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Self-propelled motion in a viscous compressible fluid

Václav Mácha and **Šárka Nečasová**
Institute of Mathematics AS CR, Žitná 25, 11567 Praha 1, Czech Republic (macha@math.cas.cz; matus@math.cas.cz)

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In this paper we focus on the existence of a weak solution to a system describing a self-propelled motion of a single deformable body in a viscous compressible fluid that occupies a bounded domain in the three-dimensional Euclidean space. The governing system considered for the fluid is the isentropic compressible Navier–Stokes equation. We prove the existence of a weak solution up to a collision.

Keywords: self-propelled motion; compressible fluid; deformable structure

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1. Introduction

This paper is devoted to the self-propelled motion of a body S in a viscous compressible fluid that is contained in a bounded domain $\Omega \subset \mathbb{R}^3$.

Self-propelled motion, or self-propulsion, is a common means of locomotion of macroscopic objects. Typical examples are motions performed by birds, fishes, aeroplanes, rockets and submarines. In the microscopic world, many minute organisms, such as flagellates and ciliates, move by self-propulsion; these have been studied by many authors. Even though the hydrodynamical mechanisms of self-propulsion may be different for macroscopic and microscopic bodies (see [32]), the self-propelled motion of a body in a viscous liquid is essentially due to the interaction between the boundary of the body and a liquid. Hence, the boundary of the body serves as the driver, and the distribution V_* of the velocity on the boundary of the body serves as its thrust. The thrust can be generated by muscular action, as in animal locomotion, or by mechanical device, as in an aeroplane.

In a famous experiment by Taylor [33], a mechanical fish can happily swim in water but makes no progress in a very viscous liquid, e.g. corn syrup. The fish consists of a cylindrical body with a plane tail that flaps to and fro. Due to the reversibility of flow in a liquid with no inertia or, mathematically, due to the linearity of the equations, whatever impetus the fish achieves by one tail flap, it will immediately lose with the next flap. In a commonly accepted model of ciliata, the layer model, the motion of the cilia produces a velocity distribution on a surface enclosing the layer of cilia, which serves to propel the animal [1, 23]. A principal characteristic of ‘flight’ is that a significant part of the aerodynamic force is needed to cancel the weight of the organism. Thus, certain features of flying apply to

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buoyant fish. In forward flight such a force can be obtained by creating horizontal vorticity, this being the main purpose of the lifting surface of the body. The soaring and gliding of birds provides a familiar example, where the classical aerodynamics of fixed-wing aircraft can be applied at once. The observations of birds led to the Lanchester–Prandtl wing theory (the notion of circulation and induced drag of wings). For more detail see [4].

The system relating to a swimming or flying creature can be considered as a fluid–structure system. In recent years, many mathematical works have been published in the field of fluid–structure interaction problems, many of them tackling the well-posedness of the corresponding equations of motion. The main difficulties in obtaining the well-posedness of such systems are the nonlinearity from the fluid (Navier–Stokes or Euler) equations, the coupling between the equations of the fluid and the equations of the structure and the fact that the spatial domain of the fluid is moving and unknown. The latter problem is simplest in the case of a rigid body structure, since in this case the motion of the structure is completely described by its rotation and translation. In the case where the structure is deformable, e.g. for an elastic structure, the existence of weak solutions could be very difficult to obtain: if the displacement of the structure is not regular, neither is the domain of the fluid. In [2, 5, 7] some approximated models are considered for the motion of an elastic structure in a viscous incompressible fluid. More precisely, the equations for the elasticity are modified in order to gain some regularity for the elastic deformation. Note that in the case of plate equations it is possible to obtain the existence of a weak solution without these approximations [19] (see also [3]). For the case of non-Newtonian fluids with elastic structure see [22].

Concerning the mathematical theory of compressible fluids, the fundamental results in the Newtonian case have been obtained in the last two decades by Lions [24] (the barotropic case with $p(\rho) = \rho^{\gamma}$) and Feireisl *et al*. [13] (the generalization to a larger class of exponents, γ), Feireisl [10] and Feireisl and Novotný [11] (heat conductive fluids, singular limits). Based on the entropy inequality, the concept was further generalized to the notions of dissipative solutions and weak–strong uniqueness (see [12, 15]).

The case of two dimensions was studied, for example, in [28]. Except for an existence result, San Martín *et al*. prove the uniqueness of the solution and provide some numerical simulations. For three spatial dimensions, Starovoitov [31] studies the motion of several rigid bodies, whereas Nečasová *et al*. [25] provide an existence result of the equation describing the self-propelled motion of a body in an incompressible fluid. The existence problem of the strong solution of self-propelled motion was studied by both Galdi and Silvestre: in [29] Silvestre studied the Stokes approximation of the self-propelled motion of a rigid body in a viscous liquid filling all the three-dimensional space outside the body. Precisely, the existence and uniqueness of the strong solution to the coupled systems of equations describing the motion of the body–liquid system were proved for any time and any regular distribution of velocity on the boundary of the body. In [30] Silvestre investigated the motion of a self-propelled rigid body through a Navier–Stokes fluid filling the whole exterior domain. The existence of a weak solution that is defined globally in time, provided that the net flux across the boundary, for the prescribed boundary values for the velocity, is zero. The works of Galdi [16, 17] were devoted to the self-

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propulsion of a rigid body at vanishing Reynolds number. Galdi considered that the shape of the body is constant during the motion; the thrust is produced because the body generates a non-zero momentum flux through its boundary and/or because it moves portions of its boundary. As already mentioned, in the limit of zero Reynolds number the importance of inertia in determining the motion of the fluid, and consequently the motion of the body, becomes negligible. The motion of the body is therefore completely determined by its geometry and by the distribution of the velocity on its boundary. In fact, it was shown in [16] that, in the steady case, the motion of the body can be completely decoupled from that of the liquid, and the method used in [17] can also be extended to unsteady self-propelled motion in order to separate the motions of the body and the liquid.

The main aim of this paper is to provide a similar result to that in [25] for the case of a compressible fluid surrounding a body. In order to prove our main theorem, we use a method presented in [9]. This is based on an approximate system with a high-viscosity limit that simulates a rigid body. Many parts of the proof are similar to those in [9], and thus they are only sketched without any rigorous details. However, there are some problems that appear to be due to the self-propelled motion and coupling with the compressible fluid. In this paper we focus on the differences coming from the non-rigid motion rather than on the problems solved in [9].

Remark 1.1.

- It is more natural to consider the incompressible case than the compressible in case of flying birds or insects. Our problem can be seen as a preparation for the singular limit and rigorous justification of the model described in [25].
- In [9] the existence of a global weak solution up to collisions was proven, which means only a local solution, similarly to [25].
- Note the differences in the case of collisions in compressible and incompressible cases: collisions can occur only in the case of slip–boundary conditions for incompressible fluids (see [18]), and in the compressible case collisions can occur even with Dirichlet conditions [9]. In the incompressible case it was proved that collisions cannot occur with Dirichlet conditions [20, 21].

This paper is organized as follows. In § 2 we introduce the setting and a governing system. The main theorem is presented in § 3. Furthermore, we introduce an approximate system in § 4. The deformable body in an approximate system is treated as a part of a fluid that has tremendous viscosity. In § 5 we deal with limiting processes in order to obtain the main result. In the appendix we include some useful lemmas.

2. Setting

We consider a flying body with a deformable structure that occupies a bounded open connected set $\mathcal{S}_t$ at an instant $t \in [0, T]$. The body is surrounded by a viscous compressible fluid in a bounded domain $\Omega \subset \mathbb{R}^3$, i.e. the fluid fills a domain $\mathcal{F}_t := \Omega \setminus \mathcal{S}_t$ at an instant t. A function $\rho_{\mathcal{S}_t} : \mathcal{S}_t \mapsto (0, \infty)$ stands for the density of the body. We consider that S_t and Ω are locally Lipschitz domains in $\mathbb{R}^3$.

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The motion of the body consists of three elements: a translation described by $a \in R^3$, a rotation represented by $Q \in \mathrm{SO}(3)$ and a smooth deformation $\mathcal{A}\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$, i.e. $\mathcal{A}$ is a smooth orientation-preserving diffeomorphism that is prescribed and stands for the self-propelled motion. Thus, the domain $\mathcal{S}_t$ can be described using a function $\eta[t]\colon \mathbb{R}^3 \mapsto \mathbb{R}^3$ as follows:

$$\mathcal{S}_t = \eta[t]\mathcal{S}_0,$$

i.e. every point $\boldsymbol{x} \in \mathcal{S}_t$ can be expressed as

$$\boldsymbol{x} = \eta[t](\boldsymbol{y}) = \boldsymbol{a}(t) + Q(t)\mathcal{A}_t(\boldsymbol{y}),$$

where $\boldsymbol{y} \in \mathcal{S}_0$ (S_0 is the initial position of the body). The velocity of a point $\boldsymbol{x}$ is

$$\begin{aligned} \boldsymbol{x}'(t) &= \eta'[t](\eta^{-1}[t](\boldsymbol{x})) \\ &= \boldsymbol{a}'(t) + Q'(t)\mathcal{A}_t(\boldsymbol{y}) + Q(t)\partial_t\mathcal{A}_t(\boldsymbol{y}) \\ &= \boldsymbol{a}'(t) + \omega(t) \times (\boldsymbol{x} - \boldsymbol{a}(t)) + \boldsymbol{w}(t,\boldsymbol{x}) \\ &=: \boldsymbol{u}_{\mathcal{S}}(t,\boldsymbol{x}), \end{aligned} \tag{2.1}$$

where $\boldsymbol{w}(t,\boldsymbol{x}) = Q(t)\partial_t\mathcal{A}_t(\mathcal{A}_t^{-1}(Q^{\mathrm{T}}(t)(\boldsymbol{x} - \boldsymbol{a}(t))))$ and

$$\mathbb{S}(\boldsymbol{\omega}) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}, \qquad \mathbb{S}(\boldsymbol{\omega}(t)) = Q'(t)Q^{\mathrm{T}}(t).$$

In what follows, we use overlined letters for quantities related to the body, which is considered without any rotation and translation, i.e. in a deformed configuration. Namely,

$$\begin{aligned} \bar{\mathcal{S}}_t &= \mathcal{A}_t(\mathcal{S}_0), \\ \bar{\boldsymbol{w}}(t,\bar{\boldsymbol{x}}) &= \frac{\partial}{\partial t}\mathcal{A}_t(\mathcal{A}_t^{-1}(\bar{\boldsymbol{x}})) \quad \forall \bar{\boldsymbol{x}} \in \mathcal{S}_t. \end{aligned}$$

Moreover, there exists a smooth function $\bar{\boldsymbol{\Lambda}}$ that coincides with $\bar{\boldsymbol{w}}$ on a set $\bar{\mathcal{S}}_t$ and is supported on a neighbourhood of $\bar{\mathcal{S}}_t$, i.e.

$$\bar{\boldsymbol{\Lambda}}(t,\bar{\boldsymbol{x}}) = \begin{cases} \bar{\boldsymbol{w}}(t,\bar{\boldsymbol{x}}) & \text{for } \bar{\boldsymbol{x}} \in \bar{\mathcal{S}}_t, \\ 0 & \text{if } \operatorname{dist}(\bar{\boldsymbol{x}},\bar{\mathcal{S}}_t) \geqslant \sigma, \end{cases}$$

where σ is sufficiently small.

We define

$$\boldsymbol{\Lambda}(t,x) = Q(t)\bar{\boldsymbol{\Lambda}}(t,(Q^{\mathrm{T}}(t)(\boldsymbol{x} - \boldsymbol{a}(t)))).$$

We denote the density of a deformable structure at an instant $t \in [0,T]$ by $\rho_{\mathcal{S}} := \rho_{\mathcal{S}}(\cdot\,,t)\colon \mathcal{S}_t \mapsto (0,\infty)$. This is given by

$$\rho_{\mathcal{S}}(t,\boldsymbol{x}) = \frac{\rho_{\mathcal{S}_0}(\mathcal{A}_t^{-1}(Q(t)^{\mathrm{T}}[x - a(t)]))}{\det(\nabla\mathcal{A}_t(\mathcal{A}_t^{-1}(Q(t)^{\mathrm{T}}[x - a(t)])))}. \tag{2.2}$$

Consequently, the density in a deformed configuration may be expressed as

$$\bar{\rho}_{\mathcal{S}}(t,\boldsymbol{x}) = \frac{\rho_{\mathcal{S}_0}(\mathcal{A}_t^{-1}(\bar{\boldsymbol{x}}))}{\det(\nabla\mathcal{A}_t(\mathcal{A}_t^{-1}(\bar{\boldsymbol{x}})))}.$$

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In what follows, we assume that $\mathcal{A}_t$ is prescribed, and establish equations for $a(t)$ and $Q(t)$. Moreover, we assume that $\mathcal{A}_t$ satisfies the following hypotheses, presented in [25].

(H1) For every $t \geqslant 0$, the mapping $\boldsymbol{y} \mapsto \mathcal{A}(t, \boldsymbol{y})$ is a smooth diffeomorphism from $\mathbb{R}^3$ onto $\mathbb{R}^3$. Moreover, for every $\boldsymbol{y} \in \mathbb{R}^3$, the mapping $t \mapsto \mathcal{A}(t, \boldsymbol{y})$ is smooth.

(H2) The total volume of the body remains constant, i.e.

$$|\bar{\mathcal{S}}_t| = |\mathcal{S}_0|.$$

(H3) The centre of gravity and the angular momentum of the body cannot be changed by interior forces:

$$\int_{\bar{\mathcal{S}}_t} \bar{\rho}_S(t, \bar{\boldsymbol{x}}) \bar{\boldsymbol{w}}(t, \bar{\boldsymbol{x}}) \, \mathrm{d}\bar{\boldsymbol{x}} = 0,$$

$$\int_{\bar{\mathcal{S}}_t} \bar{\rho}_S(t, \bar{\boldsymbol{x}}) [\bar{\boldsymbol{x}} \times \bar{\boldsymbol{w}}(t, \bar{\boldsymbol{x}})] \, \mathrm{d}\bar{\boldsymbol{x}} = 0.$$

For $\boldsymbol{x} \in \Omega$ and $t \in [0, T]$, we set[1]

$$\begin{aligned} \boldsymbol{u}(t, \boldsymbol{x}) &= \chi_{\mathcal{F}_t} \boldsymbol{u}_{\mathcal{F}}(t, \boldsymbol{x}) + \chi_{\mathcal{S}_t} \boldsymbol{u}_{\mathcal{S}}(t, \boldsymbol{x}), \\ \rho(t, \boldsymbol{x}) &= \chi_{\mathcal{F}_t} \rho_{\mathcal{F}}(t, \boldsymbol{x}) + \chi_{\mathcal{S}_t} \rho_{\mathcal{S}}(t, \boldsymbol{x}), \end{aligned}$$

where $\boldsymbol{u}_{\mathcal{F}}$ and $\rho_{\mathcal{F}}$ are the velocity and density, respectively, of the surrounding fluid. We assume that the following equations hold.

- Balance of mass:

$$\partial_t \rho_{\mathcal{F}} + \operatorname{div}(\rho_{\mathcal{F}} \boldsymbol{u}_{\mathcal{F}}) = 0 \quad \text{on } \mathcal{F}_t. \tag{2.3}$$

- Balance of linear momentum:

$$\partial_t(\rho_{\mathcal{F}} \boldsymbol{u}_{\mathcal{F}}) + \operatorname{div}(\rho_{\mathcal{F}} \boldsymbol{u}_{\mathcal{F}} \otimes \boldsymbol{u}_{\mathcal{F}}) + \nabla p = \operatorname{div} \mathbb{T}(\boldsymbol{u}_{\mathcal{F}}) + \rho_{\mathcal{F}} \boldsymbol{g} \quad \text{on } \mathcal{F}_t, \tag{2.4}$$

  where $\boldsymbol{g}$ is the exterior force.

The stress tensor $\mathbb{T}$ is given via

$$\mathbb{T}(\boldsymbol{u}_{\mathcal{F}}) := 2\mu \mathcal{D} \boldsymbol{u}_{\mathcal{F}} + \lambda I \operatorname{div} \boldsymbol{u}_{\mathcal{F}}, \tag{2.5}$$

where $2\mathcal{D} = \nabla + \nabla^{\mathrm{T}}$ is the symmetrical part of the stress tensor, $\mu \in (0, \infty)$, $\lambda \in \mathbb{R}$ and $\mu + \lambda \geqslant 0$, μ and λ are constant coefficients of viscosity. A pressure p is given by

$$p = \alpha \rho_{\mathcal{F}}^{\gamma}, \quad \alpha > 0, \tag{2.6}$$

with $\gamma \in \mathbb{R}$ restricted below. We consider the following boundary conditions:

$$\boldsymbol{u}_{\mathcal{F}}(t, \boldsymbol{x}) = \begin{cases} 0, & \boldsymbol{x} \in \partial\Omega, \\ \boldsymbol{a}'(t) + \boldsymbol{\omega}(t) \times (\boldsymbol{x} - \boldsymbol{a}(t)) + \boldsymbol{w}(t, \boldsymbol{x}) = \boldsymbol{u}_{\mathcal{S}}(t, \boldsymbol{x}), & \boldsymbol{x} \in \partial\mathcal{S}_t. \end{cases} \tag{2.7}$$

[1] We denote by χ_M the characteristic function of a set M.

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Since the motion $\bar{\mathcal{A}}_t$ is prescribed, we have to introduce equations for the unknowns $\boldsymbol{a}(t)$ and $\boldsymbol{\omega}(t)$ that describe the movement of the body. Before we write down the equations, we set

$$\begin{aligned} M &:= \int_{\mathcal{S}_t} \rho_{\mathcal{S}}\,\mathrm{d}\boldsymbol{x}, \\ J(t) &:= \int_{\mathcal{S}_t} \rho_{\mathcal{S}}(t,\boldsymbol{x})(|\boldsymbol{x}-\boldsymbol{a}(t)|^2 - (\boldsymbol{x}-\boldsymbol{a}(t))\otimes(\boldsymbol{x}-\boldsymbol{a}(t)))\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

Finally, the functions $\boldsymbol{a}(t)$, $\boldsymbol{\omega}(t)$ should satisfy

$$\left.\begin{aligned} M\boldsymbol{a}''(t) &= -\int_{\partial\mathcal{S}_t} (\mathbb{T}-pI)\boldsymbol{n}\,\mathrm{d}\Gamma + \int_{\mathcal{S}_t} \rho_{\mathcal{S}}\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}, \\ (J\boldsymbol{\omega})'(t) &= -\int_{\partial\mathcal{S}_t} (\boldsymbol{x}-\boldsymbol{a}(t))\times(\mathbb{T}-pI)\boldsymbol{n}\,\mathrm{d}\Gamma + \int_{\mathcal{S}_t} \rho_{\mathcal{S}}(\boldsymbol{x}-\boldsymbol{a}(t))\times\boldsymbol{g}\,\mathrm{d}\boldsymbol{x}. \end{aligned}\right\} \tag{2.8}$$

The initial state is described by

$$\left.\begin{aligned} &\boldsymbol{a}(0)=0, && Q(0)=I, && \mathcal{A}_0 = I, && \rho_{\mathcal{S}}(0)=\rho_{\mathcal{S}_0}, \\ &\boldsymbol{a}'(0)=\boldsymbol{a}_0, && \boldsymbol{\omega}(0)=\boldsymbol{\omega}_0, && \rho_{\mathcal{F}}(0)=\rho_{\mathcal{F}0}, && \rho(0)\boldsymbol{u}(0)=\boldsymbol{m}_0. \end{aligned}\right\} \tag{2.9}$$

For brevity, we set $\rho_0 = \chi_{\mathcal{F}_0}\rho_{\mathcal{F}0} + \chi_{\mathcal{S}_0}\rho_{\mathcal{S}_0}$. We define

$$\begin{aligned} &\mathcal{H}_\sigma(\mathcal{S}_t) = \{\boldsymbol{v}\in L^2(\Omega);\ \boldsymbol{v}=0 \text{ on } \partial\Omega,\ \mathcal{D}(\boldsymbol{v})=0 \text{ in } \mathcal{S}_t\}, \\ &\mathcal{K}_\sigma(\mathcal{S}_t) = \mathcal{H}_\sigma(\mathcal{S}_t)\cap H_0^1(\Omega), \end{aligned}$$

where L^η and H_0^η, H^η are the classical Lebesgue and Sobolev spaces. Furthermore,

$$L_\sigma^2(\Omega) = \mathcal{H}_\sigma(\mathcal{S}_t), \qquad H_\sigma^1(\Omega) = \mathcal{K}_\sigma(\mathcal{S}_t).$$

We set

$$\begin{aligned} \rho(t,\boldsymbol{x}) &= \begin{cases} \rho_{\mathcal{F}}(t,\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\mathcal{F}_t, \\ \rho_{\mathcal{S}}(t,\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\mathcal{S}_t, \end{cases} \\ \boldsymbol{u}(t,\boldsymbol{x}) &= \begin{cases} \boldsymbol{u}(t,\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\mathcal{F}_t, \\ \boldsymbol{a}'(t)+\boldsymbol{\omega}(t)\times(\boldsymbol{x}-\boldsymbol{a}(t))+\boldsymbol{w}(t,\boldsymbol{x}) & \text{if } \boldsymbol{x}\in\mathcal{S}_t. \end{cases} \end{aligned}$$

Definition 2.1 (weak solution). We say that a pair

$$(\rho,\boldsymbol{u}) \in L^\infty(L^\gamma)\times L^2(0,T_*;\mathcal{K}_\sigma(\mathcal{S}_t)), \quad T_*>0,$$

is a weak solution of (2.3)–(2.9) if the following hold.

(i) $\rho \geqslant 0$.

(ii) The renormalized equation of continuity, i.e.

$$\partial_t b(\rho) + \operatorname{div}(b(\rho)\boldsymbol{u}) + (b'(\rho)\rho - b(\rho))\operatorname{div}\boldsymbol{u} = 0, \tag{2.10}$$

where $b\in C^1(\mathbb{R})$, holds in a weak sense.

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(iii) The balance of linear momentum holds in a weak sense, i.e.

$$\int_0^T \int_\Omega ((\rho\boldsymbol{u})\partial_t\varphi + [\rho\boldsymbol{u}\otimes\boldsymbol{u}]\colon \mathcal{D}\varphi + p\operatorname{div}\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T \int_\Omega (\mathbb{T}(\boldsymbol{u})\colon \mathcal{D}\varphi - \rho\boldsymbol{g}\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega (\boldsymbol{m_0}\varphi(0,\cdot))\,\mathrm{d}\boldsymbol{x} \quad \forall\varphi\in\mathcal{R}(\mathcal{S}_t), \tag{2.11}$$

where

$$\mathcal{R}(\mathcal{S}_t) = \{\varphi\in C_0^\infty([0,T)\times\Omega),\ \mathcal{D}\varphi(x) = 0 \text{ on an open neighbourhood of } \bar{\mathcal{S}}_t\}. \tag{2.12}$$

(iv) The energy inequality

$$\frac{1}{2}\int_\Omega \left(\rho(\tau)|\boldsymbol{u}(\tau)|^2 + \frac{\alpha}{\gamma-1}\rho^\gamma(\tau)\right)\mathrm{d}\boldsymbol{x} + \int_0^\tau\int_\Omega (2\mu|\mathcal{D}\boldsymbol{u}|^2 + \lambda(\operatorname{div}\boldsymbol{u})^2)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \leqslant C(\rho(0),\boldsymbol{u}(0),\boldsymbol{g})$$

holds for almost every (a.e.) $\tau\in[0,T]$.

(v) The movement of the body $\mathcal{S}$ is compatible with $\boldsymbol{u}$ in following sense:

$$\boldsymbol{u}_{\mathcal{F}}(t,\cdot) - \boldsymbol{u}_{\mathcal{S}}(t,\cdot) \text{ belongs locally to the space } W_0^{1,2}(\Omega\setminus\mathcal{S}_t). \tag{2.13}$$

Remark 2.2.

- The overall density and velocity satisfy this definition of the weak solution. Indeed, (2.1)–(2.3) yield (2.10). This can be verified by a straightforward calculation.

  Furthermore, let $\varphi\in\mathcal{R}(\mathcal{S}_t)$. We use this φ as a test function in (2.4). We get

$$\int_0^T\int_{\mathcal{F}_t} (\partial_t(\rho\boldsymbol{u})\varphi + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\varphi - p\operatorname{div}\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^T\int_{\mathcal{F}_t}(-\mathbb{T}(\boldsymbol{u})\mathcal{D}\varphi + \rho\boldsymbol{g}\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_0^T\int_{\partial S_t}(\mathbb{T}(\boldsymbol{u})\varphi\boldsymbol{n} - p\varphi\boldsymbol{n})\,\mathrm{d}\Gamma\,\mathrm{d}t, \tag{2.14}$$

  where $\boldsymbol{n}$ is a unit outer normal of $\mathcal{S}_t$.

  Due to (H3) we have

$$\int_{\mathcal{S}_t}\rho\boldsymbol{u}\,\mathrm{d}\boldsymbol{x} = M\boldsymbol{a}'(t), \qquad \int_{\mathcal{S}_t}(\rho(\boldsymbol{x}-\boldsymbol{a}(t))\times\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = J(t)\omega.$$

  Furthermore, the transport theorem (see [26, theorem 1.22]) yields

$$\int_{\mathcal{S}_t}\partial_t(\rho\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{S}_t}\operatorname{div}(\rho\boldsymbol{u}\times\boldsymbol{u})\,\mathrm{d}\boldsymbol{x} = M\boldsymbol{a}''(t)$$

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and

$$\int_{\mathcal{S}_t} ((\boldsymbol{x}-\boldsymbol{a}(t)) \times \partial_t(\rho\boldsymbol{u}))\,\mathrm{d}\boldsymbol{x} + \int_{\mathcal{S}_t} ((\boldsymbol{x}-\boldsymbol{a}(t)) \times \operatorname{div}(\rho\boldsymbol{u} \times \boldsymbol{u}))\,\mathrm{d}\boldsymbol{x} = (J\omega)'(t).$$

Since $\mathcal{D}\varphi = 0$ on $\mathcal{S}_t$, there exist $\boldsymbol{l}_\varphi(t) \in \mathbb{R}^3$ and $\xi_\varphi(t) \in \mathbb{R}^3$ such that $\varphi(\boldsymbol{x}) = \boldsymbol{l}_\varphi + \xi_\varphi \times (\boldsymbol{x}-\boldsymbol{a}(t))$ for every $\boldsymbol{x} \in \mathcal{S}_t$. Thus, according to (2.8),

$$\begin{aligned}\int_{\mathcal{S}_t} (\partial_t(\rho\boldsymbol{u})\varphi + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u})\varphi)\,\mathrm{d}\boldsymbol{x} &= M\boldsymbol{a}''(t)\boldsymbol{l}_\varphi + \xi_\varphi(J\omega)'(t) \\ &= -\int_{\partial\mathcal{S}_t} (\mathbb{T}(\boldsymbol{u}) - pI)\varphi\boldsymbol{n}\,\mathrm{d}\Gamma + \int_{\mathcal{S}_t} \rho_{\mathcal{S}}\boldsymbol{g}\varphi\,\mathrm{d}\boldsymbol{x}.\end{aligned} \tag{2.15}$$

Consequently, (2.11) follows from (2.14) and (2.15). Let us point out that the boundary integrals presented in (2.8) vanish in a weak formulation. For more details about the weak formulation of system we refer the reader to [28, proposition 2].

- There is no *a priori* reason to assume that the momentum $(\rho\boldsymbol{u})$ is continuous in time. We can only have that a function

  $$t \mapsto \int_{\mathbb{R}^3} (\rho\boldsymbol{u}) \cdot \psi$$

  is continuous in a certain neighbourhood of a point t_0 provided $\psi = \psi(x) \in \mathcal{D}(\Omega)$ and $\psi = 0$ on a neighbourhood of $\bar{\mathcal{S}}_0$.

- An alternative condition to the concept of compatibility of the velocity $\boldsymbol{u}$ with the rigid objects was used in [6, 27], namely

  $$\boldsymbol{u} \in L^2((0,T); W_0^{1,2} \cap V^s(\Omega)),$$

  where the sets $V^s = V^s(t)$ are defined as

  $$V^s = \{\boldsymbol{u} \in W^{1,2}(\Omega) \mid \mathcal{D}\boldsymbol{u}\rho_{\mathcal{S}}(t) = 0\}.$$

3. Main result

Theorem 3.1 (main result). *Let Ω be a $\mathcal{C}^{2+\nu}$ domain, $\nu > 0$, let $\gamma > \frac{3}{2}$ and let $\mathcal{S}_0 \subset\subset \Omega$ be a bounded open connected set. Let there exist $c_1, c_2 > 0$ and initial data ρ_0, $\boldsymbol{m}_0$ be such that*

$$\rho_0 \geqslant 0, \qquad \rho_0\chi_{\mathcal{S}_0} \in [c_1, c_2], \qquad \rho_0 \in L^\gamma(\Omega),$$

$$\boldsymbol{m}_0 = 0 \text{ almost everywhere on the set } \{x \in \Omega \mid \rho_0 = 0\}, \qquad \frac{\boldsymbol{m}_0^2}{\rho_0} \in L^1(\Omega),$$

and let $\boldsymbol{g}$ be a bounded measurable function in $L^\infty((0,T)\times\mathbb{R}^3)$.

Then there exists $T_* \in (0,\infty)$ such that there exists a weak solution $(\rho, \boldsymbol{u})$ of (2.3)–(2.9) on the interval $(0, T_*)$.

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Remark 3.2. The approximation of the problem (2.3)–(2.9) is constructed in the following way.

- *d-approximation:* we approximate the continuity equation by adding the term $d\Delta\rho$, and we also add the term $d\nabla\rho\nabla\boldsymbol{u}$ to the momentum equation.
- *β-approximation:* we introduce the artificial pressure by adding a term $b\rho^\beta$, $\beta > 2$, to the constitutive equation.
- *n-approximation:* we use the penalization method introduced by Starovoitov *et al*. [27] to consider the viscosity coefficients dependent on the distance to the boundary.
- *N-approximation:* we use Galerkin approximation in order to obtain the existence of a solution to an approximate problem. Since this method is standard (see [13, 26]), we skip it here, and the existence result to the approximate problem is given directly by lemma 4.1.

Letting $n \to \infty$, $d \to 0$ and $\beta \to 0$, we get the existence of the weak solution of the problem.

4. Approximate problem

4.1. Approximation (d, β, n)

We use an approximation scheme that is proposed in [25, remark 11] (i.e. we suppose that the viscosity of a compressible fluid rapidly increases on the body $\mathcal{S}_t$) together with a known approximation scheme [13]. The part of the velocity that is zero on a 'fluid domain' is denoted by μ_χ and that which grows rapidly on a 'body domain' is denoted by λ_χ. These viscosities are defined precisely later (see § 5.2). Now, it is enough to assume that functions $\mu_\chi\colon \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ and $\lambda_\chi\colon \mathbb{R} \times \mathbb{R}^3 \mapsto \mathbb{R}$ obey

$$\mu_\chi \geqslant 0, \quad \lambda_\chi + \mu_\chi + \mu + \lambda \geqslant 0, \tag{4.1}$$

where the variable χ depends on $\boldsymbol{u}$ and will be specified in § 5.1.

The approximate problem consists of the following equations.

- A continuity equation together with Neumann condition:

$$\left.\begin{aligned} \partial_t\rho + \operatorname{div}(\rho\boldsymbol{u}) = d\Delta\rho, \quad d > 0,\\ \nabla\rho\cdot\boldsymbol{n}|_{\partial\Omega} = 0. \end{aligned}\right\} \tag{4.2}$$

- A momentum equation (we define $\boldsymbol{v} = \boldsymbol{u} - \boldsymbol{\Lambda}$):

$$\left.\begin{aligned} \partial_t(\rho\boldsymbol{u}) + \operatorname{div}(\rho\boldsymbol{u}\otimes\boldsymbol{u}) + \nabla p(\rho) + d\nabla\rho\nabla\boldsymbol{u} &\\ = \operatorname{div}(\mu\mathcal{D}\boldsymbol{u} + \mu_\chi(\mathcal{D}\boldsymbol{v}))&\\ + \operatorname{div}(\lambda I \operatorname{div}\boldsymbol{u} + \lambda_\chi I \operatorname{div}\boldsymbol{v}) + \rho\boldsymbol{g},&\\ \boldsymbol{u}|_{\partial\Omega} = 0,& \end{aligned}\right\} \tag{4.3}$$

where $\boldsymbol{\Lambda} := \boldsymbol{\Lambda}_{\boldsymbol{u}}\colon [0,T]\times\Omega \mapsto \mathbb{R}^n$ is a given function depending on $\boldsymbol{u}$.

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- A constitutive relation for a pressure:

$$p = p(\rho) = \alpha\rho^\gamma + b\rho^\beta, \quad \alpha, b > 0,\ \beta > \max\{4, \gamma\}. \tag{4.4}$$

- The initial data, which complete this system:

$$\rho(0) = \rho_0, \qquad (\rho\boldsymbol{u})(0) = \boldsymbol{m}_0, \tag{4.5}$$

where $\rho_0 \in C^{2+\nu}(\bar{\Omega})$, $0 < c_2 \leqslant \rho_0 \leqslant c_3$, $\nabla\rho_0 \cdot \boldsymbol{n}|_{\partial\Omega} = 0$ and $\boldsymbol{m}_0 \in C^2(\bar{\Omega})$.

Lemma 4.1 (existence of a solution to an approximate system). *Let $\Omega \subset \mathbb{R}^3$ be a bounded $C^{2+\nu}$, $\nu > 0$, domain, let $\mathcal{S}_0 \subset \Omega$ be a bounded open connected set and let $\boldsymbol{g} \in \mathcal{D}((0,T) \times \Omega)$ be given. Let (4.1) hold and let $\beta > \max\{4, \gamma\}$, $\gamma > \frac{3}{2}$. Moreover, let $\boldsymbol{\Lambda_u}$ satisfy*

$$\begin{aligned} \|\partial_t \boldsymbol{\Lambda_u}\|_{L^2(0,T,L^\infty(\Omega))} &\leqslant C(1 + \|\boldsymbol{u}\|_{L^2((0,T,L^2(\Omega)))}), \\ \|\boldsymbol{\Lambda_u}\|_{L^\infty(0,T0,L^\infty(\Omega))} + \|\nabla\boldsymbol{\Lambda_u}\|_{L^\infty(0,T,L^\infty(\Omega))} & \\ + \|\Delta\boldsymbol{\Lambda_u}\|_{L^\infty(0,T,(L^\infty)(\Omega))} &\leqslant C, \\ \boldsymbol{\Lambda_u}|_{\partial\Omega} &= 0, \end{aligned}$$

and let μ_χ and λ_χ be defined as in § 5.2.

Then there exists a weak solution, $(\rho, \boldsymbol{u}) \in L^\infty(0,T,L^\beta(\Omega)) \times L^2(0,T,W_0^{1,2}(\Omega))$, to the problem (4.2)–(4.5).

Proof. The proof is similar to that in [13]. The presence of the two unknowns $\boldsymbol{\Lambda}$ and χ does not cause any significant problems. □

5. Proof of the main theorem

5.1. Average rigid motion

Let $\mathcal{S}_t$ be a set defined for all times t. Hereafter, we simply write χ instead of $\chi_{\mathcal{S}_t}$:

$$\begin{aligned} M_{[\chi,\rho]} &= \int_{\operatorname{supp}\chi} \rho \,\mathrm{d}\boldsymbol{x}, \\ \boldsymbol{a}_{[\chi,\rho]} &= \frac{1}{M_{[\chi,\rho]}} \int_{\operatorname{supp}\chi} \rho(\boldsymbol{x})\boldsymbol{x} \,\mathrm{d}\boldsymbol{x}, \\ I_{[\chi,\rho]} &= \int_{\operatorname{supp}\chi} \rho(\boldsymbol{x})(|\boldsymbol{x} - \boldsymbol{a}_{[\chi,\rho]}|^2 - (\boldsymbol{x} - \boldsymbol{a}_{[\chi,\rho]}) \otimes (\boldsymbol{x} - \boldsymbol{a}_{[\chi,\rho]}))\,\mathrm{d}\boldsymbol{x}, \\ \boldsymbol{l}_{[\chi,\rho,\boldsymbol{u}]} &= \frac{1}{M_{[\chi,\rho]}} \int_{\operatorname{supp}\chi} \rho\boldsymbol{u} \,\mathrm{d}\boldsymbol{x}, \\ \omega_{[\chi,\rho,\boldsymbol{u}]} &= (I_{[\chi,\rho]})^{-1} \int_{\operatorname{supp}\chi} \rho(\boldsymbol{x})((\boldsymbol{x} - \boldsymbol{a}_{[\chi,\rho]}) \times \boldsymbol{u}(\boldsymbol{x}))\,\mathrm{d}\boldsymbol{x}. \end{aligned}$$

The quantities $\boldsymbol{l}_{[\chi,\rho]}$ and $\omega_{[\chi,\rho,\boldsymbol{u}]}$ express an average transition and rotation of $\mathcal{S}_t$, respectively. Thus, an average rigid motion of a body $\mathcal{S}$ can be described by a function $\boldsymbol{\Pi}_{[\chi,\rho,\boldsymbol{u}]}$, which is defined as follows:

$$\boldsymbol{\Pi}_{[\chi,\rho,\boldsymbol{u}]}(\boldsymbol{x}) = \boldsymbol{l}_{[\chi,\rho,\boldsymbol{u}]} + \omega_{[\chi,\rho,\boldsymbol{u}]} \times (\boldsymbol{x} - \boldsymbol{a}_{[\chi,\rho]}).$$

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Furthermore, we define a function $Q_{[\chi,\rho,\boldsymbol{u}]}\colon \mathbb{R} \mapsto \mathbb{R}^{n\times n}$ as a solution to the following ordinary differential equation:

$$Q'_{[\chi,\rho,\boldsymbol{u}]}(t) = \mathbb{S}(\omega_{[\chi,\rho,\boldsymbol{u}]})Q_{[\chi,\rho,\boldsymbol{u}]}(t), \quad Q_{[\chi,\rho,\boldsymbol{u}]}(0) = I,$$

and a function $\boldsymbol{c}_{[\chi,\rho,\boldsymbol{u}]}\colon \mathbb{R} \mapsto \mathbb{R}^3$ as a solution of

$$\boldsymbol{c}'_{[\chi,\rho,\boldsymbol{u}]}(t) = \omega_{[\chi,\rho,\boldsymbol{u}]}(t) \times \boldsymbol{c}_{[\chi,\rho,\boldsymbol{u}]}(t) + \boldsymbol{l}_{[\chi,\rho,\boldsymbol{u}]}(t), \quad \boldsymbol{c}(0) = 0.$$

We set

$$\boldsymbol{\Lambda}_{[\chi,\rho,\boldsymbol{u}]}(t,\boldsymbol{x}) = Q_{[\chi,\rho,\boldsymbol{u}]}(t)\bar{\boldsymbol{\Lambda}}(t, Q^*_{[\chi,\rho,\boldsymbol{u}]}(\boldsymbol{x} - \boldsymbol{c}_{[\chi,\rho,\boldsymbol{u}]})).$$

Let $\mathcal{S}_0 \in \Omega$ and $\boldsymbol{u}$, ρ_0 be given with $\rho_0(\boldsymbol{x}) \in [c_1, c_2]$ for all $\boldsymbol{x} \in \mathcal{S}_0$. We prescribe the movement of body $\mathcal{S}_t$ by the following system of equations:

$$\left.\begin{aligned} \partial_t\tilde{\rho} + \operatorname{div}(\tilde{\rho}(\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]} + \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]})) = 0 \quad \text{on } \mathcal{S}_t,\\ \partial_t\chi + \operatorname{div}(\chi(\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]} + \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]})) = 0 \quad \text{on } \mathcal{S}_t,\end{aligned}\right\} \tag{5.1}$$

where $\mathcal{S}_t = \operatorname{supp}\chi(t)$. We complete (5.1) with the following initial conditions:

$$\tilde{\rho}(0) = \rho_0 \text{ in } \mathcal{S}_0 \quad \text{and} \quad \chi(0) = \chi_{\mathcal{S}_0}. \tag{5.2}$$

According to lemma A.4 a solution to (5.1), (5.2) exists. Moreover, since $\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]}+\boldsymbol{\Lambda}_{\chi,\tilde{\rho},\boldsymbol{u}}$ is solenoidal and $\tilde{\rho} \in [C_1, C_2]$, we use lemmas A.2 and A.3 in order to obtain

$$\|\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]}\|_{L^\infty(\Omega)} \leqslant c\|\boldsymbol{u}\|_{L^2(\Omega)}, \tag{5.3}$$

$$\|\partial_t\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}\|_{L^2(0,T,L^\infty(\Omega))} \leqslant c(1 + \|\boldsymbol{u}\|_{L^2(\Omega)}), \tag{5.4}$$

$$\begin{aligned}\|\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}\|_{L^\infty(0,T,L^\infty(\Omega))} + \|\nabla\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}\|_{L^\infty(0,T,L^\infty(\Omega))}&\\ +\|\Delta\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}\|_{L^\infty(0,T,L^\infty(\Omega))} \leqslant c.&\end{aligned} \tag{5.5}$$

Furthermore, $\boldsymbol{\Pi}$ is a linear function; thus, we get $\nabla_x\boldsymbol{\Pi} = \mathbb{S}(\omega)$. One may derive that

$$\|\nabla_x\boldsymbol{\Pi}\|_{L^2(0,T,L^\infty(\Omega))} \leqslant c\|\boldsymbol{u}\|_{L^2(0,T,L^2(\Omega))}. \tag{5.6}$$

For details we refer reader to the proof of [25, lemma 4].

From (5.3) one may also derive that there exists $T > 0$ such that $\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}(t)|_{\partial\Omega} = 0$ for all $t < T$.

PROPOSITION 5.1. *For every* $\boldsymbol{u} \in L^1(0,T,L^1)$ *it holds that*

$$\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]} = \boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}+\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}]}.$$

Consequently,

$$\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]} = \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}+\boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]}]}.$$

Proof. The first identity follows from the definition of $\boldsymbol{\Pi}$ and from hypothesis (H3). The second identity is an easy consequence of the definition of $\boldsymbol{\Lambda}$. □

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5.2. High-viscosity limit: approximation n

Let $\{\mu_{\chi_n}\}_{n=1}^{\infty}$ and $\{\lambda_{\chi_n}\}_{n=1}^{\infty}$ be sequences of viscosities specified later. Let $\boldsymbol{u}_n$ and ρ_n be corresponding weak solution to (4.1)–(4.5), where $\boldsymbol{\Lambda}_n := \boldsymbol{\Lambda}_{[\chi_n,\tilde{\rho}_n,\boldsymbol{u}_n]}$ and $\boldsymbol{\Pi}_n := \boldsymbol{\Pi}_{[\chi_n,\tilde{\rho}_n,\boldsymbol{u}_n]}$ are constructed as in § 5.1. Furthermore, we define a set $\mathcal{S}_{nt}$ as $\mathcal{S}_{nt} = \operatorname{supp}\chi_n(t,\cdot)$. We set $\boldsymbol{u}_n = \boldsymbol{v}_n + \boldsymbol{\Lambda}_n$. In what follows, we assume

$$\boldsymbol{u}_n|_{\partial\Omega} = \boldsymbol{v}_n|_{\partial\Omega} = \boldsymbol{\Lambda}_n|_{\partial\Omega} = 0,$$

at least on some time interval $(0, T_n)$. In order to proceed to the limit (letting $n \mapsto \infty$), we have to estimate norms of solutions independently on n. We integrate (4.2) over Ω in order to get

$$\|\rho_n\|_{L^\infty(0,T,L^1(\Omega))} \leqslant C.$$

We multiply (4.3) by $\boldsymbol{v}$ and we integrate over Ω and time interval $(0, t)$. We get

$$\begin{aligned}\int_\Omega \tfrac{1}{4}\rho_n(t)|\boldsymbol{u}_n(t)|^2 + \frac{\alpha}{\gamma-1}(\rho_n^\gamma)(t) + \frac{b}{\beta-1}\rho_n^\beta(t) \\ + \int_0^t\int_\Omega (\mu + 2\mu_{\chi_n})|\mathcal{D}\boldsymbol{v}_n|^2 + (\lambda+\lambda_{\chi_n})|\operatorname{div}\boldsymbol{v}_n|^2 \\ \leqslant C(\rho(0),\boldsymbol{u}(0),\boldsymbol{m}_0,\boldsymbol{\Lambda},\boldsymbol{g},\Omega) + C\int_0^t\int_\Omega \rho_n|\boldsymbol{u}_n|^2 + C(\alpha)\int_0^t\int_\Omega \rho_n^\gamma.\end{aligned} \tag{5.7}$$

Using Gronwall's inequality we obtain

$$\begin{aligned}\|\boldsymbol{v}_n\|_{L^2(0,T,W^{1,2}(\Omega))} + \|\rho_n|\boldsymbol{u}_n|^2\|_{L^\infty(0,T,L^1(\Omega))} + \|\rho_n^\beta\|_{L^\infty(0,T,L^1(\Omega))} \\ \leqslant C(T,\rho(0),\boldsymbol{u}(0),\boldsymbol{m}_0,\boldsymbol{\Lambda},\boldsymbol{g},\Omega),\end{aligned} \tag{5.8}$$

where the constant on the right-hand side is independent of n and d. Furthermore, from (5.8) and (4.1), we get

$$\|\nabla\rho_n\|_2^2 \leqslant C, \tag{5.9}$$

where again the right-hand side does not depend on n and d. According to (5.3)–(5.5) the quantities $\boldsymbol{\Lambda}_n$ and $\boldsymbol{\Pi}_n$ are estimated uniformly and thus there exists $T_* > 0$ such that

$$\boldsymbol{\Lambda}_n(t)|_{\partial\Omega} = 0$$

for all $t \in (0, T_*)$ and for all $n \in \mathbb{N}$. From now we will work on this time interval unless stated otherwise.

We define viscosities $\mu_n := \mu_{\chi_n}$ and $\lambda_n := \lambda_{\chi_n}$ by the following formula:

$$\lambda_n = \mu_n = n\chi_n.$$

We also define the distance

$$db_{\mathcal{S}}(\boldsymbol{x}) = d_{\overline{R^N\setminus\mathcal{S}}}(\boldsymbol{x}) - d_{\bar{\mathcal{S}}}(\boldsymbol{x}),$$

where $d_K(\boldsymbol{x}) = \min_{\boldsymbol{y}\in K}|\boldsymbol{x}-\boldsymbol{y}|$, provided $K \subset \mathbb{R}^n$ is a closed set. We use $db_{\mathcal{S}}$ to define the convergence of sets. We write $\mathcal{S}_{nt} \xrightarrow{b} \mathcal{S}_t$ if and only if $db_{\mathcal{S}_{nt}} \to db_{\mathcal{S}_t}$ in $C_{\mathrm{loc}}(\mathbb{R}^3)$.

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We proceed to a limit as n tends to ∞ (passing to a subsequence if needed). Since the limiting process is the same as in [9], we present here only conclusions without detailed proof. From (5.7)–(5.9) it follows, passing to a subsequence if necessary, that

$$\begin{aligned} \rho_n &\to \rho && \text{in } L^\beta((0,T_*)\times\Omega), \\ \nabla\rho_n &\to \nabla\rho && \text{weakly in } L^2((0,T_*)\times\Omega), \\ \boldsymbol{v}_n &\to \boldsymbol{v} && \text{weakly in } L^2(0,T_*,W_0^{1,2}(\Omega)), \\ \rho_n\boldsymbol{u}_n &\to \rho\boldsymbol{u} && \text{weakly in } L^2((0,T_*)\times\Omega), \\ \rho_n\boldsymbol{v}_n\otimes\boldsymbol{v}_n &\to \mathbb{P} && \text{weakly in } L^{2N/(2N-1)}((0,T_*)\times\Omega). \end{aligned}$$

From (4.2) one may derive

$$\|\rho_n(\tau)\|^2_{L^2(\Omega)} + 2d\int_0^\tau\int_\Omega \|\nabla\rho_n\|^2 = -\int_0^\tau\int_\Omega \operatorname{div}\boldsymbol{u}_n|\rho_n|^2 + \|\rho_0\|^2_{L^2(\Omega)}$$

and also

$$\|\rho(\tau)\|^2_{L^2(\Omega)} + 2d\int_0^\tau\int_\Omega \|\nabla\rho\|^2 = -\int_0^\tau\int_\Omega \operatorname{div}\boldsymbol{u}|\rho|^2 + \|\rho_0\|^2_{L^2(\Omega)}.$$

Thus, $\|\nabla\rho_n\|_{L^2(0,T_*,L^2(\Omega))} \to \|\nabla\rho\|_{L^2(0,T_*,L^2(\Omega))}$, and since L^2 is a strictly convex space we get $\nabla\rho_n \to \nabla\rho$ strongly in $L^2((0,T_*)\times\Omega)$. Consequently,

$$\nabla\boldsymbol{v}_n\nabla\rho_n \to \nabla\boldsymbol{v}\nabla\rho \quad \text{in } \mathcal{D}'((0,T_*)\times\Omega).$$

According to (5.3), (5.5) and (5.6), $\boldsymbol{\Pi}_n + \boldsymbol{\Lambda}_n$ is bounded in $L^2(0,T_*,W^{1,\infty}(\Omega))$ independently of n. Thus, the hypotheses of lemma A.1 are satisfied and one may derive that

$$\boldsymbol{\Pi}_n + \boldsymbol{\Lambda}_n \to \boldsymbol{\Pi}_{[\chi,\tilde\rho,\boldsymbol{u}]} + \boldsymbol{\Lambda}_{[\chi,\tilde\rho,\boldsymbol{u}]} \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^3) \text{ uniformly in } t,$$

and also

$$\mathcal{S}_{nt} \xrightarrow{b} \mathcal{S}_t \quad \text{uniformly in } t.$$

We define

$$\begin{aligned} P^s &= \{(t,\boldsymbol{x}),\ \boldsymbol{x}\in\mathcal{S}_t\}, \\ P^f &= ([0,T_*]\times\Omega)\setminus\overline{P^s}. \end{aligned}$$

Both P^s and P^f are open. Thus, for a point $(t,\boldsymbol{x})\in P^f$ there exist the open intervals $J\subset[0,T]$ and $U\subset P^f$ such that

$$(t,\boldsymbol{x})\in J\times U\subset\overline{J\times U}\subset P^f.$$

We have $\partial_t\rho_n$ bounded in $L^q(J,W^{-k,q}(U))$ for some $q>1$, $k\geqslant 1$ (see [13, lemma 2.4]) and, consequently,

$$(\rho_n\boldsymbol{u}_n)\to(\rho\boldsymbol{u}) \quad \text{in } C(\bar J, L^{2\beta/(\beta+1)}(U)).$$

Due to a compact embedding $L^{2\beta/(\beta+1)}\subset W^{-1,2}$ we get

$$\rho_n\boldsymbol{u}_n\otimes\boldsymbol{u}_n \to \rho\boldsymbol{u}\otimes\boldsymbol{u} \quad \text{weakly in } L^{6/5}(J\times U).$$

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Thus, $\mathbb{P} = \rho\boldsymbol{u}\otimes\boldsymbol{u}$ on P^f. Moreover, since μ_{Ψ_n} and λ_{Ψ_n} tend to infinity on every compact $\mathcal{K}^s \subset P^s$, we derive from (5.7) that $D\boldsymbol{v}_n \to 0$ in $L^2(\mathcal{K}^s)$. Therefore,

$$\int_0^{T_*}\int_\Omega ((\rho\boldsymbol{u})\partial_t\boldsymbol{\varphi} + [\rho\boldsymbol{u}\otimes\boldsymbol{u}] : \mathcal{D}\boldsymbol{\varphi} + p\operatorname{div}\boldsymbol{\varphi} + d\nabla\boldsymbol{u}\nabla\rho\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t \\ = \int_0^{T_*}\int_\Omega (\mu\mathcal{D}\boldsymbol{u}\mathcal{D}\boldsymbol{\varphi} + \lambda\operatorname{div}\boldsymbol{u}\operatorname{div}\boldsymbol{\varphi} + \rho\boldsymbol{g}\boldsymbol{\varphi})\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t + \int_\Omega \boldsymbol{m_0}\varphi(0,\cdot)\,\mathrm{d}\boldsymbol{x}$$

whenever $\boldsymbol{\varphi} \in \mathcal{R}(\mathcal{S}_t)$ (see (2.12) for a definition).

This proves the following lemma.

Lemma 5.2. *Let $\Omega \subset \mathbb{R}^3$ be a bounded $\mathcal{C}^{2+\nu}$ domain with $\nu > 0$. Let p be given by (4.4) with $\beta > \max\{4,\gamma\}$, $\gamma > \frac{3}{2}$. Let (4.5) hold and let $\mathcal{S}_0 \subset \Omega$ be a bounded open connected set. Then there exist a time T_* and functions $\rho \in L^\infty(0,T_*,L^\beta)$, $\boldsymbol{u} \in L^2(0,T_*,W_0^{1,2}) \cap L^\infty(0,T_*L^2)$, $\tilde\rho \in L^\infty(0,T_*,L^\infty)$ and $\chi \in \mathrm{Char}(0,T_*,\mathbb{R}^3)$ such that*

- *$\rho,\boldsymbol{u}$ satisfy (4.2), (2.11) and initial condition (4.5) in a weak sense for $\rho \in C([0,T_*],L^1)$,*
- *$\tilde\rho$ and χ satisfy (5.1).*

5.3. Vanishing-viscosity limit

In this subsection, we proceed to a limit with the parameter d. Let $d_n \to 0$ and let $\boldsymbol{u}_n$ and ρ_n be corresponding weak solutions to (4.2)–(4.5) that are constructed as in lemma 5.2. Furthermore, let $\mathcal{S}_{nt}$ be bodies with corresponding motion described by $\boldsymbol{\Pi}_n = \boldsymbol{\Pi}_{[\chi_n,\tilde\rho_n,\boldsymbol{u}_n]}$ and $\boldsymbol{\Lambda}_n = \boldsymbol{\Lambda}_{[\chi_n,\tilde\rho_n,\boldsymbol{u}_n]}$. From estimates (5.8) and (5.9) we get following convergences:

$$\begin{aligned} d_n\nabla\boldsymbol{u}_n\nabla\rho_n &\to 0 \quad \text{in } L^1((0,T_*)\times\Omega),\\ d_n\Delta\rho_n &\to 0 \quad \text{in } L^2(0,T_*,W^{-1,2}(\Omega)),\\ \rho_n &\to \rho \quad \text{in } C([0,T_*],L^\beta_{\text{weak}}),\\ \boldsymbol{u}_n &\to \boldsymbol{u} \quad \text{weakly in } L^2(0,T_*,W_0^{1,2}), \end{aligned}$$

and, consequently,

$$(\rho_n\boldsymbol{u}_n) \to (\rho\boldsymbol{u}) \quad \text{weakly}^* \text{ in } L^\infty(0,T_*,L^{2\beta/(\beta+1)}(\Omega)).$$

Thus, ρ and $\boldsymbol{u}$ satisfy the continuity equation in $\mathcal{D}'((0,T_*)\times\Omega)$ and, using the same regularization procedure as in [8], we can derive that ρ and $\boldsymbol{u}$ also satisfy the renormalized continuity equation.

According to lemmas A.2 and A.3, it holds that

$$\|\boldsymbol{\Pi}_{[\chi_n,\rho_n,\boldsymbol{u}_n]} + \boldsymbol{\Lambda}_{[\chi_n,\rho_n,\boldsymbol{u}_n]}\|_{L^2(L^\infty)} + \|\nabla(\boldsymbol{\Pi}_{[\chi_n,\rho_n,\boldsymbol{u}_n]} + \boldsymbol{\Lambda}_{[\chi_n,\rho_n,\boldsymbol{u}_n]})\|_{L^2(L^\infty)} \leqslant C.$$

It follows from lemma A.1 that

$$\mathcal{S}_{nt} \xrightarrow{b} \mathcal{S}_t$$

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and

$$\boldsymbol{\Pi}_{[\chi_n,\rho_n,\boldsymbol{u}_n]} + \boldsymbol{\Lambda}_{[\chi_n,\rho_n,\boldsymbol{u}_n]} \to \boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]} + \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]} \quad \text{weakly}^* \text{ in } L^2(0,T,W^{1,\infty}(\Omega)).$$

Furthermore, we define

$$P^s = \{(t,\boldsymbol{x}),\ x \in \mathcal{S}_t\},$$
$$P^f = ((0,T_*) \times \Omega) \setminus P^s.$$

Similarly to the result of the previous subsection (see also [9, § 8]) we have

$$\rho_n \boldsymbol{u}_n \otimes \boldsymbol{u}_n \to \mathbb{P} \quad \text{in } L^{6/5}((0,T)\times\Omega)$$

and

$$\mathbb{P} = \rho\boldsymbol{u}\otimes\boldsymbol{u} \quad \text{on } P^f.$$

Following the procedure in [9, § 8] step by step, we derive that $p(\rho_n) \to p(\rho)$ weakly in $L^{(\beta+1)/\beta}(K^f)$ for any compact $K^f \subset P^f$.

Precisely, we claim that the pressure $p(\rho_n)$ is locally bounded in $L^{(\beta+1)/\beta}_{\text{loc}}(P^f)$.

One may derive the following lemma in a similar way to that in [9].

Lemma 5.3. *For any compact $K^f \subset P^f$, there exists a constant c independent of d, such that*

$$\|\rho_n\|_{L^{\beta+1}(K^f)} + \|\rho_n\|_{L^{\gamma+1}(K^f)} \leqslant c(K^f).$$

This implies that

$$p(\rho_n) \to \overline{p(\rho)} \text{ weakly in } L^{(\beta+1)/\beta}(K_f) \quad \text{for any compact } K^f \subset P^f.$$

Then we can pass to the limit

$$\int_0^{T_*}\int_{\mathbb{R}^N} ((\rho\boldsymbol{u})\partial_t\varphi + [\rho\boldsymbol{u}\otimes\boldsymbol{u}] \colon \mathcal{D}\varphi + \overline{p(\rho)}\operatorname{div}\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^{T_*}\int_{\mathbb{R}^N} (\mathbb{T}(\boldsymbol{u})\colon \mathcal{D}(\varphi) - \rho\boldsymbol{g}\varphi)\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t.$$

Our final aim is the strong convergence of density. Similarly to [9], we apply the following result.

Lemma 5.4. *Let $\beta > 7$. Then*

$$\lim_{n\to\infty}\int_0^{T_*}\int_{R^n} \phi(p(\rho_n) - (\lambda+2\mu)\operatorname{div}\boldsymbol{u}_n)\rho_n\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t = \int_0^{T_*}\int_{R^n} \phi(\overline{p(\rho)} - (\lambda+2\mu)\operatorname{div}\boldsymbol{u})\rho\,\mathrm{d}\boldsymbol{x}\,\mathrm{d}t$$

for any $\phi \in \mathcal{D}(Q^f)$.

We can conclude that $\overline{(\rho\operatorname{div}u)} \leqslant \rho\operatorname{div}u$, where $\overline{(\rho\operatorname{div}u)} := \lim\rho_n\operatorname{div}u_n$ in, say, $L^1(\Omega\times(0,T_*))$.

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Furthermore, (5.4), (5.5) and the Aubin–Lions lemma yield $\operatorname{div} \Lambda_n \to \operatorname{div} \Lambda$ strongly in $L^p(L^q)$ for any $p, q \in (1, \infty)$. Thus, for any $K^S \subset \mathcal{S}_t$ compact, it holds that

$$\rho_n \operatorname{div} u_n = \rho_n \operatorname{div} \Lambda_n \to \rho \operatorname{div} \Lambda = \rho \operatorname{div} u = \overline{\rho \operatorname{div} u} \quad \text{on } K^S.$$

It follows that

$$\overline{\rho \operatorname{div} u} \geqslant \rho \operatorname{div} u \quad \text{on } (0, T_*) \times \mathbb{R}^3.$$

This, together with renormalized continuity equation, yields that

$$\rho_n \to \rho \quad \text{in } L^1((0, T_*) \times \Omega)$$

and, consequently,

$$\overline{p(\rho)} = p(\rho) \quad \text{on } P^f$$

(see [13, § 4.6]).

Thus, the functions $\boldsymbol{u}$ and ρ satisfy (2.10) and (2.11).

We are now in the best position to prove $\boldsymbol{u}\chi = (\boldsymbol{\Pi}_{[\chi,\rho,\boldsymbol{u}]} + \boldsymbol{\Lambda}_{[\chi,\rho,\boldsymbol{u}]})\chi$. We point out that $D\boldsymbol{v}\chi = 0$ almost everywhere, and thus $\boldsymbol{v}$ is a rigid velocity on a body $\mathcal{S}$. According to considerations in [25, § 3.1], it holds that $\boldsymbol{v}\chi = (\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{v}]})\chi$. By proposition 5.1 we have

$$\boldsymbol{u}\chi = (\boldsymbol{v} + \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]})\chi = (\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{v}]} + \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]})\chi = (\boldsymbol{\Pi}_{[\chi,\tilde{\rho},\boldsymbol{u}]} + \boldsymbol{\Lambda}_{[\chi,\tilde{\rho},\boldsymbol{u}]})\chi.$$

Moreover, from the uniqueness of a solution to the transport equation, we get $\tilde{\rho}\chi = \rho\chi$. To conclude this subsection, we formulate all results into the following lemma.

Lemma 5.5. *Let $\Omega \subset \mathbb{R}^{\mathbb{N}}$ be a bounded $\mathcal{C}^{2+\nu}$ domain with $\nu > 0$. Let p be given by (4.4) with $\beta > \max\{4, \gamma\}$, $\gamma > \frac{3}{2}$. Then there exist a time T_* and functions*

$$\rho \in L^\infty(0, T_*, L^\beta), \quad \boldsymbol{u} \in L^2(0, T_*, W_0^{1,2}) \cap L^\infty(0, T_*, L^2), \quad \chi \in \operatorname{Char}(0, T_*, \mathbb{R}^3)$$

such that ρ and $\boldsymbol{u}$ solve (2.10), (2.11) and the compatibility condition (2.13) is satisfied.

5.4. Limit in pressure and domain

Our final task is to prove an existence of a solution for a pressure given by (2.6) and for a general domain Ω. We take a sequence of real numbers $b_n \to 0$, a sequence of domains Ω_n, $\Omega_n \subset \Omega_{n+1}$, $\Omega_n \xrightarrow{b} \Omega$ and the sequence of weak solutions $\boldsymbol{u}_n$, ρ_n constructed in lemma 5.5. This idea is summarized in the following lemma.

Lemma 5.6. *Let $\Omega_n, \Omega \subset \mathbb{R}^3$ be bounded domains such that*

$$\Omega_n \subset \Omega_{n+1}, \qquad \Omega_n \xrightarrow{b} \Omega \quad \text{as } n \to \infty.$$

The pressure $p = p_n$ is given by

$$p_n(\rho) = \alpha\rho^\gamma + b_n\rho^\beta$$

with

$$\gamma > \tfrac{3}{2}, \qquad \beta > 1, \qquad b_n \to 0 \quad \text{as } n \to \infty.$$

Let ρ_n and $\boldsymbol{u}_n$ be solutions to (2.3)–(2.5) and (2.6)–(2.9), respectively.

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Then there is a subsequence such that

$$\begin{aligned}\rho_n \to \rho \quad & \text{in } C([0,T_*], L^1(\Omega)),\\ \boldsymbol{u}_n \to \boldsymbol{u} \quad & \text{weakly in } L^2(0,T_*, W_0^{1,2}(\Omega)),\end{aligned}$$

where ρ and $\boldsymbol{u}$ are weak solutions to (2.3)–(2.9).

Proof. The proof is similar to that of [9, theorem 9.1], since there is no difficulty arising from a self-deformation of the body. □

Proof of theorem 3.1. We approximate a general bounded domain Ω by a sequence of smooth domains Ω_n, $\Omega_n \overset{b}{\to} \Omega$, $\Omega_n \subset \Omega_{n+1}$. This approximation exists according to [14, lemma 7.1]. According to lemma 5.5 there exist solutions ρ_n, $\boldsymbol{u}_n$ on Ω_n that satisfy the hypothesis of lemma 5.6. In order to prove the main result, it suffices to proceed to a limit with $n \to \infty$. □

Appendix A.

Lemma A.1 (Feireisl [9, proposition 5.1]). *Let $\boldsymbol{u}_n(t,\boldsymbol{x})$ be a family of functions such that $t \to \boldsymbol{u}_n(t,\cdot)$ is continuous from $[0,T]$ to $\mathbb{R}^3$, $\boldsymbol{x} \to \boldsymbol{u}_n(\cdot,\boldsymbol{x})$ is measurable from $\mathbb{R}^3$ to $\mathbb{R}^3$ and*

$$t \to \|\boldsymbol{u}_n(t,\cdot)\|_{L^\infty(\mathbb{R}^3)} + \|\nabla \boldsymbol{u}_n(t,\cdot)\|_{L^\infty(\mathbb{R}^3)}$$

is bounded in $L^2(0,T)$.

Let $\boldsymbol{\eta}_n[t]\colon \mathbb{R}^3 \to \mathbb{R}^3$ be the solution of the problem

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}_n[t](\boldsymbol{x}) = \boldsymbol{u}_n(t, \boldsymbol{\eta}_n[t](\boldsymbol{x})), \quad \boldsymbol{\eta}_n[0](\boldsymbol{x}) = \boldsymbol{x},\ \boldsymbol{x} \in \mathbb{R}^3.$$

Let also $\boldsymbol{B}_n \subset \mathbb{R}^3$ be a sequence such that $\boldsymbol{B}_n \overset{b}{\to} \boldsymbol{B}$, and denote by $\boldsymbol{B}_n(t) = \boldsymbol{\eta}_n[t](\boldsymbol{B}_n)$ the image of $\boldsymbol{B}_n$ by the flow $\boldsymbol{u}_n$.

Then, passing to subsequences,

$$\boldsymbol{\eta}_n[t] \to \boldsymbol{\eta}[t] \quad \text{in } C_{\mathrm{loc}}(\mathbb{R}^3) \text{ as } n \to \infty \text{ uniformly in } [0,T],$$

where $\boldsymbol{\eta}[t]$ solves

$$\frac{\mathrm{d}}{\mathrm{d}t}\boldsymbol{\eta}[t](\boldsymbol{x}) = \boldsymbol{u}(t, \boldsymbol{\eta}[t](\boldsymbol{x})), \quad \eta[0](\boldsymbol{x}) = \boldsymbol{x},\ \boldsymbol{x} \in \mathbb{R}^3,$$

and $\boldsymbol{u}_n \to \boldsymbol{u}$ weakly* in $L^2(0,T;W^{1,\infty}(\mathbb{R}^3))$.

Moreover, $\boldsymbol{B}_n(t) \overset{b}{\to} \boldsymbol{B}(t)$ uniformly in $[0,T]$, where $\boldsymbol{B}(t) = \boldsymbol{\eta}[t](\boldsymbol{B})$.

Lemma A.2 (Nečasová *et al.* [25, lemma 4]). *Assume that ψ_0 is the characteristic function of $\mathcal{S}_0$. Then, there exists a positive constant $C = C(\Omega, \mathcal{S}_0, C_1, C_2, \mathcal{A})$ such that, for all $\rho \in L^\infty((0,T)\times\Omega)$, $\boldsymbol{v} \in L^\infty(0,T;L^2(\Omega))$ and $\rho(t,\cdot) \in [C_1, C_2]$ for a.e. $t \in [0,T]$, we have*

$$\|\boldsymbol{\Pi}_{[\psi,\rho,\boldsymbol{v}]}\|_{L^\infty(\Omega)} \leqslant C\|\boldsymbol{v}\|_{L^2(\Omega)},$$

where χ is the solution of (5.1).

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Lemma A.3 (Nečasová *et al.* [25, lemma 5]). *Assume that* ψ_0 *is the characteristic function of* $\mathcal{S}_0$. *Then, there exists a positive constant* $C = C(\Omega, \mathcal{S}_0, C_1, C_2, \mathcal{A})$ *such that, for all* $\rho \in L^\infty((0,T)\times\Omega)$, $\boldsymbol{v} \in L^\infty(0,T;L^2(\Omega))$ *such that* $\rho(t,\cdot) \in [C_3, C_4] \subset (0,\infty)$ *for a.e.* $t \in [0,T]$, *we have*

$$\left\|\frac{\partial \boldsymbol{\Lambda}_{[\psi,\rho,\boldsymbol{v}]}}{\partial t}\right\|_{L^2(0,T;L^\infty(\Omega))} \leqslant C(1 + \|\boldsymbol{v}\|_{L^2(0,T;L^2(\Omega))})$$

and

$$\|\boldsymbol{\Lambda}_{[\psi,\rho,\boldsymbol{v}]}\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\nabla\boldsymbol{\Lambda}_{[\psi,\rho,\boldsymbol{v}]}\|_{L^\infty(0,T;L^\infty(\Omega))} + \|\Delta\boldsymbol{\Lambda}_{[\psi,\rho,\boldsymbol{v}]}\|_{L^\infty(0,T;L^\infty(\Omega))} \leqslant C.$$

Lemma A.4 (Nečasová *et al.* [25, lemma 8]). *Assume* $\boldsymbol{u} \in L^\infty(0,T;L^2(\Omega))$, $\rho_{0\varepsilon} \in C^\infty(\mathbb{R}^3)$, $\rho_{0\varepsilon} \in [C_1, C_2] \subset (0,\infty)$ *for a.e.* $x \in \mathbb{R}^3$, $\psi_0 \in \mathrm{char}(\mathbb{R}^3)$, *and* $\mathcal{S}(\psi_0)$ *is bounded and of non-empty interior. Then the problem 5.1 admits a unique solution* $(\rho,\psi) \in L^\infty((0,T)\times\mathbb{R}^3)$. *Moreover, for a.e.* $t \in (0,T)$,

$$\rho(t) \in [C_3, C_4] \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{A 1}$$

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