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Inverse position analysis, workspace determination and position synthesis of parallel manipulators with 3-RSR topology

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SUMMARY
Manipulators with 3-RSR topology are three-degree-of-freedom parallel manipulators that may be either spherical or mixed-motion manipulators. The inverse position analysis (IPA) and the workspace determination of 3-RSR manipulators are addressed by means of a new approach. The new approach is centered on a particular form of the closure equations called compatibility equations. The compatibility equations contain only the six coordinates (end-effector coordinates) which locates the end-effector pose (position and orientation) with respect to the frame, and the geometric constants of the manipulator. When the manipulator geometry is assigned, the common solutions of the compatibility equations are the end-effector coordinates which identify the end-effector poses belonging to the manipulator workspace. Moreover, they can be the starting point to easily solve the IPA. The presented compatibility equations can be also used to solve the position synthesis of the 3-RSR manipulator. This way of solving the position synthesis will demonstrate that only approximated solutions exist when more than eight end-effector poses are given.

KEYWORDS: Kinematics; Parallel mechanisms; Position analysis; Workspace.

1. INTRODUCTION

Parallel manipulators (PMs) with less than six degrees of freedom (dof) have recently attracted the attention of the industrial and academic world. Three-dof PMs are an important subset of less-than-six-dof PMs.

Three-dof manipulators are usually classified according to the type of motion their end-effector can perform. This classification separates the 3-dof PMs into four groups: translational manipulator,[1–4] spherical manipulators,[5–8] planar manipulators[9] and mixed-motion manipulators.[10-12] Translational, spherical and planar manipulators make their end-effector perform only translational, spherical and planar motion respectively, whereas the end-effector of mixed-motion manipulators is not constrained to perform any pure motion (i.e. either translational or spherical or planar) during operation. The studies on 3-dof PMs showed that some topologies can be used to obtain manipulators belonging to any type of the above-listed manipulator groups.[13]

Only three out of the six coordinates of the end-effector space (i.e. the six geometric parameters required to identify the end-effector pose (position and orientation) with respect to the frame) can be arbitrarily chosen during the end-effector's path planning of a 3-dof PM. The choice of the three coordinates is easy for either translational or spherical or planar manipulators, whereas it may present some difficulties for mixed-motion manipulators since the independent coordinates are not easy to identify and, sometimes, any three coordinates cannot be chosen.

The solution of the inverse position analysis (IPA), which is the determination of the actuated-joint variables necessary to obtain a given end-effector pose, and has to be implemented during the end-effector path planning, depends on the chosen set of three end-effector coordinates in the 3-dof PMs. The IPA solution may be very difficult since the closure equation system to be solved may not be linear for some choices of end-effector coordinates.[12] Moreover, if any three end-effector coordinates can be chosen, the IPA problems that can be formulated are

$$\binom{6}{3},$$

i.e. 20, and as many ways to represent the manipulator workspace exist.

A family of 3-dof PMs is the one collecting all the manipulators with 3-RSR topology (Figure 1). 3-RSR

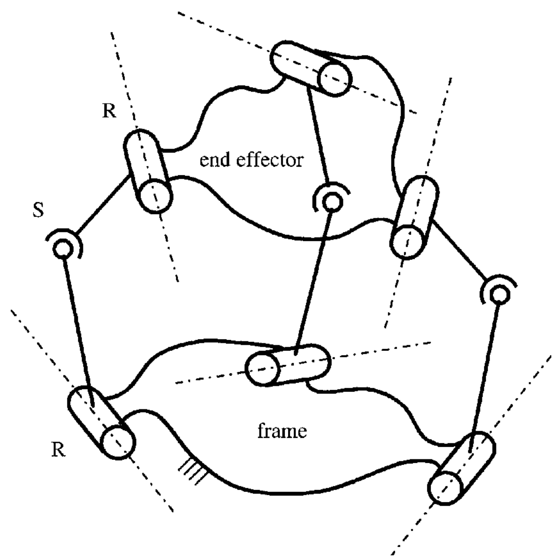

Fig. 1. Parallel manipulator with 3-RSR topology.

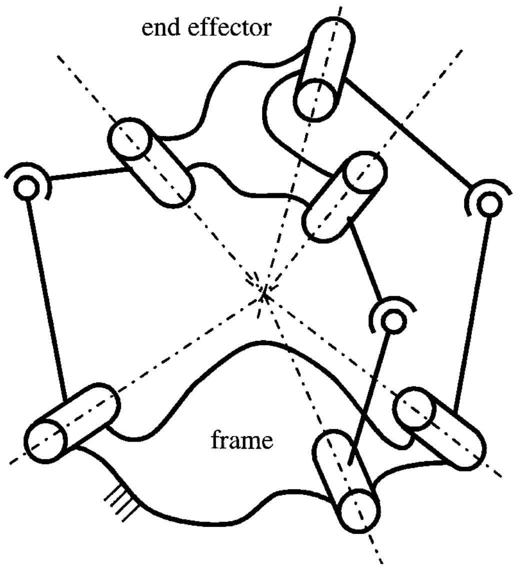

Fig. 2. The 3-RSR wrist.

manipulators have the end-effector connected to the frame by means of three serial kinematic chains (limbs) of type RSR (R and S stand for revolute pair and spherical pair, respectively). Most of such manipulators are mixed-motion manipulators. Nevertheless, if they are manufactured and assembled so that all the six revolute pair axes meet themselves at a unique point (Figure 2) a spherical manipulator is obtained.[14]

Dunlop and Jones[11] studied the position analysis of a mixed-motion 3-RSR manipulator which has three coplanar revolute pair axes both in the end-effector and in the frame (Figure 3). Moreover, they reconstruct the history of their manipulator as far as the attribution of the original mechanism to a 1968 development by Phillips and Sherwood. Their paper highlights that the inverse position analysis of a 3-RSR manipulator with general geometry is a complex problem that has not been solved yet.

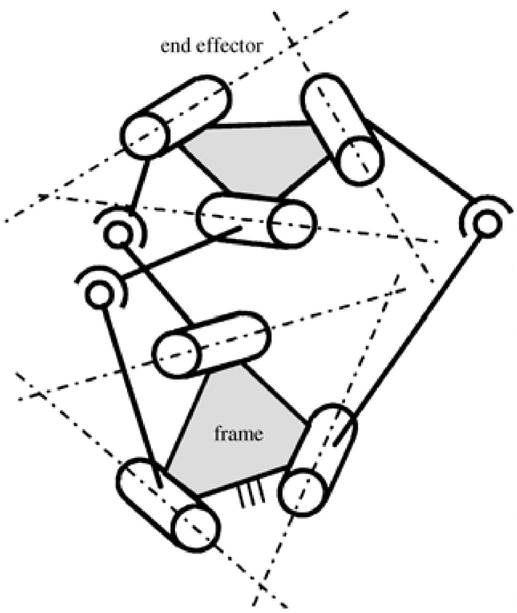

Fig. 3. Dunlop and Jones' 3-RSR manipulator.

In this paper, the inverse position analysis and the workspace determination of the manipulators with 3-RSR topology are addressed by means of a new approach. The new approach is centered on a particular form of manipulator's closure equations called compatibility equations. The compatibility equations contain only the six coordinates of the end-effector space and the geometric constants of the manipulator. When the manipulator geometry is assigned, the common solutions of the compatibility equations are the end-effector coordinates which identify the end-effector poses belonging to the manipulator workspace. Moreover, they can be the starting point to easily solve the IPA.

The presented compatibility equations can also be used to solve the position synthesis of the 3-RSR manipulator, that is the determination of the geometric constant values of the 3-RSR manipulators whose end-effector can reach a set of given poses. This way of solving the position synthesis will demonstrate that only approximated solutions exist when more than eight end-effector poses are given.

2. CLOSURE EQUATIONS

Figure 4 shows the i-th limb of type RSR and the notations that will be used. With reference to Figure 4, A_i is the center of the spherical pair. B_i is the foot of the perpendicular through A_i to the revolute pair axis fixed to the end-effector. C_i is the foot of the perpendicular through A_i to the revolute pair axis fixed to the frame. q_i and r_i are the lengths of the segments A_iB_i and A_iC_i respectively. $\mathbf{u}_i$ and $\mathbf{v}_i$ are the unit vectors of the revolute pair axes fixed to the frame and to the end-effector respectively. $\mathbf{m}_i$ and $\mathbf{n}_i$ are two mutually orthogonal unit vectors that are fixed to the frame and perpendicular to $\mathbf{u}_i$. Angle θ_i is the joint variable of the revolute pair adjacent to the frame which is the only actuated pair. S_f and S_e are two reference systems fixed to the frame and to the end-effector respectively. Points O and P are the origins of S_f and S_e, respectively.

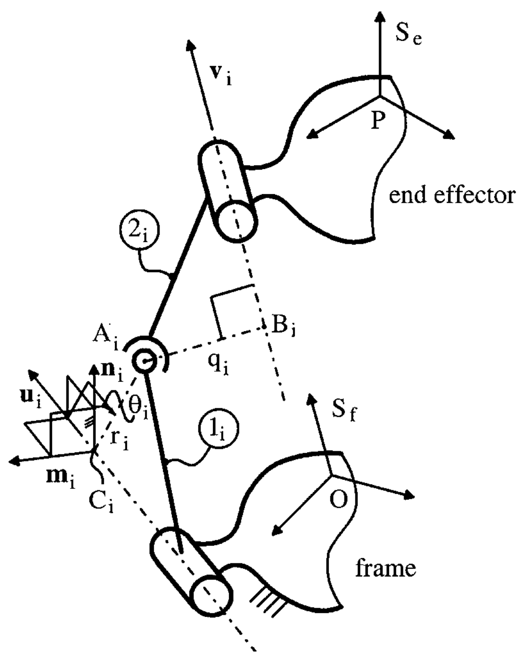

Fig. 4. The i-th limb of type RSR.

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During the end-effector motion, point A_i must lie both on the circumference, fixed to the end-effector and perpendicular to $\mathbf{v}_i$, with radius q_i and center B_i, and on another circumference, fixed to the frame and perpendicular to $\mathbf{u}_i$, with radius r_i and center C_i since it belongs both to link 2_i and to link 1_i (see Figure 4). This geometric condition can be expressed in analytical form by the equations

$$(\mathbf{A}_i - \mathbf{B}_i)^2 = q_i^2, \quad i = 1,2,3, \tag{1a}$$

$$(\mathbf{A}_i - \mathbf{B}_i) \cdot \mathbf{v}_i = 0, \quad i = 1,2,3, \tag{1b}$$

$$(\mathbf{A}_i - \mathbf{C}_i)^2 = r_i^2, \quad i = 1,2,3, \tag{1c}$$

$$(\mathbf{A}_i - \mathbf{C}_i) \cdot \mathbf{u}_i = 0, \quad i = 1,2,3, \tag{1d}$$

where the position vectors $\mathbf{A}_i$, $\mathbf{B}_i$ and $\mathbf{C}_i$ and the unit vectors $\mathbf{u}_i$ and $\mathbf{v}_i$ are measured in S_f. Position vector $\mathbf{A}_i$ can be written as follows (all the vector are measured in S_f)

$$\mathbf{A}_i = x_i\mathbf{u}_i + y_i\mathbf{m}_i + z_i\mathbf{n}_i, \quad i = 1, 2, 3, \tag{2}$$

where x_i, y_i and z_i are coefficients.

The dot product of the i-th equation (2) by $\mathbf{u}_i$ yields

$$x_i = \mathbf{A}_i \cdot \mathbf{u}_i, \quad i = 1, 2, 3. \tag{3}$$

The introduction of (3) into (1d) gives

$$x_i = \mathbf{C}_i \cdot \mathbf{u}_i, \quad i = 1, 2, 3. \tag{4}$$

Finally, the introduction of expression (2) into equation (1b), after the substitution of the right-hand side of (4) for x_i, gives

$$y_i(\mathbf{m}_i \cdot \mathbf{v}_i) + z_i(\mathbf{n}_i \cdot \mathbf{v}_i) = \mathbf{B}_i \cdot \mathbf{v}_i - (\mathbf{C}_i \cdot \mathbf{u}_i)(\mathbf{u}_i \cdot \mathbf{v}_i), \quad i = 1,2,3. \tag{5}$$

On the other side, by expanding the expressions at the left-hand side, equations (1a) and (1c) become

$$\mathbf{A}_i^2 + \mathbf{B}_i^2 - 2\,\mathbf{A}_i \cdot \mathbf{B}_i = q_i^2, \quad i = 1, 2, 3, \tag{6a}$$

$$\mathbf{A}_i^2 + \mathbf{C}_i^2 - 2\,\mathbf{A}_i \cdot \mathbf{C}_i = r_i^2, \quad i = 1, 2, 3, \tag{6b}$$

The substitution of the difference between equations (6a) and (6b) for equation (6b) into system (6) yields the equivalent system

$$\mathbf{A}_i^2 + \mathbf{B}_i^2 - 2\,\mathbf{A}_i \cdot \mathbf{B}_i = q_i^2, \quad i = 1, 2, 3, \tag{7a}$$

$$2\,\mathbf{A}_i \cdot (\mathbf{C}_i - \mathbf{B}_i) = q_i^2 - r_i^2 + \mathbf{C}_i^2 - \mathbf{B}_i^2, \quad i = 1, 2, 3. \tag{7b}$$

Finally, the introduction of (2) and (4) into (7) yields

$$y_i^2 + z_i^2 - 2\,y_i(\mathbf{m}_i \cdot \mathbf{B}_i) - 2\,z_i(\mathbf{n}_i \cdot \mathbf{B}_i) = q_i^2 - \mathbf{B}_i^2 - (\mathbf{C}_i \cdot \mathbf{u}_i)^2 + 2\,(\mathbf{C}_i \cdot \mathbf{u}_i)(\mathbf{B}_i \cdot \mathbf{u}_i), \quad i = 1, 2, 3, \tag{8a}$$

$$2\,y_i[\mathbf{m}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)] + 2\,z_i[\mathbf{n}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)] = q_i^2 - r_i^2 + \mathbf{C}_i^2 - \mathbf{B}_i^2 - 2(\mathbf{C}_i \cdot \mathbf{u}_i)[\mathbf{u}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)], \quad i = 1, 2, 3. \tag{8b}$$

Equations (5) and (8) are equivalent to equations (1) and are one expression of the system of closure equations of the 3-RSR manipulator with general geometry.

3. COMPATIBILITY EQUATIONS

The system of closure equations (5) and (8) is composed of six linear equations (equations (5) and (8b)) and three quadratic equations (equations (8a)) in y_i and z_i for $i = 1, 2, 3$. By using the linear equations, the following explicit expressions can be found

$$y_i = \frac{d_i[\mathbf{n}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)] - e_i(\mathbf{n}_i \cdot \mathbf{v}_i)}{h_i}, \quad i = 1, 2, 3, \tag{9a}$$

$$z_i = \frac{e_i[\mathbf{m}_i \cdot \mathbf{v}_i) - d_i(\mathbf{m}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)]}{h_i}, \quad i = 1, 2, 3, \tag{9b}$$

where

$$d_i = \mathbf{B}_i \cdot \mathbf{v}_i - (\mathbf{C}_i \cdot \mathbf{u}_i)(\mathbf{u}_i \cdot \mathbf{v}_i), \quad i = 1, 2, 3, \tag{10a}$$

$$e_i = 0.5(q_i^2 - r_i^2 + \mathbf{C}_i^2 - \mathbf{B}_i^2) - (\mathbf{C}_i \cdot \mathbf{u}_i)[\mathbf{u}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)], \quad i = 1, 2, 3, \tag{10b}$$

$$h_i = (\mathbf{m}_i \cdot \mathbf{v}_i)[\mathbf{n}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)] - (\mathbf{n}_i \cdot \mathbf{v}_i)[\mathbf{m}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)], \quad i = 1, 2, 3. \tag{10c}$$

The introduction of (10) into (8a) yields

$$\{d_i[\mathbf{n}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)] - e_i(\mathbf{n}_i \cdot \mathbf{v}_i)\}^2 + \{e_i(\mathbf{m}_i \cdot \mathbf{v}_i) - d_i[\mathbf{m}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)]\}^2 - 2h_i(\mathbf{m}_i \cdot \mathbf{B}_i)\{d_i[\mathbf{n}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)] - e_i(\mathbf{n}_i \cdot \mathbf{v}_i)\} - 2h_i(\mathbf{n}_i \cdot \mathbf{B}_i)\{e_i(\mathbf{m}_i \cdot \mathbf{v}_i) - d_i[\mathbf{m}_i \cdot (\mathbf{C}_i - \mathbf{B}_i)]\} - h_i^2[q_i^2 - \mathbf{B}_i^2 - (\mathbf{C}_i \cdot \mathbf{u}_i)^2 + 2\,(\mathbf{C}_i \cdot \mathbf{u}_i)(\mathbf{B}_i \cdot \mathbf{u}_i)] = 0, \quad i = 1, 2, 3, \tag{11}$$

where d_i, e_i and h_i are given by definitions (10).

In equations (11), q_i, r_i, $\mathbf{u}_i$, $\mathbf{m}_i$, $\mathbf{n}_i$ and $\mathbf{C}_i$ are geometric constants of the manipulator, whereas $\mathbf{v}_i$ and $\mathbf{B}_i$ depend on end-effector's pose and geometry and have the expressions (see Figure 4).

$$v_i = \mathbf{R}_{fe}\,{}^e\mathbf{v}_i, \quad i = 1, 2, 3, \tag{12a}$$

$$\mathbf{B}_i = \mathbf{P} + \mathbf{R}_{fe}\,{}^e(\mathbf{B}_i - \mathbf{P}), \quad i = 1, 2, 3, \tag{12b}$$

where ${}^e\mathbf{v}_i$ and ${}^e(\mathbf{B}_i - \mathbf{P})$ are the vectors $\mathbf{v}_i$ and $(\mathbf{B}_i - \mathbf{P})$, respectively, measured in S_e, and are geometric constants of the end effector. $\mathbf{P}$ is the position vector in S_f of the origin of S_e, and $\mathbf{R}_{fe}$ is the rotation matrix transforming vector components measured in S_e into the components of the same vector measured in S_f. $\mathbf{P}$ and $\mathbf{R}_{fe}$ depend on the six coordinates of the end-effector space.

In conclusion, equations (11) are a particular form of the closure equation system which contains only the geometric constants of the manipulator and the six coordinates of the end-effector space. Equations (11) will be called compatibility equations.

4. USE OF THE COMPATIBILITY EQUATIONS

Equations (11) can be easily managed by an algebraic manipulator and can be used to solve various problems regarding the design and the operation of the manipulators with 3-RSR topology. The following part of this section will show how to use the compatibility equations for (i) the solution of the inverse position analysis, (ii) the determination of the workspace and (iii) the position synthesis.

4.1. Inverse position analysis

The manipulator geometric constants are known in the position analysis.

The inverse position analysis of a 3-dof manipulator of mixed-motion type consists in the determination of the actuated-joint variables once three out of the six coordinates of the end-effector space are given. Since three coordinates can be chosen among six coordinates in twenty different ways, twenty different IPA problems can be formulated.

In the case of the manipulators with 3-RSR topology, the availability of the compatibility equations allows the same procedure to be used for solving any problem out of the twenty ones. In fact, the following algorithm can be implemented:

(a.1) By means of an algebraic manipulator, the geometric constants of the 3-RSR manipulator are introduced into equations (11) which become a system of three equations in six unknowns: the six end-effector coordinates.

(a.2) The values of the three end-effector coordinates, that are assigned in the IPA problem to be solved, are introduced into the three equations obtained from the step (a.1), and the resulting system is solved to determine the other three end-effector coordinates.

(a.3) The vectors $\mathbf{v}_i$ and $\mathbf{B}_i$ measured in S_f are computed by using formulas (12) and the six values of the three assigned end-effector coordinates and of the other three end-effector coordinates computed in the step (a.2). Then, the vectors $\mathbf{A}_i$, i = 1, 2, 3, measured in S_f, are computed by using the formulas (2) together with the values of the coefficients x_i, y_i and z_i computed by means of formulas (4) and (9).

(a.4) The values of the actuated-joint variables θ_i for i = 1, 2, 3 are computed by introducing the geometric constants of the manipulators and the vectors $\mathbf{A}_i$ for i = 1, 2, 3, computed in the step (a.3), into the formulas

$$\theta_i = \mathrm{Atan2}[\mathbf{n}_i \cdot (\mathbf{A}_i - \mathbf{C}_i),\ -\mathbf{m}_i \cdot (\mathbf{A}_i - \mathbf{C}_i)], \quad i = 1, 2, 3, \tag{13}$$

where Atan2[c, s] is a function that evaluates arctan(s/c) by taking into account which quadrant the point (c, s) is in.

4.2. Workspace determination

When the geometric constants of the 3-RSR manipulator are assigned, equations (11) become a system of three equations whose unknowns are the six coordinates of the end-effector space. In this case, the solutions of equations (11) are a three-dimensional subspace of the end-effector space, and are the coordinates which individuate all the poses, the end effector would be able to assume if the mobility of the kinematic pairs was not limited by link interference. Therefore, system (11) is the analytic expression of the unconstrained workspace of the 3-RSR manipulator.

The limitations on the mobility of the kinematic pairs can be transformed into inequalities whose only unknowns are the six coordinates of the end-effector space by exploiting the relationships that were used to deduce equations (11). For instance, the limitations on the actuated-joint variables θ_i for i = 1, 2, 3 are expressed by

$$\theta_{i,min} \le \theta_i \le \theta_{i,max}, \quad i = 1, 2, 3, \tag{14}$$

where $\theta_{i,min}$ and $\theta_{i,max}$ are the minimum and maximum values, respectively, θ_i can assume. Such limitations can be transformed, by using relationships (13), into

$$\theta_{i,min} \le \mathrm{Atan2}[\mathbf{n}_i \cdot (\mathbf{A}_i - \mathbf{C}_i),\ -\mathbf{m}_i \cdot (\mathbf{A}_i - \mathbf{C}_i)] \le \theta_{i,max}, \quad i = 1, 2, 3, \tag{15}$$

where $\mathbf{C}_i$, $\mathbf{n}_i$ and $\mathbf{m}_i$ are geometric constants, whereas $\mathbf{A}_i$ is expressed by using relationships (2), (4), (9) and (12), and is function only of the six coordinates of the end-effector space. The limitations of the joint variables of the three passive revolute pairs can be treated in an analogous way. Finally, the limitations on the mobility of the three spherical pairs can be transformed into limitations on the relative orientation between the vectors $(\mathbf{A}_i - \mathbf{C}_i)$ and $(\mathbf{B}_i - \mathbf{A}_i)$ which are functions only of the six coordinates of the end-effector space.

Once the limitations on the mobility of the kinematic pairs have been transformed into a system of inequalities whose only unknowns are the six end-effector coordinates, the workspace (constrained workspace) which takes into account such limitations can be determined by implementing the following algorithm:

(b.1) Three out of the six coordinates of the end-effector space are chosen as generalized coordinates of the manipulator to be used in the computation of the constrained workspace.

(b.2) A three-dimensional grid of values of the generalized coordinates chosen in step (b.1) is defined so that the range of variation of each coordinate is uniformly covered.

(b.3) A set of three values (one for each generalized coordinate) belonging to the grid defined in step (b.2) is substituted for the generalized coordinates into system (11). Then, the resulting system is solved to compute the values of the three remaining end-effector coordinates.

(b.4) The set of six values of end-effector coordinates (the assigned three and the computed three in the step (b.3)) is introduced into the inequality system which takes into account the limitations on the mobility of the kinematic pairs. If the system of inequalities is satisfied, the set of values of end-effector coordinates is stored since it belongs to the constrained workspace; otherwise, it is discarded.

(b.5) The steps (b.3) and (b.4) are repeated for all the sets of three values (one for each generalized coordinate) belonging to the grid chosen in step (b.2).

4.3. Position synthesis

The geometry of the frame (end-effector) is fully defined when the relative positions of the three revolute pair axes fixed in the frame (end-effector), together with the positions of the points C_i (B_i) for $i=1, 2, 3$ (see Figure 4) on those axes are assigned. These data can be assigned by providing the values of nine geometric parameters. For instance, the ones indicated in Figure 5a (Figure 5b), where the meaning of the symbols is immediately evident. Moreover, the geometry of the i-th limb (Figure 4) is assigned when the distances r_i and q_i are given. Therefore, the geometry of a 3-RSR manipulator is assigned when 24 suitable geometric parameters (9 for the frame, 9 for the end-effector and 6 for the limbs (2 per limb)) are given.

The position synthesis of a 3-RSR manipulator consists in determining the values of the 24 geometric parameters of the manipulator so that its end effector can assume a number of assigned poses.

When the six coordinates of the end-effector space are assigned, the compatibility equations (11) become three equations in 24 unknowns: the geometric parameters which define the geometry of the manipulator. Thus, a set of three equations, whose unknowns are the 24 geometric parameters, can be written for each assigned end-effector pose by introducing into equations (11) the values of the six end-effector coordinates corresponding to that pose. If p is the number of the given poses the end-effector has to assume, a system of $3\times p$ non-linear equations in the 24 geometric parameters of the manipulator will result. The solutions of such a system are the solutions of the position synthesis. The number of solutions of the position synthesis depends on the value of p and three cases are possible: p less than 8, p equal to 8 and p greater than 8. If p is less than 8, there will be $\infty^{(24-3\times p)}$ solutions. If p is equal to 8, there will be a finite number of solutions. Finally, if p is greater than 8, the number of equations will be greater than the number of unknowns and only approximate solutions can be found by using optimization techniques, e.g. the least squares method.

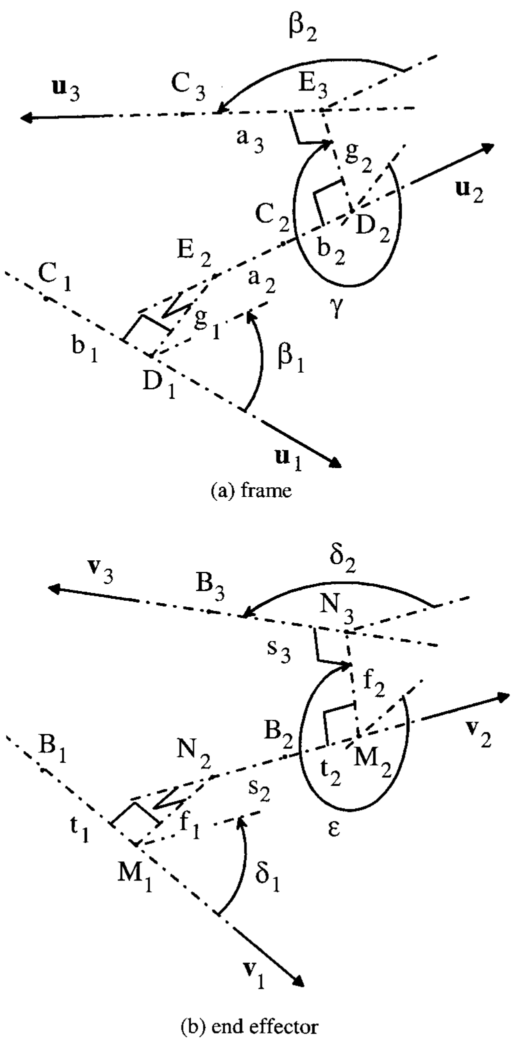

Fig. 5. Geometric parameters which identify the geometry of the frame (a) and of the end effector (b).

5. CONCLUSIONS

Manipulators with 3-RSR topology are three-degree-of-freedom parallel manipulators that may be either spherical or mixed-motion manipulators.

A new approach has been proposed to address the inverse position analysis (IPA), the workspace determination and the position synthesis of 3-RSR manipulators. The new approach is centered on a particular form of the closure equations called compatibility equations. The compatibility equations contain only the six coordinates (end-effector coordinates) which locates the end-effector pose (position and orientation) with respect to the frame, and the geometric constants of the manipulator.

When the manipulator geometry is assigned, the common solutions of the compatibility equations are the end-effector coordinates which identify the end-effector poses belonging to the manipulator workspace. Moreover, they can be the starting point to easily solve the IPA.

Finally, the compatibility equations can be used to generate the system of equations necessary for finding the solutions of the position synthesis of the 3-RSR manipulator (i.e. the determination of the geometry of the 3-RSR manipulators whose end effector can assume a set of given poses). This way of addressing the position synthesis has demonstrated that only approximated solutions exist when more than eight end-effector poses are given.

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