



COMPOSITIO MATHEMATICA

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Compositio Math. **158** (2022), 1563–1594.

[doi:10.1112/S0010437X22007515](https://doi.org/10.1112/S0010437X22007515)



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ABSTRACT

We prove that any nef b -divisor class on a projective variety defined over an algebraically closed field of characteristic zero is a decreasing limit of nef Cartier classes. Building on this technical result, we construct an intersection theory of nef b -divisors, and prove several variants of the Hodge index theorem inspired by the work of Dinh and Sibony. We show that any big and basepoint-free curve class is a power of a nef b -divisor, and relate this statement to the Zariski decomposition of curves classes introduced by Lehmann and Xiao. Our construction allows us to relate various Banach spaces contained in the space of b -divisors which were defined in our previous work.

Introduction

The notion of a b -divisor was introduced by Shokurov and has found striking applications in algebraic geometry and singularity theory [Sho96, Sho03, Amb05, Cor07, Fuj12, BdFF12, Zha14, BFJ14, HL21], and in algebraic dynamical systems [BFJ08a, Can11, GR14, Xie15, BC16, DF21]. It provides the right setting to construct Zariski decomposition of divisors [BFJ09, KM13] and of higher codimension cycles [FL17b]. It is also deeply tied with some recent developments in pluripotential analysis on non-Archimedean analytic varieties (see [Zha95, Gub98, Cha06, BFJ16, BJ18, GJKM19] and the references therein).

A b -divisor class is by definition a collection of Néron–Severi classes in all birational models of a fixed projective variety satisfying natural compatibility conditions under pushforward morphisms. It is thus in essence an ‘infinite’ object and defining the intersection product of two such objects always requires a limiting process which may be guaranteed only under special circumstances.

Such an intersection theory has been successfully developed for special classes of b -divisors: Cartier b -divisors in [KK14], relatively nef b -divisors over a closed point in [BFJ08b], and over the spectrum of a valuation ring [BFJ16, BJ18, GJKM19]. In this paper, we focus on nef b -divisors over a projective variety, and show that one can develop a natural intersection calculus for them. We proceed with proving a Hodge index theorem in this context, and invert the operator $\alpha \mapsto \alpha^{d-1}$ (d being the dimension of the ambient variety) which may be viewed as an analog of the complex Monge–Ampère operator.

We work under the following setup, and borrow the terminology from [BFJ08a, DF21].

Let X be any smooth projective variety of dimension d defined over an algebraically closed field K of characteristic 0. A Weil b -divisor class α is a family of real Néron–Severi classes

Received 27 July 2021, accepted in final form 22 November 2021, published online 5 September 2022.

2020 Mathematics Subject Classification 14C20 (primary), 14E05 (secondary).

Keywords: b -divisors, birational geometry.

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$\alpha_{X'} \in \text{NS}(X')$ that are compatible under pushforward. Here X' runs over all smooth birational models lying over X .

The space $\text{w-N}^1(\mathcal{X})$ (where \mathcal{X} stands for the Riemann–Zariski space of X) of Weil b -divisor classes forms an infinite-dimensional vector space. It can be identified with the projective limit of the real Néron–Severi spaces of all models, hence carries a natural locally convex topology in which every bounded set is relatively compact.

Cartier b -divisors are Weil b -divisors for which there exists a model X' such that $\alpha_{X''} = \pi^* \alpha_{X'}$ for any model such that the induced birational map $\pi: X'' \rightarrow X'$ is regular. In this situation we say that α is determined by $\alpha_{X'}$ in X' . The space $\text{c-N}^1(\mathcal{X})$ of Cartier b -divisors is dense in $\text{w-N}^1(\mathcal{X})$.

As the nef property is stable by pullback, one can define nef Cartier classes as those determined by nef classes in some model. We say that a class $\alpha \in \text{w-N}^1(\mathcal{X})$ is nef if it lies in the closure of the set of nef classes in $\text{w-N}^1(\mathcal{X})$.

Our main aim is to explore in depth the structure of the convex cone $\text{Nef}^1(\mathcal{X})$ of nef b -divisors. To that end, we first prove a fundamental approximation result for arbitrary nef classes.

THEOREM A. *Let $\alpha \in \text{Nef}^1(\mathcal{X})$ be any nef class. Then there exists a sequence of nef Cartier b -divisor classes $\alpha_n \in \text{c-Nef}^1(\mathcal{X})$ decreasing to α .*

The crucial point in this statement is the fact that the approximating sequence is decreasing meaning that it satisfies the condition $\alpha_n \geq \alpha$ in the sense that $\alpha_n - \alpha$ is pseudo-effective (psef) for all n . The proof relies on the asymptotic construction of multiplier ideals and is completely analogous to (and, in fact, simpler than) the approximation results of relatively nef b -divisors (over a smooth point [BFJ08b], or over a curve [BFJ16, BJ18, GJKM19]). A similar approximation result is proved in a toroidal context in [BBG22, Lemma 5.9] which does not use multiplier ideals techniques.

After we released a first version of this paper, and using the same technics based on multiplier ideal sheaves, Boucksom and Jonsson have been able to prove an approximation result for plurisubharmonic functions on the Berkovich analytification of projective varieties defined over a trivially valued field. As explained in [BJ21, § 10.3], their far-reaching result implies our theorem.

In order to intersect nef b -divisors, it is necessary to discuss an appropriate notion of positivity for classes of arbitrary codimension.

We first define a b -numerical class of codimension k as a collection of numerical classes $\alpha_{X'} \in \text{N}^k(X')$ for any proper birational model $X' \rightarrow X$ that are invariant under pushforward morphisms. Cartier b -classes are defined analogously as in codimension one. We obtain a locally convex topological space $\text{w-N}^k(\mathcal{X})$ containing $\text{c-N}^k(\mathcal{X})$ as a dense subset and we have a natural perfect intersection pairing $\text{w-N}^k(\mathcal{X}) \times \text{c-N}^{d-k}(\mathcal{X}) \rightarrow \mathbb{R}$.

Several notions of positivity have been explored in $\text{N}^k(X')$, see [FL17a]. Among those, we shall use the notion of basepoint-free (BPF) classes which seems appropriate in several contexts, in particular for applications to dynamical systems, see [Dan20, DF21].

By definition, a numerical class $\text{N}^k(X')$ is BPF if it can be approximated by images under flat morphisms of complete intersection ample classes. As previously, this notion leads to the definition of codimension- k BPF numerical b -classes.

Note that a class in $\text{N}^1(X')$ is BPF if and only if it is nef, so that $\text{BPF}^1(\mathcal{X}) = \text{Nef}^1(\mathcal{X})$. We further prove in § 5.1 that a b -numerical class $\alpha \in \text{w-N}^k(\mathcal{X})$ is BPF iff for all models X' the class $\alpha_{X'}$ is movable (see Theorem 5.2). These results generalize [BdFF12, Lemma 2.10], and suggest

the following duality conjecture that is proved by Lehmann [Leh11, Theorem 1.5] when $k = 1$, and that we extend to the case $k = d - 1$ (Proposition 5.6).

CONJECTURE 1.

- (i) A b -numerical class $\alpha \in w\text{-N}^k(\mathcal{X})$ is psef iff for any b -numerical class $\beta \in c\text{-BPF}^{d-k}(\mathcal{X})$ we have $(\alpha \cdot \beta) \geq 0$.
- (ii) A b -numerical class $\alpha \in w\text{-N}^k(\mathcal{X})$ belongs to $\text{BPF}^k(\mathcal{X})$ iff for any psef Cartier b -numerical class $\beta \in c\text{-N}^{d-k}(\mathcal{X})$, we have $(\alpha \cdot \beta) \geq 0$.

With these notions at hand and using Theorem A, one can follow the construction of positive intersection classes developed in [BFJ09, BDPP13], and define a generalized intersection product of nef b -divisor classes which takes the following form.

Given any collection of nef classes $\{\alpha_i\}_{1 \leq i \leq k} \in \text{Nef}^1(\mathcal{X})$, we define $(\alpha_1 \cdot \dots \cdot \alpha_k) \in \text{BPF}^k(\mathcal{X})$ as the infimum $(\beta_1 \cdot \dots \cdot \beta_k)$ over all nef Cartier b -classes $\beta_i \geq \alpha_i$. That this intersection product is multilinear and extends the natural one on Cartier classes is a consequence of Theorem A.

Of crucial importance to applications is the fact that the intersection product satisfies various forms of the Hodge index theorem. In particular, we shall prove the following result.

THEOREM B. *Any two b -divisor classes $\alpha, \beta \in \text{Nef}^1(\mathcal{X})$ such that $\alpha \cdot \beta = 0$ are proportional.*

We also obtain a far-reaching generalization of [DS04a, Corollary 3.5] for b -divisors, see Theorem 3.14 below. We hope that this result will find applications to the study of birational group actions on projective varieties.

Let ω be any Cartier b -divisor class determined by an ample class in X . For any $k \geq 1$, let us call big any class $\alpha \in w\text{-N}^k(\mathcal{X})$ such that $\alpha \geq c\omega^k$ for some $c > 0$. As two ample classes are comparable, the notion of bigness does not depend on the choice of ω . It follows from Siu's inequalities that for any two big BPF classes $\alpha, \beta \in \text{BPF}^k(\mathcal{X})$ there exists a constant $C > 1$ such that $C^{-1}\alpha \leq \beta \leq C\alpha$. We prove that any big BPF class of dimension one can be realized as a $(d - 1)$ th power of a nef b -divisor. Our precise result reads as follows.

THEOREM C. *The map $\alpha \mapsto \alpha^{d-1}$ induces a bijection from the convex cone of big b -divisor classes $\alpha \in \text{Nef}^1(\mathcal{X})$ onto the convex cone of big b -numerical curve classes in $\text{BPF}^{d-1}(\mathcal{X})$.*

Our proof is variational in nature, and follows the same line of arguments as in [BFJ15] (whose proof is itself inspired by [Ale96, BBGZ13]). A key point in the proof is the differentiability of the volume function, see [BFJ09, LM09, WN19]. Over a toric variety, a big curve (torus-invariant) class can be identified to a positive measure on a sphere, and our statement reduces in that case to Minkowski's theorem which states the existence of a convex body with a prescribed surface area measure [Sch14, Theorem 8.2.2].

Recently several decompositions of sufficiently positive curve classes on projective varieties have been introduced [LX16, FL17b, LX19] which generalizes the Zariski decomposition of psef divisors. Theorem C gives a new perspective on these decompositions as we now explain.

First Lehmann and Xiao [LX16] have shown that for any big curve class $\alpha \in \text{N}^{d-1}(X)$ there exists a unique nef class $\text{LX}(\alpha) \in \text{N}^1(X)$ such that

$$\alpha \geq \text{LX}(\alpha)^{d-1} \quad \text{and} \quad \text{LX}(\alpha) \cdot (\alpha - \text{LX}(\alpha)^{d-1}) = 0.$$

We prove (Theorem 5.8) that for any big BPF class α the family of nef Cartier classes $\text{LX}(\alpha_{X'})$ where X' ranges over all models X converges to the unique nef class γ solving $\gamma^{d-1} = \alpha$. As a consequence of [LX16, Theorem 5.29], we also obtain a characterization of big curve classes in terms of the functional $\widehat{\text{vol}}$ that was introduced by Xiao in [Xia17] (Theorem 5.11).

We then discuss a second decomposition based on the functional \mathfrak{M} defined on movable curve classes and introduced by Xiao. We prove (Theorem 5.12) yet another convergence result $Mv(\alpha_{X'}) \rightarrow \gamma$ where $Mv(\alpha_{X'}) \in N^1(X')$ is the movable divisor class computing $\mathfrak{M}(\alpha_{X'})$ given by [LX19, Theorem 3.14].

The intersection theory of b -divisors and the techniques presented in this paper lead to a better understanding of the structure of the space of b -classes. The most natural norm on $c\text{-}N^k(\mathcal{X})$ is arguably defined by $\|\alpha\|_\omega := \inf\{C > 0, -C\omega^k \leq \alpha \leq C\omega^k\}$, and we may thus look at the Banach space $N^k_+(\mathcal{X})$ obtained as the completion of $c\text{-}N^k(\mathcal{X})$ with respect to this norm. The underlying topological space of $N^k_+(\mathcal{X})$ is independent of the choice of ω as all such norms are equivalent.

In a previous paper [DF21], we also introduced other Banach spaces $N^k_{\text{BPF}}(\mathcal{X}), N^{k,\vee}_{\text{BPF}}(\mathcal{X}) \subset w\text{-}N^k(\mathcal{X})$ for $0 \leq k \leq d$ and $N^1_\Sigma(\mathcal{X}) \subset w\text{-}N^1(\mathcal{X})$ which played a crucial role in our approach to analyze the degree growth of rational self-maps.

First $N^k_{\text{BPF}}(\mathcal{X})$ is obtained as the completion of the space of Cartier b -divisor classes for the norm induced by the BPF cone. The space $N^{k,\vee}_{\text{BPF}}(\mathcal{X})$ is similarly defined as the completion of $c\text{-}N^k(\mathcal{X})$ for the dual norm induced by the BPF cone.

Let us review briefly the definition of $N^1_\Sigma(\mathcal{X})$ referring to § 1.4 for more details. When X is a surface, then $N^1_\Sigma(\mathcal{X})$ is the Picard–Manin space introduced by Cantat [Can11] and [BFJ08a]. It is a Hilbert space which is equipped with a natural intersection form $\langle \cdot, \cdot \rangle$ of Minkowski’s type for which the Hodge index theorem extends naturally. It contains $N^1_{\text{BPF}}(\mathcal{X})$ and is contained in $N^{1,\vee}_{\text{BPF}}(\mathcal{X})$, and its importance stems from the fact that birational surface maps induce isometries for the pairing $\langle \cdot, \cdot \rangle$.

In higher dimension, $N^1_\Sigma(\mathcal{X})$ is defined as the completion of $c\text{-}N^1(\mathcal{X})$ with respect to a family of semi-norms obtained by restricting classes to sufficiently general surfaces in birational models of X . In [DF21], we proved that a version of the Hodge index theorem still holds in this Banach space.

We use Theorem A and a version of Diskant inequalities for b -divisors extending [BFJ09, Theorem F] to prove the following inclusions.

THEOREM D. *One has the following sequence of continuous injections:*

$$N^1_{\text{BPF}}(\mathcal{X}) \hookrightarrow \text{Vect}(\text{Nef}^1(\mathcal{X})) \hookrightarrow N^1_\Sigma(\mathcal{X}) \hookrightarrow N^1_+(\mathcal{X}) = N^{1,\vee}_{\text{BPF}}(\mathcal{X}) \hookrightarrow w\text{-}N^1(\mathcal{X}). \tag{1}$$

The first and last injections were proved in [DF21, Proposition 2.10] whereas the equality $N^1_+(\mathcal{X}) = N^{1,\vee}_{\text{BPF}}(\mathcal{X})$ and the two injections $\text{Vect}(\text{Nef}^1(\mathcal{X})) \hookrightarrow N^1_\Sigma(\mathcal{X}) \hookrightarrow N^{1,\vee}_{\text{BPF}}(\mathcal{X})$ are new.

CONJECTURE 2. For any projective manifold X of dimension at least two, all inclusions in (1) are strict.

The injection $N^1_\Sigma(\mathcal{X}) \hookrightarrow N^{1,\vee}_{\text{BPF}}(\mathcal{X})$ can be viewed as an analog of the Sobolev injection of $W^{1,2}([0, 1])$ into $\mathcal{C}^0([0, 1])$. This analogy takes its root in the context of toric geometry, and we plan to discuss these links in detail in a forthcoming work.

Let us discuss some of the restrictions of our approach. As in our companion paper [DF21], we suppose K to be of characteristic zero. We use the assumption on the characteristic for the existence of smooth models and for the uniform generation of multiplier ideals which eventually relies on Kodaira vanishing theorem. We expect though that our results extend verbatim to positive characteristics with multiplier ideals being replaced by test ideals as in [GJKM19].

It is very tempting to extend our results to compact Kähler manifolds. In this case, the natural objects to consider are projective limits of Dolbeault cohomology spaces $H^k_{\mathbb{R}}(X')$ where $\mu: X' \rightarrow X$ ranges over all bimeromorphic proper holomorphic maps. Observe that one obtains

a space which is, in general, much bigger than $w\text{-N}^k(\mathcal{X})$ even when X is projective as one needs to deal with potentially transcendental classes.

The notion of volume and the Zariski decomposition of big $(1, 1)$ -cohomological classes (which in the projective case play important roles in our approach) have been defined by Boucksom [Bou04] and further characterized by DiNezza, Floris, and Trapani [DNFT17] (see also [Nak04, DDNL21, Tos19]). Even so, we face a serious obstacle in extending our results because the differentiability of volume function is not known for arbitrary Kähler manifolds. This property is equivalent to the so-called transcendental Morse inequalities (that we call Siu’s inequalities here) by an argument of Xiao, see also [WN19, Appendix A]. Much progress have been made recently towards the proof of these inequalities (see [Xia15, Tos16, Pop16, Chi16, Pop17]) but they remain open in general. Note that the duality between the movable curves and psef classes is also very much related to these problems, and is not known except in the projective case (see [WN19]).

It would be interesting to investigate analogs of Theorem A for a class $\alpha \in \text{BPF}^k(\mathcal{X})$ with $k \geq 2$. The work of Dinh and Sibony [DS04b] on the regularization of positive closed currents suggests that one might expect the existence of a sequence $\alpha_n^\pm \in c\text{-BPF}^k(\mathcal{X})$ such that $\alpha = \lim_n(\alpha_n^+ - \alpha_n^-)$, and $\sup_n(\alpha_n^\pm \cdot \omega^{d-k}) \leq C(\alpha \cdot \omega^{d-k})$ for some universal $C > 0$.

One might also want to generalize Theorem C and try to solve the equation $(\alpha^k \cdot \omega^{d-k-1}) = \theta$ where $\omega \in \text{Nef}^1(\mathcal{X})$ and $\theta \in \text{BPF}^{d-1}(\mathcal{X})$ are fixed and α lie in an appropriate subspace of numerical divisor b -classes. This would be analogous to solving mixed Monge–Ampere equations in the complex domain (see [DK14, LN15, BZ20, BZ21] and the references therein for recent developments on this problem).

Finally one can look at relative situations where X is a (non-necessarily Noetherian) scheme, Z is a strict subscheme of X , and consider all proper morphisms $X' \rightarrow X$ that are isomorphisms over $X \setminus Z$. A case of interest arises when X is a flat projective scheme over $\text{spec}(K^\circ)$ where K° is the ring of integers of a complete non-Archimedean metrized field. Then Theorem A follows from the so-called continuity of envelopes, a key problem in the development of non-Archimedean pluripotential theory. We refer to [BFJ16, BJ18, BJ21, GJKM19] for a detailed discussion of this important problem.

1. Basics on b -classes

In this section, we review briefly some notions on b -classes following the discussion in [DF21], see also [BFJ08a, BFJ08b, KM13]. We fix a smooth projective variety X of dimension d defined over an algebraically closed field K of characteristic 0.

1.1 Numerical classes of cycles

Let $Z^k(X)$ be the \mathbb{R} -vector space freely generated by irreducible subvarieties of pure codimension k in X . Given any two cycles α, β of complementary dimension in X , we denote by $(\alpha \cdot \beta) \in \mathbb{R}$ their intersection number as defined in [Ful98].

The numerical space of cycles of codimension k , denoted by $\text{N}^k(X)$, is defined as the quotient of $Z^k(X)$ by the vector space of cycles z such that $(\alpha \cdot z) = 0$ for all cycle α of dimension k . It is a finite-dimensional \mathbb{R} -vector space, and the pairing $\text{N}^k(X) \times \text{N}^{d-k}(X) \rightarrow \mathbb{R}$ is perfect.

The space $\text{N}^1(X)$ is the tensor product of the Néron–Severi group of X with \mathbb{R} . Intersection products of k divisors define numerical cycle classes of codimension k but these classes do not span $\text{N}^k(X)$ in general.

A class in $N^k(X)$ is called psef, if it belongs to the closure of the convex cone spanned by effective cycle classes. When $\alpha \in N^k(X)$ is psef, we write $\alpha \geq 0$. The set of psef classes forms a closed salient convex cone $\text{Psef}^k(X)$ inside $N^k(X)$ with non-empty interior.

A class lying in the interior of the cone of psef classes in $N^1(X)$ is said to be big. An ample class ω is big. A class $\alpha \in N^1(X)$ is big iff one can find $\epsilon > 0$ such that $\alpha \geq \epsilon\omega$.

A class $\alpha \in N^1(X)$ is nef if its intersection with any psef curve class is non-negative. The nef cone $\text{Nef}^1(X)$ is the interior of the cone of real ample classes. A nef class α is big iff $(\alpha^d) > 0$.

The negativity lemma (see, e.g., [Fuj11, Lemma 4.16]) states that for any proper birational morphism $\pi: X \rightarrow Y$ and any nef class $\alpha \in \text{Nef}^1(X)$, we have $\pi^*(\pi_*\alpha) \geq \alpha$.

A class $\alpha \in N^k(X)$ with $k \geq 1$ is called strongly BPF if it is the pushforward under a flat proper morphism of relative dimension e of the intersection of $k + e$ ample divisor classes of the intersection of $k + e$ ample divisors.¹ The closure of the cone generated by strongly BPF classes is called the BPF cone of codimension k . We denote it by $\text{BPF}^k(X)$. When $k = 1$, BPF classes coincide with nef classes (i.e. $\text{Nef}^1(X) = \text{BPF}^1(X)$). Any BPF class of arbitrary codimension is both psef and nef (i.e. intersects non-negatively the fundamental class of any irreducible subvariety V of codimension $d - k$), and the pullback of BPF classes by a proper morphism remains BPF. Note, however, that a nef class need not be psef when $d - 2 \geq k \geq 2$, see [DELV11].

A class $\alpha \in N^k(X)$ is big iff one can find $\epsilon > 0$ such that $\alpha \geq \epsilon\omega^k$ for some ample class ω . A class is big iff it belongs to the interior of the psef cone, see [FL17a, Lemma 2.12].

The image under a proper birational map of a BPF class is not BPF in general. This motivates the introduction of the notion of ‘movable’ classes. We discuss this notion in detail in §5.1 following Fulger and Lehmann [FL17b, §3]. We present here the case of curve classes which is needed in the proof of Proposition 1.9. We insist on working with closed cones, and therefore introduce a slight twist in the usual terminology by introducing first ‘strongly movable’ classes.

One says that a class $\alpha \in N^{d-1}(X)$ is strongly movable if there exists a birational proper morphism $\pi: X \rightarrow Y$, and real ample classes $\beta_1, \dots, \beta_{d-1} \in N^{d-1}(Y)$ such that $\alpha = \pi_*(\beta_1 \cdots \beta_{d-1})$. A class in $N^{d-1}(X)$ is movable if it is the limit of a sequence of positive linear combinations of strongly movable classes. By [BDPP13, Corollary 2.5], a class $\alpha \in N^{d-1}(X)$ is movable iff $(\alpha \cdot \gamma) \geq 0$ for all nef divisor classes $\gamma \in N^1(X)$.

We now present a key estimate, called Siu’s inequalities, which allows one to compare a BPF class with a complete intersection class. We refer to [Dan20, Proposition 3.4.6] for a proof.

PROPOSITION 1.1. *There exists a constant $C_d > 0$ such that for all classes $\alpha \in \text{BPF}^k(X)$ and all big and nef divisor classes β on X , one has*

$$\alpha \leq C_d \frac{(\alpha \cdot \beta^{d-k})}{(\beta^d)} \beta^k. \tag{2}$$

1.2 Weil and Cartier numerical b-classes

A model over X is a projective birational morphism $\pi: X' \rightarrow X$ from a smooth projective variety X' . Any such model is obtained as the blow-up of some ideal sheaf on X so that the category \mathcal{M}_X of all models over X is naturally a poset for which $X' \geq X''$ iff the canonical map $X' \dashrightarrow X''$ is a morphism. When $X' \geq X''$ we say that X' dominates X'' . The poset \mathcal{M}_X is inductive in the sense that any two models are simultaneously dominated by a third. The projective limit over the inductive system defined by \mathcal{M}_X where each model is endowed with the Zariski topology is a quasi-compact topological space called the Riemann–Zariski space of X that we denote by \mathcal{X} . We refer to [Vaq00] for an interpretation of this space in terms of valuations.

¹ This definition is equivalent to the original given in [FL17a], see [Dan20, Corollary 3.3.4].

A Weil numerical b -class α of codimension k is a map which assigns to any smooth model $X' \in \mathcal{M}_X$ a numerical class $\alpha_{X'} \in N^k(X')$, such that $\pi_*(\alpha_{X'}) = \alpha_{X''}$ for any pair of smooth models $X' \geq X''$ with $\pi = (\pi'')^{-1} \circ \pi'$. In the rest of the paper, we also refer to codimension-one b -classes as b -divisor classes. The class $\alpha_{X'}$ is called the incarnation of α in X' .

A Cartier b -class is a numerical b -class α for which one can find a model X'' such that $\alpha_{X'} = \pi^*(\alpha_{X''})$ for any model $X' \geq X''$. When it is the case, we say that α is determined in X'' . Conversely for any class $\alpha \in N^k(X')$, we let $[\alpha]$ be the Cartier b -class determined by α in X' .

The space of Weil numerical b -classes is an infinite-dimensional real vector space, which we denote by $w-N^k(\mathcal{X})$. It contains the set of Cartier numerical b -classes $c-N^k(\mathcal{X})$ as a subspace, and for each smooth model X' the map $\alpha \mapsto [\alpha]$ induces an injective linear map $N^k(X') \rightarrow c-N^k(\mathcal{X})$.

The space $w-N^k(\mathcal{X})$ is a locally convex topological vector space when endowed with its natural product topology, and such that every bounded set is relatively compact. Note that when K is countable this space is also separable and metrizable, but these properties do not hold when K is uncountable.

In this topology, a sequence $\alpha_n \in w-N^k(\mathcal{X})$ converges to $\beta \in w-N^k(\mathcal{X})$ iff for any (smooth) model X' we have $(\alpha_n)_{X'} \rightarrow \beta_{X'}$ in $N^k(X')$. As for any $\alpha \in w-N^k(\mathcal{X})$ the sequence of Cartier b -classes $[\alpha_{X'}]$ converges to α , the space $c-N^k(\mathcal{X})$ is dense in $w-N^k(\mathcal{X})$.

Pick $\alpha \in c-N^k(\mathcal{X})$ and $\beta \in w-N^l(\mathcal{X})$. Suppose α is determined in a model X_0 . For any other smooth model $X' \geq X_0$, we set $(\alpha \cdot \beta)_{X'} := \alpha_{X'} \cdot \beta_{X'} \in N^{k+l}(X')$. By the projection formula, $\pi_*(\alpha \cdot \beta)_{X''} = (\alpha \cdot \beta)_{X'}$ for any smooth model $X'' \geq X'$ so that we may define the class $\alpha \cdot \beta \in w-N^{k+l}(\mathcal{X})$ as the unique Weil numerical b -class whose incarnation in any smooth model dominating X_0 is equal to $(\alpha \cdot \beta)_{X'}$.

The ring $c-N^\bullet(\mathcal{X}) = \bigoplus c-N^k(\mathcal{X})$ is a graded ring over which $w-N^\bullet(\mathcal{X}) = \bigoplus w-N^k(\mathcal{X})$ is a graded module.

LEMMA 1.2. *The pairing $c-N^\bullet(\mathcal{X}) \times w-N^\bullet(\mathcal{X}) \rightarrow \mathbb{R}$ is perfect in the sense that for any integer $k \in \{0, \dots, d\}$:*

- (i) *the canonical map from $w-N^{d-k}(\mathcal{X})$ to the dual space $c-N^k(\mathcal{X})^*$ is an isomorphism;*
- (ii) *and the canonical map from $c-N^{d-k}(\mathcal{X})$ to the space of weakly continuous linear forms on $w-N^k(\mathcal{X})$ is also an isomorphism.*

We thank the referee for pointing out to us the following neat proof of the previous lemma.

Proof. The first statement follows from Poincaré duality. Pick any linear form ℓ on $c-N^{d-k}(\mathcal{X})$. For each model X' , there exists a unique class $\alpha_{X'} \in N^k(X')$ such that $\ell([\beta]) = (\alpha_{X'} \cdot \beta)$ for all $\beta \in N^{d-k}(X')$. For any proper birational morphism $\pi: X'' \rightarrow X'$ and any class $\beta \in N^{d-k}(X')$, we have $(\alpha_{X'} \cdot \beta) = \ell([\beta]) = (\alpha_{X''} \cdot \pi^*\beta)$ hence $\pi_*\alpha_{X''} = \alpha_{X'}$. The collection of numerical classes $(\alpha_{X'})$ thus defines a Weil class α which satisfies $\ell(\gamma) = (\alpha \cdot \gamma)$ for any Cartier class γ .

Before we prove the second statement, note that a basis of neighborhoods of 0 for the weak topology on $w-N^{d-k}(\mathcal{X})$ is given by sets of the form $[U, X'] := \{\alpha \in w-N^{d-k}(\mathcal{X}) \mid \alpha_{X'} \in U\}$ where X' is a smooth model and U is an open neighborhood of 0 in $N^{d-k}(X')$. As the latter space is finite dimensional, it is isomorphic to the dual of $N^k(X')$. It follows that the family of open sets

$$V(\{\beta_i\}, \{r_i\}) := \{\alpha \in w-N^{d-k}(\mathcal{X}) \mid |(\alpha_{X'} \cdot \beta_i)| < r_i\}$$

also forms a basis of neighborhood of 0, where β_i are classes in $N^k(X')$ for some model X' and r_i are positive real numbers.

Now take any weakly continuous linear form ℓ on $w-N^{d-k}(\mathcal{X})$. By what precedes, we may find a model X' , classes $\beta_i \in N^k(X')$ and $r_i > 0$ such that $|\ell(\alpha)| < 1$ if $|(\alpha \cdot [\beta_i])| < r_i$ for all i . Write $\ell_i(\alpha) := (\alpha \cdot [\beta_i])$. Then we first infer $|\ell| \leq C \max |\ell_i|$ with $C = \max r_i^{-1}$, hence $\ell = 0$ on

$\cap_i \{\ell_i = 0\}$. This implies ℓ is a linear combination of the ℓ_i 's (see, e.g., [Rud91, Lemma 3.9]). We obtain a family of real numbers a_i such that

$$\ell(\alpha) = \left(\alpha \cdot \sum a_i [\beta_i] \right)$$

for all $\alpha \in w\text{-N}^{d-k}(\mathcal{X})$ as required. □

1.3 Positive cones of b -numerical classes

We discuss here some positivity notions on b -classes that are inherited from those defined on particular models.

A class $\alpha \in w\text{-N}^k(\mathcal{X})$ is said to be psef (and we write $\alpha \geq 0$) when $\alpha_{X'} \geq 0$ for any model X' . Note that the pseudo-effectivity is only preserved by pushforward, and not by pullback, so that it may happen that a Cartier b -class determined by a psef class in a smooth model X is not psef in $w\text{-N}^k(\mathcal{X})$. However a Cartier b -divisor class $\alpha \in c\text{-N}^1(\mathcal{X})$ is psef iff its incarnation in one (or all) of its determination on a smooth model is psef.

The notion of BPF b -class is defined as follows. We let $c\text{-BPF}^k(\mathcal{X})$ be the convex cone in $c\text{-N}^k(\mathcal{X})$ generated by Cartier numerical b -classes $[\alpha]$ with $\alpha \in \text{BPF}(X')$ for some model X' . As BPF classes are stable by pullback, a class $\alpha \in c\text{-N}^k(\mathcal{X})$ is BPF iff it is BPF in one of its determination.

DEFINITION 1.3. The cone $\text{BPF}^k(\mathcal{X})$ is the (weak) closure in $w\text{-N}^k(\mathcal{X})$ of the cone $c\text{-BPF}^k(\mathcal{X})$.

As the cone $\text{BPF}^1(X')$ coincides with the nef cone for all model X' , we write $\text{Nef}^1(\mathcal{X}) = \text{BPF}^1(\mathcal{X})$, and $c\text{-Nef}^1(\mathcal{X}) = c\text{-BPF}^1(\mathcal{X})$. The next lemma is clear but we provide a proof for completeness.

LEMMA 1.4. A class $\alpha \in w\text{-N}^k(\mathcal{X})$ is BPF iff there exists a net of classes $\alpha_i \in c\text{-BPF}^k(\mathcal{X})$ such that $(\alpha_i)_{X'} \rightarrow \alpha_{X'}$ for all model X' .

Proof. Pick any class $\alpha \in w\text{-N}^k(\mathcal{X})$. For any model X' , and for any open subset U of $\text{N}^k(X')$, let $[U, X'] = \{\beta \in w\text{-N}^k(\mathcal{X}), \beta_{X'} \in U\}$ as previously. Then the collection I_α of all sets containing α and of the form $[U, X']$ forms a basis of neighborhood of α . It is also an inductive set for the order relation $[U_0, X_0] \leq [U_1, X_1]$ iff $X_0 \leq X_1$ and the preimage of U_0 by the pullback morphism $\text{N}^k(X_0) \rightarrow \text{N}^k(X_1)$ contains U_1 .

Suppose that α is BPF. Then for each $i = [U, X'] \in I_\alpha$ we may choose a Cartier BPF b -class α_i in $[U, X']$. By construction, we have $(\alpha_i)_{X'} \rightarrow \alpha_{X'}$ for all model X' . The converse statement is clear. □

LEMMA 1.5. For any class $\alpha \in \text{Nef}^1(\mathcal{X})$, and for any model $X'' \geq X'$ we have

$$\alpha \leq [\alpha_{X''}] \leq [\alpha_{X'}].$$

Proof. This is a rephrasing of the negativity lemma alluded to previously. □

1.4 Banach spaces $\text{N}_{\text{BPF}}^k(\mathcal{X})$, $\text{N}_{\text{BPF}}^{k,\vee}(\mathcal{X})$, and $\text{N}_{\Sigma}^1(\mathcal{X})$

Let ω be any fixed big and nef Cartier b -divisor class. For any class $\alpha \in c\text{-N}^k(\mathcal{X})$, we define

$$\|\alpha\|_{\text{BPF}} := \inf_{\substack{\alpha = \alpha_+ - \alpha_- \\ \alpha_{\pm} \in \text{BPF}^k(\mathcal{X})}} (\alpha_+ \cdot \omega^{d-k}) + (\alpha_- \cdot \omega^{d-k}), \tag{3}$$

$$\|\alpha\|_{\text{BPF}}^{\vee} := \sup_{\substack{\gamma \in c\text{-BPF}^{d-k}(\mathcal{X}) \\ (\omega^k \cdot \gamma) = 1}} |(\alpha \cdot \gamma)|. \tag{4}$$

Note that $\|\cdot\|_{\text{BPF}}^\vee$ is the dual norm of $\|\cdot\|_{\text{BPF}}$ on $c\text{-N}^{d-k}(\mathcal{X})$. When $\alpha \in c\text{-N}^1(\mathcal{X})$, we also set

$$\|\alpha\|_\Sigma := \sup_{\substack{\gamma \in c\text{-BPF}^{d-2}(\mathcal{X}) \\ (\gamma \cdot \omega^2)=1}} (2(\alpha \cdot \omega \cdot \gamma)^2 - (\alpha^2 \cdot \gamma))^{1/2}. \tag{5}$$

We showed in [DF21, §§ 2–3] that these functions induce norms on $c\text{-N}^k(\mathcal{X})$ and $c\text{-N}^1(\mathcal{X})$, respectively.

One defines the spaces $N_{\text{BPF}}^k(\mathcal{X})$, $N_{\text{BPF}}^{k,\vee}(\mathcal{X})$ as the completions of $c\text{-N}^k(\mathcal{X})$ with respect to $\|\cdot\|_{\text{BPF}}$ and $\|\cdot\|_{\text{BPF}}^\vee$, respectively. The space $N_\Sigma^1(\mathcal{X})$ is similarly obtained as the completion of $c\text{-N}^1(\mathcal{X})$ with respect to the norm $\|\cdot\|_\Sigma$.

In fact, all three norms only depend on ω up to equivalence, so that the corresponding completions are independent of the choice of ω (for the norm $\|\cdot\|_\Sigma$, we refer to [DF21, Theorem 3.3]). We also obtained in [DF21, Theorem 4.11] the following statement.

THEOREM 1.6. *The intersection product $c\text{-N}^1(\mathcal{X}) \times c\text{-N}^1(\mathcal{X}) \rightarrow c\text{-N}^2(\mathcal{X})$ extends continuously to a symmetric bilinear map:*

$$N_\Sigma^1(\mathcal{X}) \times N_\Sigma^1(\mathcal{X}) \rightarrow N_{\text{BPF}}^{2,\vee}(\mathcal{X}). \tag{6}$$

In [DF21, Propositions 2.10 and 2.11 and Theorem 3.16], we proved the following continuous injections:

$$\begin{aligned} N_{\text{BPF}}^k(\mathcal{X}) &\hookrightarrow N_{\text{BPF}}^{k,\vee}(\mathcal{X}) \hookrightarrow N_{\text{BPF}}^{d-k}(\mathcal{X})^* \hookrightarrow w\text{-N}^k(\mathcal{X}), \\ N_{\text{BPF}}^1(\mathcal{X}) &\hookrightarrow N_\Sigma^1(\mathcal{X}) \hookrightarrow w\text{-N}^1(\mathcal{X}). \end{aligned} \tag{7}$$

Let us record the following useful lemma.

LEMMA 1.7. *For any $\alpha \in \text{Nef}^1(\mathcal{X})$, one has $(\alpha \cdot \omega^{d-1}) = \|\alpha\|_{\text{BPF}}$.*

Proof. Fix a class $\alpha \in \text{Nef}^1(\mathcal{X})$, and note that it follows from the definition that

$$\|\alpha\|_{\text{BPF}} \leq (\alpha \cdot \omega^{d-1}).$$

Conversely, write $\alpha = \alpha_+ - \alpha_-$ where $\alpha_\pm \in \text{Nef}^1(\mathcal{X})$ are chosen so that $(\alpha_+ \cdot \omega^{d-1}) + (\alpha_- \cdot \omega^{d-1}) \leq \|\alpha\|_{\text{BPF}} + \epsilon$ with $\epsilon > 0$. Then $(\alpha \cdot \omega^{d-1}) \leq (\alpha_+ \cdot \omega^{d-1}) + (\alpha_- \cdot \omega^{d-1}) \leq \|\alpha\|_{\text{BPF}} + \epsilon$, and this implies the reverse inequality. \square

1.5 Banach space $N_+^k(\mathcal{X})$

As in the previous section, ω is a big and nef Cartier b -divisor class. For any class $\alpha \in c\text{-N}^k(\mathcal{X})$, we set

$$\|\alpha\|_\omega := \inf\{C \geq 0, -C\omega^k \leq \alpha \leq C\omega^k\}.$$

As the psef cone is closed in any model, note that the infimum is actually attained, and the function $\|\cdot\|_\omega$ defines a norm on $c\text{-N}^k(\mathcal{X})$.

LEMMA 1.8. *There exists a positive constant $C_d > 0$ depending only on d such that the following hold: for any class $\alpha \in c\text{-BPF}^k(\mathcal{X})$,*

$$\frac{1}{(\omega^d)}(\alpha \cdot \omega^{d-k}) \leq \|\alpha\|_\omega \leq C_d(\alpha \cdot \omega^{d-k}); \tag{8}$$

and for any class $\alpha \in c\text{-N}^k(\mathcal{X})$,

$$\|\alpha\|_{\text{BPF}}^\vee \leq \|\alpha\|_\omega \leq C_d\|\alpha\|_{\text{BPF}}.$$

Proof. Suppose α is BPF. By Siu’s inequalities (Proposition 1.1), we obtain $0 \leq \alpha \leq C_d(\alpha \cdot \omega^{d-k})\omega^k$ so that $\|\alpha\|_\omega \leq C_d(\alpha \cdot \omega^{d-k})$. Conversely, $\alpha \leq C\omega^k$ implies $(\alpha \cdot \omega^{d-k}) \leq C(\omega^d)$ hence (8) holds.

Pick any Cartier b -class α , and write $\alpha = \alpha_+ - \alpha_-$ with $\alpha_\pm \in \text{BPF}^k(\mathcal{X})$. The previous arguments show

$$\alpha = \alpha_+ - \alpha_- \leq \alpha_+ \leq C_d(\alpha_+ \cdot \omega^{d-k}) \leq C_d\|\alpha\|_{\text{BPF}}$$

and, similarly, $\alpha \geq -C_d\|\alpha\|_{\text{BPF}}$, hence $\|\alpha\|_\omega \leq C_d\|\alpha\|_{\text{BPF}}$.

Suppose that $-C\omega \leq \alpha \leq C\omega$ for some positive constant $C > 0$, and pick any class $\gamma \in \text{BPF}^{d-1}(\mathcal{X})$. We obtain $|(\alpha \cdot \gamma)| \leq C(\omega \cdot \gamma)$, hence $\|\alpha\|_{\text{BPF}}^\vee \leq C$. \square

Observe that replacing ω by another ample class gives an equivalent norm. We denote by $N_+^k(\mathcal{X})$ the completion of $c\text{-N}^k(\mathcal{X})$ for the norm $\|\cdot\|_\omega$.

The previous estimate proves that we have the following continuous injections:

$$N_{\text{BPF}}^k(\mathcal{X}) \hookrightarrow N_+^k(\mathcal{X}) \hookrightarrow N_{\text{BPF}}^{k,\vee}(\mathcal{X}). \tag{9}$$

We observe that both in dimension and codimension one, the norm $\|\cdot\|_\omega$ is identical to the dual BPF norm.

PROPOSITION 1.9. *For every class $\alpha \in c\text{-N}^1(\mathcal{X}) \cup c\text{-N}^{d-1}(\mathcal{X})$, we have*

$$\|\alpha\|_{\text{BPF}}^\vee = \|\alpha\|_\omega.$$

In particular, $N_+^1(\mathcal{X}) = N_{\text{BPF}}^{1,\vee}(\mathcal{X})$ and $N_+^{d-1}(\mathcal{X}) = N_{\text{BPF}}^{d-1,\vee}(\mathcal{X})$.

Proof. Suppose α is a Cartier class determined in some smooth model X' . We only need to show that $\|\alpha\|_\omega \leq \|\alpha\|_{\text{BPF}}^\vee$.

Let us first treat the case $\alpha \in c\text{-N}^{d-1}(\mathcal{X})$. For any nef class $\gamma \in \text{Nef}(X')$ such that $(\gamma \cdot \omega^{d-1}) = 1$ we have

$$((\alpha + \|\alpha\|_{\text{BPF}}^\vee \omega^{d-1}) \cdot [\gamma]) \geq 0,$$

hence the class $(\alpha + \|\alpha\|_{\text{BPF}}^\vee \omega^{d-1})_{X'}$ is psef by duality. Similarly, we obtain $(-\alpha + \|\alpha\|_{\text{BPF}}^\vee \omega^{d-1})_{X'} \geq 0$, hence $\|\alpha\|_\omega \leq \|\alpha\|_{\text{BPF}}^\vee$ as required.

Let us now treat the case where α is a b -divisor. Recall from [BDPP13] that a class $\alpha_0 \in N^1(X)$ is psef if and only if $(\alpha_0 \cdot \gamma_0) \geq 0$ for any strongly movable curve class $\gamma_0 \in N^{d-1}(X)$.

Suppose first that α is a Cartier b -divisor determined in some smooth model X' . For any strongly movable class $\gamma_0 \in N^{d-1}(X')$, there exists a class $\gamma \in c\text{-BPF}^{d-1}(\mathcal{X})$ such that $\gamma_{X'} = \gamma_0$. It follows that

$$((\alpha + \|\alpha\|_{\text{BPF}}^\vee \omega) \cdot \gamma_0) = ((\alpha + \|\alpha\|_{\text{BPF}}^\vee \omega) \cdot \gamma) \geq 0,$$

hence $\alpha + \|\alpha\|_{\text{BPF}}^\vee \omega \geq 0$ as required. \square

2. Approximation of nef b -divisor classes

In this section, we prove a version of our approximation result from the introduction in terms of nets. In § 3.2 we show that we can replace nets by sequences and obtain Theorem A.

THEOREM 2.1. *Let $\alpha \in \text{Nef}^1(\mathcal{X})$ be any nef class. There exists an inductive set I and a non-increasing net of nef Cartier b -divisor classes $(\alpha_i)_{i \in I} \in c\text{-Nef}^1(\mathcal{X})$ that is converging to α .*

In §§ 2.1 and 2.2 we first collect some general facts on ideal sheaves, and nef envelopes. The proof of Theorem 2.1 is then given in § 2.3. In § 2.4, we give a first application of this theorem and prove that a nef divisor b -class which is Cartier is determined by a nef class in some model.

2.1 Coherent ideal sheaves

To any coherent ideal sheaf \mathfrak{A} on X , we attach a Cartier b -divisor $[\mathfrak{A}]$ as follows. Choose a log-resolution X' of \mathfrak{A} so that one can write $\mathcal{O}_{X'}(-D) = \mathfrak{A} \cdot \mathcal{O}_{X'}$ for some divisor D . Then we let $[\mathfrak{A}]$ be the Cartier b -divisor determined by $[D]$ in X' . Note that this definition does not depend on the choice of resolution, and because D is effective that $[\mathfrak{A}] \geq 0$.

For any pair of ideal sheaves $\mathfrak{A}, \mathfrak{B}$ on X , we have:

- (i) $[\mathfrak{A}] \geq [\mathfrak{B}]$ when $\mathfrak{A} \subset \mathfrak{B}$; and
- (ii) $[\mathfrak{A} \cdot \mathfrak{B}] = [\mathfrak{A}] + [\mathfrak{B}]$.

The notion of multiplier ideal sheaf will play an essential role in the paper. We briefly discuss the basic definitions and properties that are necessary for our purposes, referring the interested reader to [Laz04] for an extensive treatment of this notion.

Let \mathfrak{A} be any coherent ideal sheaf on X , and let $c > 0$ be any positive constant. If $X' \xrightarrow{\pi} X$ is a resolution of \mathfrak{A} as previously, then we let $\mathcal{J}(\mathfrak{A}^c) = \pi_*(K_{X'/X} - \lfloor c \cdot D \rfloor)$ where $K_{X'/X} = \pi^*K_X - K_{X'}$ is the relative canonical sheaf, and $\lfloor c \cdot D \rfloor$ is the round-down of the divisor $c \cdot D$. This definition does not depend on the resolution.

A graded sequence of coherent ideal sheaves $\mathfrak{A}_\bullet = (\mathfrak{A}_m)$ is, by definition, a sequence of coherent ideal sheaves such that $\mathfrak{A}_0 = \mathcal{O}_X$ and $\mathfrak{A}_n \cdot \mathfrak{A}_m \subset \mathfrak{A}_{n+m}$ for all $n, m \geq 0$. For any $c > 0$, one can show that $\mathcal{J}(\mathfrak{A}_m^{c/m}) \subset \mathcal{J}(\mathfrak{A}_n^{c/n})$ whenever m divides n , and one defines the multiplier ideal sheaf $\mathcal{J}(\mathfrak{A}_\bullet^c)$ as the unique (coherent) ideal sheaf equal to $\mathcal{J}(\mathfrak{A}_m^{c/m})$ for all sufficiently divisible m . This definition makes sense by Noetherianity. Moreover, we have

$$\mathfrak{A}_m \subset \mathcal{J}(\mathfrak{A}_\bullet^m) \tag{10}$$

for all $m > 0$ by [Laz04, Theorem 11.1.19], and the following fundamental sub-additivity property holds (see [Laz04, Theorem 11.2.3]):

$$\mathcal{J}(\mathfrak{A}_\bullet^{c(m+n)}) \subset \mathcal{J}(\mathfrak{A}_\bullet^{cn}) \cdot \mathcal{J}(\mathfrak{A}_\bullet^{cm}) \tag{11}$$

for any graded sequence of coherent ideal sheaves \mathfrak{A}_\bullet .

Suppose now that $L \rightarrow X$ is a big line bundle. For n large enough, the space of global sections $H^0(X, L^{\otimes n})$ is non-zero, and we may consider the base ideal $\mathfrak{b}_n(L)$ which is the coherent ideal sheaf locally defined by the vanishing of global sections of $L^{\otimes n}$. The sequence $\mathfrak{b}_\bullet(L) := \{\mathfrak{b}_n(L)\}$ forms a graded sequence of ideal sheaves and Nadel’s vanishing theorem implies the following theorem, see [Laz04, Corollary 11.2.13].

THEOREM 2.2. *Fix any very ample line bundle A on X . For any big line bundle $L \rightarrow X$, and for any $m \geq 1$, the sheaf*

$$\mathcal{O}_X(K_X + (d + 1)A) \otimes L^m \otimes \mathcal{J}(\mathfrak{b}_\bullet^m) \tag{12}$$

is globally generated.

2.2 Nef envelopes

In this section, we gather some facts on the notion of nef envelopes following [BFJ09, BdFF12, KM13].

Let us fix any big Cartier b -divisor class $\beta \in \text{c-Nef}^1(\mathcal{X})$. Consider the set $\mathcal{D}(\beta)$ of classes $\gamma \in \text{c-Nef}^1(\mathcal{X})_{\mathbb{Q}}$ such that $\gamma \leq \beta$. By [BFJ09, Lemma 2.6] or [KM13, Lemma 3.4], for any $\gamma_1, \gamma_2 \in \mathcal{D}(\beta)$ one can find a third element $\gamma_3 \in \mathcal{D}(\beta)$ such that $\gamma_3 \geq \gamma_1$ and $\gamma_3 \geq \gamma_2$. For any model X' over X , we may thus define $P(\beta)_{X'}$ as the least upper bound of the set $\{\gamma_{X'} \in \text{Nef}^1(X'), \gamma \in \mathcal{D}(\beta)\}$ (see [BFJ09, Lemma 2.7]). The collection of classes $\{P(\beta)_{X'}\}$ defines a nef b -divisor class (called the nef envelope of β) such that for any $\gamma \in \text{c-Nef}^1(\mathcal{X})$ satisfying $\gamma \leq \beta$, we have $\gamma \leq P(\beta)$.

LEMMA 2.3. For any $\gamma \in \text{Nef}^1(\mathcal{X})$ such that $\gamma \leq \beta$, we have $\gamma \leq P(\beta)$; and the nef envelope is the least nef class satisfying this property (in particular, this class is unique).

Remark 2.4. The nef envelope coincides with the positive product intersection $\langle \beta \rangle$ of [BFJ09, §2.2]. It is also the nef part of the Zariski decomposition defined in [KM13].

Proof. Suppose $\beta \geq \gamma \in \text{Nef}^1(\mathcal{X})$, and β is determined in X' . Then $\beta_{X'} \geq \gamma_{X'}$ and by Lemma 1.4 there exists a net $\gamma_i \in \text{c-Nef}^1(\mathcal{X})$ such that $\gamma_{i,X'} \rightarrow \gamma_{X'}$. As $\beta_{X'}$ is big, we have $\gamma_{i,X'} \leq (1 + \epsilon)\beta_{X'}$ for any $\epsilon > 0$ and for $i \in I_\epsilon$ in a tail set. We infer from Lemma 1.5 $\gamma_i \leq [\gamma_{i,X'}] \leq (1 + \epsilon)\beta$, hence $\gamma_i \leq (1 + \epsilon)P(\beta)$ for $i \in I_\epsilon$. We conclude that $\gamma \leq P(\beta)$ by taking the limit of the net and then $\epsilon \rightarrow 0$.

Suppose that β' is a nef b -divisor class such that $\gamma \leq \beta'$ for any $\gamma \in \text{Nef}^1(\mathcal{X})$ satisfying $\gamma \leq \beta$. Applying these inequalities to Cartier b -divisor classes, we get $P(\beta)_{X'} \leq \beta'_{X'}$ in each model, which concludes the proof. □

We shall use the following characterization of the nef envelope.

PROPOSITION 2.5. Let $L \rightarrow X$ be any big line bundle. Then

$$P([c_1(L)]) = \lim_{n \rightarrow \infty} [c_1(L)] - \frac{1}{n}[\mathfrak{b}_n(L)]$$

where $\mathfrak{b}_n(L)$ denotes, as above, the base ideal of the line bundle $L^{\otimes n}$.

See [KM13, Proposition 3.7] for the proof.

2.3 Proof of Theorem 2.1

We fix any Cartier b -divisor class ω determined by an ample class in X . We rely on the following proposition whose proof is given below.

PROPOSITION 2.6. For any nef b -divisor class α , and any big class $\gamma \in \text{N}^1(X')$ in a model X' , there exists a nef Cartier b -divisor class β such that

$$\alpha \leq \beta \quad \text{and} \quad \beta_{X'} \leq \alpha_{X'} + \gamma.$$

Pick any class $\alpha \in \text{Nef}^1(\mathcal{X})$. Denote by I the set of all nef Cartier b -divisor classes β such that $\beta \geq \alpha + \epsilon\omega$ for some $\epsilon > 0$. We endow the set I with a partial order by declaring that β is less or equal to β' iff $\beta' \leq \beta$.

Let us first justify that I is an inductive set. Pick any two nef Cartier b -divisor classes $\beta, \beta' \in I$. We need to exhibit a class $\beta'' \in I$ such that $\beta'' \leq \beta$ and $\beta'' \leq \beta'$.

We may suppose that both classes are determined by a class in the same model X' and that $\beta \geq \alpha + \epsilon\omega$ and $\beta' \geq \alpha + \epsilon\omega$ for the same constant $\epsilon > 0$. By Proposition 2.6, there exists a nef Cartier b -divisor class β'' such that $\alpha + (\epsilon/2)\omega \leq \beta''$ and $\beta''_{X'} \leq \alpha_{X'} + \epsilon\omega_{X'}$. By Lemma 1.5, this implies $\beta'' \leq \beta$ and $\beta'' \leq \beta'$ proving that I is an inductive set.

In order to make a distinction between nef classes and the poset I parameterizing the approximants of α , let us denote by β_i the class associated with the element $i \in I$. Pick any model X' , and choose an open neighborhood U of $\alpha_{X'}$ in $\text{N}^1(X')$. One can find $\epsilon > 0$ such that any class $\gamma \in \text{N}^1(X')$ satisfying $\alpha_{X'} \leq \gamma \leq \alpha_{X'} + 3\epsilon\omega_{X'}$ belongs to U . By Proposition 2.6, there exists a nef Cartier b -divisor class β_i such that $\alpha + \epsilon\omega \leq \beta_i$ and $\beta_{i,X'} \leq \alpha_{X'} + 3\epsilon\omega_{X'}$. For any index $j \geq i$, we have

$$\alpha_{X'} \leq (\beta_j)_{X'} \leq (\beta_i)_{X'} \leq \alpha_{X'} + 3\epsilon\omega_{X'},$$

hence $(\beta_j)_{X'}$ belongs to U . This proves the net β_i is converging to α . We conclude by observing that the net is decreasing by construction.

Proof of Proposition 2.6. As the big cone is open in $N^1(X')$, one can find a big line bundle $L \rightarrow X'$ and an integer $l \in \mathbb{N}$ such that $\alpha_{X'} \leq (1/l)c_1(L) \leq \alpha_{X'} + \gamma/2$. By Proposition 2.7, there exists a sequence of nef Cartier b -divisor classes β_n decreasing to $P((1/l)c_1(L))$. Set $\beta := \beta_N$ for a sufficiently large integer N so that $\beta_{X'} \leq \alpha_{X'} + \gamma$. We infer from Lemma 2.3 that $\beta \geq P((1/l)c_1(L)) \geq \alpha$ and the proof is complete. \square

PROPOSITION 2.7. *Let $L \rightarrow X$ be any big line bundle and set $\alpha := [c_1(L)] \in c\text{-}N^1(\mathcal{X})$. Then there exists a non-increasing sequence of Cartier b -divisor classes $\beta_n \in c\text{-}Nef^1(\mathcal{X})$ converging to the envelope $P(\alpha)$. Moreover, for any $\epsilon > 0$ we have $P(\alpha) \leq \beta_n \leq (1 + \epsilon)P(\alpha)$ for all n large enough.*

Proof of Proposition 2.7. Replacing L by one of its multiple, we may (and shall) find an effective divisor D such that $L = \mathcal{O}_X(D)$. Let $\mathbf{b}_\bullet := \{\mathbf{b}_n(L)\}$ be the graded sequence of ideal sheaves defined by the base loci of $L^{\otimes n}$. Fix any very ample line bundle A on X such that $K_X + (d + 1)A$ is effective. By Theorem 2.2, the sheaf

$$\mathcal{O}_X(K_X + (d + 1)A + mD) \otimes \mathcal{J}(\mathbf{b}_\bullet^m)$$

is globally generated so that we may find a nef divisor D_m on a log-resolution X_m of the sheaf of ideals $\mathcal{J}(\mathbf{b}_\bullet^m)$ such that

$$\mathcal{O}_{X_m}(K_X + (d + 1)A + mD) \otimes \mathcal{J}(\mathbf{b}_\bullet^m) = \mathcal{O}_{X_m}(D_m).$$

If β'_m denotes the nef Cartier b -divisor class determined by D_m in X_m , then the incarnation of β'_m in X is less effective than $K_X + (d + 1)A + mD$, hence

$$\frac{1}{m}\beta'_m \leq P\left(\alpha + \frac{[K_X + (d + 1)A]}{m}\right) \leq P((1 + \epsilon)\alpha) \tag{13}$$

for any fixed $\epsilon > 0$, and any m large enough.

Now by the sub-additivity of multiplier ideals (11), we have $\mathcal{J}(\mathbf{b}_\bullet^{n+m}) \subset \mathcal{J}(\mathbf{b}_\bullet^n) \cdot \mathcal{J}(\mathbf{b}_\bullet^m)$, hence $-\mathcal{J}(\mathbf{b}_\bullet^{n+m}) \leq -\mathcal{J}(\mathbf{b}_\bullet^n) - \mathcal{J}(\mathbf{b}_\bullet^m)$ and the sequence $-(1/2^m)[\mathcal{J}(\mathbf{b}_\bullet^{2^m})]$ is decreasing. It follows that $\beta_n := (1/2^n)\beta'_{2^n}$ is also decreasing. By (10), we have $\mathbf{b}_m \subset \mathcal{J}(\mathbf{b}_\bullet^m)$, and we infer using the effectivity of the class $K_X + (d + 1)A$

$$\beta_n = \frac{1}{2^n}[K_X + (d + 1)A] + \alpha - \frac{1}{2^n}[\mathcal{J}(\mathbf{b}_\bullet^{2^n})] \geq \alpha - \frac{1}{2^n}[\mathbf{b}_{2^n}].$$

By Proposition 2.5, we have $\alpha - (1/2^m)[\mathbf{b}_{2^m}] \rightarrow P(\alpha)$ and combining this information with (13) we conclude that $\beta_n \rightarrow P(\alpha)$. \square

2.4 Application of the approximation theorem

Our next result is an analog of [BFJ08b, Proposition 2.2], [BFJ16, Theorem 5.11], [BJ18, Theorem 5.19], [GM19, Theorem 1.3], and [BBG22, Lemma 4.24].

THEOREM 2.8. *We have*

$$c\text{-}N^1(\mathcal{X}) \cap Nef^1(\mathcal{X}) = c\text{-}Nef^1(\mathcal{X}).$$

In other words any Cartier numerical b -divisor class lying in the weak closure of nef classes is determined by a nef class in some model.

Proof. Pick any $\alpha \in c\text{-}N^1(\mathcal{X}) \cap Nef^1(\mathcal{X})$. Without loss of generality, we may suppose that α is determined in X , and we need to show that α_X is nef. By adding a small real ample class of X , we may suppose that $\alpha_X = c_1(L)$ for some big line bundle $L \rightarrow X$. As α is nef,

we have $P(\alpha_X) = \alpha$. Redoing the proof of Proposition 2.7, we find a sequence of elements

$$\beta_m = \frac{1}{2^m} \beta'_{2^m} = \frac{1}{2^m} [K_X + (d + 1)A] + \alpha - \frac{1}{2^m} [\mathcal{J}(\mathfrak{b}_\bullet^{2^m})] \in \text{c-Nef}^1(\mathcal{X}),$$

which converges weakly to α . This implies $-(1/2^m)[\mathcal{J}(\mathfrak{b}_\bullet^{2^m})]$ to tend to 0 weakly. As the sequence $-(1/2^m)[\mathcal{J}(\mathfrak{b}_\bullet^{2^m})] \leq 0$ is decreasing, we obtain $\mathcal{J}(\mathfrak{b}_\bullet^{2^m}) = 0$ for all m , and β_m is thus a nef class determined in X . This implies α_X to be nef as required. \square

Remark 2.9. We prove an analogous statement for BPF curve b -numerical classes, see Proposition 5.4. The equality $\text{c-N}^k(\mathcal{X}) \cap \text{BPF}^k(\mathcal{X}) = \text{c-BPF}^k(\mathcal{X})$ remains however unclear when $2 \leq k \leq d - 2$.

3. Intersection of nef b -divisor classes

3.1 Definition

PROPOSITION-DEFINITION 3.1. *Let $\alpha_1, \dots, \alpha_k$ be any family of nef b -divisor classes, and pick $\gamma \in \text{BPF}^l(\mathcal{X})$. Then there exists a unique class $\Delta \in \text{BPF}^{k+l}(\mathcal{X})$ satisfying the following properties.*

(i) *For any nef Cartier b -divisor classes $\beta_i \geq \alpha_i$, we have*

$$\Delta \leq (\beta_1 \cdot \beta_2 \cdot \dots \cdot \beta_k \cdot \gamma). \tag{14}$$

(ii) *For any net of nef Cartier b -divisor classes $\beta_1^{(i)}, \dots, \beta_k^{(i)}$ decreasing to $\alpha_1, \dots, \alpha_k$, we have*

$$(\beta_1^{(i)} \cdot \beta_2^{(i)} \cdot \dots \cdot \beta_k^{(i)} \cdot \gamma) \downarrow \Delta. \tag{15}$$

This intersection product was defined in [BFJ09, §2] following the construction in the complex setting of positive intersection products introduced by Boucksom in [Bou02, §3.2], see also [BDPP13]. It bears a strong analogy with the definition of wedge products of currents $dd^c u_i$ in complex geometry.

Proof of Proposition–Definition 3.1. By Theorem 2.1 one can find a net of nef Cartier b -divisor classes $\alpha_l^{(i)} \downarrow \alpha_l$ for all $l = 1, \dots, k$. As γ is BPF, for any $j \geq i$ we have

$$(\alpha_1^{(i)} \cdot \alpha_2^{(i)} \cdot \dots \cdot \alpha_k^{(i)} \cdot \gamma) \geq (\alpha_1^{(j)} \cdot \alpha_1^{(i)} \cdot \dots \cdot \alpha_k^{(i)} \cdot \gamma) \geq (\alpha_1^{(j)} \cdot \alpha_1^{(j)} \cdot \dots \cdot \alpha_k^{(j)} \cdot \gamma),$$

hence the net $(\alpha_1^{(i)} \cdot \alpha_2^{(i)} \cdot \dots \cdot \alpha_k^{(i)} \cdot \gamma)$ is decreasing. Let Δ be its (weak) limit.

To prove (14), take nef Cartier b -divisor classes $\beta_l \geq \alpha_l$, $l = 1, \dots, k$. Choose a model X' in which all classes β_1, \dots, β_k are determined. Fix any Cartier b -divisor class ω determined in X by an ample class and a positive constant $C > 0$ such that $\beta_l \leq C\omega$ for all $l = 1, \dots, k$. Pick any $\epsilon > 0$. We have $(\alpha_l^{(i)})_{X'} \rightarrow \alpha_{l,X'}$ in $N^1(X')$, hence $(\alpha_l^{(i)})_{X'} \leq \alpha_{l,X'} + \epsilon\omega_{X'} \leq (\beta_l)_{X'} + \epsilon\omega_{X'}$ for all i large enough, hence $\alpha_l^{(i)} \leq \beta_l + \epsilon\omega_{X'}$ by Lemma 1.5. This proves

$$\Delta \leq (\alpha_1^{(i)} \cdot \alpha_2^{(i)} \cdot \dots \cdot \alpha_k^{(i)} \cdot \gamma) \leq (\beta_1 \cdot \beta_2 \cdot \dots \cdot \beta_k \cdot \gamma) + \epsilon k C^{k-1} (\omega^k \cdot \gamma)$$

and we conclude by letting $\epsilon \rightarrow 0$.

Finally, pick any net of nef Cartier b -divisor classes $\beta_1^{(i)}, \dots, \beta_k^{(i)}$ decreasing to $\alpha_1, \dots, \alpha_k$, and let Δ' be the limit of the net of decreasing classes $(\beta_1^{(i)} \cdot \beta_2^{(i)} \cdot \dots \cdot \beta_k^{(i)} \cdot \gamma)$. By (14) we have $\Delta \leq \Delta'$. However, the same argument applies when reversing the role of the nets $\alpha_l^{(i)}$ and $\beta_l^{(i)}$ which implies $\Delta = \Delta'$, and proves (15).

We conclude by observing that because any nef b -divisor class is the limit of a decreasing net of nef Cartier b -divisor classes, (14) and (15) characterize a unique class. \square

The following result is the main statement of the section. It can be viewed as a vague ‘algebraic’ analog of Bedford and Taylor’s intersection theory.

THEOREM 3.2.

- (i) *The intersection product of nef divisor classes coincides with the usual intersection when all classes are Cartier. It is non-negative, symmetric, multilinear in the variables $\alpha_1, \dots, \alpha_k \in \text{Nef}(\mathcal{X})$, linear in the variable γ and increasing in each variable.*
- (ii) *The intersection product of nef classes is upper-semicontinuous: for any net of divisor classes such that $\alpha_i^{(j)} \rightarrow \alpha_i$ in the weak topology, then we have*

$$\limsup_j (\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) \leq (\alpha_1 \cdots \alpha_k \cdot \gamma)$$

for any class $\gamma \in \text{BPF}^{d-k}(\mathcal{X})$.

- (iii) *The intersection product of nef divisor classes is continuous along decreasing sequences: if $\alpha_i^{(j)}$ is a net of divisor classes decreasing to α_i , then*

$$\lim_j (\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) = (\alpha_1 \cdots \alpha_k \cdot \gamma)$$

in $N_+^{k+l}(\mathcal{X})$ where $\gamma \in \text{BPF}^l(\mathcal{X})$.

- (iv) *The intersection product of nef classes is continuous in the $\|\cdot\|_\omega$ -norm. In other words, if $\|\alpha_i^{(j)} - \alpha_i\|_\omega \rightarrow 0$, then we have*

$$\lim_j (\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) = (\alpha_1 \cdots \alpha_k \cdot \gamma)$$

in $N_+^{k+l}(\mathcal{X})$.

Example 3.3. This intersection product is not continuous with respect to the weak topology. Take $X = \mathbb{P}^2$ and a general line L and choose a sequence of point $p_n \in \mathbb{P}^2$ converging to the generic point (i.e. for any curve C , the set $\{n, p_n \in C\}$ is finite). For each n , consider the blow-up $\pi_n: X_n \rightarrow \mathbb{P}^2$ at p_n , and let α_n be the nef Cartier class attached to $\pi_n^*(c_1(\mathcal{O}(1))) - [E_n]$ where E_n is the exceptional divisor of π_n . Then the sequence α_n converges weakly to the Cartier class attached to $c_1(\mathcal{O}(1))$ as $n \rightarrow +\infty$, but $\alpha_n^2 = 0$ whereas $c_1(\mathcal{O}(1))^2 = 1$.

Proof. Compatibility with the intersection of Cartier b -divisor classes is obvious from the definition. The symmetry and multilinearity follows from the corresponding properties for the intersection product of divisors. The fact that it is increasing is a consequence of the observation that $(\alpha'_1 \cdot \alpha_2 \cdots \alpha_k \cdot \gamma) \geq (\alpha_1 \cdot \alpha_2 \cdots \alpha_k \cdot \gamma)$ as soon as $\alpha'_1 \geq \alpha_1$. These arguments prove part (i).

We fix a class $\omega \in \text{Nef}^1(\mathcal{X})$ determined by an ample class in X , and prove part (ii). Suppose $\alpha_l^{(i)} \rightarrow \alpha_l$ weakly. Pick any class Δ in the accumulation locus of the net $(\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma)$. We may replace the net I by one indexed by the family of open neighborhoods of Δ (endowed with the partial ordering given by the inclusion) and assume that $(\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) \rightarrow \Delta$.

Let $\beta_l^{(i)}$ be nets of nef Cartier b -divisor classes decreasing to α_l . Fix any large index i , and choose a model X' in which $\beta_l^{(i)}$, $l = 1, \dots, k$ are all determined. As $(\alpha_l^{(j)})_{X'} \rightarrow (\alpha_l)_{X'}$ in $N^1(X')$ we have $(\alpha_l^{(j)})_{X'} \leq (\beta_l^{(i)})_{X'} + \epsilon\omega$ for an arbitrary small $\epsilon > 0$ and all j large enough, hence $\alpha_l^{(j)} \leq \beta_l^{(i)} + \epsilon\omega$ by Lemma 1.5. We infer

$$(\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) \leq (\beta_1^{(i)} \cdots \beta_k^{(i)} \cdot \gamma) + \epsilon k C^{k-1} (\omega^k \cdot \gamma),$$

where $C > 0$ is a any constant such that $\beta_l^{(i_0)} \leq C\omega$, $l = 1, \dots, k$, and i_0 is a fixed index. We now let successively j tend to infinity in the net I , then $\epsilon \rightarrow 0$, and finally let l tend to infinity in I . We conclude that $\Delta \leq (\alpha_1 \cdots \alpha_k \cdot \gamma)$ as required.

Now suppose that $\alpha_i^{(j)}$ is a net of nef b -divisor classes (not necessarily Cartier) decreasing to α_i . We claim that there exists a constant $C > 0$ such that for any $\epsilon > 0$ one has

$$(\alpha_1 \cdots \alpha_k \cdot \gamma) \leq (\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) \leq (\alpha_1 \cdots \alpha_k \cdot \gamma) + \epsilon k C^{k-1} (\omega^k \cdot \gamma)$$

for all j large enough. Indeed, the first inequality is a consequence of the fact that the intersection product is increasing in each variable, and the second inequality follows from the previous proof. These estimates imply the convergence $(\alpha_1^{(j)} \cdots \alpha_k^{(j)} \cdot \gamma) \rightarrow (\alpha_1 \cdots \alpha_k \cdot \gamma)$ to hold in $N_+^{k+l}(\mathcal{X})$ which proves part (iii).

Finally, we observe that if α and α' are two nef classes such that $\|\alpha - \alpha'\|_\omega \leq \epsilon$, then $-\epsilon\omega \leq (\alpha - \alpha') \leq \epsilon\omega$ so that part (iv) follows from part (iii). □

Proposition 1.1 together with Theorem 2.1 yield the next two results.

COROLLARY 3.4. *There exists a constant $C_d > 0$ depending only on d such that for any set of classes $\alpha_1, \dots, \alpha_k, \beta \in \text{Nef}^1(\mathcal{X})$ with $(\beta^d) > 0$ and any $\gamma \in \text{BPF}^l(\mathcal{X})$ we have*

$$(\alpha_1 \cdots \alpha_k \cdot \gamma) \leq C_d \frac{(\alpha_1 \cdots \alpha_k \cdot \gamma \cdot \beta^{d-k-l})}{(\beta^d)} \beta^{k+l}.$$

We obtain a ‘numerical’ or cohomological version of the Chern–Levine–Nirenberg inequalities [CLN69] and [Dem97, (3.3)].

COROLLARY 3.5. *There exists a constant $C_{d,\omega} > 0$ depending only on d and on the choice of ω such that for any $\alpha_1, \dots, \alpha_k \in \text{Nef}^1(\mathcal{X})$ and for any $\gamma \in \text{BPF}^{d-k}(\mathcal{X})$, we have*

$$0 \leq (\alpha_1 \cdots \alpha_k \cdot \gamma) \leq C_{d,\omega} \|\alpha_1\|_{\text{BPF}} \cdots \|\alpha_k\|_{\text{BPF}} \|\gamma\|_{\text{BPF}}.$$

3.2 Diskant inequalities and applications

THEOREM 3.6. *Pick any two classes $\alpha, \beta \in \text{Nef}^1(\mathcal{X})$ such that $(\alpha^d) > 0$ and $(\beta^d) > 0$ (i.e. both classes α, β are big). Let $s > 0$ be the largest positive number such that $\alpha - s\beta \geq 0$. Then we have*

$$(\alpha^{d-1} \cdot \beta)^{d/(d-1)} - (\alpha^d)(\beta^d)^{1/(d-1)} \geq ((\alpha^{d-1} \cdot \beta)^{1/(d-1)} - s(\beta^d)^{1/(d-1)})^d. \tag{16}$$

Proof. Pick any two nets $\alpha_j, \beta_j \in \text{c-Nef}^1(\mathcal{X})$ parametrized by an inductive set J and decreasing to α and β , respectively. Fix any $\epsilon > 0$.

Observe that $(\alpha_j^d) \geq (\alpha^d) > 0$ and $(\beta_j^d) \geq (\beta^d) > 0$ so that α_j and β_j are big. The largest non-negative number s_j such that $\alpha_j - s_j\beta_j \geq 0$ is, thus, positive. By [BFJ09, Theorem F], we have

$$(\alpha_j^{d-1} \cdot \beta_j)^{d/(d-1)} - (\alpha_j^d)(\beta_j^d)^{1/(d-1)} \geq ((\alpha_j^{d-1} \cdot \beta_j)^{1/(d-1)} - s_j(\beta_j^d)^{1/(d-1)})^d.$$

In other words, $s_j \geq \tau_j$ where

$$\tau_j = \frac{1}{(\beta_j^d)^{1/(d-1)}} (\alpha_j^{d-1} \cdot \beta_j)^{1/(d-1)} - \frac{1}{(\beta_j^d)^{1/(d-1)}} ((\alpha_j^{d-1} \cdot \beta_j)^{d/(d-1)} - (\alpha_j^d)(\beta_j^d)^{1/(d-1)})^{1/d}.$$

By Theorem 3.2(iii), we have

$$\lim_J \tau_j = \tau := \frac{1}{(\beta^d)^{1/(d-1)}} (\alpha^{d-1} \cdot \beta)^{1/(d-1)} - \frac{1}{(\beta^d)^{1/(d-1)}} ((\alpha^{d-1} \cdot \beta)^{d/(d-1)} - (\alpha^d)(\beta^d)^{1/(d-1)})^{1/d},$$

and we obtain $\alpha - \tau\beta \geq 0$ as required. □

We now complete the proof of Theorem A.

Proof of Theorem A. Let α be any nef b -divisor class. By adding $(1/n)\omega$ with ω a fixed Cartier b -divisor class determined in X by an ample divisor, we may suppose that α is big so that $(\alpha^d) > 0$.

Let α_j be a net of nef Cartier b -divisor classes decreasing to α . Diskant inequalities yield $\alpha \geq s_j \alpha_j$ with

$$s_j := \frac{1}{(\alpha_j^d)^{1/(d-1)}} (\alpha^{d-1} \cdot \alpha_j)^{1/(d-1)} - \frac{1}{(\alpha_j^d)^{1/(d-1)}} ((\alpha^{d-1} \cdot \alpha_j)^{d/(d-1)} - (\alpha^d)(\alpha_j^d)^{1/(d-1)})^{1/d}.$$

As $(\alpha_j^d) \rightarrow (\alpha^d)$, and $(\alpha^{d-1} \cdot \alpha_j) \rightarrow (\alpha^d)$, we may extract a subsequence i_n such that α_{i_n} is still decreasing, and $s_{i_n} \rightarrow 1$ which implies $\alpha_{i_n} \rightarrow \alpha$. □

The same arguments yield the following result.

COROLLARY 3.7. *For any class $\alpha \in \text{Nef}^1(\mathcal{X})$ such that $(\alpha^d) > 0$ and for any $\epsilon > 0$, there exists a class $\alpha' \in \text{c-Nef}^1(\mathcal{X})$ such that $(1 - \epsilon)\alpha' \leq \alpha \leq \alpha'$.*

This corollary allows one to obtain better continuity properties of the intersection product.

COROLLARY 3.8. *For any $\alpha_1, \dots, \alpha_k \in \text{Nef}^1(\mathcal{X})$, the map*

$$\gamma \in \text{BPF}^l(\mathcal{X}) \mapsto (\alpha_1 \cdot \alpha_2 \cdots \alpha_k \cdot \gamma) \in \text{BPF}^{k+l}(\mathcal{X})$$

is weakly continuous.

Proof. When α_i are Cartier, the result is clear. By multilinearity and by replacing α_i by $\alpha_i + \omega$, we may suppose that $(\alpha_i^d) > 0$ for all i . Pick any $\epsilon > 0$. By the previous corollary, one can find $\alpha'_i \in \text{c-Nef}^1(\mathcal{X})$ such that $(1 - \epsilon)\alpha'_i \leq \alpha_i \leq \alpha'_i$. It follows that

$$\begin{aligned} 0 \leq (\alpha'_1 \cdots \alpha'_k \cdot \gamma) - (\alpha_1 \cdots \alpha_k \cdot \gamma) &\leq ((1 - \epsilon)^{-k} - 1)(\alpha_1 \cdots \alpha_k \cdot \gamma) \\ &\leq C\epsilon(\omega^k \cdot \gamma). \end{aligned}$$

Suppose $\gamma_i \rightarrow \gamma$ is a net of classes converging weakly in $\text{BPF}^l(\mathcal{X})$. Then

$$\begin{aligned} \liminf_i (\alpha_1 \cdots \alpha_k \cdot \gamma_i) &\geq \liminf_i (\alpha'_1 \cdots \alpha'_k \cdot \gamma_i) - C\epsilon(\omega^k \cdot \gamma_i) \\ &= (\alpha'_1 \cdots \alpha'_k \cdot \gamma) - C\epsilon(\omega^k \cdot \gamma) \geq (\alpha_1 \cdots \alpha_k \cdot \gamma) - C\epsilon(\omega^k \cdot \gamma), \end{aligned}$$

which implies $\liminf_i (\alpha_1 \cdots \alpha_k \cdot \gamma_i) \geq (\alpha_1 \cdots \alpha_k \cdot \gamma)$.

We conclude using Theorem 3.2(i). □

Recall the definition of $\text{N}_{\Sigma}^1(\mathcal{X})$ from § 1.4.

COROLLARY 3.9. *The following inclusion holds:*

$$\text{Nef}^1(\mathcal{X}) \subset \text{N}_{\Sigma}^1(\mathcal{X}).$$

More precisely, for any nef class $\alpha \in \text{Nef}^1(\mathcal{X})$ there exists a sequence of nef Cartier b -divisor classes $\alpha_n \in \text{c-Nef}^1(\mathcal{X})$ such that $\|\alpha - \alpha_n\|_{\Sigma} \rightarrow 0$.

In particular, we have following continuous injection

$$(\text{Vect}(\text{Nef}^1(\mathcal{X})), \|\cdot\|_{\text{BPF}}) \hookrightarrow (\text{N}_{\Sigma}^1(\mathcal{X}), \|\cdot\|_{\Sigma}).$$

Proof. Let α be any nef class. As any Cartier class belongs to $\text{N}_{\Sigma}^1(\mathcal{X})$, by adding ω if necessary, we may assume that $(\alpha^d) > 0$. By Corollary 3.7, for each n , one can take a nef Cartier

class α_n such that $(1 + 1/n)^{-1}\alpha_n \leq \alpha \leq \alpha_n$. Using Corollary 3.5, we infer for all integers $m \geq n$:

$$\begin{aligned} \|\alpha_n - \alpha_m\|_{\Sigma}^2 &= \sup_{\substack{\gamma \in \text{c-BPF}^{d-2}(\mathcal{X}) \\ (\gamma \cdot \omega^2)=1}} 2((\alpha_n - \alpha_m) \cdot \omega \cdot \gamma)^2 - ((\alpha_n - \alpha_m)^2 \cdot \gamma) \\ &\leq \sup_{\substack{\gamma \in \text{c-BPF}^{d-2}(\mathcal{X}) \\ (\gamma \cdot \omega^2)=1}} \frac{2}{n^2}(\alpha_m \cdot \omega \cdot \gamma)^2 + \frac{2}{n}(\alpha_m^2 \cdot \gamma) \leq \frac{C}{n} \end{aligned}$$

for some constant $C > 0$. This shows that (α_n) is a Cauchy sequence in $N_{\Sigma}^1(\mathcal{X})$. However, because it converges weakly to α we conclude that α belongs to $N_{\Sigma}^1(\mathcal{X})$ and $\|\alpha_n - \alpha\|_{\Sigma} \rightarrow 0$.

By linearity, we obtain $\text{Vect}(\text{Nef}^1(\mathcal{X})) \subset N_{\Sigma}^1(\mathcal{X})$ and the continuity follows from the fact that $\|\alpha\|_{\text{BPF}} = (\alpha \cdot \omega^{d-1})$ for all $\alpha \in \text{Nef}^1(\mathcal{X})$ by Lemma 1.7. □

We now relate the intersection product of nef divisors with the intersection product $N_{\Sigma}^1(\mathcal{X}) \times N_{\Sigma}^1(\mathcal{X}) \rightarrow N_{\text{BPF}}^{2,\vee}(\mathcal{X})$ recalled in Theorem 1.6. Denote by $\iota: \text{Nef}^1(\mathcal{X}) \rightarrow N_{\Sigma}^1(\mathcal{X})$ the natural inclusion map given by the previous corollary.

COROLLARY 3.10. *For any two nef b-divisor classes α, β , the class $\alpha \cdot \beta \in \text{BPF}^2(\mathcal{X})$ defined by Proposition–Definition 3.1 is identical to the intersection product $\iota(\alpha) \cdot \iota(\beta) \in N_{\text{BPF}}^{2,\vee}(\mathcal{X})$ given by Theorem 1.6.*

Proof. Take two sequences of nef Cartier b -divisor classes such that $\|\alpha_n - \alpha\|_{\Sigma} \rightarrow 0$ and $\|\beta_n - \beta\|_{\Sigma} \rightarrow 0$. By continuity of the intersection product in $N_{\Sigma}^1(\mathcal{X})$, we have $\iota(\alpha) \cdot \iota(\beta) = \lim_n \lim_m \alpha_n \cdot \beta_m$. By Corollary 3.8, we obtain $\lim_m \alpha_n \cdot \beta_m = \alpha_n \cdot \beta$; and then $\lim_n \alpha_n \cdot \beta = \alpha \cdot \beta$ which proves the result. □

3.3 Variations on the Hodge index theorem

THEOREM B. *Any two b-divisor classes $\alpha, \beta \in \text{Nef}^1(\mathcal{X})$ such that $\alpha \cdot \beta = 0$ are proportional.*

Proof. In view of Corollary 3.10, the statement is a direct consequence of the version of Hodge index theorem holding in $N_{\Sigma}^1(\mathcal{X})$, see [DF21, Theorem 3.13]. □

PROPOSITION 3.11. *For any two nef classes α, β , the sequence $e_k := (\alpha^k \cdot \beta^{d-k})$ is log-concave. Moreover, if α or β are nef and big, and the sequence $(\log e_k)_k$ is affine, then α and β are proportional.*

Proof. Write α and β as decreasing limits of sequences of nef Cartier classes $\alpha_n \downarrow \alpha$ and $\beta_n \downarrow \beta$. By the Khovanskii–Teissier inequalities, the sequence $(\alpha_n^k \cdot \beta_n^{d-k})$ is log-concave which implies by Theorem 3.2 the log-concavity of the sequence $e_k = (\alpha^k \cdot \beta^{d-k})$ by letting $n \rightarrow \infty$.

Suppose now that α and β are big and nef, and that the sequence e_k is linear. Scale both classes such that $(\alpha^d) = (\beta^d) = 1$ so that $(\alpha^{d-1} \cdot \beta) = 1$ too. Then $\alpha = \beta$ by Diskant’s inequalities. □

Recall from § 1.4 that given any two classes $\theta, \theta' \in N_{\Sigma}^1(\mathcal{X})$ the pairing $\alpha \cdot \beta$ belongs to $N_{\text{BPF}}^{2,\vee}(\mathcal{X})$ which is included in $N_{\text{BPF}}^{d-2}(\mathcal{X})^*$. It follows that for any class $\gamma \in \text{BPF}^{d-2}(\mathcal{X})$, the pairing $\theta \cdot \theta' \cdot \gamma \in \mathbb{R}$ is well-defined.

PROPOSITION 3.12. *Let $\gamma \in \text{BPF}^{d-2}(\mathcal{X})$, and $\alpha \in \text{Nef}^1(\mathcal{X})$. Suppose that $\alpha \cdot \gamma \neq 0$, and that there exists a sequence $\gamma_n \in \text{c-BPF}^{d-2}(\mathcal{X})$ decreasing to γ .*

Then the quadratic form

$$q_\gamma(\theta) := (\theta^2 \cdot \gamma)$$

is semi-negative on the hyperplane $H_\alpha := \{\theta \in N_\Sigma^1(\mathcal{X}), (\theta \cdot \alpha \cdot \gamma) = 0\}$. In particular, we obtain

$$\ker(q_\gamma|_{H_\alpha}) := \{\theta \in H_\alpha \mid \forall v \in H_\alpha, (\theta \cdot v \cdot \gamma) = 0\} = \{\theta \in H_\alpha \mid q_\gamma(\theta) = 0\}.$$

Remark 3.13. We believe the assumption on the existence of a sequence of Cartier BPF classes approximating γ to be superfluous. Note that it is satisfied when γ is a positive linear combination of intersections of nef classes or when the codimension of γ is one by Theorem A.

Proof. Observe that the last statement is a consequence of Cauchy–Schwarz inequality. Let us prove that q is semi-negative on H_α .

Pick any decreasing sequence $\alpha_n \in \text{c-Nef}^1(\mathcal{X})$ converging to α . Pick any sequence of models $X_{n+1} \geq X_n$ such that for each n , the classes γ_n and α_n are determined in X_n . By perturbing slightly γ_n , we may assume it is represented by a surface Σ in X_n . We may also assume that $\gamma_n \leq \omega^{d-2}$ for some fixed ample class ω in X . Note that

$$0 < c := (\omega \cdot \alpha \cdot \gamma) \leq (\omega \cdot \alpha_n \cdot \gamma_n) \leq C := (\omega \cdot \alpha_1 \cdot \gamma_1).$$

Pick any class $\theta \in N_\Sigma^1(\mathcal{X})$ such that $(\theta \cdot \alpha \cdot \gamma) = 0$. Take any sequence of Cartier classes $\theta^{(j)} \rightarrow \theta$ in $N_\Sigma^1(\mathcal{X})$. By Theorem 1.6, we have $(\theta^{(j)} \cdot \beta) \rightarrow (\theta \cdot \beta)$ in $N_{\text{BPF}}^{2,\vee}(\mathcal{X})$ for any class $\beta \in \text{Nef}^1(\mathcal{X})$.

Fix $\epsilon > 0$, and take n large enough such that $|(\theta \cdot \alpha_n \cdot \gamma_n)| \leq \epsilon$. For all $j \gg 1$, we have $|(\theta^{(j)} \cdot \alpha_n \cdot \gamma_n)| \leq 2\epsilon$. Set $\tilde{\theta}_j = \theta^{(j)} - \eta_j \omega$ so that $(\tilde{\theta}_j \cdot \alpha_n \cdot \gamma_n) = 0$, and observe that

$$0 \leq |\eta_j| := \frac{|(\theta^{(j)} \cdot \alpha_n \cdot \gamma_n)|}{(\omega \cdot \alpha_n \cdot \gamma_n)} \leq 2c^{-1}\epsilon.$$

By the Hodge index theorem, we obtain

$$(\tilde{\theta}_j^2 \cdot \gamma_n) = (\tilde{\theta}_j^2|_\Sigma) \leq 0.$$

As the sequence η_j is bounded, we may extract a subsequence converging to some $0 < \eta \leq 2c^{-1}\epsilon$ and letting $j \rightarrow \infty$, we obtain

$$((\theta - \eta\omega)^2 \cdot \gamma_n) \leq 0$$

We conclude by letting first $n \rightarrow \infty$ and then $\epsilon \rightarrow 0$. □

An analog of the Hodge index theorem was used by Dinh and Sibony in [DS04a] to study commutative subgroups of the group of automorphisms on compact Kähler manifolds. Following [DS04a, Corollary 3.5], we extend their result to arbitrary b -divisor classes.

THEOREM 3.14. *Pick any class $0 \neq \gamma \in \text{BPF}^k(\mathcal{X})$ such that there exists a sequence $\gamma_n \in \text{c-BPF}^k(\mathcal{X})$ decreasing to γ .*

If $\theta, \theta' \in N_\Sigma^1(\mathcal{X})$ satisfy $(\theta^2 \cdot \gamma) \geq 0$, $((\theta')^2 \cdot \gamma) \geq 0$, and $(\theta \cdot \theta' \cdot \gamma) = 0$, then there exist $(a, a') \neq (0)$ such that $(a\theta + a'\theta') \cdot \gamma = 0$.

Remark 3.15. Note that the conditions $(\theta^2 \cdot \gamma) \geq 0$, $((\theta')^2 \cdot \gamma) \geq 0$ are automatically satisfied when $\theta, \theta' \in \text{Nef}^1(\mathcal{X})$.

Proof. We may suppose that $\theta \cdot \gamma \neq 0$, $\theta' \cdot \gamma \neq 0$.

We claim that

$$\theta^2 \cdot \gamma = (\theta')^2 \cdot \gamma = 0.$$

To see this pick any class $\beta \in \text{c-BPF}^{d-k-2}(\mathcal{X})$ such that the curve classes $\theta \cdot \gamma \cdot \beta$ and $\theta' \cdot \gamma \cdot \beta$ are both non-zero. Observe that

$$q_\gamma(a\theta + a'\theta') := ((a\theta + a'\theta')^2 \cdot \gamma \cdot \beta) = a^2(\theta^2 \cdot \gamma \cdot \beta) + (a')^2((\theta')^2 \cdot \gamma \cdot \beta) \geq 0.$$

Pick any big and nef class $\alpha \in \text{c-Nef}^1(\mathcal{X})$ such that $(\theta \cdot \gamma \cdot \beta \cdot \alpha) \neq 0$, and $(\theta' \cdot \gamma \cdot \beta \cdot \alpha) \neq 0$. By Proposition 3.12, we have $q_\gamma \leq 0$ on the hyperplane $H := (\gamma \cdot \beta \cdot \alpha)^\perp$. One can thus find a pair $a, a' \neq 0$ such that $a\theta + a'\theta' \in H$ is non-zero and such that $q_\gamma(a\theta + a'\theta') = 0$, and it follows that $(\theta^2 \cdot \gamma \cdot \beta) = ((\theta')^2 \cdot \gamma \cdot \beta) = 0$. As β is arbitrary, the claim is proved.

Pick any class $\beta \in \text{c-BPF}^{d-k-2}(\mathcal{X})$ such that both curve classes $\theta \cdot \Omega$ and $\theta' \cdot \Omega$ are non-zero with $\Omega = \beta \cdot \gamma$. We shall prove that $\theta \cdot \gamma \cdot \beta = c(\beta)\theta' \cdot \gamma \cdot \beta$ for some $c(\beta) > 0$. By adding a small multiple of ω^k to γ_n and ω^{d-k-2} to β , we may assume that $\Omega_n := \gamma_n \cdot \beta$ is strongly BPF, and determined in some model by a surface Σ_n which is the image under a flat map of a complete intersection of ample divisors. We may also impose that $\theta \cdot \Omega \cdot \omega \neq 0$.

Choose any t_n such that $(\theta_n^2 \cdot \Omega_n) = 0$ with $\theta_n = \theta + t_n\omega$. Observe that

$$\begin{aligned} 0 &= (\theta_n^2 \cdot \Omega_n) = (\theta^2 \cdot \Omega_n) + 2t_n(\omega \cdot \theta \cdot \Omega_n) + t_n^2(\omega^2 \cdot \Omega_n) \\ &= (\theta^2 \cdot (\Omega_n - \Omega)) + 2t_n(\omega \cdot \theta \cdot \Omega_n) + t_n^2(\omega^2 \cdot \Omega_n). \end{aligned}$$

As $(\omega^2 \cdot \Omega_n)$ is uniformly bounded from below by $(\omega^2 \cdot \gamma \cdot \beta)$, and $(\theta^2 \cdot \Omega_n) \sim (\theta^2 \cdot \Omega) = 0$, such a constant t_n exists and we may choose $t_n \rightarrow 0$ as $n \rightarrow \infty$.

Now pick another big and nef class ω' such that $\omega \cdot \Omega$ and $\omega' \cdot \Omega$ are not proportional, normalized by $(\omega \cdot \Omega \cdot \theta) = (\omega' \cdot \Omega \cdot \theta) \neq 0$. Choose r_n, r'_n such that

$$0 = (\theta'_n \cdot \theta_n \cdot \Omega_n), \tag{17}$$

$$0 = ((\theta'_n)^2 \cdot \Omega_n) \tag{18}$$

with $\theta'_n = \theta' + r_n\omega + r'_n\omega'$.

LEMMA 3.16. *One can find constants r_n, r'_n satisfying (17) and (18) converging to zero as $n \rightarrow \infty$.*

From $\theta_n^2 \cdot \Sigma_n = (\theta'_n)^2 \cdot \Sigma_n = \theta_n \cdot \theta'_n \cdot \Sigma_n = 0$ we infer that $\theta_n \cdot \Sigma_n$ and $\theta'_n \cdot \Sigma_n$ are proportional by the usual Hodge index theorem. Letting $n \rightarrow \infty$ proves that $\theta \cdot \gamma \cdot \beta = c(\beta)\theta' \cdot \gamma \cdot \beta$ for some $c(\beta) > 0$.

We claim that the proportionality constant $c(\beta)$ is, in fact, independent on β so that $\theta \cdot \gamma = c\theta' \cdot \gamma$ for some $c > 0$ as required.

To prove our claim we suppose first that $\beta = \omega_1 \cdot \omega_2 \cdots \omega_{d-k-2}$, and $\beta' = \omega'_1 \cdot \omega_2 \cdots \omega_{d-k-2}$ for some big and nef classes $\omega'_1, \omega_1, \dots, \omega_{d-k-2}$. Then

$$c(\beta)(\theta' \cdot \gamma \cdot \beta \cdot \omega'_1) = (\theta \cdot \gamma \cdot \beta \cdot \omega'_1) = (\theta \cdot \gamma \cdot \beta' \cdot \omega_1) = c(\beta')(\theta' \cdot \gamma \cdot \beta \cdot \omega'_1)$$

hence $c(\beta) = c(\beta')$ in this case. As flat pushforwards of complete intersection classes of ample divisors generate $\text{c-Nef}^1(\mathcal{X})$, we get the result (compare with the proof of [DF21, Theorem 3.13]). □

Proof of Lemma 3.16. Write $r = r_n$ and $r' = r'_n$. Note that (17) implies $r' = \epsilon_n - \kappa_n r$ with

$$\epsilon_n = -\frac{(\theta' \cdot \theta_n \cdot \Omega_n)}{(\omega' \cdot \theta_n \cdot \Omega_n)} \rightarrow 0$$

and

$$\kappa_n = \frac{(\omega \cdot \theta_n \cdot \Omega_n)}{(\omega' \cdot \theta_n \cdot \Omega_n)} \rightarrow 1.$$

Replacing r' in (18), we obtain the following second-degree equation:

$$\begin{aligned} & r^2 [(\omega^2 \cdot \hat{\Omega}_n) + \kappa_n^2((\omega')^2 \cdot \hat{\Omega}_n) - 2\kappa_n(\omega \cdot \omega' \cdot \hat{\Omega}_n)] \\ & + 2r [(\omega \cdot \theta' \cdot \hat{\Omega}_n) + \epsilon_n(\omega \cdot \omega' \cdot \hat{\Omega}_n) - \kappa_n(\omega' \cdot \theta' \cdot \hat{\Omega}_n)] \\ & + [((\theta')^2 \cdot \hat{\Omega}_n) + \epsilon_n^2((\omega')^2 \cdot \hat{\Omega}_n) + 2\epsilon_n(\omega' \cdot \theta' \cdot \hat{\Omega}_n)] = 0 \end{aligned}$$

with $\hat{\Omega}_n = \theta_n \cdot \Omega_n$. As the constant term tends to zero and the leading term is bounded away from zero by Hodge index theorem (using that $\omega \cdot \Omega$ and $\omega' \cdot \Omega$ are not proportional), we may choose a solution $r_n \rightarrow 0$, and this forces $r'_n \rightarrow 0$. □

4. Minkowski’s problem and applications

Let $\omega \in c\text{-Nef}^1(\mathcal{X})$ be any Cartier b -divisor class determined by an ample class in X .

4.1 Inverting $\alpha \mapsto \alpha^{d-1}$

THEOREM C. *The map $\alpha \mapsto \alpha^{d-1}$ induces a bijection from the convex cone of big b -divisor classes $\alpha \in \text{Nef}^1(\mathcal{X})$ onto the convex cone of big b -numerical curve classes in $\text{BPF}^{d-1}(\mathcal{X})$.*

Remark 4.1. The map $\alpha \mapsto \alpha^{d-1}$ is continuous for the $\|\cdot\|_{\text{BPF}}^{\vee}$ -norm (hence, for the $\|\cdot\|_{\Sigma}$ -norm too) and along decreasing limits. It is not continuous in the weak topology (pullback Example 3.3 by a map $C \times \mathbb{P}^2 \rightarrow \mathbb{P}^2$ where C is any projective curve).

Proof. Let us prove that $M(\alpha) := \alpha^{d-1}$ is injective on the cone of big and nef classes. Recall that by Corollary 3.4, a nef class α satisfies $\alpha \geq c\omega$ for some $c > 0$ iff $(\alpha^d) > 0$.

If α and β are two nef classes such that $\alpha^{d-1} = \beta^{d-1}$, then $(\alpha^d) = (\alpha \cdot \beta^{d-1})$, and $(\alpha^{d-1} \cdot \beta) = (\beta^d)$. By the log-concavity of the sequence $k \mapsto \log(\alpha^k \cdot \beta^{d-k})$ we obtain $0 < (\alpha^d) = (\beta^d) = (\alpha^k \cdot \beta^{d-k})$ for all k . This implies the equality between α and β by Proposition 3.11.

We now prove the surjectivity. Fix any class $\gamma \in \text{BPF}^{d-1}(\mathcal{X})$ such that $\gamma \geq c\omega^{d-1}$ for some $c > 0$. We follow the classical variational method to solve a Monge-Ampère equation. Let $K := \{\alpha \in \text{Nef}^1(\mathcal{X}) \mid (\alpha \cdot \gamma) = 1\}$. Note that K is compact for the weak topology of $w\text{-N}^1(\mathcal{X})$ by [DF21, Proposition 2.13], as $K \subset \{\alpha \in \text{Nef}^1(\mathcal{X}), (\alpha \cdot \omega^{d-1}) \leq c^{-1}\}$. As the function $\alpha \mapsto (\alpha^d)$ is upper-semicontinuous by Theorem 3.2(ii), there exists a class $\alpha_{\star} \in K$ such that $(\alpha_{\star}^d) = \sup_K(\alpha^d)$.

Observe that $(\alpha_{\star}^d) > 0$. Pick any Cartier class $\beta \in c\text{-N}^1(\mathcal{X})$. For t small enough, the class $\alpha_{\star} + t\beta$ is psef by Siu’s inequality. Consider the nef envelope $P(\alpha_{\star} + t\beta)$ as defined in § 2.2.

Pick any big and nef class $\omega' \in c\text{-Nef}^1(\mathcal{X})$ such that $\alpha_{\star} \leq \omega'$, and $\omega' \pm \beta$ are both nef (any sufficiently large ample class in a model in which β is determined does the job). Using [BFJ09, Corollary 3.4] and by taking limits of Cartier nef classes decreasing to α_{\star} , we have

$$P(\alpha_{\star} + t\beta)^d \geq (\alpha_{\star}^d) + dt(\alpha_{\star}^{(d-1)} \cdot \beta) - Ct^2$$

for some $C > 0$ which depends only on (ω^d) and for any $0 \leq t \leq 1$. Observe now that

$$\alpha_t := \frac{P(\alpha_{\star} + t\beta)}{(P(\alpha_{\star} + t\beta) \cdot \gamma)} \in K$$

which implies

$$\begin{aligned} (\alpha_{\star}^d) & \geq \frac{P(\alpha_{\star} + t\beta)^d}{(P(\alpha_{\star} + t\beta) \cdot \gamma)^d} \geq \frac{(\alpha_{\star}^d) + dt(\alpha_{\star}^{(d-1)} \cdot \beta) - Ct^2}{((\alpha_{\star} + t\beta) \cdot \gamma)^d} \\ & \geq (\alpha_{\star}^d) + dt(\alpha_{\star}^{(d-1)} \cdot \beta) - dt(\beta \cdot \gamma)(\alpha_{\star}^d) + O(t^2), \end{aligned}$$

hence $(\alpha_\star^{(d-1)} \cdot \beta) \leq (\beta \cdot \gamma)(\alpha_\star^d)$. Since this is true for $\pm\beta$ we infer $(\alpha_\star^{(d-1)} \cdot \beta) = (\beta \cdot \gamma)(\alpha_\star^d)$. Rescaling α_\star , we find a nef class such that $(\alpha^{d-1}) = \gamma$ as required. \square

4.2 An example of the $(d - 1)$ th root of a Cartier class which is not Cartier

For the sake of completeness, we include the following clever observation of Li [Li21, Lemma 4.2]. We are indebted to Boucksom and Jonsson for pointing this reference to us, and for letting us include it here.

PROPOSITION 4.2 (Li). *For any big line bundle $L \rightarrow X$, the b -numerical curve class*

$$P([c_1(L)])^{d-1} \in \text{BPF}^{d-1}(\mathcal{X})$$

is Cartier.

Proof. Let $\pi : X' \rightarrow X$ be any composition of blow-ups of smooth centers, and let E be any π -exceptional prime divisor in X' . It follows from [BFJ09, Theorem B] that

$$P([\pi^*c_1(L)])^{d-1} \cdot E = \lim_{n \rightarrow \infty} \frac{(d-1)!}{n^{d-1}} \dim_{\mathbb{C}}(V_n),$$

where

$$V_n := \text{Im}(H^0(X', n\pi^*L) \rightarrow H^0(E, n\pi^*L|_E)).$$

Observe that

$$V_n \subset H^0(E, n\pi^*L|_E) = H^0(\pi(E), nL|_{\pi(E)})$$

and because $\dim(\pi_*(E)) < d - 1$, the asymptotic Riemann–Roch theorem implies $\dim_{\mathbb{C}}(V_n) = o(n^{d-1})$, hence $P([\pi^*c_1(L)])^{d-1} \cdot E = 0$.

Define the class $\alpha := P([c_1(L)])^{d-1}$. The previous computation shows

$$(\alpha_{X'} \cdot E) = (\alpha \cdot [E]) = (P([c_1(L)])^{d-1} \cdot [E]) = (P([\pi^*c_1(L)])^{d-1} \cdot [E]) = 0$$

for any π -exceptional divisor E . As π is a composition of blow-ups with smooth centers, there is a decomposition $N^1(X') = \pi^*N^1(X) \oplus F$ where F is the subspace spanned by classes of exceptional divisors. As the intersection of $\alpha_{X'}$ with any element in F is zero, we have $\alpha_{X'} = \pi^*\alpha_X$, hence α is a Cartier class determined in X . \square

Example 4.3. It follows from the work of Nakayama [Nak04, Theorem IV.2.10] that there exists a big line bundle on a projective 4-fold $L \rightarrow X$ such that $P(L)$ is a nef and big b -divisor class which is *not* Cartier. The b -numerical curve class $\alpha = P(L)^3$ is BPF and Cartier whereas its unique third root is not.

Remark 4.4. It is not difficult to find a nef class α that is not Cartier and such that $\alpha^{d-1} = 0$, for example in a toric situation.

4.3 Proof of Theorem D

Recall that we aim at proving the inclusions

$$N_{\text{BPF}}^1(\mathcal{X}) \subset \text{Vect}(\text{Nef}^1(\mathcal{X})) \subset N_{\Sigma}^1(\mathcal{X}) \subset N_{\text{BPF}}^{1,\vee}(\mathcal{X}) \subset \text{w-}N^1(\mathcal{X}).$$

The first and last inclusions hold by definition, see [DF21, § 2]. The second inclusion is Corollary 3.9. It thus remains to prove the continuous inclusion $N_{\Sigma}^1(\mathcal{X}) \subset N_{\text{BPF}}^{1,\vee}(\mathcal{X})$. This fact follows from the next theorem of independent interest.

THEOREM 4.5. *For any $\alpha \in \text{c-}N^1(\mathcal{X})$, and for any $\gamma \in \text{BPF}^{d-1}(\mathcal{X})$, we have*

$$|(\alpha \cdot \gamma)| \leq 5(\gamma \cdot \omega^{d-1})\|\alpha\|_{\Sigma}. \tag{19}$$

Let $\alpha_n \in c\text{-N}^1(\mathcal{X})$ be any Cauchy sequence for the norm $\|\cdot\|_\Sigma$. Then (19) implies that α_n is a Cauchy sequence for $\|\cdot\|_{\text{BPF}}^\vee$, proving the continuous injection $\text{N}_\Sigma^1(\mathcal{X}) \subset \text{N}_{\text{BPF}}^{1,\vee}(\mathcal{X})$.

Proof. Pick any $\gamma \in \text{BPF}^{d-1}(\mathcal{X})$. By Theorem C, for any $t > 0$ we may find a class $\delta_t \in \text{Nef}^1(\mathcal{X})$ such that $\gamma + t\omega^{d-1} = (\delta_t^{d-1})$. Let $s = (\delta_t^{d-2} \cdot \omega^2)$. We now estimate using [DF21, Theorem 3.11]:

$$\begin{aligned} |(\alpha \cdot (\gamma + t\omega^{d-1}))|^2 &= |(\alpha \cdot \delta_t^{d-1})|^2 \\ &\leq 9q_{\omega,(1/s)\delta_t^{d-2}}(\alpha)q_{\omega,(1/s)\delta_t^{d-2}}(s\delta_t) \\ &\leq 18((\gamma + t\omega^{d-1}) \cdot \omega)^2 \|\alpha\|_\Sigma^2, \end{aligned}$$

and we conclude by letting $t \rightarrow 0$. □

COROLLARY 4.6. *For any $\alpha, \beta \in \text{N}_\Sigma^1(\mathcal{X})$, the maps*

$$\gamma \in \text{BPF}^k(\mathcal{X}) \mapsto \alpha \cdot \beta \cdot \gamma \in \text{BPF}^{k+2}(\mathcal{X}) \text{ and } \gamma \in \text{BPF}^k(\mathcal{X}) \mapsto \alpha \cdot \gamma \in \text{BPF}^{k+1}(\mathcal{X})$$

are weakly continuous.

Proof. Pick $\alpha \in \text{N}_\Sigma^1(\mathcal{X})$ and choose a sequence $\alpha_n \in c\text{-N}^1(\mathcal{X})$ such that $\|\alpha - \alpha_n\|_\Sigma \rightarrow 0$. Suppose $\gamma_i \in \text{BPF}^k(\mathcal{X})$ is a net of BPF classes converging weakly to γ . We may suppose that $C := \sup_i(\gamma_i \cdot \omega^{d-k}) < \infty$. Pick any BPF Cartier b -numerical class $\beta \in c\text{-BPF}^{d-k-1}(\mathcal{X})$. By (19) and Corollary 3.4, we have

$$|(\alpha \cdot \gamma_i \cdot \beta) - (\alpha_n \cdot \gamma_i \cdot \beta)| \leq 5(\gamma_i \cdot \beta \cdot \omega^{d-k})\|\alpha - \alpha_n\|_\Sigma \leq CC_d(\beta \cdot \omega^{k+1})\|\alpha - \alpha_n\|_\Sigma.$$

Pick any $\epsilon > 0$. Choose n_0 large enough such that $CC_d(\beta \cdot \omega^{k+1})\|\alpha - \alpha_{n_0}\|_\Sigma \leq \epsilon$. As the class α_{n_0} is Cartier, the map $\eta \mapsto \alpha_{n_0} \cdot \eta$ is weakly continuous, hence for i large enough we have

$$|(\alpha \cdot \gamma_i \cdot \beta) - (\alpha \cdot \gamma \cdot \beta)| \leq 2\epsilon + |(\alpha_{n_0} \cdot \gamma_i \cdot \beta) - (\alpha_{n_0} \cdot \gamma \cdot \beta)| \leq 3\epsilon.$$

This proves $\gamma \in \text{BPF}^k(\mathcal{X}) \mapsto \alpha \cdot \gamma$ is weakly continuous.

The fact that $\gamma \mapsto \alpha \cdot \beta \cdot \gamma$ is continuous follows by the same token and [DF21, Theorem 3.16]. □

4.4 Approximation of classes in $\text{BPF}^{d-1}(\mathcal{X})$

The resolution of the operator $\alpha \mapsto \alpha^{d-1}$ yields interesting consequences for the structure of $\text{BPF}^{d-1}(\mathcal{X})$.

COROLLARY 4.7. *For any $\alpha \in \text{BPF}^{d-1}(\mathcal{X})$ there exists a sequence of classes $\alpha_n \in c\text{-BPF}^{d-1}(\mathcal{X})$ which is decreasing to α . If, moreover, α is big, then for any $\epsilon > 0$ there exists a class $\alpha' \in c\text{-BPF}^{d-1}(\mathcal{X})$ such that $(1 - \epsilon)\alpha' \leq \alpha \leq \alpha'$.*

Proof. We may suppose that $\alpha \leq \omega^{d-1}$. We claim that for each integer n , there exists a class $\alpha_n \in c\text{-BPF}^{d-1}(\mathcal{X})$ such that

$$\alpha + \frac{1}{n}\omega^{d-1} \leq \alpha_n \leq \left(1 + \frac{1}{2^n}\right)\left(\alpha + \frac{1}{n}\omega^{d-1}\right). \tag{20}$$

As for all $n \geq 2$

$$\begin{aligned} \alpha_{n+1} &\leq \left(1 + \frac{1}{2^{n+1}}\right)\left(\alpha + \frac{1}{n+1}\omega^{d-1}\right) \\ &\leq \alpha_n - \frac{1}{n(n+1)}\omega^{d-1} + \frac{1}{2^{n+1}}\left(\alpha + \frac{1}{n+1}\omega^{d-1}\right) \leq \alpha_n \end{aligned}$$

the sequence α_n is decreasing to α as required. When α is big, then for any $\epsilon > 0$, we have $\omega^{d-1}/n \leq \epsilon\alpha$ for all n large enough, hence it remains to prove (20).

By Theorem C, one can find a nef and big class $\beta' \in \text{Nef}^1(\mathcal{X})$ such that $(\beta')^{d-1} = \alpha + (1/n)\omega^{d-1}$. By Corollary 3.7, one can find a nef Cartier class $\beta(\epsilon)$ such that $\beta' \leq \beta(\epsilon) \leq (1 + \epsilon)\beta'$ and we conclude by taking $\alpha_n := \beta(\epsilon)^{d-1}$ with $(1 + \epsilon)^{d-1} \leq 1 + 1/n$. \square

Recall from the definition of the space $N^1_\Sigma(\mathcal{X})$ [DF21, § 3.2] that the topological dual of the Banach space $N^1_\Sigma(\mathcal{X})$ is characterized as follows:

$$N^1_\Sigma(\mathcal{X})^* = \{\alpha \in \text{w-N}^{d-1}(\mathcal{X}), \exists C > 0, |(\alpha \cdot \beta)| \leq C\|\beta\|_\Sigma \text{ for all } \beta \in \text{c-N}^1(\mathcal{X})\}.$$

As (19) holds for any class $\alpha \in N^1_\Sigma(\mathcal{X})$ by density, we obtain the following.

COROLLARY 4.8. *We have*

$$\text{BPF}^{d-1}(\mathcal{X}) \subset N^1_\Sigma(\mathcal{X})^*. \tag{21}$$

5. Movable cones and Lehmann–Xiao decompositions of curve classes

In § 5.1, we discuss the notion of movable classes as defined in § 1.1 and relate it to positivity properties of b -numerical classes. In the last two sections §§ 5.2 and 5.3, we compare the $(d - 1)$ th root of a BPF b -curve class given by Theorem C to two divisor classes introduced by Lehmann and Xiao in [LX16, LX19].

5.1 Movable classes

Let us first introduce the notion of movable classes in the sense of Fulger and Lehmann, see [FL17b, § 3]. An effective k -cycle Z in X is said to be strictly movable if there exists a variety Λ and a reduced closed subscheme V of $X \times \Lambda$ such that for each irreducible component V_i of V :

- (i) the first projection map $V_i \rightarrow X$ is dominant for each i ;
- (ii) the second projection map $\pi_i: V_i \rightarrow \Lambda$ is flat and dominant of relative dimension k for each i ; and
- (iii) $Z = \sum a_i[\pi_i^{-1}(\lambda)]$ for some $\lambda \in \Lambda$ and some $a_i \in \mathbb{N}$.

A class $\alpha \in N^k(X)$ is strictly movable if it is the class of a strictly movable $(d - k)$ -cycle. Roughly speaking, a strictly movable class is one represented by a k -cycle which moves in a family covering X .

DEFINITION 5.1. The movable cone is the closed convex cone in $N^k(X)$ generated by strictly movable classes.

It is not difficult to see that we have the following characterization of movable divisor classes. A class $\alpha \in N^1(X)$ is movable if it lies in the closure of the convex cone generated by integral classes $\alpha = c_1(L) \in N^1(X)$ such that the base locus of a multiple of the line bundle $L \rightarrow X$ has codimension ≥ 2 .

It also follows from [FL17b, Proposition 3.13] that the movable cone is equal to the closure of $\sum_\pi \pi_* \text{BPF}^k(X')$ where $\pi: X' \rightarrow X$ varies over all (smooth) birational models over X .

We have the following interpretation of movable classes in terms of b -numerical classes.

THEOREM 5.2. *For any class $\alpha \in N^k(X)$ the following are equivalent:*

- (i) α is movable;
- (ii) there exists a b -numerical class $\beta \in \text{BPF}^k(\mathcal{X})$ such that $\beta_X = \alpha$.

In particular, a b -numerical class $\beta \in \text{w-N}^k(\mathcal{X})$ is BPF iff $\beta_{X'}$ is movable for any model X' .

Remark 5.3. As the cone $\text{BPF}^{d-1}(\mathcal{X})$ is the (weak) closure of classes $[\alpha_{X'}]^{d-1}$ with $\alpha_{X'} \in N^1(X')$ ample, we see that Definition 5.1 is also consistent with that given in § 1.1 in the case $k = d - 1$.

Proof. Suppose $\alpha \in N^k(X)$ is movable. As recalled previously, by [FL17b, Proposition 3.13], we can find a sequence of models $\pi_n: X_n \rightarrow X$ and classes $\beta_n \in \text{BPF}^k(X_n)$ such that $(\pi_n)_*\beta_n \rightarrow \alpha$. Note that $(\beta_n \cdot \omega^{d-k})$ is uniformly bounded, hence the Banach–Alaoglu theorem implies that the set $F_p = \{\beta_n, n \geq p\}$ is weakly compact for all p . Pick any class $\beta \in \bigcap_{p \geq 1} F_p$. Then β lies in $\text{BPF}^k(\mathcal{X})$ and we have $\beta_X = \alpha$ which proves (i) \Rightarrow (ii).

Conversely, suppose $\beta \in \text{BPF}^k(\mathcal{X})$. By definition, we can find a sequence of classes $\beta_n \in \text{c-BPF}^k(X)$ such that $(\beta_n)_X \rightarrow \beta_X$. As $(\beta_n)_X$ is the image under a proper birational morphism of a BPF class, it is movable, hence β_X is movable too. This proves (ii) \Rightarrow (i).

Finally pick any class $\beta \in \text{w-N}^k(\mathcal{X})$, and suppose that $\beta_{X'}$ is movable for any model X' . By what precedes, for each X' there exists a class $\tilde{\beta}(X') \in \text{BPF}^k(\mathcal{X})$ such that $\tilde{\beta}(X')_{X'} = \beta_{X'} \in N^k(X')$. This implies that the net $\tilde{\beta}(X')$ converges (as b -numerical classes) to β , hence β is BPF. □

An easy consequence of the previous theorem is the following analog of Theorem 2.8.

PROPOSITION 5.4. *We have*

$$\text{c-N}^{d-1}(\mathcal{X}) \cap \text{BPF}^{d-1}(\mathcal{X}) = \text{c-BPF}^{d-1}(\mathcal{X}).$$

In other words any Cartier b -numerical curve class that is BPF as a b -numerical curve class is determined by a BPF class in some model.

Proof. Pick any $\alpha \in \text{c-BPF}^{d-1}(\mathcal{X})$, and suppose that α is determined in some model X' and $\alpha_{X'} \in N^{d-1}(X')$ is not BPF. By the previous result, $\alpha_{X'}$ is not movable, hence by the duality theorem [BDPP13, Corollary 2.5], one can find a psef class $\beta \in N^1(X')$ such that $(\alpha_{X'} \cdot \beta) < 0$.

Observe that the Cartier b -divisor class $[\beta]$ is psef because the pullback of any psef divisor remains psef. We then obtain $(\alpha \cdot [\beta]) < 0$ which implies that α cannot belong to $\text{BPF}^{d-1}(\mathcal{X})$. □

Recall that a b -numerical class α is psef iff $\alpha_{X'} \geq 0$ for all models X' . When it is the case, we write $\alpha \geq 0$. Beware that, in general, a Cartier class which is determined by a psef class is not psef (as a b -numerical class).

We conclude this section by exploring duality statements between the BPF and the psef cone in the space of b -numerical classes.

PROPOSITION 5.5. *Pick any b -numerical class $\alpha \in \text{w-N}^k(\mathcal{X})$. If α is psef (respectively, BPF), then for any BPF (respectively, psef) Cartier b -numerical class β we have $(\alpha \cdot \beta) \geq 0$.*

Proof. Suppose α is psef and pick any BPF Cartier b -numerical class β determined in a model X' . Then $(\alpha \cdot \beta) = (\alpha_{X'} \cdot \beta_{X'}) \geq 0$.

Pick now any psef Cartier b -numerical class β determined in a model X' . If α is a Cartier BPF class determined in a model X'' , then $(\alpha_{X'} \cdot \beta_{X'}) = (\alpha \cdot \beta) = (\alpha_{X''} \cdot \beta_{X''}) \geq 0$. We obtain $(\alpha \cdot \beta) = (\alpha_{X'} \cdot \beta_{X'}) \geq 0$ by density for any nef b -numerical classes. □

Recall from the introduction that we stated a duality between the cones of BPF b -classes and psef b -classes (Conjecture 1). We prove it here for curve and divisor classes.

PROPOSITION 5.6. For $k = 1$ and $k = d - 1$, the following two statements hold.

- (i) A b -numerical class $\alpha \in \text{w-N}^k(\mathcal{X})$ is psef iff for any b -numerical class $\beta \in \text{c-BPF}^{d-k}(\mathcal{X})$ we have $(\alpha \cdot \beta) \geq 0$.
- (ii) A b -numerical class $\alpha \in \text{w-N}^k(\mathcal{X})$ belongs to $\text{BPF}^k(\mathcal{X})$ iff for any psef Cartier b -numerical class $\beta \in \text{c-N}^{d-k}(\mathcal{X})$, we have $(\alpha \cdot \beta) \geq 0$.

Remark 5.7. The fact that a b -divisor class is nef if it intersects non-negatively psef b -numerical curve classes is due to Lehmann who formulated this fact in terms of movable classes, see [Leh11, Theorem 1.5].

Proof. In view of Proposition 5.5, we need to prove four implications.

Suppose first that a class $\alpha \in \text{w-N}^1(\mathcal{X})$ intersects non-negatively all b -numerical classes $\beta \in \text{c-BPF}^{d-1}(\mathcal{X})$. By Theorem 5.2, for any model X' the class $\alpha_{X'}$ lies in the dual cone of movable curve classes, hence it is psef. This proves $\alpha \geq 0$.

Suppose next that $\alpha \in \text{w-N}^{d-1}(\mathcal{X})$ intersects non-negatively all nef Cartier b -divisor classes. As the nef cone $\text{Nef}^1(X')$ is dual to the psef cone in $\text{N}^{d-1}(X')$, the class $\alpha_{X'}$ is effective for all X' , hence $\alpha \geq 0$.

Take now a class $\alpha \in \text{w-N}^{d-1}(\mathcal{X})$ intersecting non-negatively all psef b -divisor classes. Pick any model X' and any effective divisor class $\gamma \in \text{N}^1(X')$. As the pullback of an effective divisor remains effective, the Cartier class $[\gamma]$ is psef, hence $(\alpha_{X'} \cdot \gamma) = (\alpha \cdot [\gamma]) \geq 0$. As the psef cone is dual to the movable cone in $\text{N}^{d-1}(X')$, we conclude that $\alpha_{X'}$ is movable for all X' , hence α is BPF by Theorem 5.2.

It remains to prove that a b -divisor class is nef iff it intersects non-negatively all psef Cartier b -numerical curve classes. Denote by $\mathcal{C} \subset \text{w-N}^1(\mathcal{X})$ the dual of the cone of psef Cartier b -numerical curve classes. It is weakly closed and contains $\text{Nef}^1(\mathcal{X})$. Suppose there exists a class $\alpha \in \mathcal{C}$ which is not nef. By the Hahn–Banach theorem there exists a linear form $\ell \in \text{w-N}^1(\mathcal{X})^*$ which is continuous for the weak topology and such that $\ell(\alpha) < 0$ and $\ell|_{\text{Nef}^1(\mathcal{X})} \geq 0$.

By Lemma 1.2, there exists a Cartier curve class β such that $\ell(\alpha) = (\beta \cdot \alpha)$ for all $\alpha \in \text{w-N}^1(\mathcal{X})$.

Then we have $(\alpha \cdot \beta) < 0$ and $(\theta \cdot \beta) \geq 0$ for all $\theta \in \text{Nef}^1(\mathcal{X})$. By duality, we infer $\beta_{X'} \geq 0$ for all model X' which proves $\beta \geq 0$. This gives a contradiction and concludes the proof. \square

5.2 Lehmann–Xiao decompositions of curve classes

Our objective is to relate our Theorem C to the results of Lehmann and Xiao [LX16]. Let us review briefly their construction.

Let α be any big class in $\text{N}^{d-1}(X')$ for some model X' so that $\alpha \geq c\omega^{d-1}$ for some $c > 0$. Then [LX16, Theorem 1.3] states the existence of a unique class $\text{LX}(\alpha) \in \text{Nef}^1(X')$ such that

$$\alpha \geq \text{LX}(\alpha)^{d-1} \quad \text{and} \quad \text{LX}(\alpha) \cdot (\alpha - \text{LX}(\alpha)^{d-1}) = 0.$$

Our next result relies on the extension to b -classes of the functional $\widehat{\text{vol}}$ introduced in [Xia17]. Recall that for any psef class $\alpha \in \text{N}^{d-1}(X')$, Xiao defined the following quantity:

$$\widehat{\text{vol}}_{X'}(\alpha) := \left(\inf_{\substack{\beta \in \text{Nef}^1(X') \\ (\beta^d) > 0}} \frac{(\alpha \cdot \beta)}{(\beta^d)^{1/d}} \right)^{d/(d-1)} \in \mathbb{R}_+. \tag{22}$$

We extend this functional to any psef class $\alpha \in \text{Vect}(\text{BPF}^{d-1}(\mathcal{X}))$ by setting

$$\widehat{\text{vol}}(\alpha) := \left(\inf_{\substack{\beta \in \text{Nef}^1(\mathcal{X}) \\ (\beta^d) > 0}} \frac{(\alpha \cdot \beta)}{(\beta^d)^{1/d}} \right)^{d/(d-1)} \in \mathbb{R}_+. \tag{23}$$

THEOREM 5.8. *For any big curve class $\alpha \in \text{BPF}^{d-1}(\mathcal{X})$, the family of Cartier b -divisor classes $[\text{LX}(\alpha_{X'})]$ converges weakly to β which is the unique nef b -divisor class satisfying $\alpha = (\beta^{d-1})$.*

In other words, for any $\gamma \in \text{c-N}^{d-1}(\mathcal{X})$ and for any $\epsilon > 0$, there exists a model X' such that for any model $X'' \geq X'$ we have

$$|([\text{LX}(\alpha_{X''})] \cdot \gamma) - (\beta \cdot \gamma)| \leq \epsilon.$$

Proof. Suppose that $\alpha \in \text{BPF}^{d-1}(\mathcal{X})$ is big. By Theorem C, one can write $\alpha = \gamma^{d-1}$ for some big curve class $\gamma \in \text{Nef}^1(\mathcal{X})$. We claim that γ is the unique minimizer computing $\widehat{\text{vol}}(\alpha)$.

Pick any class $\beta \in \text{Nef}^1(\mathcal{X})$ such that $(\beta^d) > 0$, and normalize it by $(\beta \cdot \omega^{d-1}) = +1$. By the proof of Theorem C, the class $\gamma/(\alpha \cdot \gamma)$ is the unique class in $\{\beta' \in \text{Nef}^1(\mathcal{X}), (\beta' \cdot \omega^{d-1}) = +1\}$ maximizing (β'^d) . It follows that

$$(\beta^d) \leq \frac{(\gamma^d)}{(\alpha \cdot \gamma)^d} = (\gamma^d)^{1-d},$$

hence by Proposition 3.11, we obtain

$$\frac{(\alpha \cdot \beta)}{(\beta^d)^{1/d}} = \frac{(\gamma^{d-1} \cdot \beta)}{(\beta^d)^{1/d}} \geq (\gamma^d)^{(d-1)/d} = \frac{(\alpha \cdot \gamma)}{(\gamma^d)^{1/d}},$$

which proves the claim.

For any model X' , we have

$$\begin{aligned} \widehat{\text{vol}}_{X'}(\alpha_{X'}) &= \left(\inf_{\substack{\beta \in \text{Nef}^1(X') \\ (\beta^d) > 0}} \frac{(\alpha_{X'} \cdot \beta)}{(\beta^d)^{1/d}} \right)^{d/(d-1)} = \left(\inf_{\substack{\beta \in \text{Nef}^1(X') \\ (\beta^d) > 0}} \frac{(\alpha \cdot [\beta_{X'}])}{(\beta^d)^{(1/d)}} \right)^{d/(d-1)} \\ &\geq \left(\inf_{\substack{\beta \in \text{Nef}^1(\mathcal{X}) \\ (\beta^d) > 0}} \frac{(\alpha \cdot \beta)}{(\beta^d)^{1/d}} \right)^{d/(d-1)} = \widehat{\text{vol}}(\alpha). \end{aligned}$$

LEMMA 5.9. *For any $\epsilon > 0$, there exists a model X' such that for any $X'' \geq X'$ we have $\widehat{\text{vol}}_{X''}(\alpha_{X''}) \leq \widehat{\text{vol}}(\alpha) + \epsilon$.*

Fix any model X' as in the lemma. As $\alpha_{X'}$ is big, it follows from [LX16, Theorem 1.3] that $\text{LX}(\alpha_{X'})$ is the unique minimizer for $\widehat{\text{vol}}_{X'}(\alpha_{X'})$. From the previous lemma, and the relation $(\alpha_{X'} - \text{LX}(\alpha_{X'})^{d-1}) \cdot \text{LX}(\alpha_{X'}) = 0$, we obtain

$$|(\gamma^d) - (\text{LX}(\alpha_{X'})^d)| = |\widehat{\text{vol}}(\alpha) - \widehat{\text{vol}}_{X''}(\alpha_{X''})| \leq \epsilon, \tag{24}$$

$$|(\gamma^{d-1} \cdot \text{LX}(\alpha_{X'})) - (\text{LX}(\alpha_{X'})^d)| = |(\alpha \cdot \text{LX}(\alpha_{X'})) - (\text{LX}(\alpha_{X'})^d)| = 0. \tag{25}$$

We conclude using the Diskant inequality (16). □

Proof of Lemma 5.9. Pick any sequence $\gamma_n \in \text{c-Nef}^1(\mathcal{X})$ decreasing to γ . By Theorem 3.2(iii), we have

$$\lim_{n \rightarrow \infty} \left(\frac{(\alpha \cdot \gamma_n)}{(\gamma_n^d)^{1/d}} \right)^{d/(d-1)} = \widehat{\text{vol}}(\alpha).$$

Choose n large enough such that

$$\left(\frac{(\alpha \cdot \gamma_n)}{(\gamma_n^d)^{1/d}} \right)^{d/(d-1)} \leq \widehat{\text{vol}}(\alpha) + \epsilon.$$

Then in any model X'' dominating a model in which γ_n is determined, we obtain $\widehat{\text{vol}}_{X''}(\alpha_{X''}) \leq \widehat{\text{vol}}(\alpha) + \epsilon$. □

Remark 5.10. Suppose that $\alpha_n \rightarrow \alpha \geq \omega^{d-1}$. Then we can write $\alpha_n = \gamma_n^{d-1}$ and $\alpha = \gamma^{d-1}$ for some unique $\gamma_n, \gamma \in \widehat{\text{Nef}}^1(\mathcal{X})$. If $\widehat{\text{vol}}(\alpha_n) \rightarrow \widehat{\text{vol}}(\alpha)$, then it is possible to show that $\gamma_n \rightarrow \gamma$ in the weak topology. It would be interesting to find more general criteria ensuring the convergence $\gamma_n \rightarrow \gamma$.

We conclude this section by relating the big curve b -classes with the positivity of the functional $\widehat{\text{vol}}$. The following result is a natural generalization of [LX16, Theorem 5.2] to b -classes. A similar statement holds in codimension one, which is a direct application of Siu’s inequalities.

THEOREM 5.11. *Pick any curve b -numerical class $\alpha \in \text{BPF}^{d-1}(\mathcal{X})$, and let ω be any big nef Cartier b -divisor class. Then the following are equivalent:*

- (i) $\widehat{\text{vol}}(\alpha) > 0$;
- (ii) α is big.

Proof. The implication (ii) \Rightarrow (i) is easy because $\widehat{\text{vol}}$ is increasing. Suppose conversely that $\widehat{\text{vol}}(\alpha) > 0$. Pick $\epsilon > 0$ and write $\alpha_\epsilon = \alpha + \epsilon\omega^{d-1}$. We apply [LX16, Theorem 5.19] to the big class $(\alpha_\epsilon)_{X'}$ and the movable class ω^{d-1} . Write $B := \text{LX}([\alpha_\epsilon]_{X'}) \in \text{Nef}^1(X')$ so that $(B^d) = \widehat{\text{vol}}_{X'}([\alpha_\epsilon]_{X'})$. We obtain

$$[(\alpha_\epsilon)_{X'}] \geq \frac{\widehat{\text{vol}}([\alpha_\epsilon]_{X'})}{d(B \cdot \omega^{d-1})} \omega^{d-1}$$

By Khovanskii–Teissier inequalities, we obtain $(B \cdot \omega^{d-1}) \leq (B^{d-1} \cdot \omega)^{d-1} / (B^d)^{d-2}$, hence

$$[(\alpha_\epsilon)_{X'}] \geq \omega^{d-1} \frac{(B^d)^{d-1}}{d(B^{d-1} \cdot \omega)^{d-1}} \stackrel{(25)}{\geq} \omega^{d-1} \frac{\widehat{\text{vol}}(\alpha_\epsilon)^{d-1}}{d(\alpha_\epsilon \cdot \omega)^{d-1}},$$

because $B^{d-1} = \text{LX}([\alpha_\epsilon]_{X'})^{d-1} \leq (\alpha_\epsilon)_{X'}$. Now $[(\alpha_\epsilon)_{X'}]$ converges weakly to α_ϵ , so that

$$\alpha_\epsilon \geq \omega^{d-1} \frac{\widehat{\text{vol}}(\alpha_\epsilon)^{d-1}}{d(\alpha_\epsilon \cdot \omega)^{d-1}} \geq \omega^{d-1} \frac{\widehat{\text{vol}}(\alpha)^{d-1}}{d(\alpha \cdot \omega)^{d-1}}.$$

Letting $\epsilon \rightarrow 0$, we get $\alpha \geq \omega^{d-1} \widehat{\text{vol}}(\alpha)^{d-1} / d(\alpha \cdot \omega)^{d-1}$ which concludes the proof. □

5.3 Positive intersections and $(d - 1)$ th root

Xiao associated to any movable class $\alpha \in \text{N}^{d-1}(X)$ the following quantity:

$$\mathfrak{M}_X(\alpha) = \inf_{\substack{\gamma \in \text{N}^1(X) \\ \text{big and movable}}} \left(\frac{(\gamma \cdot \alpha)}{\text{vol}(\gamma)^{1/d}} \right)^{d/(d-1)}, \tag{26}$$

where $\text{vol}(\gamma)$ is the standard volume function on the cone of big divisor classes (which is defined by $\text{vol}(\gamma) := \langle P(\gamma)^d \rangle$).

Lehmann and Xiao then proved that if $\mathfrak{M}_X(\alpha) > 0$, then there exists a unique class $\text{Mv}_X(\alpha) \in \text{N}^1(X)$ minimizing the right-hand side of (26), and that this class satisfies $\alpha = \langle P([\text{Mv}_X(\alpha)])^{d-1} \rangle_X$, see [LX19, Theorem 3.14].

If $\alpha \in \text{BPF}^{d-1}(X)$ is now a b -numerical curve class, then $\alpha_{X'}$ is movable for any model X' and we may set $\mathfrak{M}_{X'}(\alpha) := \mathfrak{M}_{X'}(\alpha_{X'})$.

THEOREM 5.12. *Pick any big curve class $\alpha \in \text{BPF}^{d-1}(\mathcal{X})$. Then for any model X' , the class $\alpha_{X'}$ is movable and $\mathfrak{M}_{X'}(\alpha_{X'}) > 0$. The family of Cartier b -divisor classes $[\text{Mv}_{X'}(\alpha)]$ converges weakly to β which is the unique nef b -divisor class satisfying $\alpha = (\beta^{d-1})$.*

Proof. Choose any big curve class $\alpha \in \text{BPF}^{d-1}(\mathcal{X})$, and write $\alpha = (\beta^{d-1})$ with $\beta \in \text{Nef}^1(\mathcal{X})$. Pick any model X' and any big and movable class $\gamma \in N^1(X')$. As γ is movable, we have $P([\gamma])_{X'} = \gamma$, and $\text{vol}(P([\gamma])) = \text{vol}(\gamma)$. We thus obtain

$$\left(\frac{(\gamma \cdot \alpha_{X'})}{\text{vol}(\gamma)^{1/d}}\right)^{d/(d-1)} = \left(\frac{(P([\gamma]) \cdot [\alpha_{X'}])}{\text{vol}(P([\gamma]))^{1/d}}\right)^{d/(d-1)} \stackrel{\text{Lemma 1.5}}{\geq} \left(\frac{(P([\gamma]) \cdot \alpha)}{\text{vol}(P([\gamma]))^{1/d}}\right)^{d/(d-1)} \geq \widehat{\text{vol}}(\alpha),$$

which proves $\mathfrak{M}_{X'}(\alpha) \geq \widehat{\text{vol}}(\alpha) > 0$.

On the other hand, we claim that

$$\lim_{X'} \left(\frac{([\beta_{X'}] \cdot \alpha)}{\text{vol}(\beta_{X'})^{1/d}}\right)^{d/(d-1)} = \widehat{\text{vol}}(\alpha). \tag{27}$$

Indeed, because $\text{vol}(\beta_{X'}) = \langle P([\beta_{X'}])^d \rangle$ and $P([\beta_{X'}])$ is a decreasing net of nef divisor classes converging to β , we have $\text{vol}(\beta_{X'}) \rightarrow \text{vol}(\beta)$ by Theorem 3.2. We also have $([\beta_{X'}] \cdot \alpha) \rightarrow (\beta \cdot \alpha)$ as X' tends to infinity in the net of all models. Indeed this fact is obvious when α is Cartier, and the general case follows from Corollary 4.7. This proves (27). As

$$\mathfrak{M}_{X'}(\alpha) \leq \left(\frac{([\beta_{X'}] \cdot \alpha)}{\text{vol}(\beta_{X'})^{1/d}}\right)^{d/(d-1)}$$

we conclude that $\lim_{X'} \mathfrak{M}_{X'}(\alpha) = \widehat{\text{vol}}(\alpha)$.

We obtain

$$|(\beta^d) - (P([\text{Mv}_{X'}(\alpha)])^d)| = |\widehat{\text{vol}}(\alpha) - \mathfrak{M}_{X'}(\alpha)| \rightarrow 0$$

and

$$\begin{aligned} (\beta^{d-1} \cdot P([\text{Mv}_{X'}(\alpha)])) &= (\alpha \cdot P([\text{Mv}_{X'}(\alpha)])) \leq ([\alpha_{X'}] \cdot P([\text{Mv}_{X'}(\alpha)])) \\ &\leq (P([\text{Mv}_{X'}(\alpha)])^d), \end{aligned}$$

and Diskant’s estimates (16) imply $P([\text{Mv}_{X'}(\alpha)]) \rightarrow \beta$. As $P([\text{Mv}_{X'}(\alpha)])_{X'} = \text{Mv}_{X'}(\alpha)$, it follows that $[\text{Mv}_{X'}(\alpha)] \rightarrow \beta$. □

Remark 5.13. Fulger and Lehmann [FL17b] have also introduced a notion of Zariski decomposition of a big class $\alpha \in N^k(X)$ for any k . This decomposition is based on the mobility functional which is not invariant under birational pullbacks. We do not know how to extend this functional to the space of Cartier b -numerical cycles, not to mention to the BPF cone.

ACKNOWLEDGEMENTS

We would like to thank Sébastien Boucksom, Mattias Jonsson for their comments and for letting us include in § 4.2 an example of a BPF curve class which is Cartier whereas its root is not. We also thank Ana Maria Botero and Jian Xiao for their numerous remarks on a first version of this paper. We express our gratitude to the anonymous referee for the careful reading and thoughtful suggestions.

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