COMPOSITIO MATHEMATICA

BPS invariants from p-adic integrals

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Abstract

We define *p*-adic BPS or *p*BPS invariants for moduli spaces $\mathrm{M}_{\beta,\chi}$ of one-dimensional sheaves on del Pezzo and K3 surfaces by means of integration over a non-archimedean local field F. Our definition relies on a canonical measure μ_{can} on the F-analytic manifold associated to $\mathrm{M}_{\beta,\chi}$ and the *p*BPS invariants are integrals of natural $\mathbb{G}_m$ gerbes with respect to μ_{can}. A similar construction can be done for meromorphic and usual Higgs bundles on a curve. Our main theorem is a χ-independence result for these *p*BPS invariants. For one-dimensional sheaves on del Pezzo surfaces and meromorphic Higgs bundles, we obtain as a corollary the agreement of *p*BPS with usual BPS invariants through a result of Maulik and Shen [*Cohomological χ-independence for moduli of one-dimensional sheaves and moduli of Higgs bundles*, Geom. Topol. **27** (2023), 1539–1586].

1. Introduction

Donaldson–Thomas (DT) invariants, first introduced in [Tho00], count stable sheaves with some fixed Chern character $\gamma \in H^*(X, \mathbb{Z})$ on a smooth Calabi–Yau 3-fold X. While with the original machinery the invariants could only be defined for moduli of sheaves where no strictly semi-stables occur, Joyce and Song [JS12] and Kontsevich and Soibelman [KS08] independently developed a generalized theory allowing the definition of DT invariants for moduli of objects (in CY3 categories) also in those cases where strictly semi-stables are present. The two approaches have some differences and the resulting generalized numerical invariants, denoted in the above references by $\bar{\mathrm{DT}}_\gamma$ and $\hat{\mathrm{DT}}_\gamma$ or $\Omega(\gamma)$ respectively do not coincide. The relation between the two is however understood and explained for example in [JS12, §6.2] or [DM15, §6.7].

Both theories admit, at least in some specific geometries, refinements to motivic, cohomological and sheaf theoretic invariants. Let $\mathbb{M}_\gamma(X)$ denote some moduli stack of semi-stable sheaves on a smooth Calabi–Yau 3-fold; Joyce–Song theory is refined by the cohomology of a certain perverse sheaf $\mathcal{DT}_\gamma$ on the moduli stack obtained (given an orientation data) gluing locally defined vanishing cycle sheaves (see [BBBJ15, BBDJS15]). Kontsevich–Soibelman theory should instead be refined by the cohomology $\mathrm{H}^*(\mathrm{M}_\gamma(X), \Phi_\gamma)$ for Φ_γ the so-called *BPS sheaf*, named after Bogomol'nyi, Prasad and Sommerfield, defined on the moduli space $\mathrm{M}_\gamma(X)$ of S-equivalence classes. For X a compact CY3 threefold the existence of the sheaf Φ_γ is still conjectural; see [Tod23] for a conjectural definition inspired to the case of moduli spaces of quiver representation with potential. In the latter situation, the BPS sheaf Φ_γ, the sheaf on the moduli stack $\mathcal{DT}_\gamma$ as well their relation are understood [DM20].

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Following [JS12] and [DM20] in this paper we call the generalized DT invariants of Kontsevich and Soibelman BPS invariants, since as suggested by the authors themselves, their invariants should count BPS states.

Particularly interesting from the perspective of enumerative geometry is the case of one-dimensional sheaves on X, i.e. sheaves with Chern character $\gamma = (0, 0, \beta, \chi)$. In this case the invariants arising from the moduli spaces of Gieseker semi-stable sheaves are (conjecturally) related to Gopakumar–Vafa invariants [Kat08, MT18, Tod23].

Our main focus is the special case where the CY3-fold is a del Pezzo or a K3 local surface $X = \mathrm{Tot}(S)$. In these cases the BPS sheaf is better understood. For a local del Pezzo case a result of Meinhardt [Mei15] states that $\Phi_{\beta,\chi}$ exists and coincides with the intersection complex $IC_{\mathrm{M}_{\beta,\chi}}$ of the moduli space of S-equivalence classes. For a local K3 the situation is more complicated and an explicit description of the BPS sheaf has only recently been given in [DHM23].

The main result of the present paper gives a surprising relation between the BPS invariants of local del Pezzo surfaces and certain non-archimedean (or p-adic) integrals. The relation is indirect and relies on results of Maulik and Shen [MS23] which we explain next.

1.1 Cohomological χ independence and trace of Frobenius

Let S be a smooth surface over $\mathbb{C}$ and β an ample base point free class. We denote by $\mathbb{M}_{\beta,\chi}(S)$ the moduli stack of Gieseker semi-stable (with respect to some fixed polarization H) one-dimensional sheaves, and by $\mathrm{M}_{\beta,\chi}(S)$ the moduli space of S-equivalence classes. When S is del Pezzo, the stack of semi-stables is smooth. Under this hypothesis Maulik and Shen [MS23] have recently proved the independence of the intersection cohomology from the Euler characteristic:[1]

$$\mathrm{IH}^*(\mathrm{M}_{\beta,\chi}(S)) \cong \mathrm{IH}^*(\mathrm{M}_{\beta,\chi'}(S)) \quad \forall\, \chi, \chi'.$$

In fact, they prove a stronger statement, namely that the pushforward of the intersection complexes along the Hilbert–Chow morphisms $h_\chi, h_{\chi'}$ are isomorphic. We denote by h_χ the morphism

$$h_\chi : \mathrm{M}_{\beta,\chi}(S) \to \mathsf{B} = \mathbb{P}H^0(S, \mathcal{O}_S(\beta)),$$

associating to a sheaf its Fitting support.

The result proves a refinement of a special case of a more general and long-standing conjecture in the enumerative geometry of smooth CY3-fold, known as the Pandharipande–Thomas strong rationality conjecture [PT10] and later reformulated by Toda [Tod14] as the multiple cover formula conjecture for generalized DT invariants of moduli spaces of one dimensional sheaves. Translated in the Kontsevich–Soibelmain theory, the conjecture predicts the independence of the BPS invariants from the Euler characteristic.

The conjecture is expected to hold more in general at the refined level.

Now, choosing a spreading out, we can assume that the moduli spaces $\mathrm{M}_{\beta,\chi}(S)$ we are considering are defined over some large finite field $k = \mathbb{F}_q$ and that χ independence of $\Phi_{\beta,\chi}$ holds over k (see, for example, [BBDG18, §6] for precise definitions and results on spreading out of constructible complexes). We can then look at the function:

$$\mathrm{BPS}_{\beta,\chi} : \mathrm{M}_{\beta,\chi}(k) \to \mathbb{C}, \quad x \mapsto q^{-\dim \mathcal{M}_{\beta,\chi}} \mathrm{Tr}(\mathrm{Fr}, \Phi_{\beta,\chi,x}).$$

We call a pair (β, χ) generic (with respect to H), if any Gieseker semi-stable sheaf in $\mathbb{M}_{\beta,\chi}$ is stable. In this case $\mathbb{M}_{\beta,\chi} \to \mathrm{M}_{\beta,\chi}$ is a $\mathbb{G}_m$ gerbe and, in particular, $\mathrm{M}_{\beta,\chi}$ is smooth. Then the function $\mathrm{BPS}_{\beta,\chi}$ will have the following two properties:

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[1] Strictly speaking in [MS23] S is assumed to be toric, but with the recent results of [Yua23] this assumption can be dropped, see [MS23, Remark 0.2] and Corollary 2.2.4.

(i) $\mathrm{BPS}_{\beta,\chi} \equiv q^{-\dim \mathrm{M}_{\beta,\chi}}$ if (β,χ) is generic;

(ii) for all $\chi, \chi' \in \mathbb{Z}$ and for all $y \in B(k)$ we have

$$\sum_{x \in h_\chi^{-1}(y)(k)} \mathrm{BPS}_{\beta,\chi}(x) = \sum_{x \in h_{\chi'}^{-1}(y)(k)} \mathrm{BPS}_{\beta,\chi'}(x),$$

the second being a consequence of [MS23].

The main content of the paper is the construction of a function $p\mathrm{BPS}_{\beta,\chi}$ via non-archimedean integration satisfying these two properties.

1.2 p-adic BPS function and its invariance

Consider S a smooth del Pezzo surface over $\mathsf{Spec}(\mathcal{O})$, where $\mathcal{O}$ denotes the ring of integers of a non-archimedean local field F with residue field $k \cong \mathbb{F}_q$.

As recalled previously, the del Pezzo hypothesis ensures that the moduli stack is smooth, which allows us to construct in §3 a *canonical measure* μ_{can} on the F-analytic manifold $\mathrm{M}_{\beta,\chi}(\mathcal{O})^\natural$ associated to $\mathrm{M}_{\beta,\chi}$.

In fact, the construction of μ_{can} works more generally for normal, generically stabilizer-free, Artin stacks $\mathcal{M}$ admitting a universally closed morphism $\mathcal{M} \xrightarrow{\pi} \mathrm{M}$ to a quasi-projective variety M such that π is generically an isomorphism. The existence of a canonical measure in this context should be of independent interest.

Importantly, these hypothesis are also satisfied by moduli stacks $\mathrm{M}_{\beta,\chi}$ of semi-stable sheaves on $S \to \mathsf{Spec}(\mathcal{O})$ a smooth projective K3 over $\mathcal{O}$ for many choices of (β,χ) (see §3.2 for details), as well as by moduli stacks of (usual) Higgs bundles.

Once defined the canonical measure on $\mathrm{M}_{\beta,\chi}(\mathcal{O})^\natural$, we then define the non-archimedean BPS function $p\mathrm{BPS}_{\beta,\chi}$ as follows: for any $x \in \mathrm{M}_{\beta,\chi}(k)$ denote by $\mathrm{M}_{\beta,\chi}(\mathcal{O})_x \subset M_{\beta,\chi}(\mathcal{O})^\natural$ the ball of $\mathcal{O}$-rational points specializing to x over k. Then

$$p\mathrm{BPS}_{\beta,\chi} : \mathrm{M}_{\beta,\chi}(k) \to \mathbb{C}$$

is given by

$$p\mathrm{BPS}_{\beta,\chi}(x) = q^{-\dim M_{\beta,\chi}} \int_{M_{\beta,\chi}(\mathcal{O})_x} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu_{\mathrm{can}}, \tag{1.2.1}$$

where $\varphi_{\beta,\chi}$ is a certain complex-valued function associated with the natural $\mathbb{G}_m$ gerbe coming from the $\mathbb{G}_m$-rigidification of $\mathbb{M}_{\beta,\chi}$ (see §4 for the precise definition) and g is the arithmetic genus of the curves in the linear system $|\beta|$.

Our main result says that the non-archimedean BPS function enjoys the two properties above.

THEOREM 1.2.2 (Theorem 5.0.2). *Let $S \to \mathsf{Spec}(\mathcal{O})$ be either a smooth projective del Pezzo surface or a K3 surface satisfying Assumption 3.2.1. The function $p\mathrm{BPS}_{\beta,\chi} : \mathrm{M}_{\beta,\chi}(k) \to \mathbb{C}$ satisfies the following two properties:*

(i) *$p\mathrm{BPS}_{\beta,\chi} \equiv q^{-\dim \mathrm{M}_{\beta,\chi}}$ if (β,χ) is generic;*

(ii) *for all $\chi, \chi' \in \mathbb{Z}$ and for all $y \in B(k)$ we have*

$$\sum_{x \in h_\chi^{-1}(y)(k)} p\mathrm{BPS}_{\beta,\chi}(x) = \sum_{x \in h_{\chi'}^{-1}(y)(k)} p\mathrm{BPS}_{\beta,\chi'}(x).$$

1.3 Relation to previous works, consequences and final considerations

The original motivation for Theorem 1.2.2 is a degree-independence result for Higgs bundles of rank coprime to the degree [GWZ20b, Theorem 7.15], which in turn is a special case of a conjecture by Mozgovoy and Schiffmann [MS14].[2] Thus, not surprisingly, the proof of Theorem 1.2.2 also relies on the same idea used in [GWZ20b, GWZ20a], which boils down to a Fubini argument along the Hilbert–Chow morphism h_χ and the fact that up to measure 0, the fibers of h_χ over a non-archimedean local field are Picard schemes of smooth curves.

As a corollary of Theorem 1.2.2, we obtain from [MS23] (combined with the results in [Yua23]) that after pushforward along h_χ, the function $p\mathrm{BPS}_{\beta,\chi}$ agrees with the trace of Frobenius on the BPS sheaf for the case of del Pezzo surfaces. The χ-independence conjecture for BPS cohomology suggests, that this continues to hold in the K3 case.

A more concrete application of Theorem 1.2.2, via the Weil conjectures, is the χ independence of the Betti numbers for $\mathrm{M}_{\beta,\chi}$ as long as χ and β are coprime. For $\mathbb{P}^2$ this is also a consequence of a more general result of Bousseau [Bou23, Conjecture 0.4.2, Theorem 0.4.5].

When $\mathrm{M}_{\beta,\chi}$ is smooth, the first part of Theorem 1.2.2 implies

$$p\mathrm{BPS}(x) = \mathrm{BPS}(x)$$

for all $x \in \mathrm{M}_{\beta,\chi}(k)$.

While for more general pairs (β, χ) Definition (1.2.1) has no obvious connection with the definition of the BPS function as trace of Frobenius of some suitable cohomology, Theorem 1.2.2 suggests that the above equality still holds.

For the case of S del Pezzo, this intriguing identity is the subject of an ongoing project of the third named author with Michael Groechenig and Paul Ziegler.

The refined χ independence for BPS invariants of moduli of sheaves on a K3 surface is instead still conjectural (see [MT19] for a proof of the numerical version), so is the identification of $p\mathrm{BPS}$ and BPS invariants.

We expect that a conceptual proof (not relying on the χ independence) for the equality between $p\mathrm{BPS}$ and BPS invariants will be easier to obtain in the Fano case, due to the smoothness of the moduli stacks.

Finally, as in [MS23], we can prove the same statements for moduli of (meromorphic or not) Higgs bundles; that is, pairs of a vector bundle $\mathcal{E}$ on a curve C of genus $g(C) \geq 2$ and a morphism $\Theta \colon \mathcal{E} \to \mathcal{E}(D)$ for D a fixed effective divisor of degree $\deg(D) \geq 2g(C) - 2$.

In this case, the Hilbert–Chow morphism is replaced by the Hitchin fibration

$$H \colon \mathrm{M}_{r,\chi}(C) \to \bigoplus_{i=1}^{r} \mathrm{H}^0(C, \mathcal{O}_C(iD))$$

associating to $(\mathcal{E}, \Theta)$ the characteristic polynomial of Θ. The χ independence for meromorphic Higgs bundles was also proven combinatorially in [MO19]. Moreover, during the writing of this article, χ independence for BPS invariants of Higgs bundles has been proved in [KK21]. This indirectly implies that our non-archimedean $p\mathrm{BPS}$ function agrees with the geometric BPS function beyond the Fano cases.

This is in sharp contrast with the intersection complex on the coarse moduli space, which does depend on the Euler characteristic χ in the $K3$ and Higgs bundle case, see [MS23, § 0.4].

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[2] There is an alternative proof by Yu [Yu23] and proofs of the complete conjecture by Mellit [Mel20] and Kinjo and Koseki [KK21].

2. Moduli spaces of sheaves and Higgs bundles

In this section we recall some properties of moduli spaces of sheaves on smooth surfaces and moduli spaces of (meromorphic or not) Higgs bundles relative to a base scheme T.

The case $T = \mathsf{Spec}\, k$ with k algebraically closed of $\mathrm{char}(k) = 0$ is classical and all the statements can be found for example in [HL97, § 4] for moduli of sheaves on surfaces, and in [Nit91] for moduli of Higgs bundles.

For T a field of positive or mixed characteristic, moduli spaces of sheaves were first constructed by Langer in [Lan04b, Lan04a]. More general moduli stacks and moduli spaces of sheaves and complexes over a smooth projective family X/T were studied in [Lie06] and [HT10]. Recently the study of moduli stacks in the relative context has been vastly generalized in [BLM$^+$21].

We often cite the latter reference, even though the results of Langer, Lieblich and of Huybrechts–Thomas would suffice for our purposes.

2.1 Relative moduli spaces of sheaves on surfaces

Let $S \xrightarrow{r} T$ be a smooth family of projective surfaces, i.e. r is smooth and projective. We are mostly interested in the case where the relative anti-canonical bundle $-K_{S/T}$ is r-ample or trivial; we then say that S is del Pezzo, respectively K3, over T.

For us T will be either $\mathsf{Spec}(\mathcal{O})$, or $\mathsf{Spec}\, F$, or $\mathsf{Spec}\, k$ where F is a p-adic field, $\mathcal{O}$ its ring of integers, and k a field, possibly of positive characteristic.

2.1.1 *Construction of the moduli spaces.* Let us fix β on S the class of an r-effective divisor, ample and base point free, i.e. β_t is ample and base point free for each $t \in T$. Given $\mathcal{F}$ a sheaf on S, flat over T, we say that $\mathcal{F}$ is *one-dimensional* over T of class β if

$$\mathrm{Supp}(\mathcal{F}) := \mathrm{Fit}_0(\mathcal{F}) = C$$

with $C \in \mathbb{P}_T(g_* \mathcal{O}_S(\beta))$. Since the Fitting support is compatible with pull-back [Sta, Tag 07Z6] this means that $\mathcal{F}_t$ is supported on a curve of class β_t on S_t for each $t \in T$.

Let $\mathcal{O}_S(1)$ be a r-polarization. Given $\mathcal{F}$ a sheaf on S flat over T, the Hilbert polynomial

$$P_r(n) := \chi(S_t, \mathcal{F}_t \otimes \mathcal{O}_{S_t}(n)) = (\beta_t \cdot H_t)n + \chi(\mathcal{F}_t)$$

for $H_t \in |\mathcal{O}_{S_t}(1)|$ is locally constant on T (see, for example, [Har77, Theorem 9.9]); in particular, $\chi(\mathcal{F}_t)$ is locally constant.

We denote by $p_r(n)$ the reduced *relative Hilbert polynomial* obtained from $P_r(n)$ dividing by the coefficient of the leading term.

DEFINITION 2.1.1 (see [BLM$^+$21, Definition 21.11]). Let β, χ and $\mathcal{O}_S(1)$ be as above. We denote by

$$\mathbb{M}_{\beta,\chi}(S) \colon (\mathrm{Sch}\,/T)^{\mathrm{op}} \to \mathrm{Gpds}$$

the functor whose value on $W \to T$ consist of sheaves $\mathcal{F}$ on $S \times_T W$, flat over W such that for all $w \in W$, $\mathcal{F}_w$ is a Gieseker semi-stable one-dimensional sheaf of class β_w and Euler characteristic $\chi \in \mathbb{Z}$. We recall that $\mathcal{F}_w$ is said Gieseker semi-stable if for all proper subsheaves $\mathcal{G} \in \mathcal{F}_w$ we have

$$n + \frac{\chi(\mathcal{G})}{[\mathrm{Supp}(\mathcal{F}_w)] \cdot H_w} \leq p_r(n), \tag{2.1.2}$$

and Gieseker stable if the inequality is strict. Finally, we say that $\mathcal{F}_w$ is geometrically Gieseker stable if its pull-back $\mathcal{F}_{\overline{w}}$ to the algebraic closure is Gieseker stable.

An object $\mathcal{F} \in \mathbb{M}_{\beta,\chi}(S)(W)$ is called a *family of Gieseker semi-stable* sheaves of class (β, χ) over W.

Here $\mathbb{M}_{\beta,\chi}^{\mathrm{st}}(S)$ is the sub-functor of families of Gieseker geometrically stable objects.

THEOREM 2.1.3 [Lan04b, Lan04a, BLM$^+$21]. *In the notation above:*

(i) $\mathbb{M}_{\beta,\chi}(S)$ *and* $\mathbb{M}_{\beta,\chi}^{\mathrm{st}}(S)$ *are algebraic stacks of finite type over T; the same statement hold for the $\mathbb{G}_m$-rigidification $\mathcal{M}_{\beta,\chi}(S)$ of $\mathbb{M}_{\beta,\chi}(S)$;*

(ii) *there exists a projective T-scheme $\mathrm{M}_{\beta,\chi}(S)$ which uniformly represent the set valued functor over T-schemes associating to W the set of S-equivalence classes of families of Gieseker semi-stable sheaves on the fibers of $S \times_T W \to W$ with Hilbert polynomial $p_r(n)$;*

(iii) $\mathbb{M}_{\beta,\chi}^{\mathrm{st}}(S)$ *is a $\mathbb{G}_m$ gerbe over its coarse moduli space $\mathrm{M}_{\beta,\chi}^{\mathrm{st}}(S)$, which is a quasi-projective scheme over T representing isomorphism classes of Gieseker geometrically stable sheaves on the fibers of $S \times_T W \to W$ with Hilbert polynomial $p_r(n)$;*

(iv) $\mathbb{M}_{\beta,\chi}(S)$ *is universally closed over T and, thus, universally closed over $\mathrm{M}_{\beta,\chi}(S)$.*

Proof. (i) A far more general version of this statement is proved in [BLM$^+$21, Theorem 21.24]. For the benefit of the reader, we recall the key steps of the proof in the special case of Langer [Lan04a], which is the only one we need.

Let E be a coherent sheaf on S flat on T, and let $\mathrm{Quot}_{T,p_r(n)}(E) \colon (\mathrm{Sch}/T)^{\mathrm{op}} \to \mathrm{Sets}$ be the relative Quot functor, whose $W \to T$ valued points are quotients $E_W \to Q \to 0$ with Q a coherent sheaf on $S \times_T W$, flat over W, with Hilbert polynomial $p_r(n)$ along the fibers of $S \times_T W \xrightarrow{r_W} W$. As proved for example in [Sta, Tag 09TQ], $\mathrm{Quot}_{T,p_r(n)}(E)$ is represented by an algebraic space over T locally of finite presentation.

By Langer's results [Lan04a, §3], families of Gieseker semi-stable sheaves on the fibers of $S \xrightarrow{r} T$ with fixed Hilbert polynomial $p_r(n)$ are bounded, i.e. any $\mathcal{F} \in \mathbb{M}_{\beta,\chi}(S)$ is a quotient of $E = \mathcal{O}_S(-N)^{\oplus F(N)}$ for some fixed N.

Furthermore, as shown in [Lan04a, §4], the open subsets $\mathcal{R}^{\mathrm{st}} \subseteq \mathcal{R}^{\mathrm{ss}} \subset \mathrm{Quot}_{T,p_r(n)}(E)$ of Gieseker stable/semi-stable sheaves on the fibers of $S \xrightarrow{r} T$, correspond to the opens of stable/semi-stable points with respect to the natural action of the T-smooth group $\mathrm{GL}(F(N))$.

It thus follows that $\mathbb{M}_{\beta,\chi}(S) \cong [\mathcal{R}^{\mathrm{ss}}/\mathrm{GL}(F(N))]$ and $\mathbb{M}_{\beta,\chi}^{\mathrm{st}}(S) \cong [\mathcal{R}^{\mathrm{st}}/\mathrm{GL}(F(N))]$ as stacks over Sch/T. The statement for the rigidification $\mathcal{M}_{\beta,\chi}(S)$ now follows immediately as the latter is simply given by $[\mathcal{R}^{\mathrm{ss}}/\mathrm{PGL}(F(N))]$.

(ii) The projective T-scheme $\mathrm{M}_{\beta,\chi}(S)$ is the GIT quotient $\mathcal{R}^{\mathrm{ss}}//\mathrm{GL}(F(N))$ this exists and has the claimed property by [Ses77, Theorem 4], as also argued by Langer in the proof of [Lan04a, Theorem 4.1].

(iii) The statement follows by the construction explained in statement (i).

(iv) The universal closedness over T follows from a relative version of Langton's theorem. The reader can see [Lan75] or [HL97, Theorem 2.B.1] for the proof over an algebraic closed field, and [BLM$^+$21, Lemma 21.22] for the general case.

Given the GIT construction of the moduli space recalled above, the universal closedness of $\mathbb{M}_{\beta,\chi}(S) \to \mathrm{M}_{\beta,\chi}(S)$ also follows from [AHH23, Theorem A.8]. $\qquad \square$

2.1.2 *Deformation theory.* Let $S \xrightarrow{r} T$ be as in the previous section, and let $\mathcal{F}$ be a coherent sheaf on S flat over T. We denote by $\mathrm{Ext}_S^i(\mathcal{F}, \mathcal{F})$ the T sheaf obtained by taking the ith cohomology of the complex $Rr_* \mathrm{R}\,\mathcal{H}om(\mathcal{F}, \mathcal{F})$, i.e.

$$\mathrm{Ext}_S^i(\mathcal{F}, \mathcal{F}) := \mathrm{H}^i(Rr_* \mathrm{R}\,\mathcal{H}om(\mathcal{F}, \mathcal{F})).$$

By [Lie06, § 3] (see also [HT10, § 4]) the sheaves $\operatorname{Ext}_S^i(\mathcal{F}, \mathcal{F})$ control the deformation theory of $\mathcal{F}$ over T, namely:

(i) there is a natural isomorphism $\operatorname{Ext}_S^0(\mathcal{F}, \mathcal{F}) \cong \operatorname{End}(\mathcal{F}, \mathcal{F})$;
(ii) the first-order deformations of $\mathcal{F}$ are given by $\operatorname{Ext}_S^1(\mathcal{F}, \mathcal{F})$;
(iii) the obstruction to deformations are contained in $\operatorname{Ext}_S^2(\mathcal{F}, \mathcal{F})$.

LEMMA 2.1.4. *Let $S \xrightarrow{r} T$ be a smooth family of del Pezzo surfaces. Then $\mathbb{M}_{\beta,\chi}(S)$ is smooth over T.*

Proof. It suffices to show that the obstruction $\operatorname{Ext}_S^2(\mathcal{F}, \mathcal{F})$ is zero. We argue that $\operatorname{Ext}_S^2(\mathcal{F}, \mathcal{F}) \otimes k(\bar{t}) = 0$ for any geometric point $\bar{t} \in T$. By [AK80, Theorem 1.9], the cohomology and base change property for Ext sheaves implies

$$\operatorname{Ext}_S^2(\mathcal{F}, \mathcal{F}) \otimes k(\bar{t}) = \operatorname{Ext}_{S_{\bar{t}}}^2(\mathcal{F}_{\bar{t}}, \mathcal{F}_{\bar{t}}),$$

where $\mathcal{F}_{\bar{t}}$ is, by definition, a Gieseker semi-stable sheaf of class $\beta_{\bar{t}}$ and Euler characteristic χ on $S_{\bar{t}}$. The vanishing of the right-hand side is then classical; see, for example, [MS23, Lemma 2.5]. $\square$

2.1.3 *Singularities in the K3 case.* For $S \to T$ a K3 surface, it will never be the case that $\operatorname{Ext}_S^2(\mathcal{F}, \mathcal{F})$ vanishes. Nonetheless, the following generalization of a classical result of Mukai [Muk84] holds.

THEOREM 2.1.5 [Ina11, Theorem 3.2]. *Let S/k be a smooth projective K3 surface over an algebraically closed field k and let $\mathcal{F}$ be a simple sheaf, i.e. $\operatorname{Hom}_S(\mathcal{F}, \mathcal{F}) = k$. Then the deformations of $\mathcal{F}$ are unobstructed. In particular, the open sub-stack $\operatorname{Spl}(S/k)_{(\beta,\chi)} \subseteq \mathbb{M}_{\beta,\chi}(S)$ of simple semi-stable sheaves is smooth over k.*

COROLLARY 2.1.6. *Let $S \to T$ be a smooth projective relative K3 surface and T a smooth base with perfect residue fields. Then the moduli stack $\operatorname{Spl}_{(\beta,\chi)}(S/T)$ of simple semi-stable sheaves is smooth over T.*

Proof. By Inaba's result, Theorem 2.1.5, the closed fibers of $\operatorname{Spl}_{(\beta,\chi)}(S/T) \to T$ are smooth of dimension $\beta_{\bar{t}}^2 + 2$, and since $S \to T$ is smooth this number is constant on T (see [Ful16, § 20.3]). It then follows from [Sch10, Theorem 3.3.27] that $\operatorname{Spl}_{(\beta,\chi)}(S/T) \to T$ is flat and, thus, in fact smooth since all of the closed fibers are. $\square$

This immediately implies that $\mathbb{M}_{\beta,\chi}^{\mathrm{st}}(S)$ is smooth for $S \to T$ as in Corollary 2.1.6. Furthermore, for $T = \mathbb{C}$, the singularities of the moduli stack $\mathbb{M}_{\beta,\chi}(S)$ have been studied intensively [AS18, BZ19, Dav23, Cra01]. In the following theorem, we state, in a form convenient for our purposes, some consequences of the results obtained in the quoted references.

THEOREM 2.1.7. *Let $S/\mathbb{C}$ be a smooth projective K3 surface and (β, χ) a dimension vector such that the locus of $\operatorname{Spl}_{\beta,\chi}^{\mathrm{ss}}(S) \subseteq \mathbb{M}_{\beta,\chi}(S)$ of simple semi-stable sheaves has complement of codimension at least two. Then $\mathbb{M}_{\beta,\chi}(S)$ is normal with local complete intersection singularities.*

Proof. For a morphism $Y \to X$ he property of being local complete intersection is étale local on source and target [Sta, Tag 06C3]. Thus, it is enough to argue that for any point $q \in \mathbb{M}_{\beta,\chi}(S)$ there exist an étale neighborhood U_q with local complete intersection singularities.

By [Dav23, Theorem 5.11], an étale local model for $\mathbb{M}_{\beta,\chi}(S) \to \mathrm{M}_{\beta,\chi}(S)$ is given by the moduli stack $\mathfrak{M}_{\mathbf{d}}(\Pi_Q) = [\mu_{\mathbf{d}}^{-1}(0)/\operatorname{GL}_{\mathbf{d}}]$ (see [Dav23, § 3] for notation) of representations of the preprojective algebra of a certain explicit quiver, already considered in [AS18].

On the other hand, by [Cra01, Theorem 1.2], $\mu_{\mathbf{d}}^{-1}(0)$ is a local complete intersection as soon as there exists a simple representation of the preprojective algebra with given dimension vector.

By our assumptions, $\mathrm{Spl}^{\mathrm{ss}}_{\beta,\chi}(S)$ is dense in $\mathbb{M}_{\beta,\chi}(S)$. This ensures the existence of a simple representation in U_q for each q. Thus, since étale morphisms preserve the dimension of the automorphism groups and automorphism groups for quiver representations are connected, there exists a simple representation for Π_Q for each Q appearing in the local description.

Once we know that $\mathbb{M}_{\beta,\chi}(S)$ is local complete intersection, and thus Gorenstein, the normality simply follows from the assumption on the codimension of the locus of simple sheaves and Mukai and Inaba's Theorem 2.1.5. □

Remark 2.1.8. We show in § 2.2 that the hypothesis on the codimension of the complement of $\mathrm{Spl}^{\mathrm{ss}}_{\beta,\chi}(S)$ is satisfied as soon as: β is ample and base point free, and the locus V of non-reduced curves in the linear system $|\beta|$ is of codimension at least 2.

For example, let S be the K3 surface obtained as the 2:1 cover of $S \xrightarrow{\tau} \mathbb{P}^2$ branched along a general sextic Z. Consider $\beta = \tau^*[dL]$ for $d \geq 1$. Then a curve C in $|\beta|$ is the 2:1 cover of a curve $D \in |dL|$ branched over $D \cap Z$. Then C can be non-reduced only if, either D itself is non-reduced or there is a component of D contained in Z, i.e. Z is a component of D. Both loci have high codimension in $|dL|$.

2.1.4 *Hilbert–Chow map.* We keep the notation of the previous section. Let us denote by $B_T = \mathbb{P}_T(g_*\mathcal{O}_S(\beta))$, and by $\mathcal{C} \to B_T$ the universal curve. As for the case $T = \mathsf{Spec}(\mathbb{C})$ there is a morphism

$$\mathbf{h}_\chi \colon \mathbb{M}_{\beta,\chi}(S) \to B_T,$$

which associate to $\mathcal{F}$ its fitting support and factors through the moduli space

$$h_\chi \colon \mathrm{M}_{\beta,\chi}(S) \to B_T.$$

We call h_χ the Hilbert–Chow morphism and $\mathbf{h}_\chi$ the stacky Hilbert–Chow morphism. Since both $\mathrm{M}_{\beta,\chi}(S)$ and B_T are projective over T, h_χ is proper over T.

For $b \colon T \to B_T$ a section such that $\mathcal{C}_b/T$ is smooth, the fiber $\mathbf{h}_\chi^{-1}(b)$ is the Picard stack $\mathfrak{Pic}^{\chi+g-1}_{\mathcal{C}_b/T}$ where g is the genus of the curve, and $h_\chi^{-1}(b)$ the Picard space $\mathrm{Pic}^{\chi+g-1}_{\mathcal{C}_b/T}$, i.e. the $\mathbb{G}_m$-rigidification.

For $b \colon T \to B_T$, such that $\mathcal{C}_b/T$ is integral, the fiber of the stacky Hilbert–Chow morphism is some universally closed stack over T containing $\mathfrak{Pic}^{\chi+g-1}_{\mathcal{C}_b/T}$ as an open sub-stack, and $h_\chi^{-1}(b)$ is the corresponding GIT quotient, which is the compactification of the Picard space $\mathrm{Pic}^{\chi+g-1}_{\mathcal{C}_b/T}$ constructed by Altman and Kleiman [AK80].

In general, for $\mathcal{C}_b/T$ non-integral not all line bundles of total degree $\mathrm{totdeg} = \chi + g - 1$ are Gieseker semi-stable and one cannot say much about the fiber, except that it is a moduli stack of semi-stable sheaves of pure dimension 1. In the reduced case, moduli stacks and moduli spaces of rank-one semi-stable torsion free sheaves on $\mathcal{C}_b/T$ have been studied, and in the following we recall some known results.

2.2 The open sub-stack of invertible sheaves
We denote by

$$\mathfrak{U}_{\beta,\chi} \subseteq \mathfrak{Pic}^{\chi+g-1}_{\mathcal{C}/B_T}$$

the open sub-stack corresponding to line bundles of total degree $\chi + g - 1$, which are Gieseker semi-stable on the fibers of $\mathcal{C}/B_T$.

The rigidification $\mathcal{U}_{\beta,\chi} \subseteq \mathcal{M}_{\beta,\chi}(S)$ is stabilizer free, and $\mathfrak{U}_{\beta,\chi} \to \mathcal{U}_{\beta,\chi}$ is simply a $\mathbb{G}_m$ gerbe. It follows from Theorem 2.1.3, that if we further restrict to geometrically stable line bundles

$\mathfrak{U}_{\beta,\chi}^{\mathrm{st}}$, the rigidification $\mathcal{U}_{\beta,\chi}^{\mathrm{st}}$ is a good moduli space for the moduli stack and it is isomorphic to an open in $\mathrm{M}_{\beta,\chi}(S)$.

The opens $\mathcal{U}_{\beta,\chi}^{\mathrm{st}} \subseteq \mathcal{U}_{\beta,\chi}$ will play a crucial role in the construction of the canonical measure. To this end, we need to prove the following codimension estimate.

PROPOSITION 2.2.1. (i) *Let $S \xrightarrow{r} T$ be a relative del Pezzo surface and β a base point free and ample curve class. Then the complement of $\mathcal{U}_{\beta,\chi}$ in $\mathcal{M}_{\beta,\chi}(S)$, or equivalently the complement of $\mathfrak{U}_{\beta,\chi}$ in $\mathbb{M}_{\beta,\chi}(S)$, has codimension at least two.*

(ii) *Let $T = \mathrm{Spec}(\mathbb{C})$ and S a smooth projective K3 surface. Suppose furthermore that β is ample, base point free and the codimension of the non-reduced locus $V \subset B$ is at least 2. Then the complement of $\mathcal{U}_{\beta,\chi}$ in $\mathcal{M}_{\beta,\chi}(S)$ has codimension at least two.*

The proof of this statement is somewhat involved and it will occupy the rest of the section. The reader who wish to get to the core of the paper quickly may skip this in a first reading.

Proof of Proposition 2.2.1. In case (i), by Lemma 2.1.4, $\mathbb{M}_{\beta,\chi}(S) \to T$ is smooth, so it is sufficient to estimate the codimension of the locus of line bundles on each geometric fiber $\mathbb{M}_{\beta_{\bar{t}},\chi}(S_{\bar{t}}) \to \mathrm{Spec}(\bar{k})$. In case (ii) we are already by hypothesis working on an algebraically closed field. This allows us to reduce to the following classical setting.

Setting 2.2.2. The surface S is a smooth del Pezzo or K3 surface over $\bar{k}$ an algebraically closed field; H is a polarization on S, β is a base point free and ample curve class and $B = |\beta|$ the linear system of curves in class β.

Working on $T = \mathrm{Spec}(\bar{k})$ allows us to appeal to the results of [MS23, Yua23]. It is proved in [Yua23], with techniques inspired by [Sch16], that for smooth projective surfaces S over an algebraically closed field $\bar{k}$ with canonical $K_S \leq 0$ the stacky Hilbert–Chow morphism $\mathbf{h}_\chi$ (and, thus, also h_χ) is equidimensional.

THEOREM 2.2.3 [Yua23, Theorem 1.2, Corollary 1.3]. *Let S be a smooth projective surface and $\beta = \sum_{i=1}^n m_i \beta_i$ an ample effective curve class with β_i pairwise distinct. If $K_S \cdot \beta_i \leq 0$ for all i, then*

$$\dim(\mathbf{h}_\chi^{-1}(b)) \leq g - 1$$

for any $b \in |\beta|$, where $g = \frac{1}{2}\beta \cdot (\beta + K_S) + 1$.

Using the dimension estimate, Maulik and Shen proved the following.

COROLLARY 2.2.4 [MS23, Theorem 2.3]. *For S del Pezzo, Hilbert–Chow morphisms $\mathbf{h}_\chi$ and h_χ are equidimensional. In particular, $\mathrm{M}_{\beta,\chi}(S)$ is irreducible.*

COROLLARY 2.2.5. *For S a K3 over an algebraically closed field, Hilbert–Chow morphisms $\mathbf{h}_\chi$ and h_χ are equidimensional. In particular, $\mathrm{M}_{\beta,\chi}(S)$ is irreducible.*

Proof. This is a word-by-word repetition of the argument in [MS23, Theorem 2.3] given the dimension estimate in Theorem 2.2.3. □

Remark 2.2.6. The dimension estimate $\dim(\mathbf{h}_\chi^{-1}(b)) \leq g - 1$ had previously been proved in [MS23, Proposition 2.6] under the additional hypothesis that S is a toric del Pezzo, by reducing the estimate to curves with underlying support a union of possibly non-reduced toric divisors, for which the results of [CL16] apply.

We use the equidimensionality of the (stacky) Hilbert–Chow as follows. First, by Bertini's theorem, since β is ample and base point free, there is a dense open $B^{sm} \subset B$ such that for

$b \in B^{sm}$ the curve $\mathcal{C}_b$ is a non-singular, integral curve of genus $g = \frac{1}{2}\beta \cdot (\beta + K_S) + 1$. In this case, $\mathbf{h}_\chi^{-1}(b) \cong \mathfrak{Pic}_{\mathcal{C}_b}^{\chi+g-1}$, which is simply a $\mathbb{G}_m$ gerbe over the degree $\chi + g - 1$ component of the Picard group.

In particular, $\mathbf{h}_\chi^{-1}(B^{sm})$ is contained in the open sub-stack of (stable) line bundles. Note, furthermore, that since the number of connected components is lower semi-continuous in a proper flat family (see, for example, [DM69, Theorem 4.17]) all the curves in the linear system are connected.

To prove the codimension estimate, we now look at the fibers of the Hilbert–Chow morphism over the locus of singular curves.

The complement of B^{sm} in B has codimension 1, and by equidimensionality of $\mathbf{h}_\chi$ so does $\mathbf{h}_\chi^{-1}(B \setminus B^{sm})$.

Let us denote by $V \subseteq B \setminus B^{sm}$ the locus of non-reduced curves. We prove in §2.2.1 that this has always codimension at least 2 in B for S del Pezzo, provided β is a base point free ample class. For S a K3, the codimension of V is at least two by assumption; see Remark 2.1.8 for examples where the assumption is verified. Then $\mathbf{h}_\chi^{-1}(V)$ has codimension at least two, and it can be discarded for the purposes of estimating the codimension of the complement of $\mathfrak{U}_{\beta,\chi}$.

Finally, we study the fibers over $(B \setminus B^{sm}) \setminus V$ in §2.2.2 and prove that the locus of line bundles in $\mathbf{h}_\chi^{-1}(b)$ is dense for all $b \in (B \setminus B^{sm}) \setminus V$.

In conclusion, the complement of $\mathfrak{U}_{\beta,\chi}$ is contained in the union of $\mathbf{h}_\chi^{-1}(V)$ and a divisor $\mathfrak{D} \subset \mathbf{h}_\chi^{-1}(B \setminus B^{sm})$. These have both codimension at least two, which allows us to conclude.

2.2.1 *Excluding the non-reduced locus.* Consider the stratification of B given by

$$B = \bigsqcup_{\underline{\beta}} B_{\underline{\beta}},$$

where $\underline{\beta} = \{m_1\beta_1, \ldots, m_s\beta_s\}$ with $s \geq 1$, $m_i \geq 1$, and β_i are distinct curve classes such that there exists an integral curve in $|\beta_i|$ for each i; $B_{\underline{\beta}}$ is the locally closed strata in B of curves of type $\underline{\beta}$, i.e.

$$B_{\underline{\beta}} = \left\{ C = \sum_i m_i C_i \in |\beta|, \ C_i \in |\beta_i| \right\},$$

with C_i integral.

Let Z be an irreducible component in $\overline{B}_{\underline{\beta}}$; we denote by δ_Z the dimension of the affine part of the Jacobian of $\mathcal{C}_b$ for $b \in Z^\circ \subseteq \overline{B}_{\underline{\beta}}$ generic, i.e.

$$\delta_Z = \dim(\mathrm{Pic}^0(\mathcal{C}_b))^{\mathrm{Aff}}.$$

LEMMA 2.2.7. *Let β be a base point free ample class on a smooth del Pezzo surface. The locus $V \subset B$ of non-reduced curves has codimension at least two.*

Proof. The locus V of non-reduced curves is the union of closed strata $\overline{B}_{\underline{\beta}}$ with $\underline{\beta}$ having at least one $m_i \geq 2$. We want to show that the codimension of any irreducible component Z of such a stratum $\overline{B}_{\underline{\beta}}$ is at least two in B. The proof follows from the following inequality which is proved in [MS23, Proposition 4.3]:

$$\mathrm{codim}(Z) \geq -K_S \cdot \left(\sum_i (m_i - 1)\beta_i \right) + \delta_Z \geq 1 + \delta_Z, \tag{2.2.8}$$

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where the second inequality holds since $-K_S$ is ample, $m_i - 1 \geq 0$ for each i and $m_i - 1 \geq 1$ for at least one i and, thus,

$$-K_S \cdot \left(\sum_i (m_i - 1)\beta_i \right) \geq 1;$$

since $-K_S$ is ample, the equality can hold only if all but one m_i are equal to 1.

If the latter inequality is strict, we are done; otherwise we can assume that $m_1 = 2$ and $m_i = 1$ for any other i. Again, if $\delta_Z \geq 1$ we are done.

To complete the proof we only need to look the cases where $m_1 = 2$, $m_i = 1$ for any other i, and $\delta_Z = 0$.

Let us denote by $C := \mathcal{C}_b$ for $b \in B_{\{2\beta_1, \beta_2, \ldots, \beta_s\}}$ a generic point in this stratum, by $C' \subseteq C$ the reduced subcurve, and by $\beta' = \beta - \beta_1$ its class.

We have two short exact sequences:

$$0 \to \mathcal{O}_{C_1}(-\beta' \cdot \beta_1) \to \mathcal{O}_C \to \mathcal{O}_{C'} \to 0$$

$$0 \to O_{C'} \to \oplus \bigoplus_{i=1}^s \mathcal{O}_{\widetilde{C}_i} \to \bigoplus_{p \in C'^{\mathrm{sing}}} \mathcal{O}_p^{\delta(p)} \to 0.$$

The latter is simply the normalization sequence for $\widetilde{C'} = \sqcup_{i=1}^s \widetilde{C}_i \to C'$ (see [Liu06, § 7.5]). We denoted by

$$\delta(p) = \mathrm{length}_k(\mathcal{O}_{\widetilde{C'},p}/\mathcal{O}_{C'})$$

the so-called delta invariant of the singularity.

By [Liu06, Lemma 5.11], there is a surjective morphism of abelian groups

$$\mathrm{Pic}^0(C) \to \mathrm{Pic}^0(C') \to 0$$

with kernel an affine unipotent group of dimension $\dim \mathrm{H}^1(C_1, \mathcal{O}_{C_1}(-\beta' \cdot \beta_1))$. Moreover, by [Liu06, Theorem 5.19], $\mathrm{Pic}^0(C)$ is an extension of $\times \mathrm{Pic}^0(\widetilde{C}_i)$ by an affine group of dimension

$$\left(\sum_{p \in C'^{\mathrm{sing}}} \delta(p) \right) - s + 1, \tag{2.2.9}$$

and, thus,

$$\delta_Z = \left(\sum_{p \in C'^{\mathrm{sing}}} \delta(p) \right) - s + 1 + \dim \mathrm{H}^1(C_1, \mathcal{O}_{C_1}(-\beta' \cdot \beta_1)).$$

One can prove by induction on the number s of irreducible components that $\sum_{p \in C'^{\mathrm{sing}}} \delta(p) - s + 1 \geq 0$ with equality holding only if the only singularities are nodes and C' is of compact type, namely its dual graph is a tree. For the induction step look at the partial normalization of C' given by $R \sqcup \overline{C' \setminus R}$ for R an irreducible component.

Let us also analyze the degree of $\mathcal{O}_{C_1}(-\beta' \cdot \beta_1)$. We can write $\beta' = \beta'' + \beta_1$ with β'' and effective curve class, and we can assume that $\beta'' \cdot \beta_1 > 0$ otherwise C' is not connected and we can estimate δ_Z separately on each connected component. From $0 \leq g(\beta_1) = \frac{1}{2}K_S \cdot \beta_1 + \frac{1}{2}\beta_1^2 + 1$ and the fact that K_S is anti-ample, it follows that $\beta_1^2 \geq -1$. In particular, $-\beta' \cdot \beta_1 = -\beta'' \cdot \beta_1 - \beta_1^2 \leq 0$ with equality holding only in the case $\beta_1^2 = -1$, and $\beta'' \cdot \beta_1 = 1$. However, if the equality is verified, then $\beta \cdot \beta_1 < 0$, contradicting the ampleness of β.

We can thus assume that $\mathcal{O}_{C_1}(-\beta' \cdot \beta_1)$ has strictly negative degree d. By Riemann–Roch,

$$\dim \mathrm{H}^1(C_1, \mathcal{O}_{C_1}(-\beta' \cdot \beta_1)) = g(C_1) - d - 1,$$

which is always positive unless $g(C_1) = 0$ and $d = -1$. In particular, in this case we must have: $\beta'' \cdot \beta_1 = 1$, $\beta_1^2 = 0$ and $K_S \cdot \beta_1 = -2$. We can thus give a better estimate of the codimension of the component Z in B corresponding to such curves. By [MS23, Lemma 2.1], given γ an effective curve class, the dimension of the linear system $|\gamma|$ is $\frac{1}{2}\gamma \cdot (\gamma - K_S)$. Thus, $\dim B = \frac{1}{2}\beta \cdot (\beta - K_S)$, whereas the dimension of Z is $\frac{1}{2}\beta' \cdot (\beta' - K_S)$. We compute

$$\begin{aligned}
\mathrm{codim}(Z) &= \tfrac{1}{2}\beta \cdot (\beta - K_S) - \tfrac{1}{2}\beta' \cdot (\beta - K_S) \\
&= \tfrac{1}{2}[(\beta' + \beta_1)^2 - K_S \cdot \beta - \beta'^2 + K_S \cdot \beta'] \\
&= \tfrac{1}{2}[2\beta' \cdot \beta_1 - K_S \cdot \beta_1] = 2. \qquad \square
\end{aligned}$$

The assumption in case (ii) and Lemma 2.2.7 in case (i) of Proposition 2.2.1 allows us to disregard completely the locus V of non-reduced curve.

2.2.2 *Reduced curves.* If $\mathcal{C}_b$ is integral, then any rank-one torsion free sheaf $\mathcal{F}$ is strictly stable, independently from the choice of polarization. This is easily seen noting that for any proper sub-sheaf $\mathcal{G} \hookrightarrow \mathcal{F}$, the cokernel is supported on a dimension 0 sub-scheme, which imply the strict inequality in (2.1.2). In particular, $\mathbf{h}_\chi^{-1}(b)$ is a $\mathbb{G}_m$ gerbe over the compactification $\overline{\mathrm{Pic}}_{\mathcal{C}_b}^{\chi+g-1}$ of $\mathrm{Pic}_{\mathcal{C}_b}^{\chi+g-1}$ of Altman and Kleiman [AK80]. It was proved in [Reg80, Theorem A] that for an integral curve with only planar singularities $\overline{\mathrm{Pic}}_{\mathcal{C}_b}^{\chi+g-1}$ is irreducible and, thus, the open of line bundles is dense.

We are left to study the fiber $\mathbf{h}_\chi^{-1}(b)$ for $b \in |\beta|$ a reduced but possibly reducible curve.

PROPOSITION 2.2.10. *Let C be a reduced curve with planar singularities and polarization H_C. Let $\mathbb{M}_\chi(C)$ be the moduli stack of Gieseker semi-stable rank-one torsion free sheaves on C. Then the open substack $\mathfrak{U}_\chi$ of semi-stable invertible sheaves is dense in $\mathbb{M}_\chi(C)$.*

Proof. To show that $\mathfrak{U}_\chi \subseteq \mathbb{M}_\chi(C)$ is dense, we need to show that any point F in $\mathbb{M}_\chi(C)$ is the limit of a family of line bundles on C in a one-parameter family. This now follows from the deformation theory of rank-one sheaves on planar curves. Let $q_1, \ldots q_k$ be the points (necessarily) in the singular locus C^{sing} where F fails to be a line bundle. We look at the forgetful morphism

$$\mathrm{Def}_F \xrightarrow{l} \prod_{i=1,\ldots k} \mathrm{Def}_{F_{q_i}},$$

where F_{q_i} is the stalk of F at q_i. Since we are looking at curves with planar singularities, the forgetful morphism is smooth [FGV99]. Thus, it is sufficient to show that the generic element in each local deformation space $R_{F_{q_i}}$ corresponds to a invertible module. However, this follows from the analogous statement for compactified Jacobians of integral planar curves [Reg80]. More explicitly: in order to describe $R_{F_{q_i}}$, consider an integral curve C_{q_i} with q_i as unique singular point. By [Reg80] the Picard is dense in the compactified Jacobian, which in particular implies that the generic point in $R_{F_{q_i}}$ parametrizes a line bundle. $\qquad \square$

This concludes the proof of Proposition 2.2.1. $\qquad \square$

Remark 2.2.11. When there are no strictly semi-stables, and thus $\mathbb{M}_\chi(C)$ is a $\mathbb{G}_m$ gerbe on $h_\chi^{-1}(C) = \mathrm{M}_\chi(C)$, it is stated in [MRV17, Corollary 2.20] or also in [Lop05, Theorem 4.5] that stable line bundles are dense in the moduli space.

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When there are strictly semi-stables (which is the most interesting case for us) the density statement at the *level of moduli spaces* $M_\chi(C)$ is false; the description of the moduli space in [Lop05, Theorem 4.5] explains this fact.

2.3 Moduli of Higgs bundles

Let $C \to T$ be a smooth projective curve. As for the case of sheaves on surfaces, there is an adaptation of theory of moduli stack and moduli spaces of Higgs bundles relative to a base T. The case where T is a field of positive characteristic is already considered in [Ngô06].

We refer to [GWZ20b, § 7.1] and references therein for the adaptation of the classical theory to the case of smooth curves over $T = \mathsf{Spec}(O_F), \mathsf{Spec}(F)$.

We denote by $\mathbb{M}_{r,\chi}(C)$ and $M_{r,\chi}(C)$ the moduli stack and moduli space of semi-stable Higgs bundles $(\mathcal{E}, \Theta \colon \mathcal{E} \to \mathcal{E} \otimes \mathcal{O}_C(D))$ of rank r and Euler characteristic χ on the fibers of $C \to T$. Here D is a relative effective divisor of degree $d = \deg(D) \geq 2g - 2$ on the fibers of $C \to T$. If equality holds, we assume that D is the canonical divisor and $g \geq 2$.

The stability is the (fiber-wise) Gieseker stability for sub-bundles preserved by the Higgs field, i.e. $(\mathcal{E}, \Theta)$ is semi-stable if for any $\mathcal{F} \subseteq \mathcal{E}$ such that $\Theta|_{\mathcal{F}} \colon \mathcal{F} \to \mathcal{F} \otimes \mathcal{O}_C(D)$

$$\frac{\chi}{r(\mathcal{F})} \leq \frac{\chi}{r}.$$

The Hitchin fibration is the morphism:

$$H \colon M_{r,\chi}(C) \to B_T := \bigoplus_{i=1}^{r} H^0(C, \mathcal{O}_C(id))$$

associating to $(\mathcal{E}, \Theta)$ the coefficients of its characteristic polynomial.

Via the spectral correspondence [Hit87], $\mathbb{M}_{r,\chi}(C)$ can be interpreted as a moduli stack of one-dimensional sheaves $\mathcal{F}$ on the non-compact surface

$$S_C := \mathrm{Tot}(\mathcal{O}_C(D)) \xrightarrow{\pi} C$$

with $[\mathrm{Supp}(\mathcal{F})] = r[C]$. The hypothesis on the degree of D implies that S_C is del Pezzo, which, in turn, guarantees the smoothness of the moduli stack $\mathbb{M}_{r,\chi}(C)$ and its $\mathbb{G}_m$-rigidification $\mathcal{M}_{r,\chi}(C)$ for each (r, χ) similar to Lemma 2.1.4.

Moreover, via the correspondence we can identify the fibers of the Hitchin fibration H with the Simpson compactified Jacobians of the spectral curve $C_{P(t)} \subseteq S_C$ cut out by the polynomial $\det(\pi^*\Theta - t\mathrm{Id})$.

In the following theorem we collect the analogous statements of Theorem 2.1.3 and Proposition 2.2.1 for the moduli stack of meromorphic Higgs bundles. Since the proof is essentially the same as in the previous section we do not repeat it here.

THEOREM 2.3.1. (i) *The moduli stack $\mathbb{M}_{r,\chi}(C)$ and its $\mathbb{G}_m$-rigidification $\mathcal{M}_{r,\chi}(C)$ are algebraic stacks of finite type over T.*
 (ii) *If $\deg(D) > 2g - 2$, then $\mathbb{M}_{r,\chi}(C)$ is smooth over T.*
(iii) *If $D = K_C$ and $T = \mathrm{Spec}(\mathbb{C})$, then $\mathbb{M}_{r,\chi}(C)$ is normal.*
(iv) *There exists a quasi-projective T-scheme $M_{r,\chi}(C)$, proper over B_T which uniformly represent the moduli functor of S-equivalence classes of Higgs bundles.*
 (v) *We have that $\mathbb{M}_{r,\chi}^{st}(C)$ is a $\mathbb{G}_m$ gerbe over the moduli space $M_{r,\chi}^{st}(C)$*
(vi) *We have that $\mathbb{M}_{r,\chi}^{st}(C)$ is universally closed over T and thus over $M_{r,\chi}^{st}(C)$*
(vii) *The open $\mathcal{U}_{r,\chi} \subseteq \mathcal{M}_{r,\chi}(C)$ of Higgs bundles whose associated spectral sheaf is a line bundle is dense and its complement has codimension at least two in $\mathcal{M}_{r,\chi}(C)$.*

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Remark 2.3.2. In the case of Higgs bundle, no extra assumption is necessary to show normality of the moduli stack. We can understand that from at least two points of view: in the language of [DHM22], the moduli stack of Higgs bundles satisfies the *totally negative CY2* property [DHM22, §7]; under this additional property the normality follows from the results of [Ver22].

Otherwise, we note that the locus V corresponding to non-reduced spectral curves in the Hitchin base V has always codimension at least two and, thus, as in Proposition 2.2.1, the locus of Higgs bundle whose corresponding spectral sheaf is a line bundle has codimension at least two in the stack, which allows us to argue as in Theorem 2.1.7.

The estimate on the codimension on V can be given by an explicit calculation:

By Riemann–Roch the base B of the Hitchin fibration has dimension

$$\dim B = d\frac{r(r+1)}{2} + r(1-g),$$

where r is the fixed rank and $d = \deg \mathcal{O}_C(D) \geq 2g-2$. The non-reduced locus is the (closure of) the strata $V_{kr_1,r_2} \subseteq B$ whose points correspond to polynomials in t admitting a factorization of the form

$$P(t) = P_1(t)^k P_2(t), \quad \deg(P_1(t)) = r_1 \geq 1, \ k \geq 2, \quad \deg(P_2(t)) = r_2 \geq 0, \ kr_1 + r_2 = r. \tag{2.3.3}$$

Thus, the codimension of the non-reduced locus is the minimum of the codimension of the strata V_{kr_1,r_2}. Now, the dimension of the strata is

$$\dim V_{kr_1,r_2} = d\frac{r_1(r_1+1)}{2} + r_1(1-g) + d\frac{r_2(r_2+1)}{2} + r_2(1-g),$$

and it is easy to see by direct computation that

$$\dim B - \dim\dim V_{kr_1,r_2} \geq \frac{4g+1}{2},$$

for $d \geq 2g-1$ and

$$\dim B - \dim\dim V_{kr_1,r_2} = 3(g-1),$$

for $d = 2g-2$.

3. p-adic integration

Let F be a non-archimedean local field F with ring of integers $\mathcal{O}$ and residue field $k \cong \mathbb{F}_q$. We consider the normalized Haar measure μ on the locally compact group $(F,+)$ so that $\mu(\mathcal{O}) = 1$.

Since F is a completely valued field there is a basic theory of differential geometry available. In particular, one can define analytic manifolds and differential forms as over the reals, see [Igu00, CNS18] for details. Given an n-dimensional analytic manifold M and a non-vanishing n-form ω on M, one defines a Borel measure μ_ω on M by writing in a local chart $U \subset F^n$ of M

$$\omega_{|U} = f \cdot dx_1 \wedge \cdots \wedge dx_n,$$

and integrating $|f|$ against the Haar measure on F^n.

We are interested in analytic manifolds that arise from algebraic geometry. Concretely, given a smooth, separated and finite-type F-scheme, or more generally F-algebraic space X, its set of F-points $X(F)$ carries naturally the structure of an analytic manifold by the inverse function theorem.

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3.1 The measure on the moduli spaces

Let $\mathcal{M}$ be a normal algebraic stack with a finite-type morphism to $\mathsf{Spec}(\mathcal{O})$ of dimension n. We assume that there exists a (quasi-)projective variety M and a morphism $\pi : \mathcal{M} \to M$ satisfying the following conditions: there exists a dense open substack $U' \subset \mathcal{M}$, smooth over $\mathcal{O}$, that is stabilizer free, i.e. an algebraic space, and that the complement of U' has codimension at least 2 in $\mathcal{M}$; there exists a second dense open $U \subset U'$ such that π restricted to U induces an isomorphism onto its image in M which we also denote by U.

The goal of this section is to construct a canonical measure on the analytic manifold

$$M^{\natural} = M(\mathcal{O}) \cap U(F).$$

For this we need the following extra assumption.

Assumption 3.1.1. For any $x \in M^{\natural} = M(\mathcal{O}) \cap U(F)$ there exists a finite extension L/F with ring of integers $\mathcal{O}_L$ such that the base change $x_L : \mathsf{Spec}(\mathcal{O}_L) \to M_L = M \times_{\mathcal{O}} \mathcal{O}_L$ admits a lift $x'_L : \mathsf{Spec}(\mathcal{O}_L) \to \mathcal{M}_L$.

Note that Assumption 3.1.1 is, in particular, satisfied if $\mathcal{M} \to M$ is universally closed [Sta, 01KA].

We fix a Zariski-open cover $U' = \bigcup_{i \in I} U'_i$ such that on each U'_i the relative canonical bundle $\Omega^n_{U'/\mathcal{O}}$ is trivial and pick a trivializing (i.e. non-vanishing) n-form ω_i on U'_i. Let $U_i = U \cap U'_i$. We have a decomposition

$$M^{\natural} = \bigcup_i M^{\natural}_i,$$

with $M^{\natural}_i = M(\mathcal{O}) \cap U_i(F)$ and we claim that the measures μ_{ω_i} on $M^{\natural}_i$ glue to a measure on $M^{\natural}$ that is independent of the choice of $(\omega_i)_i$.

PROPOSITION 3.1.2 (The canonical measure on $M^{\natural}$). *For all $i, j \in I$ and any measurable subset $A \subset M^{\natural}_i \cap M^{\natural}_j$ we have*

$$\mu_{\omega_i}(A) = \mu_{\omega_j}(A).$$

In particular, the family $(\mu_{\omega_i})_{i \in I}$ glues to a Borel measure $\mu_{\mathrm{can},\mathcal{M}}$ on $M^{\natural}$. Furthermore, $\mu_{\mathrm{can},\mathcal{M}}$ is independent of the choice of $(\omega_i)_{i \in I}$.

Proof. Since ω_i and ω_j are both non-vanishing sections of the same invertible sheaf over U' we have $\omega_{j|U'_i \cap U'_j} = f \cdot \omega_{i|U'_i \cap U'_j}$ with $f \in H^0(U'_i \cap U'_j, \mathcal{O}^*_{U'_i \cap U'_j})$. We claim that for every $x \in M^{\natural}_i \cap M^{\natural}_j \subset (U_i \cap U_j)(F)$ we have that $f(x) \in \mathcal{O}^*_F$. This implies the first part of the proposition, since μ_{ω_i} and μ_{ω_j} are given by locally integrating the absolute value of ω_i and ω_j.

To prove the claim, let L/F be a finite extension such that x_L lifts to an $\mathcal{O}_L$-point x'_L of $\mathcal{M}_L$ as in Assumption 3.1.1. Let $p : X \to \mathcal{M}$ be a smooth atlas. By passing to an even bigger finite extension of F if necessary, we may further assume that the closed point $\mathsf{Spec}(k_L) \to \mathcal{M}_{k_L}$ lifts to the atlas X_{k_L}. Since $X_L \to \mathcal{M}_L$ is smooth we can then lift the whole x'_L to a morphism $\tilde{x}_L : \mathsf{Spec}(\mathcal{O}_L) \to X_L$ by Hensel's lemma. Since $F^* \cap \mathcal{O}_L = \mathcal{O}^*$ it is enough to show that $f(x_L) = f \circ \pi \circ p(\tilde{x}_L) \in \mathcal{O}^*_L$.

Write $V_i = p^{-1}(U'_i)$. The point is now that $f \circ p_{|V_i \cap V_j}$ is a regular non-vanishing function on $V_i \cap V_j$ and the complement of $V_i \cap V_j$ in X has codimension at least 2 by assumption. Since X is normal, Hartogs' theorem applies and f thus extends to a regular function on X which is still

non-vanishing, as the 0-locus would have codimension 1. Therefore,

$$f(x_L) = f \circ p(\tilde{x}_L) \in \mathcal{O}_L^*,$$

as claimed.

The proof of the independence of $\mu_{\mathrm{can},\mathcal{M}}$ from $(\omega_i)_{i \in I}$ is essentially the same. □

If the stack $\mathcal{M}$ is clear from the context we often write μ_{can} instead of $\mu_{\mathrm{can},\mathcal{M}}$.

Example 3.1.3. Consider the quotient stack $\mathcal{M} = [\mathbb{A}^2/\mathbb{G}_m]$ where $\mathbb{G}_m$ acts linearly on $\mathbb{A}^2$ with weights $(1, -1)$. In the above notation we have

$$U' = [(\mathbb{A}^2 \setminus \{0\})/\mathbb{G}_m], \quad U = [(\mathbb{A}^2 \setminus \{xy = 0\})/\mathbb{G}_m] \cong \mathbb{G}_m, \quad M \cong \mathbb{A}^1.$$

Then U' is isomorphic to the affine line with doubled origin, which admits an open cover by two copies of $\mathbb{A}^1$. The standard 1-forms on these $\mathbb{A}^1$ glue to a global non-vanishing 1-form on U' and μ_{can} is given by integrating $\mathrm{d}x$ (i.e. the standard Haar measure) on $M^\natural = \mathcal{O} \cap F^* = \mathcal{O} \setminus \{0\}$.

Remark 3.1.4. If $\mathcal{M} = M$ is stabilizer-free we have $M^\natural = M(\mathcal{O})$ and the construction of μ_{can} goes back to Weil [Wei12]. The volume is related to the number of k-rational points by the formula [Wei12, Theorem 2.2.5]

$$\int_{M(\mathcal{O})} \mu_{\mathrm{can}} = \frac{|M(k)|}{q^n}.$$

Remark 3.1.5. If $\mathcal{M}$ is a Deligne–Mumford stack, then $\pi : \mathcal{M} \to M$ is proper [KM97] and, thus, Assumption 3.1.1 is automatically satisfied. In this case μ_{can} agrees with the orbifold measure constructed in [GWZ20a, §2.3], although in [GWZ20a, §2.3] there is no assumption on the codimension of the complement of $U \subset \mathcal{M}$. If $\mathcal{M}$ is tame, the total volume of $M^\natural$ can be expressed as a weighted point count of the twisted inertia stack of $\mathcal{M}$ (see [GWZ20a, Theorem 2.21]).

Remark 3.1.6. Groechenig pointed out to us the following alternative construction: let $\mathcal{M}/\mathcal{O}$ be any finite-type Artin stack of relative dimension n with a morphism $\pi : \mathcal{M} \to M$ to a scheme M. Assume, furthermore, that there exists a smooth stabilizer-free open $U \subset \mathcal{M}$ such that $\pi_{|U}$ is an isomorphism onto its image and that Assumption 3.1.1 is satisfied. Finally, suppose that there exists a line bundle $\widetilde{\mathcal{K}}$ on $\mathcal{M}$ which restricts to a power of the canonical line bundle $\mathcal{K}_{U/\mathcal{O}}$ on U. Then as in [Yas17, §4.1], integrating a suitable root of the absolute value of local sections of $\widetilde{\mathcal{K}}$ against the Haar measure defines a measure μ^{vir} on $M^\natural$ by an argument as in Proposition 3.1.2. By construction, if both μ^{vir} and μ_{can} are defined they coincide.

Interestingly, in the context of moduli spaces $\mathbb{M}$ of sheaves on a surface S, such a line bundle $\widetilde{\mathcal{K}}$ arises from the deformation theory as the determinant of $\mathbf{R}\mathcal{H}\mathbf{om}_{\mathbb{M}}(\mathcal{F}, \mathcal{F})$, where $\mathcal{F}$ is the universal sheaf on $S \times \mathbb{M}$ (see [BBBJ15] for the definition in the strictly semi-stable case, or [HL97, §8.3] for a simpler explanation on the locus of stables).

3.2 The canonical measure on moduli spaces of sheaves

Let S be a smooth projective relative surface over $\mathsf{Spec}(\mathcal{O})$ and $\mathcal{M}_{\beta,\chi}$ the $\mathbb{G}_m$-rigidified moduli stack of semi-stable one-dimensional sheaves on S with moduli space $\mathrm{M}_{\beta,\chi}$ as in §2.

We take the open substacks $\mathcal{U}_{\beta,\chi}^{st} \subset \mathcal{U}_{\beta,\chi} \subset \mathcal{M}_{\beta,\chi}$ of geometrically stable line bundles and of line bundles, respectively, as substacks $U \subset U'$ in the notation of §3.1. From now on we always assume that $S \to \mathsf{Spec}(\mathcal{O})$ is either a smooth projective del Pezzo surface or a K3 surface satisfying Assumption 3.2.1.

Assumption 3.2.1. Let $S \to \mathsf{Spec}(\mathcal{O})$ be a smooth projective K3 surface and (β, χ) a dimension vector such that the following hold.

(i) The locus of $\mathcal{U}_{\beta,\chi} \subseteq \mathbb{M}_{\beta,\chi}(S)$ of line bundles has complement of codimension at least two and $\mathcal{U}_{\beta,\chi}^{st}$ is non-empty.

(ii) The moduli stack $\mathbb{M}_{\beta,\chi}(S)$ is normal.

Remark 3.2.2. We have already commented on the restriction that Assumption 3.2.1(i) imposes in Remark 2.1.8. Over $\mathbb{C}$, it follows from Theorem 2.1.7 that Assumption 3.2.1(i) implies Assumption 3.2.1(ii), but we do not know of a proof over more general bases. However, by choosing a suitable spreading out $S \to \mathsf{Spec}(B)$ of a complex K3 surface, satisfying Assumption 3.2.1(i), over the spectrum of a finitely generated $\mathbb{Z}$-algebra B we obtain finite-type moduli stacks $\phi : \mathbb{M}_{\beta,\chi} \to \mathsf{Spec}(B)$ by Theorem 2.1.3. Then Theorem 2.1.7 implies that the generic fiber of ϕ is geometrically normal and, thus, by [Gro66, Theorem 12.1.6, Corollary 9.5.2], the fibers of ϕ are normal on a non-empty open $V \subset \mathsf{Spec}(B)$. Thus, by pulling back S along regular points $\mathsf{Spec}(\mathcal{O}) \to V$, with $\mathcal{O}$ the ring of integers of a non-archimedean local field, we obtain many examples of relative K3 surfaces over $\mathsf{Spec}(\mathcal{O})$ where Assumption 3.2.1 holds [Gro65, Proposition 6.14.1].

By Theorem 2.1.3, Lemma 2.1.4 and Proposition 2.2.1 all the assumptions for Proposition 3.1.2 to hold are satisfied and we obtain a canonical measure μ_{can} on $\mathrm{M}_{\beta,\chi}^{\natural}$.

Also, the smooth base B of the Hilbert–Chow morphism $h_{\chi} : \mathrm{M}_{\beta,\chi} \to B$ admits a canonical measure $\mu_{\mathrm{can},B}$ and it will be useful later to have an explicit description of the relative measure $\mu_{\mathrm{can}}/\mu_{\mathrm{can},B}$ on smooth fibers of h_{χ}, as in [GWZ20b, §6.3].

LEMMA 3.2.3. *Let $b \in B(\mathcal{O})$ be such that the pullback of the universal curve $\mathcal{C}_b \to \mathsf{Spec}(\mathcal{O})$ is generically smooth. Then under the identification of $h_{\chi}^{-1}(b)(F)$ with $\mathrm{Pic}_{\mathcal{C}_b}^d(F)$, for $d = \chi + g - 1$, the relative measure $\mu_{\mathrm{can}}/\mu_{\mathrm{can},B}$ on $h_{\chi}^{-1}(b)(F)$ is given by integrating the absolute valued of a translation-invariant gauge form ω_b on the $\mathrm{Pic}_{\mathcal{C}_b}^0$-torsor $\mathrm{Pic}_{\mathcal{C}_b}^d$.*

Proof. Let $\pi : \mathrm{Pic}_{\mathcal{C}/B}^{\mathrm{totdeg}=0} \to B$ be the smooth group scheme of total degree-zero line bundles on a flat family $\mathcal{C} \to B$ of curves. By [BLR12, §4.2] it admits, up to replacing B with a neighborhood of $b \in B$, a global, translation-invariant, trivializing section ω_{rel} of $\Omega_{\mathrm{Pic}^{\mathrm{totdeg}=0}/B}^g$. In particular, $\mathrm{Pic}_{\mathcal{C}/B}^{\mathrm{totdeg}=0}/\mathcal{O}_F$ has a volume form $\omega = \pi^*\omega_B \wedge \omega_{\mathrm{rel}}$, where ω_B is a volume form in a neighborhood of $b \in B$ inducing $\mu_{\mathrm{can},B}$.

By [GWZ20b, Lemma 6.13], since $\mathrm{Pic}_{\mathcal{C}/B}^{\mathrm{totdeg}=d}$ is a torsor under the group scheme of total degree-zero line bundles, the relative form ω_{rel} induces a section $\widetilde{\omega}_{\mathrm{rel}}$ of $\Omega_{\mathrm{Pic}^{\mathrm{totdeg}=d}/B}^g$; we thus have a volume form $\omega_d = \pi^*\omega_B \wedge \widetilde{\omega}_{\mathrm{rel}}$ on $\mathrm{Pic}_{\mathcal{C}/B}^{\mathrm{totdeg}=d}$.

Since $\mathcal{U}_{\beta,\chi}$ has complement of codimension at least two in $\mathcal{M}_{\beta,\chi}$ by Proposition 2.2.1, integrating the absolute value of ω_d computes μ_{can}. Therefore, on $h_{\chi}^{-1}(b)(F)$ the relative measure $\mu_{\mathrm{can}}/\mu_{\mathrm{can},B}$ is given by integrating the absolute value of $\omega_b = \widetilde{\omega}_{\mathrm{rel},b}$. $\qquad\square$

An analogous construction can be done for moduli spaces of Higgs bundles using Theorem 2.3.1.

4. Gerbes and Tate duality

4.1 $\mathbb{G}_m$ gerbes

Let $X \to V$ be a stack for the fppf-topology. Then X is said to be a *gerbe* if:

(i) for every scheme T over V and objects $x, y \in X(T)$, there exists some fppf-cover $f : U \to T$ and an isomorphism $x_{|U} \cong y_{|U}$;

(ii) there exists an fppf-cover $U \to V$ such that the groupoid $X(U)$ is non-empty.

For every $T \to V$ and isomorphism $\phi\colon x \to y$ in $X(T)$, conjugation by ϕ induces an isomorphism $\mathsf{Aut}_T(x) \to \mathsf{Aut}_T(y)$. Suppose that every object of X has *abelian* automorphism group in its fiber category; in this case we say that the gerbe $X \to V$ is *abelian*. Then for every $T \to V$ and objects $x, y \in X(T)$, there is a canonical isomorphism $\mathsf{Aut}_T(x) \cong \mathsf{Aut}_T(y)$. These isomorphisms determine a sheaf of groups $Band(X/V)$ over V.

We say that an abelian gerbe $X \to V$ is a $\mathbb{G}_m$ gerbe if there is an isomorphism $\mathbb{G}_{m,V} \to Band(X/V)$.

By [Gir20] equivalence classes of $\mathbb{G}_m$ gerbes on V are in bijection with the group $H^2(V_{fppf}, \mathbb{G}_m)$, the *Brauer group* of V. A gerbe X/V is *trivial* if it is equivalent to $[X/\mathbb{G}_{m,V}]$, where $\mathbb{G}_{m,V}$ acts trivially on X.

We recall that for F a local field, the Brauer group $H^2(F_{fppf}, \mathbb{G}_m)$ is isomorphic to $\mathbb{Q}/\mathbb{Z}$ by mean of the Hasse invariant; see, for example, [Ser68, Proposition XIII.6].

Given X/V a $\mathbb{G}_m$ gerbe, for any point $F \xrightarrow{s} V$ we get a $\mathbb{G}_m$ gerbe s^*X/F by pullback. This defines a complex valued function:

$$\varphi_V\colon V(F) \to H^2(F_{fppf}, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi i}} \mathbb{C} \tag{4.1.1}$$

4.2 Picard stacks/schemes of curves

Let V be a scheme and $c\colon X \to V$ a smooth proper curve[3] with fibers of genus $g \geq 0$.

We denote by $\mathfrak{Pic}_{X/V}$ the Picard stack of X/V; in other words, for an V-scheme T, $\mathfrak{Pic}_{X/V}(T)$ is the groupoid of invertible sheaves on $X \times_V T$. The group of isomorphism classes of objects of $\mathfrak{Pic}_{X/V}(T)$ is therefore the classical Picard group $\mathrm{Pic}(X \times_V T)$. For each integer d, we denote by $\mathfrak{Pic}^d_{X/V}$ the substack of line bundles of degree d.

We denote by $\mathrm{Pic}_{X/V}$ the sheaf $R^1_{fppf}c_*\mathbb{G}_m$ on V; it can be equivalently defined as the sheafification of the presheaf $T \mapsto \mathrm{Pic}(X \times_V T)$. The natural morphism

$$\alpha\colon \mathfrak{Pic}_{X/V} \to \mathrm{Pic}_{X/V} \tag{4.2.1}$$

is a gerbe [Sta, 0DME] banded by $\mathbb{G}_m$.

The degree-map factors through α and we denote by $\mathrm{Pic}^d_{X/V}$ the corresponding component of degree d line bundles. The degree-zero part $\mathrm{Pic}^0_{X/V}$ is represented by an abelian scheme of relative dimension g, the *relative Jacobian*, and each $\mathrm{Pic}^d_{X/V}$ is an V-torsor under it.

In general, a T-point of $\mathrm{Pic}_{X/V}$ need not correspond to a line bundle on $X \times_V T$; this property gets lost when the sheafification of $T \mapsto \mathrm{Pic}(X \times_V T)$ is taken.

Lemma 4.2.2 [BLR12, Proposition 8.4]. *For every V-scheme T, there is a canonical exact sequence of abelian groups, functorial in T,*

$$0 \to \mathrm{Pic}(T) \to \mathrm{Pic}(X \times_V T) \to \mathrm{Pic}_{X/V}(T) \to H^2(T, \mathbb{G}_m) \to H^2(X \times_V T, \mathbb{G}_m).$$

Since in our case $\mathrm{Pic}_{X/V}$ is a V-scheme, we may look at the particular case $T = \mathrm{Pic}_{X/V}$. Then the identity in $\mathrm{Pic}_{X/V}(\mathrm{Pic}_{X/V})$ gets mapped to the element of $H^2(\mathrm{Pic}_{X/V}, \mathbb{G}_m)$ corresponding to the $\mathbb{G}_m$ gerbe α of (4.2.1). By Lemma 4.2.2, α is trivial if and only if there is a universal Poincaré bundle $\mathcal{L}$ on $X \times_V \mathrm{Pic}_{X/V}$. For example, this is the case when $X \to V$ has a section, since then the map

$$H^2(\mathrm{Pic}_{X/V}, \mathbb{G}_m) \to H^2(X \times_V \mathrm{Pic}_{X/V}, \mathbb{G}_m)$$

is injective.

---

[3] That is, a smooth proper morphism flat and of finite presentation whose fibers are one-dimensional and connected.

More generally, for an V-scheme T, the map $c_T \colon \mathrm{Pic}_{X/V}(T) \to H^2(T, \mathbb{G}_m)$ sends $x \colon T \to \mathrm{Pic}_{X/V}$ to the equivalence class of the $\mathbb{G}_m$ gerbe on T obtained by pulling back α along x. Such a gerbe is trivial if and only if x comes from a line bundle on $X \times_V T$ (that is, is in the essential image of α).

4.3 Tate duality

We consider now the special case where the base V is the spectrum of a non-archimedean local field F. As recalled above, the Brauer group $H^2(F, \mathbb{G}_m)$ is identified with $\mathbb{Q}/\mathbb{Z}$ via the Hasse invariant. We denote by $f \colon \mathrm{Pic}_{X/F}(F) \to \mathbb{Q}/\mathbb{Z}$ the group homomorphism of Lemma 4.2.2, and for each integer $d \in \mathbb{Z}$, we call $f_d \colon \mathrm{Pic}_{X/F}^d(F) \to \mathbb{Q}/\mathbb{Z}$ the restriction of f.

Note that any $x \in \mathrm{Pic}_{X/F}^d(F)$ gives rise to an isomorphism $s_x \colon \mathrm{Pic}_{X/F}^0 \to \mathrm{Pic}_{X/F}^d$ and since f is a group homomorphism we have

$$f_d \circ s_x = f_d(x) + f_0. \tag{4.3.1}$$

The following proposition gives some control over f_0 in terms of torsor instead of gerbes.

PROPOSITION 4.3.2. *The image of f_0 is $((1/d_0)\mathbb{Z})/\mathbb{Z}$, where d_0 is the smallest positive integer for which $\mathrm{Pic}_{X/F}^{d_0}(F)$ is non-empty.*

In particular, f_0 is constant if and only if $\mathrm{Pic}_{X/F}^1(F)$ is non-empty.

Similar statements have been used crucially in [GWZ20b, GWZ20a] and the following discussion is essentially taken from there.

The proof of Proposition 4.3.2 relies on Tate duality over local fields. Let A be an abelian variety over F and A^t its dual.

THEOREM 4.3.3 [Mil86, Theorem 3.7.8]. *There is a canonical perfect pairing*

$$A(F) \otimes H^1(F, A^t) \to H^2(F, \mathbb{G}_m) = \mathbb{Q}/\mathbb{Z}.$$

The pairing admits the following geometric description, see [GWZ20b, Remark 3.11]. First from [Mil86, Lemma 3.1] we have an isomorphism

$$H^1(F, A^t) \cong \mathsf{Ext}^2(A, \mathbb{G}_m). \tag{4.3.4}$$

Now, working in the category of commutative group stacks considered in [Bro21], elements of $\mathsf{Ext}^2(A, \mathbb{G}_m)$ can be represented by $\mathbb{G}_m$ gerbes on A with a group structure.

Given $x \in \mathsf{A}(F)$, $\tau \in H^1(F, A^t)$ and $\alpha_\tau \in \mathsf{Ext}^2(A, \mathbb{G}_m)$ the image of τ under (4.3.4), the paring of x with τ from Theorem 4.3.3 equals the class of the $\mathbb{G}_m$ gerbe $x^* \alpha_\tau$ on $\mathsf{Spec}(F)$ under the Hasse invariant isomorphism $H^2(F, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z}$.

Thus, for $A = \mathrm{Pic}_{X/F}^0 = A^t$ the homomorphism f_0 is simply the Tate-duality pairing with the $\mathbb{G}_m$ gerbe $\alpha_0 \colon \mathfrak{Pic}_{X/F}^0 \to \mathrm{Pic}_{X/F}^0$.

Proof of Proposition 4.3.2. To simplify the notation we drop the subscript X/F. By the previous discussion and Theorem 4.3.3 we see that $d_0 \cdot f_0 \equiv 0$ if and only if $\alpha_0^{d_0}$ is equivalent to the trivial $\mathbb{G}_m$ gerbe on Pic^0. We thus need to show that the latter holds if and only if $\mathrm{Pic}^{d_0}(F)$ is non-empty.

We consider the category of dualizable commutative group stacks DCGS_F over $\mathsf{Spec}(F)$; see, for example, [Bro21] for a detailed account. The internal hom-functor $D(\cdot) = \mathcal{H}om(\cdot, B\mathbb{G}_m)$ induces an anti-equivalence on DCGS_F satisfying $D \circ D = Id$ and extending the usual duality functor on abelian varieties. Furthermore, there is an auto-equivalence $D(\mathfrak{Pic}) \cong \mathfrak{Pic}$

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(see [Tra16, § 3.2]). Thus, applying D to the short exact sequence

$$0 \to B\mathbb{G}_m \to \mathfrak{Pic} \to \mathrm{Pic} \to 0,$$

we get the sequence

$$0 \to D(\mathrm{Pic}) \to \mathfrak{Pic} \xrightarrow{\beta} \mathbb{Z} \to 0.$$

Here exactness on the right follows from [Bro21, Proposition 3.18] and the fact that $\mathcal{E}xt^2(\mathrm{Pic}) = 0$ (see [Bro21, Corollary 11.5]). Since β is an epimorphism with kernel $\mathfrak{Pic}^0$ we get an equivalence $D(\mathrm{Pic}) \cong \mathfrak{Pic}^0$. From this we deduce that the sequences

$$0 \to \mathrm{Pic}^0 \to \mathrm{Pic} \to \mathbb{Z} \to 0 \tag{4.3.5}$$

$$0 \to B\mathbb{G}_m \to \mathfrak{Pic}^0 \to \mathrm{Pic}^0 \to 0 \tag{4.3.6}$$

are exchanged by duality.

Now (4.3.5) is the extension of $\mathbb{Z}$ by Pic^0 corresponding to the Pic^0-torsor Pic^1 under the equivalence [Bro21, Proposition 5.8]. In particular, Pic^1 has a rational point if and only (4.3.5) splits. Dually (4.3.6) is the $B\mathbb{G}_m$-torsor on Pic^0 corresponding to α_0 under the equivalence [Bro21, Proposition 5.11] and splits if and only if α_0 is trivial. Since $D(\cdot)$ is an anti-equivalence, we have thus shown that Pic^1 has a rational point if and only if α_0 is trivial.

To extend this to Pic^{d_0} and $\alpha_0^{d_0}$, we note that their associated short exact sequences are given by the d_0-fold Baer sum [Sta, 010I] of (4.3.5) and (4.3.6), respectively. By definition, $D(\cdot)$ commutes with direct sums and exchanges the addition map with the diagonal, i.e. for every $G \in \mathrm{DCGS}_F$

$$D(G \oplus G \xrightarrow{+} G) \cong D(G) \xrightarrow{\Delta} D(G) \oplus D(G),$$

and, hence, also commutes with Baer sums. We thus deduce that Pic^{d_0} has a rational point if and only if $\alpha_0^{d_0}$ is trivial, which proves the proposition. $\square$

5. Main theorem

We are now ready to state and prove our main theorem. Let $S \to \mathsf{Spec}(\mathcal{O})$ be either a smooth projective del Pezzo surface or a K3 surface satisfying Assumption 3.2.1 and consider the analytic manifold $\mathrm{M}^{\natural}_{\beta,\chi}$ together with its canonical measure $\mu = \mu_{\mathrm{can}}$ as constructed in § 3.2.

The restriction of the natural $\mathbb{G}_m$ gerbe $\alpha \colon \mathbb{M}^{st}_{\beta,\chi} \to \mathcal{M}^{st}_{\beta,\chi} \cong \mathrm{M}^{st}_{\beta,\chi}$ induces a function

$$\varphi_{\beta,\chi} \colon \mathrm{M}^{\natural}_{\beta,\chi}(F) \to H^2(F, \mathbb{G}_m) \cong \mathbb{Q}/\mathbb{Z} \xrightarrow{e^{2\pi i \cdot}} \mathbb{C},$$

by pullback to the generic fiber as defined in (4.1.1).

Now for any $x \in \mathrm{M}_{\beta,\chi}(k)$ denote by $\mathrm{M}_{\beta,\chi}(\mathcal{O})_x \subset \mathrm{M}_{\beta,\chi}(\mathcal{O})^{\natural}$ the ball of $\mathcal{O}$-rational points specializing to x over k. Then define the non-archimedean BPS function $p\mathrm{BPS}_{\beta,\chi} \colon \mathrm{M}_{\beta,\chi}(k) \to \mathbb{C}$ to be

$$p\mathrm{BPS}_{\beta,\chi}(x) = q^{-\dim M_{\beta,\chi}} \int_{M_{\beta,\chi}(\mathcal{O})_x} \varphi^g_{\beta,\chi} \, \mathrm{d}\mu_{\mathrm{can}}. \tag{5.0.1}$$

THEOREM 5.0.2. *The function $p\mathrm{BPS}_{\beta,\chi} \colon \mathrm{M}_{\beta,\chi}(k) \to \mathbb{C}$ satisfies the following two properties:*

(i) $p\mathrm{BPS}_{\beta,\chi} \equiv q^{-\dim \mathrm{M}_{\beta,\chi}}$ *if (β, χ) is generic;*[4]

---

[4] Recall that a pair (β, χ) generic (with respect to H), if any Gieseker semi-stable sheaf in $\mathbb{M}_{\beta,\chi}$ is stable.

(ii) *for all $\chi, \chi' \in \mathbb{Z}$ and for all $y \in B(k)$ we have*

$$\sum_{x \in h_\chi^{-1}(y)(k)} p\text{BPS}_{\beta,\chi}(x) = \sum_{x \in h_{\chi'}^{-1}(y)(k)} p\text{BPS}_{\beta,\chi'}(x).$$

Note that, by definition,

$$\sum_{x \in h_\chi^{-1}(y)(k)} p\text{BPS}_{\beta,\chi}(x) = \int_{h_\chi^{-1}(B(\mathcal{O})_y)} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu.$$

To prove Theorem 5.0.2 we want to split up the integral using Fubini's theorem.

First let $B^{sm} \subset B$ be the open subscheme, where the universal curve $\mathcal{C} \to B$ is smooth. Then $h_\chi^{-1}(B(\mathcal{O})_y \cap B^{sm}(F))$ is contained in $h_\chi^{-1}(B(\mathcal{O})_y)$ and its complement has measure 0, since it is contained in $h_\chi^{-1}((B \setminus B^{sm})(F))$, that is, the F-points of a closed subscheme of $M_{\beta,\chi}$ (see [GWZ20b, Proposition 4.4]).

Since h_χ is proper, we can identify the fiber over a point $b \in B(\mathcal{O}) \cap B^{sm}(F)$ with $h_\chi^{-1}(b)(F)$ where we write $h_\chi^{-1}(b)$ for the F-scheme $h_\chi^{-1}(b)_F$. As in [GWZ20b, §6.3] we therefore have an equality:

$$\sum_{x \in h_\chi^{-1}(y)(k)} p\text{BPS}_{\beta,\chi}(x) = \int_{h_\chi^{-1}(B(\mathcal{O})_y)} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu = \int_{b \in B(\mathcal{O})_y \cap B^{sm}(F)} \left(\int_{h_\chi^{-1}(b)(F)} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu_{\omega_b} \right) \mathrm{d}\mu_{\text{can},B}.$$

$$(5.0.3)$$

Here ω_b is the gauge form on the $\text{Pic}_{\mathcal{C}_b}^0$-torsor $h_\chi^{-1}(b) = \text{Pic}_{\mathcal{C}_b}^{\chi+g-1}$ from Lemma 3.2.3. As the outer integral in (5.0.3) is independent of χ we only need to analyze the fiber integral $\int_{h_\chi^{-1}(b)(F)} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu_{\omega_b}$.

First we note that the gerbe $\alpha \colon \mathbb{M}_{\beta,\chi}^{st} \to \mathcal{M}_{\beta,\chi}^{st}$ restricted to the fiber over b gets identified with $\mathfrak{Pic}_{\mathcal{C}_b}^{\chi+g-1} \to \text{Pic}_{\mathcal{C}_b}^{\chi+g-1}$. Therefore, the associated function $\varphi_{\beta,\chi}$ gets identified with

$$\varphi_{\chi+g-1} = e^{2\pi i f_{\chi+g-1}},$$

where $f_{\chi+g-1}$ is the degree $d = \chi + g - 1$ component of the function f introduced in §4.3. We may rewrite the innermost integral in (5.0.3) as

$$\int_{h_\chi^{-1}(b)(F)} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu_{\omega_b} = \int_{\text{Pic}_{\mathcal{C}_b}^{\chi+g-1}(F)} \varphi_{\chi+g-1}^g \, \mathrm{d}\mu_{\omega_b}. \qquad (5.0.4)$$

Now if $\text{Pic}_{\mathcal{C}_b}^{\chi+g-1}(F) = \emptyset$ this integral vanishes as we integrate over the empty manifold. Otherwise any $x \in \text{Pic}_{\mathcal{C}_b}^{\chi+g-1}(F)$ gives an isomorphism $\text{Pic}_{\mathcal{C}_b}^0 \xrightarrow{\sim} \text{Pic}_{\mathcal{C}_b}^{\chi+g-1}$ and by (4.3.1) we have

$$\int_{h^{-1}(b)(F)} \varphi_{\beta,\chi}^g \, \mathrm{d}\mu_{\omega_b} = \varphi_{\chi+g-1}^g(x) \int_{\text{Pic}_{\mathcal{C}_b}^0(F)} \varphi_0^g \, \mathrm{d}\mu_{\omega_b} \qquad (5.0.5)$$

To complete the proof we need to further study the function $f_0 : \text{Pic}_{\mathcal{C}_b}^0(F) \to \mathbb{Q}/\mathbb{Z}$.

LEMMA 5.0.6. *The following are equivalent:*

(i) $f_0 \equiv 0$;
(ii) $\text{Pic}_{\mathcal{C}_b}^1(F)$ *is non-empty;*
(iii) $\text{Pic}_{\mathcal{C}_b}^g(F)$ *is non-empty.*

In this case the image of $f : \text{Pic}_{\mathcal{C}_b}(F) \to \mathbb{Q}/\mathbb{Z}$ is contained in $((1/g)\mathbb{Z})/\mathbb{Z}$.

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Proof. The equivalence (i) $\iff$ (ii) follows from Proposition 4.3.2. Clearly part (ii) implies part (iii) so we only need to prove that part (iii) implies part (i). Suppose then that $\mathrm{Pic}^g_{\mathcal{C}_b}(F)$ is non-empty. We claim that $f(\mathrm{Pic}^g_{\mathcal{C}_b}(F)) = 0$, where

$$f : \mathrm{Pic}_{\mathcal{C}_b}(F) \to \mathbb{Q}/\mathbb{Z}$$

is as in §4.3. Consider the moduli stack $\mathbb{M}_{\beta,1}$ of semi-stable sheaves with Euler characteristic $\chi = 1$. For this choice of χ there are no strictly semi-stable sheaves; hence every semi-stable sheaf is geometrically stable and, in particular, simple. It follows that the coarse space $M_{\beta,1}$ coincides with the $\mathbb{G}_m$-rigidification $\mathcal{M}_{\beta,1}$, hence the coarse moduli map $\mathbb{M}_{\beta,1} \to M_{\beta,1}$ gives a $\mathbb{G}_m$ gerbe α over $\mathsf{Spec}\mathcal{O}$. For every $x \in M_{\beta,1}(\mathcal{O})$, the pullback $x^*\alpha$ is a trivial gerbe, since $Br(\mathcal{O}) = 0$. In particular, the map

$$f_{\beta,1} : M_{\beta,1}(\mathcal{O}) \to \mathbb{Q}/\mathbb{Z}$$

vanishes. Now note that line bundles of degree g have Euler characteristic 1, i.e. $\mathrm{Pic}^g_{\mathcal{C}_b}(F) \subset M_{\beta,1}(\mathcal{O})$. This proves the claim. To conclude the proof, let $x \in \mathrm{Pic}^g_{\mathcal{C}_b}(F)$. For every $y \in Pic^0_{\mathcal{C}_b}(F)$, $f(y) = f(x) + f(y) = f(x+y) \in f(\mathrm{Pic}^g(F)) = 0$. Hence, f_0 is zero.

The statement about the image of f follows from the observation $f(\mathrm{Pic}^g_{\mathcal{C}_b}(F)) = 0$. $\qquad\square$

Remark 5.0.7. The equivalence of parts (ii) and (iii) in Lemma 5.0.6 is true more generally. In fact, for any geometrically integral, smooth projective curve C of genus g over a local field Pic^{g-1}_C has a rational point by [Lic69] and [PS99, Corollary 4].

Proof of Theorem 5.0.2. By the previous discussion, in particular (5.0.3) and (5.0.4), χ independence amounts to show for all $d, d' \in \mathbb{Z}$ and $b \in B(\mathcal{O}) \cap B^{sm}(F)$ the equality

$$\int_{\mathrm{Pic}^d_{\mathcal{C}_b}(F)} \varphi^g_d \, \mathrm{d}\mu_{\omega_b} = \int_{\mathrm{Pic}^{d'}_{\mathcal{C}_b}(F)} \varphi^g_{d'} \, \mathrm{d}\mu_{\omega_b}.$$

For simplicity take $d' = g$. Consider first the case where $\mathrm{Pic}^g_{\mathcal{C}_b}(F) = \emptyset$. Then the right-hand side is zero. If $\mathrm{Pic}^d_{\mathcal{C}_b}(F)$ is also empty, we are done. Otherwise, let $x \in \mathrm{Pic}^d(F)$. Then as in (5.0.5)

$$\int_{\mathrm{Pic}^d_{\mathcal{C}_b}(F)} \varphi^g_d \, \mathrm{d}\mu_{\omega_b} = \varphi^g_d(x) \int_{\mathrm{Pic}^0_{\mathcal{C}_b}(F)} \varphi^g_0 \, \mathrm{d}\mu_{\omega_b}. \tag{5.0.8}$$

As $\mathrm{Pic}^g_{\mathcal{C}_b}(F)$ is empty, so is $\mathrm{Pic}^1_{\mathcal{C}_b}(F)$ by Lemma 5.0.6, hence f_0 is a non-trivial character, i.e. a surjective homomorphism $\mathrm{Pic}^0_{\mathcal{C}_b}(F) \to \mathbb{Z}/d_0\mathbb{Z}$ for some $d_0 \geq 2$ by Lemma 4.3.2. Furthermore, d_0 is the smallest integer such that $\mathrm{Pic}^{d_0}_{\mathcal{C}_b}(F) \neq \emptyset$, hence d_0 does not divide g. The function φ^g_0 is therefore non-zero and by a character-sum argument the right-hand side of (5.0.8) vanishes, as required.

Next we consider the case $\mathrm{Pic}^g_{\mathcal{C}_b}(F) \neq \emptyset$. Then by Lemma 5.0.6 also $\mathrm{Pic}^1_{\mathcal{C}_b}(F) \neq \emptyset$ and, thus, also $\mathrm{Pic}^d_{\mathcal{C}_b}(F) \neq \emptyset$. Furthermore, the image of f is g-torsion and, thus, $\varphi^g_d = \varphi^g_g \equiv 1$. Since $\mathrm{Pic}^d_{\mathcal{C}_b}(F) \cong \mathrm{Pic}^0_{\mathcal{C}_b}(F) \cong \mathrm{Pic}^g_{\mathcal{C}_b}(F)$ we are done with the proof of part (ii).

The proof of part (i) follows from the fact that for (β, χ) generic, the stable locus is everything, thus the morphism $\mathbb{M}_{\beta,\chi} \to M_{\beta,\chi}$ is a $\mathbb{G}_m$ gerbe. In this case $f_{\beta,\chi} \equiv 0$, since the gerbe $x^*\alpha \in H^2(F, \mathbb{G}_m)$ is pulled back from $H^2(\mathsf{Spec}(\mathcal{O}), \mathbb{G}_m) = 0$ and, thus, trivial. $\qquad\square$

Remark 5.0.9. It is straightforward to extend the definition of the pBPS-function and the proof of Theorem 5.0.2 to the case of usual and meromorphic Higgs bundles using Theorem 2.3.1.

CONFLICTS OF INTEREST
None.

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