RINGS ALL OF WHOSE FACTOR RINGS ARE SEMI-PRIME

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We prove in this paper that fifteen classes of rings coincide with the class of rings named in the title. One of them is the class of rings R such that $X^2 = X$ for each R-ideal X: we shall refer to rings with this property (and thus to the rings of the title) as fully idempotent rings. The simple rings and the (von Neumann) regular rings are fully idempotent. Indeed, every finitely generated right or left ideal of a regular ring is generated by an idempotent [1, p. 42], so that $X^2 = X$ holds for every one-sided ideal X. Since the Jacobson radical of a regular ring is zero [1, p. 42], Sasiada's simple radical ring [3] is an example of a fully idempotent ring which is not regular. We prove that S is a fully idempotent ring when S is the ring of n by n matrices over a fully idempotent ring R, even when S is locally matrix over R. Every ideal and, trivially, every epimorph of a fully idempotent ring is a fully idempotent ring. We prove that the direct sum of fully idempotent rings is a fully idempotent ring. Thus if R is a simple ring and S is Sasiada's simple radical ring $R \oplus S$ is neither a simple ring nor a regular ring and is fully idempotent. We do not know of an indecomposable example of this phenomenon. The results discussed so far appear in Sections 1 and 2.

A result in Section 3 states that every ring R has an ideal W such that (1) each R-ideal not contained in W has a nonzero epimorph which is a fully idempotent ring; (2) the ring W has no nonzero epimorph which is a fully idempotent ring; and (3) the Levitzky radical [=maximal locally nilpotent ideal] L is contained in W. No nonzero epimorphs of R-ideals contained in L are fully idempotent rings (for R-ideals contained in W this has not been proved or disproved). We note that in Sasiada's simple radical ring $0 = W = L \neq J$ (the Jacobson radical), while in the ring of power series over a field, L is contained properly in $W = J$. In the examples considered we have not found an exception to the inclusion $W \subseteq J$.

We prove in Section 4 that every ring R has an ideal V such that (1) no nonzero ideal of R/V is a fully idempotent ring and (2) an R-ideal is a fully idempotent ring if and only if it is contained in V.

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1. _Characterizations of the rings which have only semi-prime factor rings._ A ring is called semi-prime if no nonzero ideal is nilpotent.

1.1 _Notation._ Let U and V be subsets of a set S on which a multiplication is defined. Then

$$(U:V) = \{s \in S \mid vs \in U \text{ for all } v \in V\}.$$

$$(U:V)' = \{s \in S \mid sv \in U \text{ for all } v \in V\}.$$

1.2 THEOREM. _For a ring_ R, _not necessarily having an identity, the sixteen statements_ (A) _through_ (P) _are equivalent._ (Convention: We denote right ideals by T, left ideals by V, and ideals by X, Y, or Z, and we adopt an abbreviated form of statement: by "$T \cap X \subseteq XT$" we mean that $T \cap X \subseteq XT$ for all ideals X and all right ideals T of R.)

(A) _Every factor ring of_ R _is a semi-prime ring._

(B) _Every factor ring of the ring_ X _is a semi-prime ring._

(C) $X \cap Y = XY$.

(D) $(Y:X) \cap X = X \cap Y$.

(E) $(Y:X)' \cap X = X \cap Y$.

(F) $X^2 = X$. (_Thus_ R _is a fully idempotent ring._)

(G) _If_ $Z \subseteq X$, _then_ $(Y:X) \cap Z = Y \cap Z$.

(H) _If_ $Z \subseteq X$, then $(Y:X)' \cap Z = Y \cap Z$.

(I) $T \cap X \subseteq XT$.

(J) _If_ $T \subseteq X$, _then_ $T \subseteq XT$.

(K) $(T:X) \cap X \subseteq X \cap T$.

(L) _If_ $T \subseteq X$, _then_ $(T:X) \cap X \subseteq T$.

(M) $V \cap X \subseteq VX$.

(N) _If_ $V \subseteq X$, _then_ $V \subseteq VX$.

(O) $(V:X)' \cap X \subseteq V \cap X$.

(P) _If_ $V \subseteq X$, _then_ $(V:X)' \cap X \subseteq V$.

Proof. If (F) holds for R, it also holds for each epimorphic image S of R, so that S has no nilpotent nonzero ideals. Thus (F) implies (A). If (F) is false, however, and if X is an ideal such that $X^2 \neq X$, then $R/(X^2)$ has a nonzero nilpotent ideal. Thus $R/(X^2)$

is not semi-prime and (A) does not hold. The equivalence of (A) and (F) has been proved.

It is clear that (B) implies (A). We assume (A). Then, by (F), $X^2 = X$ for each ideal X of R. Let K be an R-right, X-left submodule of the ideal X. Then $(K + RK) = (K + RK)^2$, so that

$$K + RK = K^2 + KRK + RK^2 + (RK)^2.$$

The right member of the equality is contained in K, since X contains K, KR, RK, and RKR. We have $RK \subseteq K$, and have proved that every R-right, X-left submodule of X is an R-ideal. Now let J be an X-ideal, so that the R-right, X-left submodule, $J + JR$, of X is an R-ideal. Thus $(J + JR)$ is equal to its square and we can prove $JR \subseteq J$ by steps similar to these just used. Evidently the R-right module J is an R-ideal. By assumption $J^2 = J$, for each X-ideal J. Clearly, the epimorphs X/J of X have no nonzero nilpotent ideals, and are by definition semi-prime rings. The equivalence of (A) and (B) has been proved.

(i) The equivalence of (A), (B), and (F) has been proved.

We prove that (D) implies (C). If X and Y are ideals, we may write, using (D)

$$(Y \cap X) \subseteq (XY\colon X) \cap X = XY \cap X = XY,$$

whence we can obtain (C): $X \cap Y = XY$ for all ideals X and Y. If (C) holds and if X and Y are ideals, we have, since $(Y\colon X)$ is an ideal,

$$(Y\colon X) \cap X = X(Y\colon X) \subseteq X \cap Y.$$

The reverse inclusion is trivial, and (D) has been obtained from (C), completing the proof of the equivalence of (C) and (D).

From (C), (F) is obtained by setting $Y = X$. If (F) is assumed and if X and Y are ideals, then

$$XY \supseteq (X \cap Y)^2 = X \cap Y.$$

Since $X \cap Y \supseteq XY$, (C) is implied by (F). Thus (C), (D) and (F) have been proved equivalent and, considering (i) F is equivalent with each of (A) through (D). Considering the symmetry of (E) and (D), we may summarize:

(ii) Statements (A) through (F) have been proved equivalent.

(D) is obtained from (G) by setting $Z = X$. Conversely, if (D) holds and if X, Y and $Z \subseteq X$ are ideals, then

$$(Y:X) \cap Z \subseteq Z \cap (Y:X) \cap X = Z \cap X \cap Y = Y \cap Z.$$

The reverse inclusion is trivial and (G) holds. Thus (D), (G), and (H) are equivalent ((G) and (H) are symmetric).

(iii) Statements (A) through (H) have been proved equivalent.

(F) is obtained from (J) by setting $T = X$. Thus (J) implies statements (A) through (H). If the first eight statements hold, then by (D) we have for any ideal X and any right ideal T

$$X \cap T \subseteq (XT:X) \cap X = XT \cap X = XT.$$

Thus (I) holds. Since (I) clearly implies (J), we have

(iv) Statements (A) through (J) have been proved equivalent.

If (L) is assumed, then we have $XT \subseteq X$ for any right ideal T and any ideal X containing T, so that

$$T = T \cap X \subseteq (XT:X) \cap X \subseteq XT.$$

The inclusion obtained is (J). Thus each of (A) through (J) is implied by (L). Conversely, let the first ten statements hold. Then for any right ideal T and any ideal X, we have by (I)

$$(T:X) \cap X \subseteq X(T:X) \subseteq T \cap X.$$

Thus (K) is implied by any of the ten preceding statements. Clearly, (K) implies (L) and we have

(v) Statements (A) through (L) have been proved equivalent.

Considering the symmetry of the pairs ((I), (M)), ((J), (N)), ((K), (O)), and ((L), (P)) the sixteen statements have been proved equivalent.

Remark. From (C), (D), and (E) of Theorem 1.2, it is evident for ideals X and Y of a fully idempotent ring R that $XY = YX$ and that $(Y:X) \cap X = (Y:X)' \cap X$. For each right ideal T, we have $T \subseteq RT$ by statement (J) of the theorem.

2. Further properties of fully idempotent rings.

Notation. If A and B are subsets of a ring R and if $x \in R$, AxB is the set $\{\Sigma a_i x b_i \mid a_i \in A, b_i \in B\}$. $(A:R)^*$ is the set $\{t \in R \mid RtR \subseteq A\}$. Evidently, $(A:R)^*$ is an R-ideal if A is a subgroup of the additive group of R.

2.1 PROPOSITION. *If A is an ideal in a fully idempotent ring R, then* $RAR = A$.

2.2 PROPOSITION. *If R is a fully idempotent ring and A is an R-ideal, then* $(A:R)^* = A$.

Proof. We need $(A:R)^* \subseteq A$. Let Y denote the ideal $(A:R)^*$. By Proposition 2.1 $Y = RYR \subseteq A$.

2.3 PROPOSITION. *If R is a fully idempotent ring and if $t \in R$, then* $t \in RtR$.

Proof. Let $X = RtR$. By Proposition 2.2 $X = \{X:R)^*$ and, clearly, $t \in (X:R)^*$.

Notation. R_n will denote the ring of n by n matrices over a ring R. For $t \in R$, $tE_{\alpha\beta}$ is the matrix with t in position (α, β) and zeros elsewhere.

We quote from [2, p. 40, Proposition 1]:

2.4 PROPOSITION. *Let B be a ring such that for every $b \in B$ we have $b \in BbB$. Then the ideals of the matrix ring B_n are of the form U_n where U is an ideal in B.*

2.5 THEOREM. *If R is a fully idempotent ring, so is* $S = R_n$.

Proof. Let Y be an ideal of S. By Propositions 2.3 and 2.4 $Y = U_n$ for some ideal U of R. To prove $Y = Y^2$ it is sufficient to show that $tE_{\alpha\beta} \in Y^2$, $1 \leq \alpha, \beta \leq n$, for each $t \in U$. Since R is fully idempotent, $t = \Sigma u_i v_i$ where the u_i and v_i belong to U. Thus $tE_{\alpha\beta} = \Sigma(u_i E_{\alpha 1})(v_i E_{1\beta})$ belongs to Y^2, as required.

Definition. A ring S is locally matrix over a ring R if, and only if, given any finite subset T of S there is a subring S' of S

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containing T, such that R_n and S' are isomorphic for some positive integer n.

2.6 THEOREM. *If a ring S is locally matrix over a fully idempotent ring R, then S is a fully idempotent ring.*

Proof. Let Y be an ideal of S, and let $y \in Y$. Let V be a subring of S such that $y \in V$ and for some positive integer n, $V \cong R_n$. Now $y \in H = V \cap Y$ and H is an ideal of V. By Theorem 2.5 V is fully idempotent so that $H = H^2$. We have $y \in H^2 \subseteq Y^2$. Thus $Y = Y^2$ for each R-ideal Y, completing the proof.

2.7 THEOREM. *Let R be the direct sum $\Sigma_{i \in I} R_i$ of rings R_i each of which is fully idempotent. Then R is a fully idempotent ring.*

Proof. If Y is an ideal of R, let y_i be the i-th component of an element y of Y. For any element $x \in R_i y_i R_i$ there is an element $y' \in RyR$ whose i-th component is x and whose j-th component for all $j \neq i$ is zero; in particular we can take $x = y_i$, using Proposition 2.3. For a fixed i, then, the set Y_i of projections on R_i of the elements of Y is contained in Y. It is easy to see that Y_i is an ideal of R and of R_i and that $Y = \Sigma_{i \in I} Y_i$. By hypothesis $Y_i = Y_i^2$ for each $i \in I$. Since Y is the direct sum of the R-ideals Y_i and since any element $w \in Y$ has only finitely many nonzero components, $w \in Y^2$ is easily obtained from $w_i \in Y_i^2$. We have $Y = Y^2$ for each R-ideal Y; R is a fully idempotent ring.

Remark. In consequence of Theorem 2.7, we can obtain a non-simple fully idempotent ring $Q = R \oplus S$, where R and S are simple rings. Since the radical of a regular ring is zero [1, p. 42], Q is not regular if one of the summands is Sasiada's simple radical ring.

3. *An ideal having no idempotent epimorphs.*

3.1 PROPOSITION. *Let R be any ring and let W be an ideal of R such that for every R-ideal $Y \subsetneq W$ W/Y fails to be a fully idempotent ring. Then W/Y fails to be fully idempotent for every W-ideal Y.*

Proof. Let $X \neq W$ be a left ideal of W and a right ideal of R and assume that W/X is a fully idempotent ring. Then the ring $W/(RX + X)$ is fully idempotent and, by hypothesis, $W = RX + X$. Since X is a right ideal of R and since $X + W^2 = W$, we have

$$(1) \qquad X + RX = X + (X + RX)^2 = X + (RX)^2 + RX^2 .$$

Since $RX \subseteq W$, $RX^2 \subseteq X$ and $(RXR)X \subseteq X$. From (1), then $X + RX = X$; X is an R-ideal, so that W/X is not fully idempotent. Thus we have proved that W has no nonzero fully idempotent epimorphs W/X where X is a W-ideal and a right or left R-ideal.

If the proposition is false, W/Y is a fully idempotent ring for some W-ideal $Y \neq W$. Then the ring emimorph $W/(Y + RY)$ is fully idempotent and, by the conclusion of the preceding paragraph, $Y + RY = W$. From $Y + W^2 = W$ we obtain

$$(2) \qquad Y + RY = Y + (Y + RY)^2 = Y + YW + (RY)^2 + RY^2 .$$

Since $YR \subseteq WR \subseteq W$, $R(YR)Y \subseteq RWY \subseteq Y$. Similarly, $RY^2 \subseteq WY \subseteq Y$. From (2), then, $Y = Y + RY$; Y is a left R-ideal. The first part of the proof contradicts that W/Y is fully idempotent.

3.2 PROPOSITION. *Let the ring R have ideals S and $T \subsetneq S$ such that S/T is a fully idempotent ring. Let $X \subseteq S$ be an R-ideal such that for each R-ideal Y properly contained in X X/Y is not a fully idempotent ring. Then $X \subseteq T$.*

Proof. By (B) of Proposition 1.2 $(X + T)/T$ is a fully idempotent ring. Thus $X/(X \cap T)$ is fully idempotent, so that, by hypothesis, $(X \cap T) = X$, as required.

Definition. An R-ideal Y is locally nilpotent if each finitely generated subideal of Y is nilpotent.

Remark and definition. Every ring R has a locally nilpotent ideal L such that (1) L contains each locally nilpotent ideal of R and (2) R/L has no locally nilpotent ideals [2, p. 197, Propositions 1 and 2]. L is called the Levitzky radical of R.

3.3 PROPOSITION. *Let the ring R have ideals S and $T \subsetneq S$ such that S/T is a fully idempotent ring, and such that the Levitzky radical L of R is contained in S. Then $L \subseteq T$.*

Proof. If L is not contained in T, let $y \notin T$ belong to L and let Y be the R-ideal generated by y. Since Y is finitely generated, Y is nilpotent. But since S/T is fully idempotent, we have the contradiction: $T \neq (T + Y) = (T + Y)^n = T + Y^n$, $n = 1, 2, \ldots,$ completing the proof.

3.4 Remark. If R is any ring, we will obtain an R-ideal W of R maximal with respect to the following property: if $B \neq W$ is a W-ideal then W/B is not a fully idempotent ring. For any R-ideal U, let $M(U)$ be the intersection of all R-ideals X contained in U such that U/X is a fully idempotent ring. If U contains the Levitzky radical L of R and contains those R-ideals H which have no non zero fully idempotent epimorphs H/Y, then by Propositions 3.2 and 3.3 $M(U)$ contains the specified ideals. In particular R, $M(R)$, $M(M(R))$, $\ldots$, contains L and the R-ideals having no nonzero fully idempotent epimorphs.

Let $W_1 = R$; for each ordinal $\alpha > 1$, define W_α as follows:

Case I. $\alpha = \beta + 1$ is not a limit ordinal. Define $W_\alpha = M(W_\beta)$.

Case II. α is a limit ordinal. Then W_α is the intersection of the ideals W_β with $\beta < \alpha$.

For some ordinal α we must have $W_\alpha = W_{\alpha+1}$; this is the R-ideal W of the following theorem:

3.5 THEOREM. Every ring R has an ideal W such that

(1) W/B fails to be a fully idempotent ring if $B \neq W$ is a W-ideal;
(2) if H is an R-ideal which is not contained in W, then, for some R-ideal Y properly contained in H, H/Y is a fully idempotent ring;
(3) the Levitzky radical L of R is contained in W.

Proof. Remark 3.4 asserts that $W = W_\alpha = W_{\alpha+1}$ satisfies (2) and (3), and it is clear that W/B is not fully idempotent if $B \neq W$ is an R-ideal and even if B is a W-ideal by Proposition 3.1. Q.E.D.

Comments concerning the ideal W.

(i) If (P) is the property "no nonzero epimorph is fully idempotent", then (P) holds for the R-ideal W and for the R-ideals contained in the Levitzky radical L by the argument in the proof of Proposition 3.3.

(ii) If F is a field and $R = F(x)$ is the ring of formal power series over F, then the only nonzero ideals are the principal ideals of

$1, x, x^2, \ldots$ (for each nonzero element $r \in R$ is such that for some non-negative integer m $x^{-m}r$ belongs to R and is invertible). Clearly, W and the radical of R coincide with the unique maximal ideal (x), while $L = 0$. In this example property (P) mentioned in comment (i) holds for the R-ideals contained in W. We have no example where (P) is false for an R-ideal contained in W.

(iii) If R is Sasiada's simple radical ring [3], then $W = L = 0$ is properly contained in the Jacobson radical.

4. The maximum fully idempotent ideal of a ring. For an arbitrary ring R we obtain an ideal V such that every nonzero ideal of R/V fails to be a fully idempotent ring and such that an R-ideal Y is a fully idempotent ring if and only if $Y \subseteq V$. We proceed by a series of remarks.

4.1 Remark. Let $\{A_i\}_{i \in I}$ be an increasing chain of R-ideals such that for each $i \in I$ $T = T^2$ holds for every R-ideal $T \subseteq A_i$. Then $T = T^2$ also holds for R-ideals $T \subseteq A$, where A is the union of the A_i. Thus: let $t \in T$ where $T \subseteq A$ is an R-ideal; then for some $i \in I$, $t \in (T \cap A_i) = (T \cap A_i)^2 \subseteq T^2$.

4.2 Remark. Let R-ideals A and $X \supseteq A$ be such that $T = T^2$ holds for every R-ideal $T \subseteq A$ and $T = A + T^2$ holds for every R-ideal T with $A \subseteq T \subseteq X$. Then $T = T^2$ holds for every R-ideal $T \subseteq X$. For $T = T^2 + (T \cap A)$ and $T \cap A = (T \cap A)^2$. (The first equality comes from the isomorphism $(A + T)/A \cong T/(T \cap A)$.)

4.3 Remark. Let A and B be R-ideals such that $T = T^2$ holds for every R-ideal contained in A or in B. Then $A + B$ has this property also. For, if T is an R-ideal with $A \subseteq T \subseteq A + B$, then $T = A + S$ for some R-ideal $S \subseteq B$, whence $S = S^2$ and $T = A + S = A + S^2 \subseteq A + T^2$. Then Remark 4.2 is applicable with $X = A + B$.

4.4 THEOREM. Every ring R has an ideal V such that (1) every nonzero ideal of R/V fails to be a fully idempotent ring and (2) an R-ideal Y is a fully idempotent ring if and only if Y is contained in V.

Proof. By Remark 4.1 and Zorn's lemma there is an R-ideal V which is maximal with respect to having each of its R-subideals equal to its square. By Remark 4.2, then, no nonzero ideal of R/V is a fully idempotent ring; (1) has been proved.

By Remark 4.3 V contains each R-ideal Y such that $T = T^2$ for each R-ideal $T \subseteq Y$, whence V contains each R-ideal Y which is a fully idempotent ring. Conversely, we show that V is fully idempotent, so that by (B) of Theorem 1.2 each V-ideal (and thus each R-ideal contained in V) is a fully idempotent ring. But V is fully idempotent if each V-ideal is an R-ideal and that was established, essentially, in the proof that (A) - equivalently (F) - of Theorem 1.2 implies (B).

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