r G) \cap A \subseteq I \cap A$, which proves that $I \cap A$ is G-prime. A very similar argument shows that $J \rtimes_r G$ is prime if $J \in \text{Prime}^G(A)$. The result then follows from Theorem 2·1.

Recall that the primitive ideals in a C*-algebra are the kernels of the irreducible representations of A. If (A, G, α) is a system, then an ideal $I \in \mathcal{I}^G(A)$ is called *G-primitive* if $I = P^G := \bigcap_{g \in G} \alpha_g(P)$ for some primitive ideal P of A. We denote by $\text{Prim}^G(A)$ the set of G-primitive ideals equipped with the Fell-topology.

Moreover, if a group G acts on a topological space X, then the *quasi-orbit space* $\mathcal{O}(X)$ is defined as the quotient space $X/\sim$ by the equivalence relation $x \sim y \Leftrightarrow \overline{Gx} = \overline{Gy}$. In what follows we denote by $\mathcal{O}(x)$ the *quasi-orbit of x*, i.e., the equivalence class of x under this relation. The following lemma is [**10**, lemma on p. 221]:

LEMMA 2·4 (Green). *Suppose that (A, G, α) is a system. Then $\mathcal{O}(\text{Prim}(A)) \to \text{Prim}^G(A); \mathcal{O}(P) \mapsto P^G$ is a homeomorphism.*

The following lemma is also well known to the experts. For completeness we give a proof, which is an adaptation of the proof of [**10**, corollary on p. 222].

LEMMA 2·5. *Let* (A, G, α) *be a system with A separable. Then* $\mathrm{Prime}^G(A) = \mathrm{Prim}^G(A)$.

Proof. We first remark that every primitive ideal is prime, since if $P = \ker \pi$ for some irreducible representation $\pi : A \to \mathcal{B}(H)$ and if $J_1, J_2 \in \mathcal{I}(A)$ such that $J_1 \cap J_2 \subseteq P$, then $J_i \subseteq P$ for some $i \in \{1, 2\}$ because otherwise π would restrict to irreducible representations on J_1 and J_2 and then $\{0\} = \overline{\pi(J_1 \cap J_2)H} = \overline{\pi(J_1)\pi(J_2)H} = H$.

Suppose now that $I = P^G$ is G-primitive. Then, if $J_1, J_2 \in \mathcal{I}^G(A)$ such that $J_1 \cap J_2 \subseteq I$, we have $J_1 \cap J_2 \subseteq P$, and hence $J_i \subseteq P$ for some $i \in \{1, 2\}$. But then $J_i = J_i^G \subseteq P^G = I$. Hence I is also G-prime.

For the converse let $I \in \mathrm{Prime}^G(A)$. We need to show that there exists a $P \in \mathrm{Prim}(A)$ with $I = P^G$. For this let $F \subseteq \mathrm{Prim}(A)$ be the hull of I, i.e., $F = \{Q \in \mathrm{Prim}(A) : I \subseteq Q\}$. This is a closed G-invariant subset of $\mathrm{Prim}(A)$ and we need to show that $F = \overline{G(P)}$ for some $P \in \mathrm{Prim}(A)$. This means that the image $C := q(F) \subseteq \mathcal{O}(\mathrm{Prim}(A))$ is the closure of a single point $\mathcal{O}(P) \in \mathcal{O}(\mathrm{Prim}(A))$, where $q : \mathrm{Prim}(A) \to \mathcal{O}(\mathrm{Prim}(A))$ denotes the quotient map. It follows from the discussion preceding [**10**, lemma on p. 222] that $\mathcal{O}(\mathrm{Prim}(A)) \cong \mathrm{Prim}^G(A)$ is a totally Baire space, and it is second countable since A is separable. By [**10**, lemma on p. 222], to conclude that $F = \overline{G(P)}$ it suffices to show that C is irreducible, which means that it cannot be written as a union of two proper closed subsets C_1, C_2. So assume that C_1, C_2 are closed subsets of $\mathcal{O}(\mathrm{Prim}(A))$ with $C = C_1 \cup C_2$. Let F_1, F_2 denote their inverse images in $\mathrm{Prim}(A)$ and let $J_i = \ker F_i = \bigcap\{Q : Q \in F_i\}$ for $i = 1, 2$. Then $F = F_1 \cup F_2$ which implies that $J_1 \cap J_2 = I$. Since I is G-prime we have $J_i \subseteq I$ for some $i \in \{1, 2\}$ which implies that $F_i = \{Q \in \mathrm{Prim}(A) : J_i \subseteq Q\}$ contains F. This completes the proof.

COROLLARY 2·6. *Suppose that* (A, G, α) *is a system such that A is separable, G is countable, the action of G on A is exact and the action of G on* $\widehat{A}$ *is essentially free. Then*

$$\mathrm{ind} : \mathrm{Prim}^G(A) \longrightarrow \mathrm{Prim}(A \rtimes_r G); \quad I \longmapsto I \rtimes_r G$$

is a homeomorphism with inverse map given by

$$\mathrm{res} : \mathrm{Prim}(A \rtimes_r G) \longrightarrow \mathrm{Prim}^G(A); \quad P \longmapsto P \cap A.$$

Proof. This follows as a direct combination of Proposition 2·3 together with Lemma 2·5 (which also implies that $\mathrm{Prim}(A \rtimes_r G) = \mathrm{Prime}(A \rtimes_r G)$ since $A \rtimes_r G$ is separable).

Recall that if (A, G, α) is a system with G discrete, then every representation $\pi : A \to \mathcal{B}(H)$ gives rise to an induced representation $\mathrm{ind}\, \pi : A \rtimes_r G \to \mathcal{B}(l^2(G, H))$ which is the integrated form $\tilde{\pi} \times \lambda$ of the covariant homomorphism $(\tilde{\pi}, \lambda)$ of (A, G, α) on $l^2(G, H)$ given by the formulas

$$\big(\tilde{\pi}(a)\xi\big)(g) = \pi(\alpha_{g^{-1}}(a))\xi(g) \quad \text{and} \quad \big(\lambda(h)\xi\big)(g) = \xi(h^{-1}g),$$

for all $\xi \in l^2(G, H), a \in A$, and $h, g \in G$.

If $J = \ker \pi$ is G-invariant, then G acts on A/J and $\ker \tilde{\pi} = J$, hence $\mathrm{ind}\, \pi$ gives a faithful representation of $A/J \rtimes_r G$. When the action of G on A is exact, we see that $\ker(\mathrm{ind}\, \pi) = J \rtimes_r G$. It can be shown (e.g., see [**10**, proposition 9]) that in general $\ker(\mathrm{ind}\, \pi)$ depends only on $\ker \pi$ hence the definition of induced ideals given earlier for G-invariant

ideals can be extended to all ideals of A by letting $\operatorname{ind} J = \ker(\operatorname{ind} \pi)$, where π is any representation of A with $\ker \pi = J$.

Moreover, it is easily checked that for a general representation $\pi : A \to \mathcal{B}(H)$ the induced representation $\operatorname{ind} \pi$ is unitarily equivalent to $\operatorname{ind}(\pi \circ \alpha_g)$ for all $g \in G$ (the equivalence being implemented by the unitary $\lambda(g)$). Thus we see that on the level of ideals we get for all $g \in G$ and for every ideal $J \in \mathcal{I}(A)$ that

$$\operatorname{ind} J = \operatorname{ind}(\alpha_g(J)) = \operatorname{ind}(J^G).$$

Combining this with Corollary 2·6 and Lemma 2·4 we get the following corollary.

COROLLARY 2·7. *Suppose that (A, G, α) is a system such that A is separable, G is countable, the action of G on A is exact and the action of G on $\widehat{A}$ is essentially free. Then*

$$\operatorname{ind} : \mathcal{O}(\operatorname{Prim}(A)) \longrightarrow \operatorname{Prim}(A \rtimes_r G); \quad \mathcal{O}(P) \longmapsto \operatorname{ind} P$$

is a homeomorphism.

3. *Orbits, stabilizers and quasi-orbits of an adelic space*

Let K be an algebraic number field with ring of algebraic integers R. The nonzero elements of R form a semigroup denoted $R^\times$, which contains the group R^* of invertible elements or *units* of R. In this section we will study the primitive ideal space of the C*-algebra $\mathfrak{T}[R]$ which was introduced in [**5**] in terms of generators and relations and shown to be isomorphic to the C*-algebra of the left regular representation of the semigroup $R \rtimes R^\times$ on $\ell^2(R \rtimes R^\times)$. Because of [**5**, propositions 5·1 and 5·2], there is an action of the group $K \rtimes K^*$ on a locally compact Hausdorff space $\Omega_{\mathbb{A}_f}$ such that $\mathfrak{T}[R]$ is isomorphic to a full corner in the transformation group C*-algebra $C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*$, and thus its primitive ideal space is naturally homeomorphic to that of $C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*$. For a specific description of the homeomorphism see [**11**, lemma 2·7]. We want to use the results from Section 2 above, for which it becomes important to understand the orbits, the stability subgroups, and the quasi-orbit space of the action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$.

We begin by recalling the definition of the space $\Omega_{\mathbb{A}_f}$ and the action of $K \rtimes K^*$ from [**5**, section 5]. Denote by $\mathbb{A}_f$ the locally compact, totally disconnected ring of finite adeles over K. The finite adeles have a maximal compact subring $\hat{R}$, the *integral adeles*, whose group of units (i.e. invertible elements) is denoted $\hat{R}^*$, the compact group of *integral ideles*. The space $\Omega_{\mathbb{A}_f}$ is the quotient of $\mathbb{A}_f \times \mathbb{A}_f$ by the equivalence

$$(r, a) \sim (s, b) \quad \Longleftrightarrow \quad a\hat{R}^* = b\hat{R}^* \text{ and } r - s \in a\hat{R}.$$

If we denote the class of $(r, a) \in \mathbb{A}_f \times \mathbb{A}_f$ by $\omega_{r,a}$, then the group $K \rtimes K^*$ acts on $\Omega_{\mathbb{A}_f}$ in the obvious way by $(x, k) \cdot \omega_{r,a} = \omega_{x+kr,ka}$.

Denote by $\mathcal{P}$ the set of all prime ideals of R. Recall from [**5**, section 5] that each point $\omega_{r,a} \in \Omega_{\mathbb{A}_f}$ has a valuation vector $\{v_P(a)\}_{P \in \mathcal{P}}$ which depends only on (and in fact characterizes) the second component. This valuation vector can be defined as follows: if $a \in \hat{R}$ and $P \in \mathcal{P}$, let $v_P(a)$ be the smallest nonnegative integer n such that the canonical projection of a in R/P^{n+1} is nonzero, and put $v_P(a) = \infty$ if a projects onto $0 \in R/P^n$ for every n. If a is a finite adele, then there exists $d \in R$ such that $da \in \hat{R}$, and we let $v_P(a) = v_P(da) - v_P(d)$, which does not depend on d.

Thus, we may regard the second component of a point of $\Omega_{\mathbb{A}_f}$ as a *superideal* for which, in analogy to the supernatural numbers, we allow infinite powers and also finitely many negative ones. Hence we write

$$\mathbb{A}_f / \hat{R}^* \cong \prod_{P \in \mathcal{P}} (P^{\mathbb{Z} \cup \{\infty\}}; P^{\mathbb{N} \cup \{\infty\}}), \qquad a \longmapsto \prod_{\mathcal{P}} P^{e_P(a)}$$

where the restriction in the direct product is that the elements of $\mathbb{A}_f / \hat{R}^*$ can have negative exponents for at most finitely many prime ideals. The topology in each coordinate is from the natural order in $\mathbb{Z}$; as usual, the factor P^∞ corresponds to $0 \in K_P$ in the P coordinate.

LEMMA 3·1. *For each $a \in \mathbb{A}_f / \hat{R}^*$, let*

$$Z(a) := \{P \in \mathcal{P} : a_P = 0\} = \{P \in \mathcal{P} : e_P(a) = \infty\}.$$

Then the closure of the orbit of $\omega_{r,a}$ is the set $\{\omega_{s,b} \in \Omega_{\mathbb{A}_f} : Z(b) \supset Z(a)\}$.

Proof. It is easy to see that the set $\{\omega_{s,b} \in \Omega_{\mathbb{A}_f} : Z(b) \supset Z(a)\}$ is closed and that every point in the orbit of $\omega_{r,a}$ has a second coordinate that vanishes on $Z(a)$ (and possibly also at other primes). Thus $\{\omega_{s,b} \in \Omega_{\mathbb{A}_f} : Z(b) \supset Z(a)\}$ contains the closure of the orbit of $\omega_{r,a}$.

To prove the reverse inclusion, consider $\omega_{r,a} \in \Omega_{\mathbb{A}_f}$ and let (s, b) be a point in $\mathbb{A}_f \times (\mathbb{A}_f / \hat{R}^*)$ with $Z(b) \supset Z(a)$. Let W_2 be typical basic neighbourhood of $b \in \mathbb{A}_f / \hat{R}^*$:

$$W_2 := \{c \in \mathbb{A}_f / \hat{R}^* : v_{P_j}(c) = v_{P_j}(b) \text{ for } P_j \notin Z(b) \text{ and } v_{P_j}(c) \geqslant n_j \text{ for } P_j \in Z(b)\}$$

where $\{P_1, P_2, \ldots, P_n\}$ is a given finite set of prime ideals and $n_j \geqslant 0$ is a given integer for every j such that $P_j \in Z(b)$. For each j such that P_j is not in $Z(b)$ choose $e_j = v_{P_j}(b) - v_{P_j}(a)$; for each j such that $P_j \in Z(b) \setminus Z(a)$, let $e_j = n_j + |v P_j(a)|$; finally let $e_j = 0$ for each j such that $P_j \in Z(a)$ (this last choice is not essential, we just make it to be thorough).

Now let Q be a prime ideal in the inverse class of the product $\prod_{j=1}^n P_j^{e_j}$ such that $Q \neq P_j$ for $j = 1, 2, \ldots, n$ and choose k to be a generator of the principal ideal $Q \prod_{j=1}^n P_j^{e_j}$. Then $ka \in W_2$.

If W_1 is any neighbourhood of s in $\mathbb{A}_f$, there exists $x \in K$ such that $x + kr \in W_1$, because K is dense in $\mathbb{A}_f$. Since (the images of) neighbourhoods of the form $V = W_1 \times W_2$ form a neighbourhood basis for $\omega_{s,b}$ in the quotient $\Omega_{\mathbb{A}_f}$, and since we have shown that every such neighbourhood contains a point of the form $\omega_{x+kr,ka}$, we conclude that $\omega_{s,b}$ is in the closure of the orbit $\{\omega_{x+kr,ka} : x \in K, k \in K^*\}$ of $\omega_{r,a}$.

It is now easy to describe the quasi-orbit space. Following [11, section 2] we consider the power set $2^{\mathcal{P}}$ of the set of prime ideals of R equipped with the power-cofinite topology, in which the basic open sets are indexed by finite $G \subseteq \mathcal{P}$ and are given by $U_G = \{T \in 2^{\mathcal{P}} : T \cap G = \varnothing\}$.

LEMMA 3·2. *The correspondence $\omega_{r,a} \mapsto Z(a)$ induces a homeomorphism of the quasi-orbit space to the power set of the prime ideals with the power-cofinite topology.*

Proof. The preceding lemma shows that two points are in the same quasi-orbit if and only if their second components vanish at exactly the same subset of prime ideals. This gives a bijection from the quasi-orbit space to $2^{\mathcal{P}}$. The topology on the quasi-orbit space is obtained from the topology on $\Omega_{\mathbb{A}_f}$ through the quasi-orbit map $\omega_{r,a} \mapsto q(\omega_{r,a})$ which is continuous and open by Lemma 2·4.

Since the first component is lost in passing from $\omega_{r,a}$ to $Z(a)$, an argument similar to that in the proof of [**11**, proposition 2·4] shows that basic neighbourhoods in $\Omega_{\mathbb{A}_f}$ are mapped exactly onto the power-cofinite neighbourhoods of $2^{\mathcal{P}}$.

Example 3·3. The action of $K \rtimes K^*$ has many points with nontrivial stabilizer. Here we give some examples of such points together with their stabilizers.

(i) Let $(x, k) \in K \rtimes K^*$. Then the point $(x, k)\omega_{0,1} = \omega_{x,k}$ coincides with $\omega_{0,1} \in \Omega_{\mathbb{A}_f}$ iff $k \in \hat{R}^* \cap K = R^*$ and $x \in \hat{R} \cap K = R$, so the stabilizer of $\omega_{0,1} \in \Omega_{\mathbb{A}_f}$ is $R \rtimes R^*$.

(ii) The points $\omega_{m,k}$ in the orbit of $\omega_{0,1}$ have stabilizers $\{(m, k)(r, w)(-mk^{-1}, k^{-1}) : r \in R, \ w \in R^*\} = \{(m(1-w) + kr, w) : r \in R, \ w \in R^*\}$.

(iii) The stabilizer of the point $\omega_{0,0}$ consists of elements (x, k) such that $x = 0 \in (\mathbb{A}_f \cap K)/0 = K$. That is, $\{0\} \times K^*$.

(iv) The points $\omega_{x,0}$ in the orbit of $\omega_{0,0}$ have stabilizers $(x, 1)(\{0\} \times K^*)(-x, 1) = \{(x(1 - k), k) : k \in K^*\}$.

Despite the abundance of nontrivial stabilizers, we shall see next that there are sufficiently many points with trivial stabilizer to generate all quasi-orbits.

LEMMA 3·4. *For every subset A of prime ideals there exists a point $\omega_{r,a}$ with trivial stabilizer and $Z(a) = A$.*

Proof. If $P \in A$ the condition $\omega_{x+kr,ka} = \omega_{r,a}$ implies $x + (k - 1)r_P = 0$, that is, either $r_P = -x/(k - 1) \in K$, or else $k = 1$ and $x = 0$. When $A \neq \varnothing$ the first case can be eliminated by choosing $\omega_{r,a}$ with $r_P \notin K$.

For $A = \varnothing$ choose any a with $Z(a) = \varnothing$ such that $\mathcal{Q} := \{P \in \mathcal{P} : v_P(a) > 0\}$ is infinite. Write $\mathcal{Q} = \mathcal{Q}_1 \sqcup \mathcal{Q}_2$ with both $\mathcal{Q}_1$ and $\mathcal{Q}_2$ infinite and choose any r such that $v_P(r) = v_P(a) - 1$ for $P \in \mathcal{Q}_1$ and $v_P(r) = v_P(a)$ for $P \in \mathcal{Q}_2$.

The group element (x, k) fixes the point $\omega_{r,a}$ if and only if $ka\hat{R}^* = a\hat{R}^*$ (so that $k \in R^*$) and $x + (k - 1)r \in a\hat{R}$. If $k \neq 1$, this last condition gives $r \in (1/(k - 1))(a\hat{R} - x)$. When $x = 0$ this is clearly impossible because the factor $1/(k - 1)$ only reduces the valuation at finitely many places. When $x \neq 0$ this is also impossible because subtracting x reduces to zero the valuation at all primes in $\mathcal{Q}$ except possibly at the finitely many primes P that satisfy $v_P(k) \neq 0$.

As a direct corollary of the above result we get the following.

COROLLARY 3·5. *The action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$ is essentially free.*

Proof. The points $\omega_{r,a}$ considered in the last part of the proof of Lemma 3·4 (for the case $A = \varnothing$) have trivial stabilizers and their orbits are dense by Lemma 3·1.

We are now ready for our main result, namely the characterization of the primitive ideal space of the C*-algebra $\mathfrak{T}[R]$ from [**5**]. Recall that $\mathfrak{T}[R]$ can be realized as the full corner in the crossed product $C_0(\Omega_{\mathbb{A}_f}) \rtimes K \rtimes K^*$ corresponding to the full projection $\mathbf{1}_{\Omega_{\hat{R}}}$ associated to the clopen subset $\Omega_{\hat{R}} \subset \Omega_{\mathbb{A}_f}$ as in [**5**, proposition 5.1].

THEOREM 3·6. *Let $2^{\mathcal{P}}$ denote the power set of the set of prime ideals of R with the power-cofinite topology. For each subset A of $\mathcal{P}$ let I_A be the kernel of the compression to $\mathfrak{T}[R]$ of the induced representation corresponding to a point $\omega_{r,a}$ with $Z(a) = A$ and trivial stabilizer. Then the map $A \mapsto I_A$ is a homeomorphism of $2^{\mathcal{P}}$ to $\mathrm{Prim}\,\mathfrak{T}[R]$.*

Proof. The compression by a full projection respects unitary equivalence and gives a homeomorphism of primitive ideals, see, e.g. [**11**, lemma 2·7]. Hence the result is a direct consequence of Lemmas 3·2 and 3·4 and of Corollary 3·5 above.

Remark 3·7. Since the action of $K \rtimes K^*$ on $\Omega_{\mathbb{A}_f}$ is essentially free, Sierakowski's theorem (Theorem 2·1) also gives a direct correspondence between the ideals of $\mathfrak{T}[R]$ and the invariant ideals in $C_0(\Omega_{\mathbb{A}_f})$, i.e., the invariant open sets in $\Omega_{\mathbb{A}_f}$, which are in bijection to the open subsets of $\mathcal{O}(\Omega_{\mathbb{A}_f}) \cong 2^{\mathcal{P}}$, since every invariant open set is the union of the quasi-orbits it contains.

Remark 3·8. In the situation of [**11**, theorem 2·8], where only the multiplicative action of $\mathbb{Q}_+^*$ on the finite (rational) adeles is considered, the nontrivial isotropy appears in the primitive ideal space through the infinite torus $\widehat{\mathbb{Q}_+^*}$ sitting above the only point with nontrivial isotropy. This had originally led us to believe that $\mathfrak{T}[R]$ would have a complicated primitive ideal structure. Thus Lemma 3·4 was a welcome surprise: the stability subgroups do not play a role in the primitive ideal space of $\mathfrak{T}[R]$, ultimately due to the additive action of R.

Remark 3·9. The description of the primitive ideal space shows that there is only one maximal ideal in $\mathfrak{T}[R]$, namely the primitive ideal $I_{\mathcal{P}}$ corresponding to the set $\mathcal{P}$ viewed as an element of $2^{\mathcal{P}}$. Therefore, the quotient $\mathfrak{T}[R]/I_{\mathcal{P}}$ is the only simple quotient of $\mathfrak{T}[R]$, which is isomorphic to the ring C*-algebra $\mathfrak{A}[R]$ considered by Cuntz and Li in [**4**].

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