



Slice-torus Concordance Invariants and Whitehead Doubles of Links

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Abstract. In this paper we extend the definition of slice-torus invariant to links. We prove a few properties of the newly-defined slice-torus link invariants: the behaviour under crossing change, a slice genus bound, an obstruction to strong sliceness, and a combinatorial bound. Furthermore, we provide an application to the computation of the splitting number. Finally, we use the slice-torus link invariants and the Whitehead doubling to define new strong concordance invariants for links, which are proven to be independent of the corresponding slice-torus link invariant.

1 Introduction

The study of knots up to concordance is extremely relevant in low-dimensional topology, and it has been extensively pursued. Two (smooth) knots in \mathbb{S}^3 are said to be *concordant* if they bound a properly (smoothly) embedded annulus in $\mathbb{S}^3 \times [0, 1]$. A knot is called *slice* if it is concordant to the unknot. The set of knots up to concordance, endowed with the operation of connected sum, is an infinitely generated Abelian group \mathcal{C} , called the *concordance group*, whose neutral element is the class of slice knots.

The advent of knot homologies (such as knot Floer homology and Khovanov–Rozansky homologies) led to the definition of a variety of new tools to study concordance, to obstruct sliceness and to compute the slice genus (*i.e.*, the minimal genus of a surface properly embedded in \mathbb{D}^4 bounding the given knot). The earlier among these tools are the Ozsváth–Szabó invariant τ [27] and the Rasmussen invariant s [29]. These invariants (once suitably normalized) share lot of properties, and Livingston [21] decided to study them in a more general framework. Livingston defined a class of invariants, which were later named by Lewark [20], as follows.

Definition 1.1 ([20, 21]) A *slice-torus invariant* is a knot concordance invariant v such that the following hold:

- ▷ $v : \mathcal{C} \rightarrow \mathbb{R}$ is a group homomorphism;
- ▷ $|v(K)| \leq g_4(K)$, where g_4 denotes the slice genus, for each knot K ;
- ▷ for each torus knot $T(p, q)$ we have that

$$v(T(p, q)) = \frac{(p-1)(q-1)}{2}.$$

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The family of the slice torus invariants also includes, aside from τ and $s/2$, (a suitable normalization of) the \mathfrak{sl}_n ($n \geq 3$) analogues of the Rasmussen invariant. These invariants, denoted by s_n , were introduced independently by Lobb [23] and Wu [35]. The s_n 's provide orthogonal information with respect to τ and s , and they were shown to be linearly independent from s and τ by Lewark ([20]).

The slice-torus invariants can be used to produce other concordance invariants. For example, using the fact that if two knots are concordant then also their Whitehead doubles are also concordant, Livingston and Naik ([22]) defined¹ the functions

$$F_v(K)(t) = \nu(W_+(K, t)) \quad \text{and} \quad \bar{F}_v(K)(t) = \nu(W_-(K, t)), \quad t \in \mathbb{Z},$$

where ν is a slice-torus invariant and $W_{\pm}(K, t)$ denotes the positive (resp. negative) t -twisted Whitehead double of K . These functions are non-increasing, non-constant, take values respectively in $[0, 1]$ and $[-1, 0]$, and assume both the maximal and the minimal possible values. In particular, if the slice-torus invariant is integer-valued all the information contained in each function can be condensed into a single integer. These integers, denoted by t_v and \bar{t}_v , are defined as the maximal value of t such that $F_v(K, t)$ and $\bar{F}_v(K, t)$, respectively, assume their maximum. It is not difficult to see that $\bar{t}_v(K) = -t_v(-K^*) - 1$, where $-K^*$ is the mirror image of K with the orientation reversed, so these invariants contain the same amount of information. At the time of writing it is still unknown whether the invariant t_v can provide new information with respect to ν . In fact, there are some hints in the opposite direction; for instance, it is known that $t_{\tau} = 2\tau - 1$ ([15, Theorem 1.5]), and it has been conjectured that $t_{s/2} = 3s/2 - 1$ ([28]).

The aim of this paper is to extend these definitions and constructions to the case of links and to describe some applications and examples. Before stating the main results of this paper, let us recall a few basic facts about link concordance. The first thing one should point out is that the definition of concordance is no longer unique. Two oriented links in \mathbb{S}^3 are said to be

- ▷ *weakly concordant* if there exists a genus 0 connected, compact, oriented surface, properly embedded in $\mathbb{S}^3 \times [0, 1]$, bounding the two links;
- ▷ *strongly concordant* if there exists a disjoint union of annuli, properly embedded in $\mathbb{S}^3 \times [0, 1]$, such that each of them bounds a component of each link.

In particular, strongly concordant links should have the same number of components. A link is said to be *weakly* (resp. *strongly*) *slice* if it is weakly (resp. strongly) concordant to an unlink. Similarly, one can define a (*weak*) *slice genus* and a *strong slice genus*. The former is just the minimal genus of any connected, compact, oriented surface properly embedded in $\mathbb{S}^3 \times [0, 1]$ bounding the link. The latter has a similar definition but one has to consider only the surfaces such that each connected component bounds exactly one component of the link. The (weak) slice genus of a link shall be denoted by g_4 .

Almost all the slice-torus invariants known to the authors can be extended to strong concordance invariants of links (see [4, 7, 18]), and they give rise to bounds on the slice genus. Thus far these invariants have been studied separately. Motivated by the common properties of these extended slice-torus invariants, in Section 2 we

¹These functions were originally defined only for integer-valued slice-torus invariants. Of course, the same definition works for all slice-torus invariants, and most of the properties proved in [22] still hold.

give the definition of slice-torus link invariants. For now, the reader should keep in mind that these are \mathbb{R} -valued strong concordance invariants, and that when restricted to knots these invariants give rise to \mathbb{R} -valued slice-torus invariants (Corollary 2.12). Moreover, the slice-torus link invariants include (once properly rescaled and translated) the extension to links of τ ([7]), s ([4]), and the s_n invariants ([18]). Some of the results in this paper concerning the slice-torus link invariants were proved separately for τ , s , and the s_n 's. We will mention whenever a result was known for one or more of the above-mentioned invariants, or if it is completely new.

The first result, which is known for τ , s and s_n ($n \geq 3$), consists of a bound on the slice genus and an obstruction to strong sliceness.

Proposition 1.2 *If v is a slice-torus link invariant and L is an ℓ -component link, then*

$$-g_4(L) \leq v(L) \leq g_4(L) + \ell - 1.$$

Furthermore, if L is a strongly slice link, then $v(L) = 0$.

Another known property of s , τ , and s_n is the detection of the 3-dimensional and, under the hypothesis of non-splitness, the slice genus of positive links. It turns out that these results holds for any slice-torus link invariant.

Theorem 1.3 *Let L be an ℓ -component positive link, and let D be a positive diagram representing L . Then*

$$v(L) = g_3(L) + \ell - \ell_s = \frac{n(D) - O(D) + \ell}{2}$$

for each slice-torus link invariant v , where $n(D)$ is the number of crossings of D , $O(D)$ denotes the number of Seifert circles, and ℓ_s is the number of split components of L . Furthermore, if L is also non-split then

$$v(L) = g_4(L) + \ell - 1.$$

Computing the value of slice-torus link invariants for non-positive links can be difficult. However we provide a combinatorial bound whose proof appears in Section 2, which allows us to compute the slice-torus link invariants for certain classes of links, namely quasi-positive links (Theorem 3.2) and negative links (Proposition 2.16). This bound was known for the s -invariant (e.g., [6, 16]), but unknown for τ and the s_n 's. Moreover, the values of the s_n 's for quasi-positive and negative links and the value of τ for negative links were unknown (the value of τ for quasi-positive links was computed in [8]).

Theorem 1.4 *Let L be an ℓ -component link, let ℓ_s be the number of split components of L , and let v be a slice-torus link invariant. For each non-splittable² diagram D representing L , the following inequality holds:*

$$(1.1) \quad \frac{w(D) - O(D) + 2s_+(D) + \ell - 2\ell_s}{2} \leq v(L).$$

²That is, the number of connected components of D is the number of split components of L . Equivalently, we cannot obtain another diagram for L with more connected components than D .

The first truly novel application of the slice-torus link invariants is the computation of the splitting number. Before proceeding further, we wish to remind the reader of the definition of the two main versions of the splitting number. Following [2], the *splitting number* \tilde{sp} is the minimal number of crossing changes (among all diagrams) necessary to transform an ℓ -component link L into the disjoint union of ℓ knots. A second version of the splitting number, which was studied, for example, in [9, 10], has a similar definition, but the only crossing changes allowed are those between different components. We denote this second version by sp , and call it the *strong splitting number*. Clearly, we have the inequality $\tilde{sp} \leq sp$, but the equality does not hold in general. We prove that the slice-torus link invariants can be used to produce a lower bound for the splitting number \tilde{sp} .

Theorem 1.5 *Suppose that v is a slice-torus link invariant and L is a link with components K_1, \dots, K_ℓ . Then we have*

$$(1.2) \quad \left| v(L) - \sum_{i=1}^{\ell} v(K_i) \right| \leq \tilde{sp}(L).$$

Furthermore, we provide an infinite family of examples where our bound on \tilde{sp} is sharp and $\tilde{sp} \neq sp$ (Proposition 3.7). To the best of our knowledge, there is no other known method to compute the value of the splitting number for this family of links. A weaker version of Theorem 1.5 featuring the strong splitting number was proved, for the s_n -invariants, in [18].

Remark 1.6 Few months after this paper was posted on the arxiv, the authors were informed that another bound on the splitting number \tilde{sp} was previously discovered by A. Conway in his Ph.D. thesis [13, Proposition 4.4.5]. Conway's bound is not published and does not appear in the arxiv and uses completely different techniques from the ones used in this paper. The main ingredients for Conway's bound are the multivariate signature and nullity. It can also be checked that Conway's bound can be used to compute the splitting number for the family L_t in Proposition 3.7. Nonetheless, we expect the two bounds to be independent. Since the comparison between the two bounds falls outside the scope of this paper, we leave the discussion of this topic to a forthcoming paper.

The final part of our paper is dedicated to the definition of new strong concordance invariants via Whitehead doubling. The notion of Whitehead double for links is not unique. We will be interested in two different kinds of Whitehead doubles. The first kind is the fully clasped Whitehead double $W_{\pm}(L, \underline{m})$, which is basically obtained by doubling all the components, where $\underline{m} \in \mathbb{Z}^{\ell}$ encodes the number of twists in the double of each component. The second type of Whitehead double we are interested in is the reduced Whitehead double $W'_{\pm}(L, m; L_1)$, which is obtained by doubling only the component L_1 and inserting $m \in \mathbb{Z}$ full twists. Notice that in the case L is a knot, the two constructions yield the same result: the m -twisted Whitehead double of L .

We use the fact that if two links are strongly concordant, then their Whitehead doubles are also strongly concordant (Theorem 4.1) to define four functions that are

strong concordance invariants. More precisely, we consider the functions

$$F_v(L)(\underline{t}) = v(W_+(L, \underline{t})) \quad \text{and} \quad F'_v(L; L_1)(t) = v(W'_+(L, t; L_1))$$

and

$$\bar{F}_v(L)(\underline{t}) = v(W_-(L, \underline{t})) \quad \text{and} \quad \bar{F}'_v(L; L_1)(t) = v(W'_-(L, t; L_1)),$$

where v is a slice-torus link invariant. These functions generalize the functions $F_v(K)(t)$ and $\bar{F}_v(K)(t)$, and thus the invariant t_v defined by Livingston and Naik [22].

Theorem 1.7 *Suppose that v is a slice-torus link invariant, L is a link, and L_1 a component of L . The functions $F_v(L)$, $F'_v(L; L_1)$, $\bar{F}_v(L)$, and $\bar{F}'_v(L; L_1)$ are non-increasing and bounded. Furthermore, $F_v(L)$ and $\bar{F}_v(L)$ are non-constant and assume the maximal possible value.*

As an application, we show that these functions can be used to obstruct the existence of a strong concordance to a split link (Theorem 6.3). We conclude the paper with some sample computations, proving the following result, which is still unknown in the case of knots.

Theorem 1.8 *There exists a 2-component link L and a slice-torus link invariant v such that the function F'_v does not depend only on the linking matrix of L and on $v(L)$.*

2 Slice-torus Link Invariants

In this section, we introduce the slice-torus link invariants and prove their first properties. We start by proving that slice-torus link invariants have a controlled behaviour with respect to the crossing changes. This will be fundamental in the last part of the paper. Afterward, we prove the bound on the slice genus, which follows from a more general bound on the difference of the slice-torus link invariants of cobordant links. Finally, we compute the value of the slice-torus link invariants of the positive links, and we use it to produce the combinatorial bound. From the combinatorial bound will follow the computation of the value of slice-torus link invariants of negative links.

2.1 Definition and First Properties

Let us start with the definition of slice-torus link invariants.

Definition 2.1 *A slice-torus link invariant is an \mathbb{R} -valued strong concordance link invariant v satisfying the following properties:*

- (A) if L_1 and L_2 are related by an oriented band move, and L_1 has one component less than L_2 (cf. Figure 1), then

$$v(L_2) - 1 \leq v(L_1) \leq v(L_2);$$

- (B) v is additive under disjoint union, that is $v(L_1 \sqcup L_2) = v(L_1) + v(L_2)$;

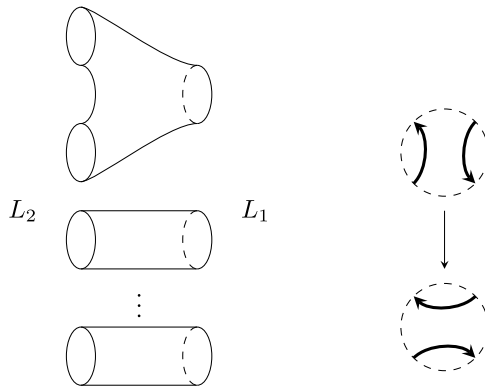


Figure 1: A schematic description of a band move between the links L_1 and L_2 (left), and the local description of an oriented band move (right).

(C) for each ℓ -component link L we have

$$0 \leq v(L) + v(-L^*) \leq \ell - 1,$$

where L^* denotes the mirror image of L , and $-L$ denotes L with the orientation reversed;

(D) if $T_{p,q}$ is the positive (p, q) -torus knot, then

$$v(T_{p,q}) = \frac{(p-1)(q-1)}{2}.$$

Property (C) in this definition can be relaxed in the following sense: we can require the inequality in the property to hold only in the case of knots. Thus, we obtain that for each knot K , we have

$$v(-K^*) = -v(K).$$

With this different definition in place, we can prove the following property:

(C') for every ℓ -component link L , we have

$$0 \leq v(L) + v(-L^*) \leq \ell,$$

and $v(-K^*) = -v(K)$ for each knot K .

This is done by observing that we can obtain a strongly slice link if we perform ℓ band moves on the link $L \sqcup -L^*$. In turn, this can be seen by putting a diagram of L , and its mirror image with reverse orientation, in a symmetric position with respect to a line. Then add an unknotted band between each pair of corresponding components. The result of this operation is a link bounding a ribbon surface that is the union of ribbon disks, and thus is a strongly slice link. Now (C') follows from Property (B). All the results in the paper, with the exception of Proposition 4.9 and some computations in Section 5, remain true replacing Property (C) with Property (C') in the definition of slice-torus link invariant.

Our choice of Property (C) is motivated by the fact that, once suitably normalised, s ([4, Lemma 6.1]), the s_n 's ([18, Theorem 3]), and τ satisfy it (see the examples below for the normalizations). For the latter invariant, Property (C) follows from the additivity with respect to the connected sum ([7, Subsection 3.3]), and from the following lemma.

Lemma 2.2 *Suppose that v is a strong concordance invariant that satisfies Properties (A), (B), and (D) and is additive under connected sums of links. Then Property (C) also holds and v is a slice-torus link invariant.*

Proof As we remarked before, we can apply ℓ band moves on $L \sqcup -L^*$, each one between a component of L and its corresponding mirror image, in the way that the result is strongly slice. We observe that the first of these moves changes $L \sqcup -L^*$ into $L' = L\# -L^*$; then we have that $v(L') = v(L) + v(-L^*)$ by assumption.

At this point, applying Property (A) $\ell - 1$ times yields to

$$0 \leq v(L') = v(L) + v(-L^*) \leq \ell - 1,$$

where we used that $v(J) = 0$ if J is strongly slice. This last claim follows from the additivity under connected sums and Property (B). ■

Example 2.3 The quantity $v_s = \frac{s+\ell-1}{2}$ is a slice-torus link invariant, where s is the extension of Rasmussen invariant (cf. [29]) to links introduced in [4], and ℓ is the number of components of the link.

Example 2.4 The Ozsváth–Szabó τ -invariant (cf. [27]), which was extended to links in [7], is a slice-torus link invariant.

Example 2.5 More generally, we can consider the \mathfrak{sl}_n version of the Rasmussen invariant, denoted with s_n , introduced by Lobb and Wu independently in [25, 35], which were extended to links in [18]. Then we have that

$$v_{s_n} = \frac{-s_n(L) + (\ell - 1)(n - 1)}{2(n - 1)}$$

is a slice-torus link invariant. In particular, if $n = 2$, then $s_n(L) = -s(L)$, and we recover the expression in Example 2.3.

Remark 2.6 The unknot can be seen as $T_{1,p}$. In particular, it follows from Property (D) that for each slice-torus link invariant, $v(\bigcirc) = 0$.

The value of a slice-torus link invariant on the Hopf link and the negative trefoil knot is constant (i.e., does not depend on the slice-torus link invariant). Since these values will be used in the follow up, we record them in the following lemma.

Lemma 2.7 *Let v be a slice-torus link invariant, and denote by H_{\pm} the positive (resp. negative) Hopf link. Then we have that $v(T_{2,3}^*) = -1$, $v(H_+) = 1$ and $v(H_-) = 0$.*

Proof The first equality follows directly from Properties (C) and (D) in the definition of slice-torus invariant, and from the fact that $T_{2,3}^* = -T_{2,3}^*$. As concerns the other

two equalities, notice that H_{\pm} can be obtained from both the positive (resp. negative) trefoil knot and the unknot via a band move. Thus, it follows from Property (A) that

$$1 = v(T_{2,3}) \leq v(H_+) \leq v(\bigcirc) + 1 = 1.$$

A similar reasoning works for the negative Hopf link. ■

Even though Property (C) implies that $v(-K^*) = -v(K)$ for each knot K , the previous lemma disproves the analogue of this result for multi-component links.

We now turn to another property of the slice torus link invariants: the additivity under connected sum of knots.

Proposition 2.8 *Let K_1 and K_2 be two oriented knots. Then for each slice-torus link invariant v , we have*

$$v(K_1 \# K_2) = v(K_1) + v(K_2),$$

where $\#$ denotes the connected sum.

Proof Since any connected sum can be obtained from a disjoint union via a band move, Properties (A) and (B) in Definition 2.1 tell us that

$$v(K_1 \# K_2) \leq v(K_1 \sqcup K_2) = v(K_1) + v(K_2).$$

The same reasoning applied to $-K_1^*$ and $-K_2^*$, together with Property (C), implies that

$$-v(K_1 \# K_2) = v(-(K_1 \# K_2)^*) \leq v(-K_1^*) + v(-K_2^*) = -v(K_1) - v(K_2),$$

and the equality follows. ■

Remark 2.9 Denote by $L_1 \#_{K_1, K_2} L_2$ the connected sum of L_1 and L_2 along the components K_1 and K_2 , respectively. Then it follows from Properties (A) and (B) that

$$v(L_1) + v(L_2) - 1 \leq v(L_1 \#_{K_1, K_2} L_2) \leq v(L_1) + v(L_2),$$

for each slice-torus link invariant v .

We conclude this subsection with the following proposition, concerning the behaviour of slice-torus link invariants under crossing changes. Recall that a *cobordism* between two oriented links L_0 and L_1 is an oriented compact surface Σ , properly embedded in $\mathbb{S}^3 \times [0, 1]$, such that

$$\Sigma \cap \{0\} = L_0, \quad \text{and} \quad \Sigma \cap \{1\} = -L_1,$$

where the orientation on the left-hand side of each equation is induced by Σ , and each connected component of Σ has boundary on both L_0 and L_1 . (This kind of cobordism is sometimes called *good cobordism* in the literature, e.g., [4, 29].)

Proposition 2.10 *Let D_+ and D_- be two link diagrams representing the links L_+ and L_- , respectively. Suppose that D_- is obtained from D_+ by replacing a positive crossing with a negative one; then*

$$v(L_-) \leq v(L_+) \leq v(L_-) + 1,$$

for each slice-torus link invariant v .

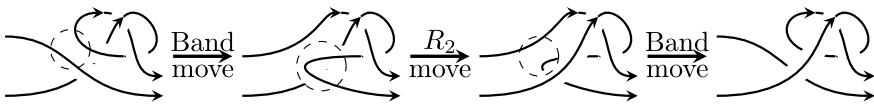


Figure 2: Two band moves relating $L_+ \#_K T_{2,3}^*$ and L_- .

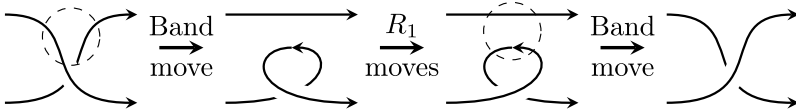


Figure 3: Two band moves relating L_+ and L_- .

Proof The links $L_+ \#_K T_{2,3}^*$ and L_- are related by two band moves, where K is a component of L_+ corresponding to a component of D_+ passing through the crossing changed. A movie describing the two band moves is shown in Figure 2.

The cobordism described in Figure 2 can have either genus 0 or genus 1, depending on whether the strands involved in the first band move belong to different components or not. In both cases, combining the inequalities given by Property (A), we obtain

$$v(L_-) - 1 \leq v(L_+ \#_K T_{2,3}^*).$$

Putting together the above inequality, the inequalities in Remark 2.9, and the computations in Lemma 2.7, we obtain

$$v(L_-) - 1 \leq v(L_+ \#_K T_{2,3}^*) \leq v(L_+) + v(T_{2,3}^*) = v(L_+) - 1,$$

which is the first inequality in the statement. To recover the second inequality, consider the cobordism in Figure 3. There are two cases to consider, depending whether or not the first band move merges two components. In both cases, Property (A) tells us that

$$v(L_+) - 1 \leq v(L_-),$$

and the result follows. ■

2.2 A Bound on the Slice-genus

In analogy with the case of the slice-torus invariants, each slice-torus link invariant gives rise to a lower bound for the slice genus. This bound is a consequence of the following more general proposition.

Proposition 2.11 *Let v be a slice-torus link invariant. Given two links L_0 and L_1 with ℓ_0 and ℓ_1 components, respectively, such that there exists a cobordism $\Sigma \subset \mathbb{S}^3 \times [0, 1]$ from L_0 to L_1 with k connected components, we have*

$$v(L_1) - g(\Sigma) - \ell_1 + k \leq v(L_0) \leq v(L_1) + g(\Sigma) + \ell_0 - k,$$

where $g(\Sigma)$ denotes the genus of Σ . In particular, when Σ is connected,

$$v(L_1) - g(\Sigma) - \ell_1 + 1 \leq v(L_0) \leq v(L_1) + g(\Sigma) + \ell_0 - 1.$$

Proof By standard arguments, up to a boundary fixing ambient isotopy we can assume that the projection onto the second factor

$$pr_2 : \mathbb{S}^3 \times [0, 1] \longrightarrow [0, 1],$$

when restricted to $\Sigma \setminus \partial\Sigma$ has only a finite number of (non-degenerate) critical values, let us denote these values by $0 < t_0 < \dots < t_h < 1$. Basic Morse theory tells us that we can assume the links $L_i^\pm = pr_2^{-1}(t_i \pm \epsilon) \cap \Sigma$, where $i \in \{0, \dots, h\}$ and $\epsilon > 0$ is sufficiently small, to be obtained one from the other by either an oriented band move (1-handle attachment), the split union with an unknot (0-handle attachment), or the removal of an unknotted split component (2-handle attachment). Furthermore, we have that $pr_2^{-1}([t_i + \epsilon, t_{i+1} - \epsilon])$ is (topologically) a disjoint union of cylinders and that L_i^+ and L_{i+1}^- are isotopic.

Thanks to [17, Theorem 3.1], up to isotopy the order of the attachments can be chosen as follows:

- (1) We start with $L_0 = \Sigma \cap \mathbb{S}^3 \times \{0\}$.
- (2) We attach all the 0-handles.
- (3) We perform a sequence of fusion 1-handles (*i.e.*, 1-handles attachments lowering the number of components) merging all the newly attached 0-handles.
- (4) We perform another sequence of $\ell - k$ fusion 1-handles until we end up with a k -component link diagram. Each fusion move merges two knots that belong to the same component of Σ .
- (5) We perform g fission 1-handles (*i.e.*, 1-handles attachments increasing the number of components), followed by g fusion 1-handles (with $g = g(\Sigma)$).
- (6) We perform a sequence of fission 1-handles (and isotopies) ending up into the link obtained as a split union of L_1 and an unlink.
- (7) We attach all 2-handles on the unlink.
- (8) We end up with $L_1 = \Sigma \cap \mathbb{S}^3 \times \{1\}$.

A schematic representation of the cobordism Σ when the surface is connected can be seen in Figure 4. As highlighted in Figure 4, the cobordism obtained from the attachments described in point 0 to 3 (resp. in points 6 and 8) is a strong concordance between L_0 and a link L'_0 (resp. a link L'_1 and L_1).

By the strong-concordance invariance of v , it follows immediately that $v(L_0) = v(L'_0)$ and $v(L_1) = v(L'_1)$. Now consider the portion of Σ between the link L'_0 and L'_1 , say Σ' ; then it follows from Property (A) that

$$v(L'_0) \leq v(L'_1) + g(\Sigma') + \ell_0 - k.$$

Since $g(\Sigma') = g(\Sigma)$, the second inequality in the statement follows. The other inequality is obtained by reversing the roles of L_0 and L_1 . ■

Now, as an easy consequence of Proposition 2.11, we obtain the desired lower bound on the slice genus.

Proof of Proposition 1.2 Consider a minimal genus surface $\Sigma \subset \mathbb{D}^4$ bounding L ($= \Sigma \cap \mathbb{S}^3$). Without altering the genus, we can assume Σ to be connected. By removing a small disk from Σ , we obtain a genus $g_4(L)$ cobordism between L and the unknot.

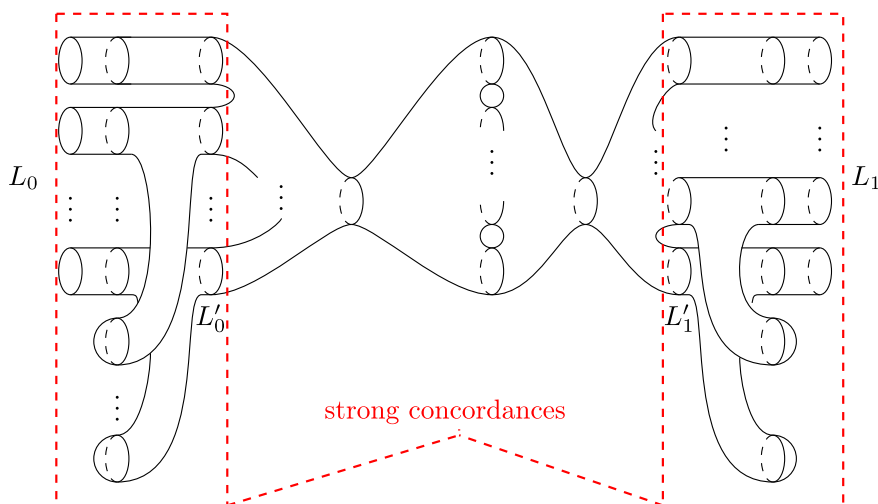


Figure 4: A schematic description of a connected cobordism Σ after the re-ordering of the handles.

Then Proposition 2.11 tells us that

$$-g_4(L) = v(\bigcirc) - g(\Sigma) - 1 + 1 \leq v(L) \leq g(\Sigma) + v(\bigcirc) + \ell - 1 = g_4(L) + \ell - 1.$$

Moreover, if L is strongly slice, then L is strongly concordant to an unlink. Since v is a strong concordance invariant, we have that $v(L) = v(\bigcirc_\ell) = 0$, where \bigcirc_ℓ denotes the unlink with ℓ components. ■

Another consequence of Proposition 2.11, together with Proposition 2.8, is the following corollary.

Corollary 2.12 *If v is a slice-torus link invariant, then the restriction of v to knots is a slice-torus invariant.* ■

At this point the following question arises naturally.

Question 2.13 Let v be a slice-torus invariant. Is there a slice-torus link invariant \tilde{v} whose restriction to knots is v ? If such \tilde{v} exists, is it unique?

The authors believe that all the known slice-torus link invariants admit such an extension, and the answer to the above question is left for future work.

2.3 Combinatorial Bounds and the Detection of the Slice Genus

Using the slice-genus bound proved in Proposition 2.11, we can adapt the arguments used by Kawamura [16] for slice-torus invariants, and Abe ([1], see also [20]) for the Rasmussen invariant s , to the case of slice-torus link invariants.

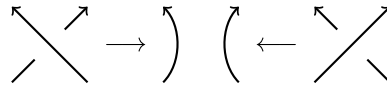


Figure 5: The oriented resolution of a crossing.

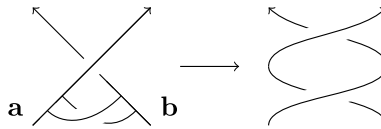


Figure 6: An oriented band move adding a positive crossing to the diagram. Notice that this procedure does not change the number of Seifert circles. Moreover, if the arcs *a* and *b* belong to different components, then the band move is a fusion move.

Before going into the details, we need to introduce some notation. Let D be an oriented link diagram representing a link L . Denote by $n(D)$, $n_+(D)$ and $n_-(D)$ the number of crossings, positive crossings, and negative crossings of D , respectively. The oriented resolution of D is the set of circles (*Seifert circles*) obtained by replacing each crossing with its oriented resolution, as shown in Figure 5. Denote by $O(D)$ the number of Seifert circles of D . If D is non-split, then a Seifert surface for the link L can be obtained by considering a disk for each Seifert circle, and for each crossing on the diagram, we add a band between the corresponding circles. This procedure goes often under the name of *Seifert algorithm*, and it is easy to see that it produces a connected, compact, oriented surface in \mathbb{S}^3 bounding the link L , which is precisely the definition of Seifert surface (see, for example, [30, Chapter 5]). Furthermore, the genus of the surface Σ_D obtained via the Seifert algorithm can be easily computed as an exercise, and it turns out that

$$g(\Sigma_D) = 1 + \frac{n(D) - O(D) - \ell}{2}.$$

Now we can take the first step towards our combinatorial bound. More precisely, we prove Theorem 1.3, which asserts that the slice-torus link invariants compute the Seifert genus of positive links, and also the slice genus in the non-split case.

Proof of Theorem 1.3 We suppose first that L is non-split. Consider the Seifert surface Σ_D obtained via the Seifert algorithm from D . We can remove a small disk from Σ_D , and apply the bound in Proposition 2.11 to obtain the following inequalities

$$v(L) \leq g_4(L) + \ell - 1 \leq g_3(L) + \ell - 1 \leq g(\Sigma_D) + \ell - 1 = \frac{n(D) - O(D) + \ell}{2}.$$

Now, starting from L , we can apply a sequence of $\ell - 1$ fusion moves as in Figure 6 to obtain a positive knot K , and a connected genus 0 cobordism from K to L .

The slice-torus link invariants detect the genus of positive knots (Corollary 2.12 and [16, Theorem 4.4]), and their value can be computed directly from a positive diagram. This leads us to the following sequence of equalities:

$$\frac{n(D) - O(D) + \ell}{2} = \frac{\overbrace{n(D)}^{n(D_K)} + \ell - 1 - O(D) + 1}{2} = v(K),$$

where D_K is the positive diagram of K obtained from D by attaching bands as shown in Figure 6. Finally, from Proposition 2.11, we obtain the inequality

$$v(K) \leq v(L),$$

and the claim follows.

The statement for split links is proved by observing that, except for the slice-genus, all the quantities involved in the equalities for non-split links are additive under disjoint unions. ■

Corollary 2.14 *Let v be a slice-torus link invariant. Then for each coherently oriented, positive torus link T of type $T_{m,n}$, we have the equalities*

$$v(T) = g_3(T) + \ell - 1 = g_4(T) + \ell - 1 = \frac{(n - 1)(m - 1) + \ell - 1}{2},$$

where $\ell = \text{GCD}(m, n)$ is the number of components of T .

Proof The statement follows immediately from Theorem 1.3. ■

Before proceeding further we need some more notation. First, we need to describe how to associate with D a graph $\Gamma(D)$, called the *Seifert graph*. The vertices of $\Gamma(D)$ are the Seifert circles, and there is an edge between two vertices for each crossing the corresponding circles share in D . An edge of the Seifert graph is *positive* (resp. *negative*) if the corresponding crossing is positive (resp. negative). Let $s_+(D)$ (resp. $s_-(D)$) denote the number of connected components of the graph obtained from $\Gamma(D)$ by removing all the negative (resp. positive) edges.

There is another graph $G(D)$ we can obtain from D . This graph has one vertex for each component of the link L represented by D , and two vertices of $G(D)$ share a crossing if there is at least a negative crossing joining the corresponding components.

Now that all the notation is set into place, we can state the following lemma, which is basically due to Kawamura. For the sake of completeness, we will spell out the proof.

Lemma 2.15 ([16, Lemma 5.5]) *If L is a link with a non-splittable diagram D , then there exists a positive link L_+ , a diagram D_+ for L_+ , and a cobordism³ Σ_+ from L to L_+ such that*

$$n(D_+) = n_+(D) + s_+(D) - 1, \quad O(D_+) = O(D)$$

³Recall that for us each cobordism between links is such that each connected component of the cobordism has boundary touching both links.

and

$$\chi(\Sigma_+) = -n_-(D) + s_+(D) - 1.$$

Furthermore, the number of components of Σ_+ is lower than or equal to the number of components of $G(D)$.

Proof Let us start from D . Via band moves (cf. the first part of Figure 3), we can eliminate all the negative crossings in D . With this procedure we end up with a collection of $s_+(D)$ link diagrams, say $D_1, \dots, D_{s_+(D)}$. Moreover, we can see the Seifert graph of each D_i as a sub-graph of $\Gamma(D)$. Consider the graph $\bar{\Gamma}$ obtained from $\Gamma(D)$ by collapsing each $\Gamma(D_i)$. Notice that each vertex of $\bar{\Gamma}$ corresponds to a D_i . Since L is non-split, the diagram D is connected (as a graph) and non-splittable. In particular, $\Gamma(D)$ (and thus $\bar{\Gamma}$) is connected. Pick a spanning tree T for $\bar{\Gamma}$. Via band moves we add a positive crossing (cf. the second part of Figure 3) between D_i and D_j if the corresponding vertices in $\bar{\Gamma}$ are joined by an edge in T . Call D_+ and Σ_+ the diagram and the surface, respectively, obtained via the procedure just described.

The computation of the number of crossings and the number of Seifert circles of D_+ , and the computation of the Euler characteristic of Σ_+ are easily done. Moreover, Σ_+ is a cobordism by construction.

All that is left is to count the number of connected components of Σ_+ . Since Σ_+ is a cobordism, each connected component touches a component of L . Moreover, if there is a negative crossing in D between two components L_1 and L_2 , then there is a band joining them. It follows that L_1 and L_2 belong to the same connected component of Σ_+ . ■

Proof of Theorem 1.4 Let us borrow the notation from the statement of Lemma 2.15. First, we want to compute the genus of Σ_+ . We can assume L non-split, since all the quantities in the statement are additive under disjoint union. From the general formula

$$\chi(\Sigma) = 2c(\Sigma) - 2g(\Sigma) - c(\partial\Sigma),$$

where c denotes the number of connected components, we obtain

$$-g(\Sigma) = \frac{\chi(\Sigma) + c(\partial\Sigma)}{2} - c(\Sigma).$$

Denoted by ℓ_+ the number of components of L_+ , plugging in Σ_+ and replacing the corresponding quantities with their value we obtain

$$-g(\Sigma_+) = \frac{-n_-(D) + s_+(D) - 1 + \ell + \ell_+}{2} - c(\Sigma_+).$$

We should argue that we can assume $G(D)$ to be connected. This is easily done by replacing D with a diagram D' such that

$$w(D) = w(D'), \quad O(D) = O(D'), \quad s_+(D) = s_+(D'), \quad \text{and} \quad c(G(D')) = 1.$$

This can be obtained by choosing a positive crossing between each pair of components of L that share only positive crossings, and perform a second Reidemeister as illustrated in Figure 7. Since $c(\Sigma_+) \leq c(G(D)) = 1$, we can assume Σ_+ to be connected.

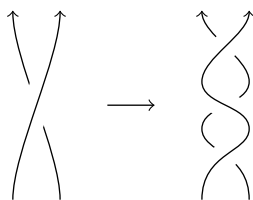


Figure 7: A second Reidemeister move near a positive crossing.

Now consider the quantity

$$v(L_+) - g(\Sigma_+) - \ell_+ + 1.$$

Since L_+ is positive, from Theorem 1.3 it follows that this is equal to

$$\frac{n_+(D) + s_+(D) - 1 - O(D) + \ell_+}{2} + \frac{-n_-(D) + s_+(D) - 1 + \ell + \ell_+}{2} - 1 - \ell_+ + 1,$$

and simple computations show that this equals

$$\frac{w(D) - O(D) + 2s_+(D) + \ell - 2}{2}.$$

Finally, Proposition 2.11 tells us that

$$v(L_+) - g(\Sigma_+) - \ell_+ + 1 \leq v(L),$$

and the result follows. ■

The combinatorial bound presented in Theorem 1.4 is analogous to the bounds presented in [6,16] (see also [1,20,24]) for the Rasmussen and Rasmussen–Beliakova–Wehrli invariants. A possible direction of work might be to find an analogue of the combinatorial bound presented in [12]. Let us leave this matter aside for now and turn to the last result of this section.

Proposition 2.16 *Let L be a negative link, and let ℓ_s be the number of its split components. Then*

$$v(L) = \frac{-n(D) + O(D) + \ell - 2\ell_s}{2}$$

for each negative diagram D and each slice-torus link invariant v .

Proof Combining Corollary 2.12 with [20, Theorem 5] (notice the different normalization, and, see also, [1]), we obtain that our claim is true for negative knots. Since the quantities v , n , O , ℓ , and ℓ_s are additive with respect to the disjoint union, we can assume L to be non-split ($\ell_s = 1$). The proof goes by induction on the number of components of L . Suppose the claim true for all $1 \leq \ell < r$, and assume $\ell = r$. Then by performing a band move similar to the one in Figure 6 between two components, we can obtain a negative link L' that has $\ell - 1$ components, is non-split, and has a negative

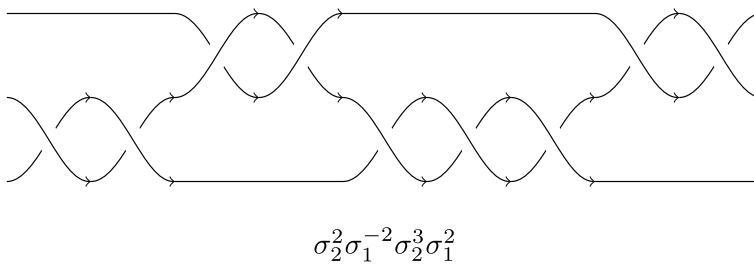


Figure 8: A quasi-positive 3-braid and its “geometrical” representation.

diagram with $n(D) + 1$ crossings and $O(D)$ Seifert circles. By Property (A), we have

$$\begin{aligned} v(L) \leq v(L') + 1 &= \frac{-n(D) - 1 + O(D) + \ell - 1 - 2}{2} + 1 \\ &= \frac{-n(D) + O(D) + \ell - 2}{2}, \end{aligned}$$

where the first equality is the inductive hypothesis. Since any negative diagram is non-splittable, the other inequality follows from Theorem 1.4. ■

This proposition allows us to prove the equivalent of Corollary 2.14 for coherently oriented, negative torus links.

Corollary 2.17 *Let T^* be the mirror image of a torus link of type $T_{m,n}$ with all the components oriented in the same direction. Then for every slice-torus link invariant v , we have the equalities*

$$v(T^*) = -g_3(T) = -g_4(T) = \frac{\ell - 1 - (n - 1)(m - 1)}{2},$$

where $\ell = \text{GCD}(m, n)$ is the number of components of T .

Proof Proposition 2.16 says that

$$v(L^*) = \ell - 1 - v(L)$$

if L is a non-split positive link. Then the claim follows from this observation, Corollary 2.14, and the fact that $g_4(L) = g_4(L^*)$ and $g_3(L) = g_3(L^*)$. ■

3 Applications

This section is dedicated to two applications. The first is an application of the combinatorial bound, and consists of the computation of the slice-torus link invariants of quasi-positive links. The second application is a lower bound on the splitting number of links.

3.1 Quasi-positive Links

Let us recall the definition of quasi-positive braid and quasi-positive link.

Definition 3.1 A quasi-positive link is any link that can be realized as the closure of a d -braid of the form

$$\prod_{i=1}^b w_i \sigma_{j_i} w_i^{-1},$$

where σ_j for $j = 1, \dots, d - 1$ are the Artin generators of the d -braids group.

Thus, quasi-positive links are closures of braids consisting of arbitrary conjugates of positive (Artin) generators.

Theorem 3.2 Consider the d -braid $B = (w_1 \sigma_{j_1} w_1^{-1}) \dots (w_b \sigma_{j_b} w_b^{-1})$, and denote by L its closure. Then, for every slice-torus link invariant v , we have the equality

$$v(L) = \frac{b - d + \ell}{2},$$

where ℓ is the number of components of L .

Proof Since all quantities involved in the statement are additive under disjoint union, we can assume L to be non-split. First, we wish to prove the inequality

$$v(L) \leq \frac{\ell - \chi(\Sigma)}{2},$$

where Σ is a compact oriented surface, properly embedded in \mathbb{D}^4 , such that $\partial\Sigma = L$. Assume Σ has k connected components. Then Proposition 2.11 tells us that

$$v(L) \leq g(\Sigma) + \ell - k = \frac{\ell - \chi(\Sigma)}{2}.$$

Since L bounds a surface Σ_B , which satisfies the previous properties and is such that $\chi(\Sigma_B) = d - b$, as it is shown in [33], we obtain that

$$v(L) \leq \frac{b - d + \ell}{2}.$$

The other inequality follows from the bound in equation (1.1). In fact, this gives that

$$\frac{b - d + 2s_+(B) + \ell - 2}{2} = \frac{w(B) - O(B) + 2s_+(B) + \ell - 2}{2} \leq v(L).$$

Since $1 \leq s_+(B)$, the statement follows. ■

3.2 Splitting Number

As we anticipated, the slice-torus link invariants can be used to obtain a lower bound for the splitting number \tilde{sp} of a link ([2]), which is sometimes called weak splitting number ([5]). Let us recall its definition first.

Definition 3.3 The *splitting number* $\tilde{sp}(L)$ of a link L is defined as the minimum number of crossing changes to perform on a diagram (for all possible diagrams) of L in order to turn the link into a disjoint union of knots.

Note that in literature, the symbol $sp(L)$ usually denotes a different version of the splitting number of L , which we called the strong splitting number in the introduction. The *strong splitting number* sp is defined exactly as \tilde{sp} , but the only crossing changes allowed are those between different components. In particular, we have that $\tilde{sp}(L) \leq sp(L)$.

Remark 3.4 For each oriented link L , we have

$$sp(L) \equiv \sum_{1 \leq i < j \leq \ell} lk(L_i, L_j) \pmod{2},$$

where L_1, \dots, L_ℓ denote the components of L . This fact can be easily proved by induction; alternatively, the reader can consult [9, Lemma 2.1].

Proof of Theorem 1.5 If $\tilde{sp}(L) = 0$, then L is a disjoint union of knots. The additivity of v (Property (B)) tells us that in this case, the left-hand side of equation (1.2) is also zero. Thus, the (in)equality holds. We claim that the quantity

$$\left| v(L) - \sum_{i=1}^{\ell} v(K_i) \right|$$

increases at most by 1 at each crossing change. The theorem is then proved by induction on the value of $\tilde{sp}(L)$ as follows; consider a minimal sequence of crossing changes from L to a split union of knots. Denote by L' the first step in this sequence; then

$$\left| v(L) - \sum_{i=1}^{\ell} v(K_i) \right| \leq \left| v(L') - \sum_{i=1}^{\ell} v(K'_i) \right| + 1 \leq \tilde{sp}(L') + 1 = \tilde{sp}(L),$$

where the first inequality is our claim, and the second inequality follows from the inductive hypothesis.

Now let us prove our claim. First, assume the crossing change to happen between different components. In particular, none of the K_i 's is modified under this crossing change, while $v(L)$ can either increase or decrease at most by 1 (cf. Proposition 2.10). Now assume the crossing change to be performed on a component of L , say $K = K_i$ for some i . This crossing change modifies both L and K , but leaves all the other components unchanged. Again from Proposition 2.10, it follows that

$$(3.1) \quad v(L_+) - 1 \leq v(L_-) \leq v(L_+)$$

and that

$$(3.2) \quad -v(K_+) \leq -v(K_-) \leq -v(K_+) + 1,$$

where the plus and minus denote the signs of the crossing, before and after the change. Adding equations (3.1) and (3.2), we obtain

$$v(L_+) - v(K_+) - 1 \leq v(L_-) - v(K_-) \leq v(L_+) - v(K_+) + 1.$$

Since either $L = L_+$ and $K = K_+$, or $L = L_-$ and $K = K_-$, and all the other components of L are left unchanged, the claim follows. ■

Corollary 3.5 *Let p and q be coprime integers, and $k > 0$. Then the following equality holds:*

$$\text{sp}(T_{k p, k q}) = \tilde{\text{sp}}(T_{k p, k q}) = \frac{k(k-1)|pq|}{2}.$$

Proof In [18, Corollary 3], the author proves that

$$(3.3) \quad \left| v_{s_n}(T) - \sum_{i=1}^k v_{s_n}(K_i) \right| = \text{sp}(T_{k p, k q}) = \frac{k(k-1)pq}{2},$$

where $p, q > 0$, T is the positive torus link of type $T_{k p, k q}$, and K_1, \dots, K_k are the components of T . Notice that by Corollary 2.14 the value of any slice-torus link invariant on positive torus links does not depend on the chosen invariant; therefore, Jeong’s computation holds for any slice-torus link invariant. From equation (3.3) and from

$$\left| v(L) - \sum_{i=1}^{\ell} v(K_i) \right| \leq \tilde{\text{sp}}(L) \leq \text{sp}(L),$$

the desired equality follows for $p, q > 0$. In the other cases, at most we recover the mirror image of $T_{k|p|, k|q|}$, and since the splitting number of a link and its mirror is the same, the corollary follows. ■

Remark 3.6 Notice that $\tilde{\text{sp}}(L)$ does not depend on the orientation of L . It follows that the inequality in Theorem 1.5 holds for every relative orientation of the link. Thus, the maximum among all these values is still a lower bound for $\tilde{\text{sp}}(L)$.

In [2, 5] some lower bounds for $\tilde{\text{sp}}$ are also given. In this paper we describe an infinite family of 2-components links for which Theorem 1.5 allows us to compute $\tilde{\text{sp}}$, where all the obstructions in [2, 5] fail.

Proposition 3.7 *Let us consider the links L_t in Figure 9. Then we have that $\tilde{\text{sp}}(L_t) = t$ and $\text{sp}(L_t) = t + 1$ for every $t \geq 1$.*

Proof We use the link version of the τ -invariant, see [7]. Since L_t is non-split alternating for every $t \geq 1$, we have that $\tau(L_t)$ is determined by the signature

$$\tau(L_t) = \frac{\ell - 1 - \sigma(L_t)}{2} = \frac{1 - \sigma(L_t)}{2};$$

where ℓ is the number of components of the link, which is always equal to two in this case.

An easy computation gives that $\sigma(L_t) = 1 - 2t$, and thus $\tau(L_t) = t$, for every $t \geq 1$. The link L_t has unknotted components; hence, Theorem 1.5 implies $t \leq \tilde{\text{sp}}(L_t)$. On the other hand, we immediately see that L_t can be unlinked (and thus reduced to the split union of knots) by changing t crossings: one crossing for each full twist except one, plus the crossing circled in Figure 9. Thus, we proved that $\tilde{\text{sp}}(L_t) = t$.

It follows that

$$\text{sp}(L_t) \geq \tilde{\text{sp}}(L_t) = t.$$

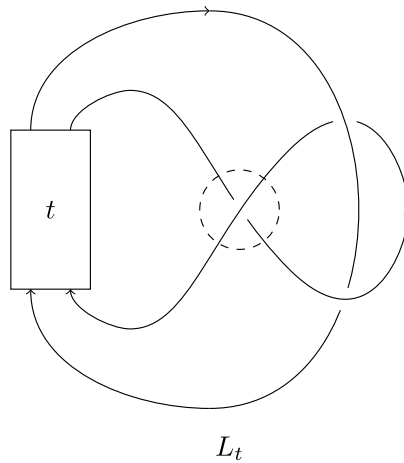


Figure 9: A diagram for the 2-component link L_t , where $t \in \mathbb{N} \setminus \{0\}$ denotes the number of positive full twists.

Moreover, a simple computation shows that

$$|lk(K_t, K'_t)| = t - 1,$$

where K_t and K'_t are the components of L_t . Therefore, by Remark 3.4, we have that $\text{sp}(L_t)$ is at least $t + 1$. Finally, a direct inspection of the diagram in Figure 9 tells us that $t + 1 \geq \text{sp}(L_t)$, and the statement follows. ■

4 Whitehead Doubles and a Concordance Invariant for Links

In this section we define some link invariants related to Livingston and Naik's invariant t_v and study some of their properties. We start by defining the fully clasped and the reduced Whitehead doubles. Then we define the functions F_v, F'_v, \bar{F}_v and \bar{F}'_v and prove their basic properties. Finally, we prove an obstruction for a link to be concordant to a split link. We recall that each strong concordance defines a bijection between the components of the two links, identifying them. Throughout this section all links are oriented.

4.1 Whitehead Doubles of Links

Unlike the case of knots, the Whitehead double of links is not uniquely defined. In this paper we use two among many possible definitions of Whitehead double. The two constructions give non-isotopic links, unless our link is a knot or the unlink.

The first family we introduce will be referred to as fully clasped Whitehead doubles, and is defined as follows. Let L be a link with ℓ components, and let $\underline{t} = (t_1, \dots, t_\ell) \in \mathbb{Z}^\ell$. The *positively* (resp. *negatively*) *fully clasped Whitehead double* $W_\pm(L, \underline{t})$ is the

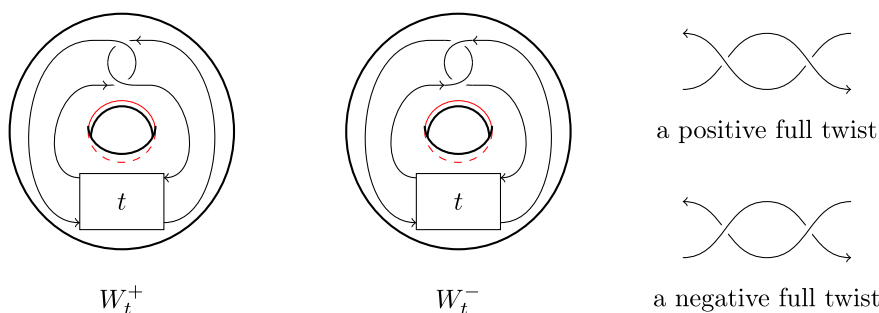


Figure 10: The patterns W_t^\pm . The box represents either $|t|$ positive full twists or $|t|$ negative full twists, depending on whether t is positive or negative.

ℓ -component link obtained by the satellite of companion L , with pattern⁴ on the i -th component given by the positively (resp. negatively) clasped t -twist knot $W_{t_i}^\pm$ (see Figure 10).

The second family of Whitehead doubles considered in this paper is called the reduced Whitehead doubles. Let L be a link. Fix $t \in \mathbb{Z}$. The *positive* (resp. *negative*) *reduced Whitehead double* $W'_\pm(L, t; L_1)$ is the ℓ -component link obtained by the satellite of companion L , with pattern on L_1 given by the positively (resp. negatively) clasped twist knot W_t^\pm (Figure 10).

For both of these families, there are diagrams which can be easily described directly from a diagram D of L . Given a diagram D , representing L , denote by D_1, \dots, D_ℓ the sub-diagrams representing the components L_1, \dots, L_ℓ . Draw a parallel copy of the diagrams D_1, \dots, D_ℓ , add $t_i - w(D_i)$ full twists between the two copies of D_i , and insert the clasps in all the components to obtain the diagram $D_\pm(L, \underline{t})$ for the fully clasped Whitehead double. The diagram $D'_\pm(L, t; L_1)$ for the reduced Whitehead double can be obtained as follows: draw a parallel copy of D_1 (the component corresponding to L_1), add $t - w(D_1)$ full twists and a clasp between the two copies of D_1 , and leave all the other components untouched. An example of such diagrams is depicted in Figure 11.

4.2 Slice-torus Link Invariants of Whitehead Doubles

Now we shall study how the slice-torus link invariants behave in the case of Whitehead doubles. Before proceeding, we observe that given a link L and a slice-torus link invariant v , there are two functions

$$F_v(L): \mathbb{Z}^\ell \longrightarrow \mathbb{R} \quad \text{and} \quad F'_v(L; L_1): \mathbb{Z} \longrightarrow \mathbb{R}$$

defined as

$$F_v(L)(\underline{t}) = v(W_+(\underline{L}, \underline{t})) \quad \text{and} \quad F'_v(L; L_1)(t) = v(W'_+(L, t; L_1)).$$

⁴The homeomorphism sending the torus containing the knot W_t^\pm to a tubular neighbourhood of each component is assumed to send the longitude drawn in red in Figure 10 to the longitude determined by a Seifert surface.

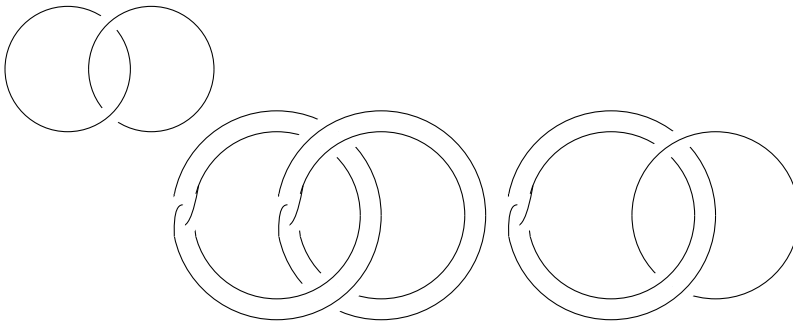


Figure 11: Diagrams of the (untwisted) fully clasped (bottom left) and (untwisted) reduced (bottom right) Whitehead doubles of the Hopf link (top left).

Similarly, we can define $\overline{F}_v(L)$ and $\overline{F}'_v(L)$ by using the negative Whitehead doubles. Since two equivalent links have equivalent Whitehead doubles, it follows immediately that these functions are link invariants (where we identify the variables corresponding to isotopic components), but we can say more. In fact, we have that all of these functions are also invariant under strong concordance.

Theorem 4.1 *Let L_1 and L_2 be two ℓ -component links which are strongly concordant. Consider $\underline{t} \in \mathbb{Z}^\ell$ and $t \in \mathbb{Z}$. Denote by L'_1 and L'_2 two components of L_1 and L_2 , respectively. Then $W_\pm(L_1, \underline{t})$ and $W'_\pm(L_1, t; L'_1)$ are strongly concordant (respecting the ordering of the components) to $W_\pm(L_2, \underline{t})$ and $W'_\pm(L_2, t; L'_2)$, respectively. In particular, the functions $F_v, \overline{F}_v, F'_v$, and \overline{F}'_v are strong concordance invariants of links.*

Proof Suppose that the strong concordance between L_1 and L_2 appears as in Figure 4. Consider a movie (i.e., a sequence of band moves, birth and death of unknotted components, and Reidemeister moves) from a diagram of L_1 into one of L_2 , describing a concordance. We start by taking the fully clasped Whitehead doubles of L_1 , obtained by doubling the given diagram of L_1 as we described before in this section. Every birth move now becomes a double birth move (see Figure 12), which corresponds to the attachment of two 0-handles. Moreover, when we have a split move, the component involved will be doubled, and then the move now consists of two band moves instead of one, as shown in Figure 13. Therefore, we have two cases, depending on whether the doubled component is clasped or not. If it is clasped, then after the band moves it will be split into three components, one containing the clasp and the other two being one the double of the other. On the other hand, if the doubled component is not clasped, then the bands will turn it into two doubled components.

Now in the case of a merge move, we observe that two clasped components cannot be joined together. In fact, otherwise our cobordism would not be a strong concordance. This implies that each merge move corresponds precisely to the inverse of a split move, and then we obtain the same conclusions as the previous case.

At this point, we can perform the death moves, which will be doubled in the same way as the birth moves. This is because the clasps and the full twists can be isotoped

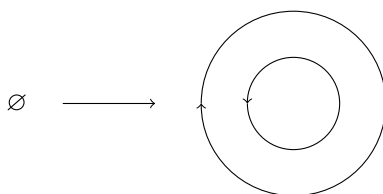


Figure 12: A double birth move.

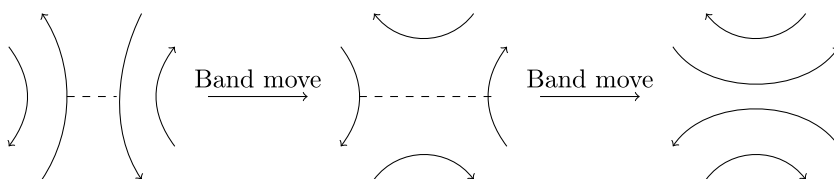


Figure 13: A double band move.

to be everywhere on their doubled component of L_1 ; therefore, we can always be sure that clasps and twists will not appear on the components that we want to cancel with the death moves.

After this procedure, we are left with a diagram of the fully clasped Whitehead double of L_2 , with the same number of twists. Moreover, the new cobordism that we have obtained is a strong concordance by construction.

For the reduced Whitehead doubles the reasoning is exactly the same, provided that we take care of two more details.

First, we only double the birth moves that will be joined with the clasped component and not the others.

Second, we have to observe that we cannot have a merge move between a component that is doubled and one that is not. In fact, we start from a strong concordance and, as we remarked before, we cannot merge different components of L_1 together. Then the claim follows from the same argument we used for fully clasped doubles. ■

Remark 4.2 In the case L is a knot, the functions $F_\nu(L)$ and $F'_\nu(L; L)$ coincide. Furthermore, when ν is \mathbb{Z} -valued these functions assume only two values, and the point where their value changes is the t_ν invariant introduced in [22].

Let us start by proving that the functions we introduced are bounded.

Theorem 4.3 For each ℓ -component link L and $\underline{m} \in \mathbb{Z}^\ell$, then

$$F_\nu(L)(\underline{m}) \in [0, \ell] \quad \text{and} \quad \overline{F}_\nu(L)(\underline{m}) \in [-\ell, 0].$$

Furthermore, given a component L_0 of L and $m \in \mathbb{Z}$, we have

$$F'_\nu(L; L_0)(m) \in [\nu_0, \nu_0 + 1] \quad \text{and} \quad \overline{F}'_\nu(L; L_0)(m) \in [\nu_0 - 1, \nu_0],$$

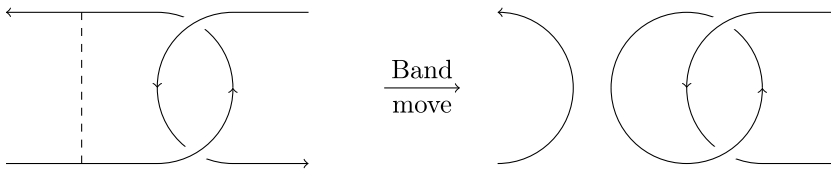


Figure 14: A band move between $W_-(L, m; L_0)$ and $(L \setminus L_0) \sqcup H_-$.

where

$$v_0 = \begin{cases} v(L \setminus L_0), & \ell \geq 2, \\ 0, & \ell = 1. \end{cases}$$

Proof Let us prove only the part of statement concerning the invariants F'_v and \overline{F}'_v . The rest of the statement can be proved by iterating the same reasoning.

Notice that we can obtain a diagram of $(L \setminus L_0) \sqcup H_+$ via a single band move on $D_+(L, m; L_0)$ (similar to the one illustrated in Figure 14). Thus, from Properties (A) and (B), and Lemma 2.7 it follows that

$$v(W_+(L, m; L_0)) \leq v((L \setminus L_0) \sqcup H_+) = v_0 + 1$$

and

$$v_0 = v_0 + 1 - 1 = v((L \setminus L_0) \sqcup H_+) - 1 \leq v(W_+(L, m; L_0)).$$

The same reasoning applies for \overline{F}'_v , the only change is that we get H_- instead of H_+ (see Figure 14). Since $v(H_-) = 0$, the result follows. ■

Moreover, there is a non-increasing property akin to the one proved in [22].

Theorem 4.4 Let L be an oriented link, and let v be a slice-torus link invariant. If \underline{m} and \underline{n} are two elements of \mathbb{Z}^ℓ such that $m_i \geq n_i$ for all i , then

$$F_v(L)(\underline{n}) - \sum_{i=1}^{\ell} (m_i - n_i) \leq F_v(L)(\underline{m}) \leq F_v(L)(\underline{n}).$$

Furthermore, if $m \geq n$ are two integers, then

$$F'_v(L; L_0)(n) - (m - n) \leq F'_v(L; L_0)(m) \leq F'_v(L; L_0)(n).$$

Moreover, the same result holds for \overline{F}_v and \overline{F}'_v .

Proof It is sufficient to prove the result for the case $m_i = n_i$, for all $i \neq i_0$, and $m_{i_0} = n_{i_0} + 1$. It is sufficient to notice that, in this case, one can obtain $W_{\pm}(L, \underline{m})$ (resp. $W'_{\pm}(L, m; L_0)$) from $W_{\pm}(L, \underline{n})$ (resp. $W'_{\pm}(L, n; L_0)$) by a second Reidemeister move and a crossing change from a positive crossing to a negative crossing (see Figure 15) and the result follows from Proposition 2.10. ■

In particular, Theorem 4.4 implies that every time we add a positive full twist on one component, the value of v cannot increase, and it can decrease at most by one.

From the fact that $v(K) = -v(-K^*)$, for each knot K and slice-torus invariant v , it follows that $\overline{F}_v(K)(t) = -F_v(K^*)(-t)$. In the case of links, this property does not

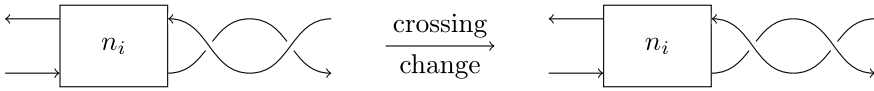


Figure 15: How obtain $W_{\pm}(L, \underline{m})$ (resp. $W'_{\pm}(L, m; L_0)$) from $W_{\pm}(L, \underline{n})$ (resp. $W'_{\pm}(L, n; L_0)$) by a second Reidemeister move and a crossing change.

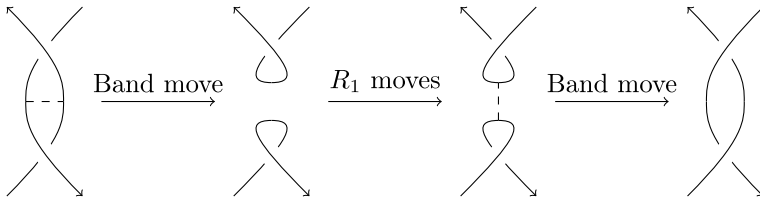


Figure 16: A sequence of band moves and isotopies relating positive and negative clasps.

hold, but the invariants F_v and \bar{F}_v (and their “reduced” variants) still share a similar, albeit weaker, symmetry property, which is made precise in the following result.

Proposition 4.5 *Let L be an ℓ -component oriented link and let v be a slice-torus link invariant. If there exists \underline{m} (resp. \underline{n}) in \mathbb{Z}^{ℓ} such that*

$$F_v(L)(\underline{m}) = \ell \quad (\text{resp. } \bar{F}_v(L)(\underline{n}) = -\ell),$$

then

$$\bar{F}_v(L)(\underline{m}) = 0 \quad (\text{resp. } F_v(L)(\underline{n}) = 0).$$

Moreover, let L_0 be a component of L , if there exists m (resp. n) in \mathbb{Z} such that

$$F'_v(L; L_0)(m) = v_0 + 1 \quad (\text{resp. } \bar{F}'_v(L; L_0)(n) = v_0 - 1),$$

then

$$\bar{F}'_v(L; L_0)(m) = v_0 \quad (\text{resp. } F'_v(L; L_0)(n) = v_0),$$

where v_0 is defined as in Theorem 4.3.

Proof We will prove only the parts of the statement concerning the functions F'_v and \bar{F}'_v . The proofs for the other functions are completely analogous and thence left to the reader. Let us start by considering the sequence of band moves (and isotopies) depicted in Figure 16. By Property (A), we obtain that

$$|v(W'_+(L, m; L_0)) - v(W'_-(L, m; L_0))| \leq 1.$$

It follows that if $F'_v(L; L_0)(n) = v_0 + 1$ (resp. $\bar{F}'_v(L; L_0)(m) = v_0 - 1$), then

$$v_0 \leq \bar{F}'_v(L; L_0)(m) \quad (\text{resp. } F'_v(L; L_0)(n) \leq v_0).$$

Theorem 4.3 provides the other half of the bound(s), and the equality follows. ■

We saw that $v(W_{\pm}(L, \underline{t}))$ is bounded for every ℓ -component link L . Now we want to prove that for some ℓ -tuples (t_1, \dots, t_{ℓ}) , the invariant v assumes the maximum value possible. To do this we equip \mathbb{S}^3 with the standard contact structure ξ_{st} and we recall that the Thurston–Bennequin number $tb(\mathcal{L})$ of a Legendrian knot is the linking number between L and the contact framing $L_{\xi_{st}}$ induced by ξ_{st} . See [14] for details.

Theorem 4.6 *Suppose that v is a slice-torus link invariant. Then for each Legendrian representative \mathcal{L} of L in (\mathbb{S}^3, ξ_{st}) , and (t_1, \dots, t_{ℓ}) such that $t_i = tb(\mathcal{L}_i)$ for every $i = 1, \dots, \ell$, we have that*

$$F_v(L)(t_1, \dots, t_{\ell}) = \ell.$$

Proof Let us consider a surface F' such that the Legendrian link \mathcal{L} in \mathbb{S}^3 equipped with the contact structure ξ_{st} is embedded in F' , and $tb(\mathcal{L})$ coincides with the Seifert framing induced by F' . This means that the (t_1, \dots, t_{ℓ}) -twisted double of the link L can be embedded in F' as the boundary of a collar neighbourhood of \mathcal{L} . Let us call this neighbourhood $F'' \subset F'$. Moreover, we can change F'' by positive Hopf plumbings in such a way that the new surface F has $W_+(L, t_1, \dots, t_{\ell})$ as boundary.

From the work of Rudolph [31, 32] and thanks to our choice of (t_1, \dots, t_{ℓ}) , we can assume that F'' is a quasi-positive surface. Therefore, since F is obtained from F'' through positive Hopf plumbings, F is also a quasi-positive surface. This also implies that $W_+(L, t_1, \dots, t_{\ell})$ is a strongly quasi-positive link.

Now, possibly after performing more positive plumbings, the surface F can be seen as a subsurface of a minimal Seifert surface G of a torus knot $T_{m,n}$, for m, n sufficiently big. (See [31, 32].) The surface $\Sigma = G \setminus \dot{F}$ is a connected cobordism between $W_+(L, t_1, \dots, t_{\ell})$ and $T_{m,n}$ (this is basically the same argument used in [22]); its genus can be computed from the fact that $\chi(F) + \chi(\Sigma) = \chi(G)$. Hence, we have

$$(-\ell) + (1 - 2g(\Sigma) - \ell) = 1 - 2g(G),$$

because F is by construction the union of ℓ copies of a torus with a disk removed, and Σ and G are both connected, which gives

$$g(\Sigma) = g(G) - \ell.$$

Each surface S in \mathbb{S}^3 with boundary subdivision $\partial_0 S \sqcup \partial_1 S$ can be seen⁵ as a surface in $\mathbb{S}^3 \times [0, 1]$ such that $S \cap (\mathbb{S}^3 \times \{i\}) = \partial_i S$. In particular, we can see Σ as a cobordism between $T_{m,n}$ and $W_+(L, t_1, \dots, t_{\ell})$. Propositions 2.11 and 1.2 imply that

$$v(T_{m,n}) \leq v(W_+(L, t_1, \dots, t_{\ell})) + g(\Sigma) = v(W_+(L, t_1, \dots, t_{\ell})) + g(G) - \ell,$$

which, in turn, gives

$$\ell \leq v(W_+(L, t_1, \dots, t_{\ell})),$$

since $v(T_{m,n}) = g(G)$ (cf. Corollary 2.14). Now the statement follows directly from Theorem 4.3. ■

The last theorem, together with Proposition 4.5, immediately implies the following corollary.

⁵Consider a Morse function $f: S \rightarrow [0, 1]$ such that $f^{-1}(i) = \partial_i S$, $i = 0, 1$ (it is easily shown that such a function always exists). The desired embedding is given by the map

$$F: S \longrightarrow \mathbb{S}^3 \times [0, 1]: p \mapsto (p, f(p)).$$

Corollary 4.7 Let L be a link. For every $\underline{t} \in \mathbb{Z}^\ell$, as in the hypotheses of Theorem 4.6, we have that $\overline{F}_v(L)(\underline{t}) = 0$.

In [22] it was proved that in the case of knots, the functions F_v and \overline{F}_v also assume the minimal value. That is, we have the following proposition.

Proposition 4.8 (Livingston and Naik, [22]) *If K is a knot, then there exists an integer t such that*

$$v(W_+(K, t)) = 0 \quad \text{and} \quad v(W_-(K, t)) = -1.$$

In general, we do not have the same result in the case of multi-component links. That is to say, we cannot prove that any of the functions we introduced reach the minimum. However, we can prove that some of them are non-constant.

Proposition 4.9 *For every slice-torus link invariant v and ℓ -component link L , there exists an ℓ -tuple (t_1, \dots, t_ℓ) such that*

$$F_v(L)(t_1, \dots, t_i, \dots, t_\ell) \in (\ell - 1, \ell],$$

$$F_v(L)(t_1, \dots, t_i + 1, \dots, t_\ell) \in (\ell - 2, \ell - 1]$$

for some i , and the same holds true for $\overline{F}_v(L)$.

Proof Suppose that $F_v(L)$ has values only in the interval $(\ell - 1, \ell]$. Then by Property (C), we have that

$$v(W_+(L, \underline{t})^*) = v(W_-(L^*, -\underline{t})) \leq -1$$

for each $\underline{t} \in \mathbb{Z}^\ell$, but this contradicts Corollary 4.7. The claim follows from Theorem 4.4. The case of $\overline{F}_v(L)$ is dealt with in the same way. ■

5 An Example: the Hopf Link

In this section, we will give compute explicitly the functions F_{v_s} and \overline{F}_{v_s} for the Hopf link, where v_s is the slice-torus link invariant associated with s . Notice that the fully clasped Whitehead doubles of the positive and negative Hopf links are isotopic, and thus the computation we achieve are valid both for H_+ and H_- . To lighten the notation, throughout the section we shall omit H_\pm from the notation unless confusion may arise.

5.1 Computations for General Slice-torus Link Invariant

Let v be a slice-torus link invariant. There are a few observations on $F_v(t_1, t_2)$ and $\overline{F}_v(t_1, t_2)$ that can be made, and allow us to partially compute these functions.

First, notice that exchanging the roles of the components of the Hopf link yields the same link. Thus, we obtain the following symmetry property:

$$F_v(H_\pm)(t_1, t_2) = F_v(H_\pm)(t_2, t_1), \quad \text{for all } (t_1, t_2) \in \mathbb{Z}^2.$$

Furthermore, with the same reasoning as in the proof of Theorem 4.3, it follows immediately that

$$v(W_+(\bigcirc, t_i)) \leq F_v(t_1, t_2) \leq v(W_+(\bigcirc, t_i)) + 1, \quad i \in \{1, 2\}.$$

Moreover, since $W_+(\bigcirc, -1) = T_{2,3}$ and $W_+(\bigcirc, 0) = \bigcirc$, we have that

$$v(W_+(\bigcirc, t)) = F_v(\bigcirc)(t) = \begin{cases} 1, & t \leq -1, \\ 0, & t \geq 0, \end{cases}$$

for each v . Putting these facts together with Theorem 4.3, and with the fact that the combinatorial bound (cf. Theorem 1.4) is sharp in the case $(t_1, t_2) = (0, 0)$, we obtain that

$$F_v(t_1, t_2) = \begin{cases} 2, & (t_1, t_2) \in (-\infty, -1] \times (-\infty, -1], \\ 1, & (t_1, t_2) \text{ or } (t_2, t_1) \in [-1, -\infty) \times (-\infty, 0] \cup \{(0, 0)\}, \\ \in [0, 1], & \text{otherwise.} \end{cases}$$

A similar reasoning works with \bar{F}_v , with the only difference that the bound is sharp in $(1, 1)$ and not in $(0, 0)$. This lead us to the following:

$$\bar{F}_v(t_1, t_2) = \begin{cases} 0, & (t_1, t_2) \in (-\infty, -1] \times (-\infty, -1] \\ \in [-1, 0], & (t_1, t_2) \text{ or } (t_2, t_1) \in \{0\} \times (-\infty, 0], \\ -1, & (t_1, t_2) \text{ or } (t_2, t_1) \in [1, +\infty) \times (-\infty, 0] \cup \{(1, 1)\}, \\ \in [-2, -1], & \text{otherwise.} \end{cases}$$

The information we gathered on the functions F_v and \bar{F}_v is summarized in Figure 17.

Remark 5.1 This amount of information is already enough to distinguish the unlink with two components, and every disjoint union of two knots, from the Hopf link by applying Theorem 6.3.

5.2 Computations with the s -invariant

The s -invariant was introduced by Rasmussen [29] in the case of knots and extended to links by Beliakova and Werhli [4]. Let us say a few words on this invariant.

Fix a field \mathbb{F} , in [19] E. S. Lee introduced a link homology theory $H_{Lee}^*(\cdot, \mathbb{F})$, which is a deformation of Khovanov homology. The homology of this theory is pretty simple: given an oriented link diagram D representing a link L there is a set of cycles, called *canonical generators*, whose homology classes generate $H_{Lee}^*(L, \mathbb{F})$. This set is indexed by the possible orientations of the underlying unoriented diagram ([19, Theorem 5.1]). Moreover, the homological degree h of each canonical generator is completely determined by the linking matrix of L . However, this theory has a natural (decreasing) filtration \mathcal{F}_* , called the *quantum filtration*, which contains non-trivial information on concordance.

Let D be an oriented link diagram. The set of the possible orientations of the underlying unoriented diagram will be denoted by $\mathbb{O}(D)$, and the canonical generator associated with a given $o \in \mathbb{O}(D)$ will be denoted by $v_o(D; \mathbb{F}) \in C_{Lee}^{h(o,L)}(D, \mathbb{F})$.

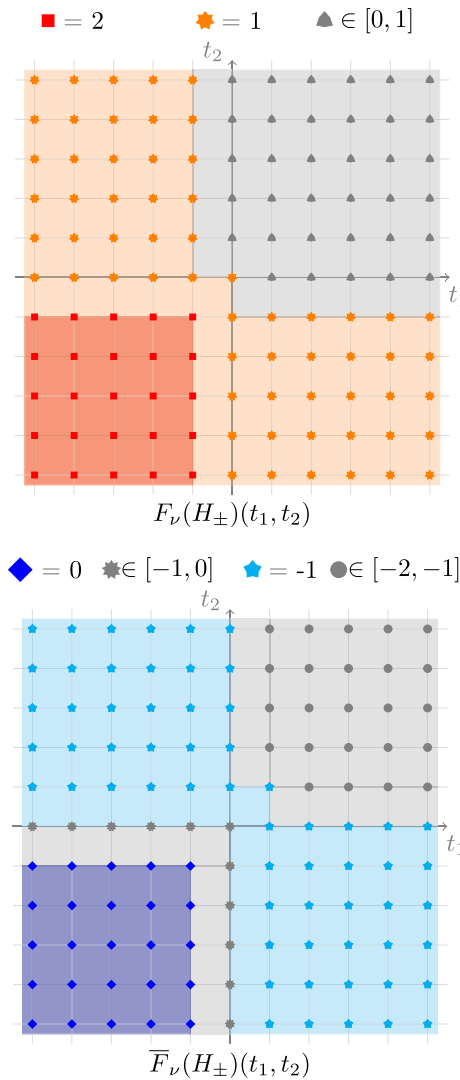


Figure 17: (Colour online.) Partial computations of the functions $F_\nu(H_\pm)(t_1, t_2)$ and $\bar{F}_\nu(H_\pm)(t_1, t_2)$ for an arbitrary slice-torus link invariant ν .

Definition 5.2 (Rasmussen [29], Beliakova–Wehrli [4]) Let D be an oriented link diagram representing an oriented link L . The *Rasmussen–Beliakova–Wehrli (RBW) invariant* associated with $o \in \mathbb{O}(D)$ is the integer

$$s(o, L; \mathbb{F}) = \frac{\text{Fdeg}([\mathbf{v}_o(D; \mathbb{F}) - \mathbf{v}_{-o}(D; \mathbb{F})]) - \text{Fdeg}([\mathbf{v}_o(D; \mathbb{F}) + \mathbf{v}_{-o}(D; \mathbb{F})])}{2},$$

where Fdeg indicates the filtered degree in $H_{T\text{Lee}}^\bullet(L, \mathbb{F})$ and $-o$ denotes the opposite orientation with respect to o . If o is exactly the orientation induced by L , we will omit o from the notation and call $s(L; \mathbb{F})$ the s -invariant or *Rasmussen invariant* of L .

Remark 5.3 The original definition of the RBW-invariants ([4, 29]) does not work over fields of characteristic 2. However, using a twisted version of Lee theory, defined by Bar-Natan in [3], one can extend the definition to characteristic 2 (of course, the two theories give the same invariants if $\text{char}(\mathbb{F}) \neq 2$, see [26]). With an abuse of notation, we call these extended invariants RBW-invariants.

Let D be an oriented link diagram representing the oriented link L . It can be easily shown (see, for instance, [12, Proposition 11] and subsequent proof) that

$$\begin{aligned} s(L; \mathbb{F}) - 1 &= \text{Fdeg}([\mathbf{v}_{-o_D}(D)]) = \text{Fdeg}([\mathbf{v}_{o_D}(D)]) \\ &= \max \{ \text{Fdeg}(x) \mid x \in [\mathbf{v}_{o_D}(D)] \}, \end{aligned}$$

where o_D is the orientation of D , and the filtered degree $\text{Fdeg}(x)$ is defined as the maximal j such that $x \in \mathcal{F}_j C_{\text{Lee}}^*(D, \mathbb{F})$. We shall make use of this alternative definition of s to prove the following results, which allows one to compute s in a number of cases.

Proposition 5.4 *Let L be an oriented link, D an oriented diagram representing L and \tilde{L} the unoriented link underlying L . If for each $o \in \mathcal{O}(D)$ such that for $\mathbf{v}_o(D) \in C_{\text{Lee}}^0(D, \mathbb{F})$, we have that L is isotopic to (\tilde{L}, o) , then*

$$s(L; \mathbb{F}) = 1 + \min \{ j \in \mathbb{Z} \mid \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(L; \mathbb{F}) \neq 0 \},$$

where $\text{Gr}_{\mathcal{F}}^*$ indicates the associated graded object corresponding to the quantum filtration.

Proof The homology classes of the canonical generators associated with the orientations satisfying the above hypothesis, generate $H_{\text{Lee}}^0(L; \mathbb{F})$. Since L is isotopic to (\tilde{L}, o) , for all o 's such that $\mathbf{v}_o(D) \in C_{\text{Lee}}^0(D, \mathbb{F})$, it follows that all the corresponding $[\mathbf{v}_o]$'s have the same filtered degree. The set of such $[\mathbf{v}_o]$'s is a basis of $H_{T\text{Lee}}^0(L; \mathbb{F})$, and the minimal filtered degree of the elements of a basis of a filtered vector space does not depend on the choice of the basis (this fact is easy to prove, but the lazy reader can consult, for example, [11, Corollary A.6]). The claim follows immediately from the fact that the minimal degree of the elements of a filtered basis⁶ of a filtered vector space V is the minimal degree where $\text{Gr}^* V$ is non-trivial. ■

Remark 5.5 The proof of the previous proposition does not imply anything about the support of the associated graded object (also called the Pardon invariant), aside from the RBW invariant being the lowest non-trivial quantum degree (plus one) in homological degree 0. In particular, we did not prove that $\text{Gr}_{\mathcal{F}}^* H_{\text{Lee}}^0(L; \mathbb{F})$ is supported only in two degrees, which is false, as we shall see in the examples.

⁶A filtered basis of a filtered vector space V is a basis $\{e_i\}_{i=1, \dots, k}$ for V such that the direct sum filtration on $V = \bigoplus_{i=1}^k \mathbb{F} \langle e_i \rangle$ coincides with the original filtration (where the filtration on $\mathbb{F} \langle e_i \rangle$ is understood). It is clear that every filtered vector space admits a filtered basis, and this choice gives an isomorphism between V and $\text{Gr}^* V$.

Using Proposition 5.4 and the knight move pairings ([11, 19]) we can determine the s -invariant from Khovanov homology in many cases, including the Whitehead doubles of the Hopf link. An essential data to perform a computation using the knight move pairing is the homological degrees on which Lee homology is supported. These are completely determined by the linking matrix. In our case, since the linking matrix of fully clasped Whitehead doubles is always vanishing, Lee homology is always concentrated in homological degree 0.

Remark 5.6 We remark that, in general, the knight move pairing is not sufficient to compute the associated graded object to Lee homology. However, if the field is of characteristic different from 2 and the pairing is unique, this method can be used (see [11, Chapter 2 & Appendix B] for more details).

In order to avoid technical difficulties we will work with $\mathbb{F} = \mathbb{Q}$. In this case we are able to complete the computation started at the beginning of the section.

We recall (see Example 2.3) that the Rasmussen invariant is not a slice-torus link invariant by itself, but we need to re-scale it and add a correction term; more specifically, recall that the slice-torus link invariant associated with the Rasmussen invariant is

$$v_s(L) = \frac{s(L) + \ell - 1}{2} \in \mathbb{Z},$$

where L is an ℓ -component link. Which means that the values that $s(W_+(H_{\pm}, t_1, t_2))$ can assume are 3, 1, and -1 . The KnotTheory package of Mathematica [34] was used to compute the Khovanov homology, and the results of these computations are collected in the appendix.

Let us start with the link $W_+(H_{\pm}, 0, 1)$. In this case, the result of our computations is the following (cf. Table A.1):

$$\dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_+(H_{\pm}, 0, 1)) = \begin{cases} 2, & \text{if } j = 0, 2, \\ 0, & \text{otherwise.} \end{cases}$$

It follows that $s(W_+(0, 1)) = 1$, and thus

$$F_{v_s}(H_{\pm})(0, 1) = \frac{1 + 2 - 1}{2} = 1.$$

Notice that the link $W_+(H_{\pm}, 0, 1)$ is *pseudo-thin*; that is, $\text{Gr}_{\mathcal{F}}^{\bullet} H_{\text{Lee}}^0$ is supported in two points (see [6]).

Now let us consider the links $W_+(H_{\pm}, 1, 1)$ and $W_+(H_{\pm}, 0, 2)$. We have that (cf. Table A.2)

$$\begin{aligned} \dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_+(H_{\pm}, 0, 2)) &= \dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_+(H_{\pm}, 1, 1)) \\ &= \begin{cases} 2, & \text{if } j = 0, \\ 1, & \text{if } j = -2, 2, \\ 0, & \text{otherwise.} \end{cases} \end{aligned}$$

It follows that $F_{v_s}(H_{\pm})(1, 1) = F_{v_s}(H_{\pm})(1, 2) = 0$. Notice that neither of these links is pseudo-thin (cf. Remark 5.5).

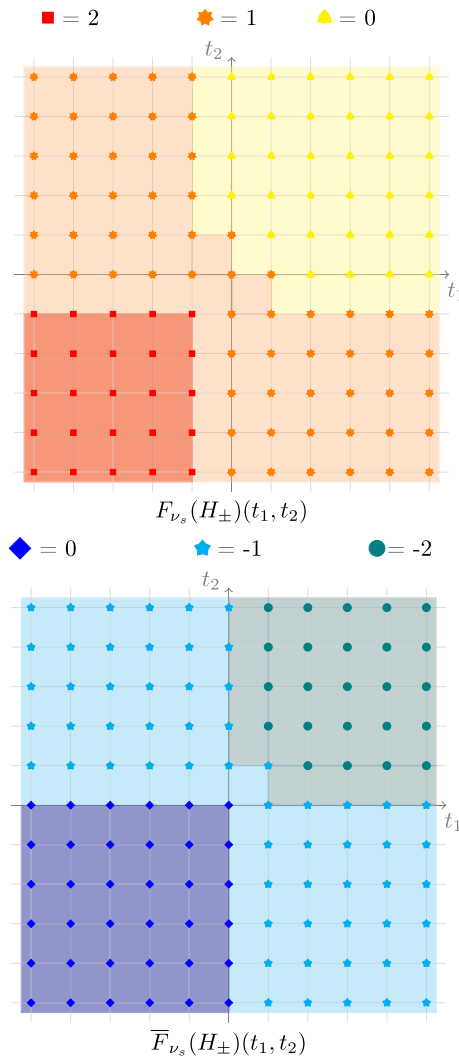


Figure 18: (Colour online.) Computations of the functions $F_{\nu_s}(H_{\pm})(t_1, t_2)$ and $\bar{F}_{\nu_s}(H_{\pm})(t_1, t_2)$, where ν_s is the slice-torus link invariant associated with the Rasmussen link invariant.

Thanks to the non-increasing property proved in Theorem 4.4, F_{ν_s} is completely determined. The result of our computations is shown in the top of Figure 18.

Now let us turn to the computation of the function \bar{F}_{ν_s} . We start by computing the associated graded object to Lee homology for the link $W_-(H_{\pm}, 0, 0)$, which yields the following result (cf. Table A.3, left)

$$\dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_-(H_{\pm}, 0, 0)) = \begin{cases} 2, & \text{if } j = -2, 0, \\ 0, & \text{otherwise.} \end{cases}$$

As a consequence, we have that $\overline{F}_{v_s}(0, 0) = 0$, and thus (by Theorem 4.4) it follows that $\overline{F}_{v_s}(0, t) = \overline{F}_{v_s}(t, 0) = 0$ for each $t \leq 0$.

Finally, we computed the associated graded object to Lee homology for the link $W_-(H_{\pm}, 1, 2)$ (cf. Table A.3, right), and we obtained that

$$\dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_-(H_{\pm}, 1, 2)) = \begin{cases} 2, & \text{if } j = -4, \\ 1, & \text{if } j = -6, -2, \\ 0, & \text{otherwise.} \end{cases}$$

Now Proposition 5.4 implies that $s(W_-(H_{\pm}, 1, 2)) = -5$, and thus $\overline{F}_{v_s}(1, 2) = -2$. This completes the computation of the function \overline{F}_{v_s} , which is summarized on the bottom of Figure 18.

Remark 5.7 Notice that while the invariants t_v and \overline{t}_v contain the same information, in the case of multi-component links the set of points where F_v and \overline{F}_v change their values are not related *a priori*. In Figure 18, we have an example of how the “jumping loci” of these the two functions are not trivially related.

6 Further Examples

In this section we explore two further examples: the split links and the link L8a9. By analysing the former example, we are able to define a new obstruction for a link to be strongly concordant to a split link. With the latter example we will see that the functions F'_v and \overline{F}'_v contain different information than the linking matrix and v .

6.1 Split Links

Let L_i be an ℓ_i -component link, for $i \in \{1, 2\}$. Since Whitehead doubling and disjoint union commute, Property (B) tells us that

$$v(W_{\pm}(L_1 \sqcup L_2, (\underline{t}, \underline{s}))) = v(W_{\pm}(L_1, \underline{t})) + v(W_{\pm}(L_2, \underline{s}))$$

for each $\underline{t} \in \mathbb{Z}^{\ell_1}$ and $\underline{s} \in \mathbb{Z}^{\ell_2}$. Denote by L the disjoint union of L_1 and L_2 . Proposition 4.9 implies that there exists a unit square in $\mathbb{Z}^{\ell_1 + \ell_2}$ with vertices

$$(\underline{t}, \underline{s}), \quad (\underline{t} + \underline{e}_i, \underline{s}), \quad (\underline{t}, \underline{s} + \underline{e}_j), \quad \text{and} \quad (\underline{t} + \underline{e}_i, \underline{s} + \underline{e}_j),$$

for some $i, j \in \{1, \dots, \ell_1 + \ell_2\}$, where \underline{e}_i denotes the vector (of the appropriate length) with i -th entry 1 and all the other entries 0, satisfying the following properties:

- ▷ $F_v(L)$ assumes at least three different values on the vertices of the square;
- ▷ the maximal and the minimal values of $F_v(L)$ on the square are attained exactly once;
- ▷ the maximal value of $F_v(L)$ on the square is equal to ℓ , where ℓ is the number of the components of L .

We call such a square a *3-valued square* for the function $F_v(L)$. The concept of 3-valued square can be generalized as follows.

Definition 6.1 Fix $\ell \geq 1$, and consider a bounded function $F: \mathbb{Z}^\ell \rightarrow \mathbb{R}$. A $(k+1)$ -valued cube for F is a k -dimensional cube in \mathbb{Z}^ℓ with edges of length 1, such that the following hold:

- ▷ F assumes at least $k+1$ different values on the vertices of the cube;
- ▷ the maximal and the minimal values of F on the cube are attained exactly once;
- ▷ the maximal value of F on the cube is equal to the maximal value of F .

Notice that a 3-valued square is a 3-valued cube. The existence of a 3-valued square for $F_\nu(L)$ and $\overline{F}_\nu(L)$ in the case L is the split union of two links can be generalized as follows.

Proposition 6.2 Suppose that L is a link with ℓ_s split components. Then $F_\nu(L)$ and $\overline{F}_\nu(L)$ have at least one $(\ell_s + 1)$ -valued cube, for each slice-torus link invariant ν .

Proof It follows from Proposition 4.9, and from the additivity of $F_\nu(L)$ and $\overline{F}_\nu(L)$ with respect to the disjoint union of links. ■

This gives a criterion to obstruct the strong concordance with split links. In particular, this criterion requires only a partial computation of either $F_\nu(L)$ or $\overline{F}_\nu(L)$ (cf. Remark 5.1).

Theorem 6.3 Let L be a link. If there exists a slice-torus link invariant ν such that either $F_\nu(L)$ or $\overline{F}_\nu(L)$ do not admit any $(r+1)$ -valued cube, then L is not strongly concordant to any link with r split components.

Proof It is an immediate consequence of Theorem 4.1, which states that F_ν and \overline{F}_ν are strong concordance invariants, and Proposition 6.2. ■

In particular, when ν is a \mathbb{Z} -valued slice-torus link invariant, we have that if an ℓ -component link L is strongly concordant to the disjoint union of ℓ knots then $F_\nu(L)$ has exactly one $(\ell+1)$ -valued cube. Moreover, in this case, F_ν also assumes the minimal value.

6.2 The Link L8a9

In this subsection we use the function F'_{ν_s} to prove that the link $L = L8a9$ (see Figure 19) is not strongly concordant to the positive Hopf link H_+ .

Notice that L is a non-split, alternating link with the same signature and linking matrix as H_+ . This means that these two links also have the same Lee homology and the same filtered link Floer homology (see [6, 7, 19]). It follows that these links have the same s and τ invariants.

Proposition 6.4 The function $F'_{\nu_s}(L; A)$ differs from $F'_{\nu_s}(H_\pm)$, and thus there is no strong concordance between the links L and H_+ .

Proof Let us consider the reduced Whitehead doubles $W'_+(L, 1; A)$ and $W'_+(H_\pm, 1)$. We omitted the component in the latter reduced Whitehead double, since the choice of the component for the Hopf link is immaterial (as the results are isotopic).

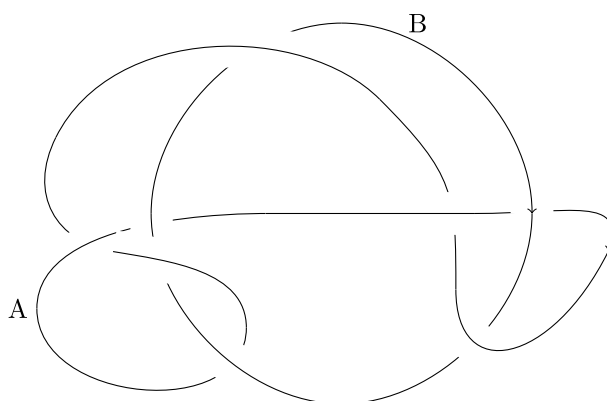


Figure 19: A diagram of the link L , which is the link $L8a9$ in KnotAtlas.

Similarly to what we did in the previous subsection, the Khovanov homologies shown in Table A.4 allow us to determine that

$$\dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_+(L, 1; A)) = \begin{cases} 2, & \text{if } j = 0, 2, \\ 0, & \text{otherwise,} \end{cases}$$

$$\dim \text{Gr}_{\mathcal{F}}^j H_{\text{Lee}}^0(W_+(H_{\pm}, 1)) = \begin{cases} 2, & \text{if } j = 0, \\ 1, & \text{if } j = -2, 2, \\ 0, & \text{otherwise.} \end{cases}$$

It follows from Theorem 5.4 that $F'_{v_s}(L; A)(1) = (s(W'_+(L, 1; A)) + 1)/2 = 1$ and $F'_{v_s}(H_{\pm})(1) = (s(W'_+(H_{\pm}, 1)) + 1)/2 = 0$. ■

Proposition 6.4 tells us that the functions F_v, \bar{F}_v, F'_v , and \bar{F}'_v can effectively give more information, as concordance invariants, than v and the linking matrix. In particular, we have also shown that $F'_{v_s}(L)$ is not determined by the Lee homology of L .

Proof of Theorem 1.8 The result follows from Proposition 6.4. ■

A Tables of Khovanov Homology

A.1 Tables Relative to Section 5

In this subsection we collect the tables of the Khovanov homology used in the computations in Section 5. We have highlighted the column corresponding to the homological degree 0, which is the homological degree where Lee homology is concentrated in these cases. All the homologies are computed with coefficients in \mathbb{Q} .

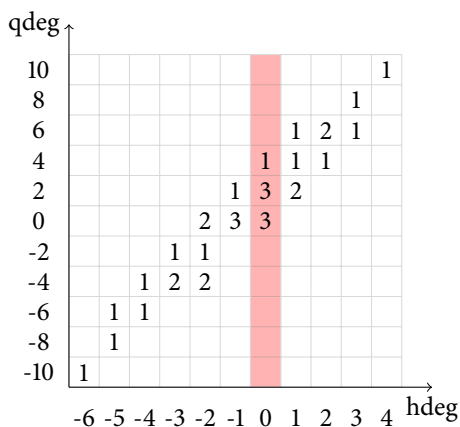


Table A.1: The Khovanov homology of the link $W_+(H_{\pm}, 0, 1)$.

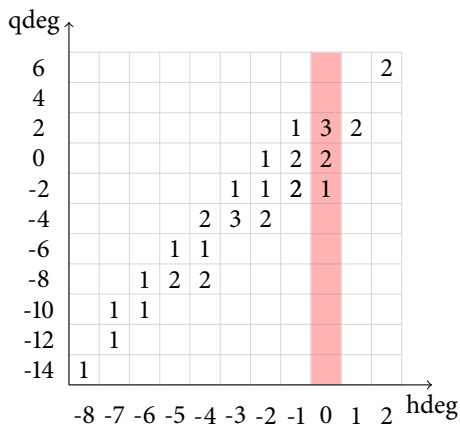
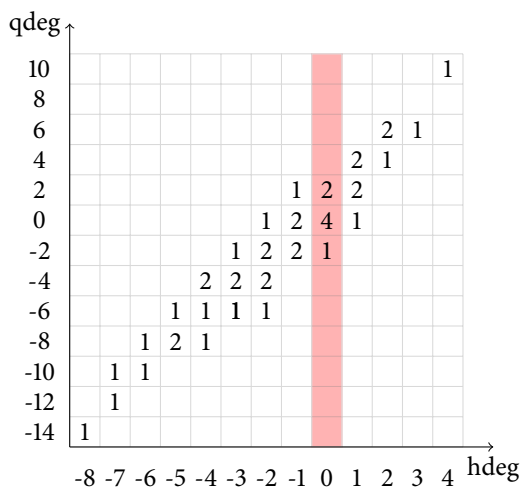


Table A.2: The Khovanov homology of the links $W_+(H_{\pm}, 1, 1)$ and $W_+(H_{\pm}, 0, 2)$.

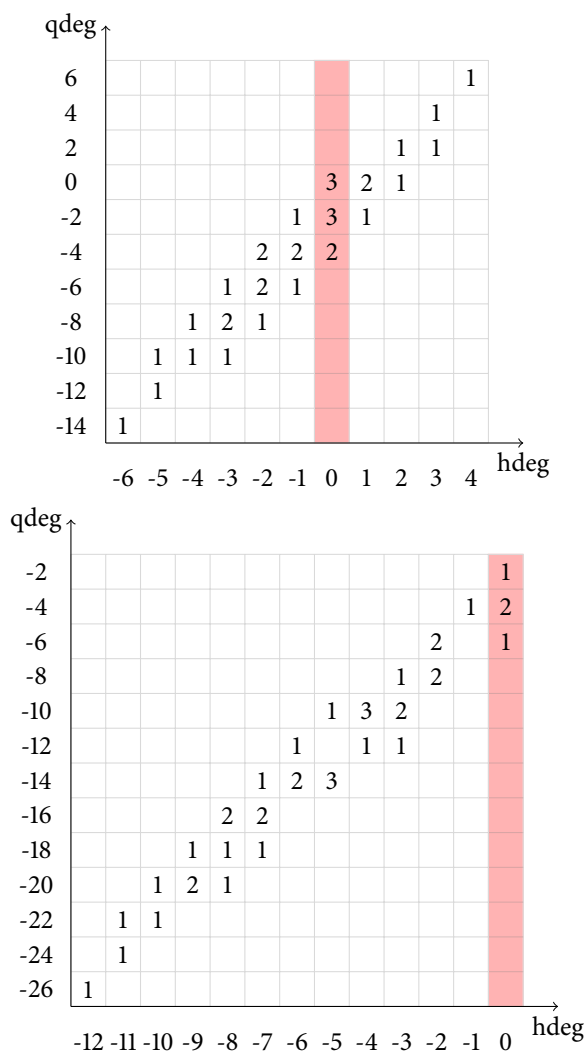


Table A.3: The Khovanov homology of the links $W_-(H_{\pm}, 0, 0)$ and $W_-(H_{\pm}, 1, 2)$.

A.2 Tables Relative to Section 6

In this subsection we collected the tables of the Khovanov homology used in the computations in Section 6.

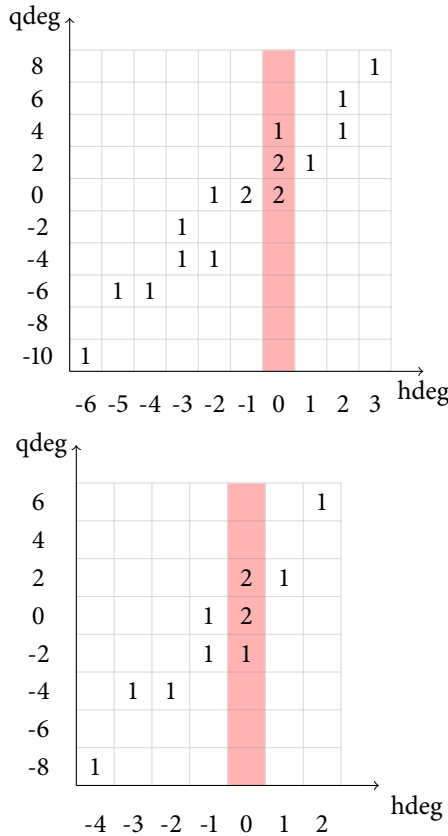


Table A.4: The Khovanov homology of the links $W'_+(L, 1; A)$ and $W'_+(H_{\pm}, 1)$.

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