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THE BNSR-INVARIANTS OF THE HOUGHTON GROUPS, CONCLUDED

MATTHEW C. B. ZAREMSKY

Department of Mathematics and Statistics, University at Albany (SUNY), Albany, NY 12222, USA (mzaremsky@albany.edu)

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Abstract We give a complete computation of the Bieri–Neumann–Strebel–Renz invariants $\Sigma^m(H_n)$ of the Houghton groups H_n. Partial results were previously obtained by the author, with a conjecture about the full picture, which we now confirm. The proof involves covering relevant subcomplexes of an associated CAT(0) cube complex by their intersections with certain locally convex subcomplexes, and then applying a strong form of the Nerve Lemma. A consequence of the full computation is that for each $1 \leq m \leq n-1$, H_n admits a map onto $\mathbb{Z}$ whose kernel is of type F_{m-1} but not F_m; moreover, no such kernel is ever of type F_{n-1}.

Keywords: Houghton group; BNSR-invariant; finiteness properties; cube complex; CAT(0)

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Introduction

The Bieri–Neumann–Strebel–Renz (BNSR) invariants $\Sigma^m(G)$ ($m \in \mathbb{N}$) of a group G are a sequence of geometric invariants, introduced in [**2**, **4**], that reveal a breadth of information about certain subgroups of G. They are notoriously difficult to compute, and a full computation has been done for only a handful of relevant groups. Most prominently, the BNSR-invariants have been fully computed for all right-angled Artin groups [**8**, **15**]. In [**20**], we computed the BNSR-invariants of the generalized Thompson groups $F_{n,\infty}$ and obtained partial results for the Houghton groups H_n. In this paper we finish the computation of the BNSR-invariants of the Houghton groups. Our main result is as follows (with the notation $m(\chi)$ explained in § 2).

Theorems 2.2 and 2.3. *For any* $0 \neq \chi \in \mathrm{Hom}(H_n, \mathbb{R})$ *we have* $[\chi] \in \Sigma^{m(\chi)-1}(H_n) \setminus \Sigma^{m(\chi)}(H_n)$.

The statement $[\chi] \in \Sigma^{m(\chi)-1}(H_n)$ was proved in [**20**] and is cited as Theorem 2.2 here; the new result is Theorem 2.3, that $[\chi] \notin \Sigma^{m(\chi)}(H_n)$. Since we always have $\Sigma^m(G) \subseteq$

$\Sigma^{m-1}(G)$ for all m and G, Theorems 2.2 and 2.3 provide a complete computation of the BNSR-invariants of the Houghton groups.

The proof makes use of the CAT(0) structure of a natural cube complex on which H_n acts. We exhibit a family of combinatorially convex subcomplexes called 'blankets' (Definition 3.1) that cover a relevant χ-superlevel complex, and use a strong form of the Nerve Lemma to conclude that the χ-superlevel complex is not $(m(\chi)-1)$-connected. This then leads reasonably quickly to the result. The geometry of the complex is crucial for getting the regions covered by the blankets to be highly connected enough to apply the Strong Nerve Lemma. We believe that this covering approach could be useful in the future for understanding BNSR-invariants of other groups acting naturally on CAT(0) cube complexes, and possibly for approaching the Σ^m-conjecture (see Conjecture 1.5) for (certain classes of) metabelian groups.

This paper is organized as follows. In § 1 we recall some general background. In § 2 we recall some specific background from [**20**] and state our main result, Theorem 2.3. In § 3 we recall the CAT(0) cube complex X_n on which H_n acts, and introduce an important family of subcomplexes called 'blankets' in X_n. Finally, in § 4 we prove Theorem 2.3.

1. Background

In this section we collect some background on Houghton groups, Morse theory and BNSR-invariants.

1.1. Houghton groups

Let $[n] := \{1, \ldots, n\}$. The *Houghton group* H_n, introduced in [**11**], is the subgroup of $\mathrm{Symm}([n] \times \mathbb{N})$ consisting of those elements η such that for each $1 \leq i \leq n$ there exists $m_i \in \mathbb{Z}$ and $N_i \in \mathbb{N}$ such that $(i, x)\eta = (i, x + m_i)$ for all $x \geq N_i$ (we will always write elements of $\mathrm{Symm}([n] \times \mathbb{N})$ to the right of their arguments, to sync with the notation in [**20**]). Intuitively such an η acts as an 'eventual translation' on each $\{i\} \times \mathbb{N}$. It is known that H_n is of type F_{n-1} but not F_n [**6**, Theorem 5.1]. Higher-dimensional versions of the Houghton groups, due to Bieri and Sach [**3**, **17**], have also been developed.

1.2. BNSR-invariants

A group is of *type* F_m if it admits a proper cocompact action on an $(m-1)$-connected CW-complex. Given a group G, call a homomorphism $\chi : G \to \mathbb{R}$ a *character*, and call two characters *equivalent* if they are positive scalar multiples of each other. The equivalence classes $[\chi]$ of non-trivial characters form the *character sphere* $\Sigma(G)$. The BNSR-invariants $\Sigma^m(G)$ $(m \in \mathbb{N})$ of a group G are certain subspaces of $\Sigma(G)$, defined whenever G is of type F_m. They were introduced for $m = 1$ in [**4**] and $m \geq 2$ in [**2**]. The definition is as follows.

Definition 1.1 (BNSR-invariant). Let G be a group acting properly cocompactly on an $(m-1)$-connected CW-complex Y, so in particular G is of type F_m. For any $0 \neq \chi \in \mathrm{Hom}(G, \mathbb{R})$ there exists a *character height function* $h_\chi : Y \to \mathbb{R}$, that is, a map satisfying $h_\chi(g.y) = \chi(g) + h_\chi(y)$ for all $g \in G$ and $y \in Y$. The mth *BNSR invariant*

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$\Sigma^m(G)$ is

$$\Sigma^m(G) := \{[\chi] \in \Sigma(G) \mid (Y^{t\le h_\chi})_{t\in\mathbb{R}} \text{ is essentially } (m-1)\text{-connected}\}.$$

Here $Y^{t\le h_\chi}$ is the full subcomplex of Y supported on those $v \in Y^{(0)}$ with $t \le h_\chi(v)$. Recall that $(Y^{t\le h_\chi})_{t\in\mathbb{R}}$ being *essentially* $(m-1)$-*connected* means that for all $t \in \mathbb{R}$ there exists $s \le t$ such that the inclusion $Y^{t\le h_\chi} \to Y^{s\le h_\chi}$ induces the trivial map in π_k for all $k \le m-1$.

As is standard, we will write $\Sigma^m(G)^c$ for the complement $\Sigma(G) \setminus \Sigma^m(G)$. Note that $\Sigma^1(G) \supseteq \Sigma^2(G) \supseteq \cdots$. There are also homological BNSR-invariants $\Sigma^m(G;\mathbb{Z})$, analogous to the homological finiteness properties FP_m, though we will not discuss the homological case much here.

Remark 1.2 (Erratum to [7, 20]). Definition 1.1 is almost identical to [20, Definition 1.1], except that there the condition on stabilizers was that each p-cell stabilizer be of type F_{m-p}, whereas here we just assume a proper action (so finite stabilizers). The reason for this change is that in order for h_χ to exist, the cell stabilizers need to lie in $\ker(\chi)$, and assuming a proper action is an easy way to assure this. This condition on χ killing the stabilizers was accidentally omitted in [20, Definition 1.1] (and in [7, Definition 8.1], which [20, Definition 1.1] followed), though since in practice it was always applied to situations with finite stabilizers, this error was irrelevant. Also note that one can always choose Y so that the action of G is proper (even free), it is just sometimes convenient (in situations other than our present one) to deal with spaces with infinite stabilizers.

The BNSR-invariants of G form a sort of catalogue describing the precise finiteness properties of subgroups of G containing the commutator subgroup $[G, G]$, namely the following.

Citation 1.3 (see [5, Theorem 1.1]). *Let G be a group of type F_m and let $[G, G] \le H \le G$. Then H is of type F_m if and only if for every $[\chi] \in \Sigma(G)$ such that $\chi(H) = 0$ we have $[\chi] \in \Sigma^m(G)$.*

In our computation of $\Sigma^m(H_n)$ for the Houghton groups H_n, it turns out that the structure is such that the invariant $\Sigma^1(H_n)$ already determines all the $\Sigma^m(H_n)$ in a natural, 'polyhedral' way. We describe this phenomenon in the following definition.

Definition 1.4 (Bieri Σ-property). For a finitely generated group G and a subset S of $\Sigma(G)$, denote by $\mathrm{conv}_{\le m} S$ the union of convex hulls in $\Sigma(G)$ of all subsets of at most m non-antipodal elements in S. Now suppose G is of type F_m and suppose that

$$\Sigma^m(G)^c = \mathrm{conv}_{\le m} \Sigma^1(G)^c.$$

Then we say G has the *Bieri Σ^m-property*. If G has the Bieri Σ^m-property for all m such that G is of type F_m then we say G has the *Bieri Σ-property*.

Every finitely generated group trivially has the Bieri Σ^1-property, but finitely presented groups need not have the Bieri Σ^2-property; in fact, Kochloukova found a solvable (even

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nilpotent-by-abelian) counterexample [12, Theorem B]. The well-known Σ^m-conjecture, which seems to originally be due to Bieri though perhaps was first stated in this form by Meinert, can be phrased (in homotopical form) as follows.

Conjecture 1.5 (Σ^m-conjecture). *Every metabelian group of type* F_m *has the Bieri* Σ^m*-property.*

One can state the Σ^m-conjecture simultaneously for all m as the 'Σ-conjecture' that every metabelian group has the Bieri Σ-property. As seen in Corollary 2.7, our results here imply that the Houghton groups have the Bieri Σ-property. Meinert proved that the Σ-conjecture holds for metabelian groups of finite Prüfer rank [16], and Harlander–Kochloukova proved the Σ^2-conjecture in general [10]. In addition to the Houghton groups, some other non-metabelian groups whose BNSR-invariants are fully computed, revealing that they have the Bieri Σ-property, include Thompson's group F [5], its relatives $F_{n,\infty}$ [13, 20] and its braided version [21]. The most prominent family of groups whose BNSR-invariants are fully computed is the family of right-angled Artin groups [8, 15], and from the computation one can see that generally speaking 'most' right-angled Artin groups do not have the Bieri Σ-property. This is essentially because it is easy for a collection of subsets of vertices of a flag complex to individually induce disconnected subcomplexes but have their union induce a highly connected (e.g. contractible) subcomplex. On the other hand, free groups and free abelian groups do have the Bieri Σ-property for trivial reasons. We leave a precise classification of which right-angled Artin groups have the Bieri Σ-property as an exercise for the reader.

1.3. Morse theory

We will use the definition of Morse function and the statement of the Morse Lemma from [18, 20]. We recall these now; see [18, 20] for any details we leave out here. Fix an affine cell complex Y in this subsection.

Definition 1.6 (Morse function). Let $(h, s) : Y \to \mathbb{R} \times \mathbb{R}$ be a map such that h and s restrict to affine functions on cells. We call (h, s) a *Morse function* provided that $|s(Y^{(0)})| < \infty$ and there exists $\varepsilon > 0$ such that whenever v and w are adjacent 0-cells in Y, either $|h(v) - h(w)| \geq \varepsilon$, or $h(v) = h(w)$ and $s(v) \neq s(w)$.

We view (h, s) as a height function via the lexicographic order on $\mathbb{R} \times \mathbb{R}$, and the conditions ensure that adjacent 0-cells have different heights. Along a given cell, (h, s) achieves its minimum and maximum values at unique 0-faces. If a 0-cell v is a 0-face of a cell at which (h, s) achieves its minimum (maximum) on that cell, that cell belongs to the *ascending (descending) star* of v. The *ascending (descending) link* of v, denoted $\mathrm{lk}^{\uparrow} v$ ($\mathrm{lk}^{\downarrow} v$) is the link of v in its ascending (descending) star. Write $Y^{p \leq h \leq q}$ for the full subcomplex of Y supported on those 0-cells v with $p \leq h(v) \leq q$.

The following is essentially [20, Lemma 1.4], phrased slightly differently.

Lemma 1.7 (Morse Lemma). *Suppose that for each 0-cell v with $p \leq h(v) < q$ ($q < h(v) \leq r$) the ascending (descending) link $\mathrm{lk}^{\uparrow} v$ ($\mathrm{lk}^{\downarrow} v$) is $(m-1)$-connected. Then the inclusion $Y^{q \leq h} \to Y^{p \leq h}$ ($Y^{h \leq q} \to Y^{h \leq r}$) induces an isomorphism in π_k for each $k \leq m - 1$ and a surjection in π_m.*

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As a remark, if s is constant and $h(Y^{(0)})$ is discrete in $\mathbb{R}$, then this definition of Morse function reduces to the original one introduced by Bestvina and Brady in [1].

We will need one more topological tool, namely the following strong version of the classical Nerve Lemma.

Citation 1.8 (see [19, Proposition 1.21]). *Let X be a CW-complex covered by subcomplexes $(X_i)_{i\in I}$ and let L be the nerve of the cover. Let $n \geq 1$. Suppose that any non-empty intersection $X_{i_1} \cap \cdots \cap X_{i_r}$ for $1 \leq r \leq n$ is $(n-r)$-connected. Then $H_k(X) \cong H_k(L)$ for all $k \leq n-1$, and $H_n(X)$ surjects onto $H_n(L)$.*

This was not phrased exactly this way in [19, Proposition 1.21], but it is straightforward to see that this is an equivalent formulation. Note that being not n-acyclic implies being not n-connected.

2. Characters of Houghton groups and statement of results

In [20] a partial computation of $\Sigma^m(H_n)$ was obtained. Before stating the result, we need some notation and background from [20] regarding characters of H_n. For each $1 \leq i \leq n$ the function $\eta \mapsto m_i$, with m_i as in the definition of the Houghton groups, defines an epimorphism $\chi_i : H_n \to \mathbb{Z}$, and $\chi_1, \ldots, \chi_n$ span $\mathrm{Hom}(H_n, \mathbb{R})$. Since elements of H_n are bijective we have $m_1 + \cdots + m_n = 0$ for each η and hence $\chi_1 + \cdots + \chi_n = 0$. In fact, $\mathrm{Hom}(H_n, \mathbb{R}) \cong \mathbb{R}^{n-1}$ with basis $\{\chi_1, \ldots, \chi_{n-1}\}$, so $\Sigma(H_n) \cong S^{n-2}$. This also implies that for an arbitrary character χ of H_n, written as $\chi = a_1\chi_1 + \cdots + a_n\chi_n$ for $a_i \in \mathbb{R}$, the coefficients $(a_1, \ldots, a_n)$ are uniquely determined up to shifting by constants $(a, \ldots, a)$. The number

$$m(\chi) := |\{i \mid a_i < \max\{a_1, \ldots, a_n\}\}|$$

is therefore well defined. This measurement will turn out to determine which BNSR-invariants contain $[\chi]$.

It is sometimes convenient to express characters in a 'standard form' with respect to the characters χ_i.

Definition 2.1 ((ascending) standard form). If $\chi = a_1\chi_1 + \cdots + a_n\chi_n$ we call this expression for χ a *standard form* if $\max\{a_i\}_{i=1}^n = 0$. We call it an *ascending standard form* if $a_1 \leq \cdots \leq a_n = 0$.

Up to shifting by $\chi_1 + \cdots + \chi_n = 0$ any χ can be put in standard form. Up to automorphisms of H_n, every χ is equivalent to one in ascending standard form. In particular, when trying to determine which BNSR-invariants contain a given character class, without loss of generality it can be expressed in ascending standard form. For χ in ascending standard form, $m(\chi)$ equals the largest i such that $a_i \neq 0$.

2.1. Statement of results

The partial results obtained in [20] are as follows.

Theorem 2.2 (see [20]). *For any $0 \neq \chi \in \mathrm{Hom}(H_n, \mathbb{R})$ we have $[\chi] \in \Sigma^{m(\chi)-1}(H_n)$.*

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It was conjectured in [20] that, moreover, $[\chi] \notin \Sigma^{m(\chi)}(H_n)$; and this is our main result here.

Theorem 2.3. *For any $0 \neq \chi \in \mathrm{Hom}(H_n, \mathbb{R})$ we have $[\chi] \notin \Sigma^{m(\chi)}(H_n)$.*

We will prove Theorem 2.3 in § 4.

2.2. Consequences

In this subsection we collect some easy consequences of the computation of $\Sigma^m(H_n)$. First we have some results on finiteness properties of kernels of characters.

Corollary 2.4. *For any $\chi : H_n \twoheadrightarrow \mathbb{Z}$ with $m = \min\{m(\chi), m(-\chi)\}$, the kernel of χ is of type F_{m-1} but not F_m.*

Proof. This follows immediately from the computation of $\Sigma^m(H_n)$ together with Citation 1.3. □

Corollary 2.5. *For each $1 \leq m \leq n-1$, H_n admits a map to $\mathbb{Z}$ whose kernel is of type F_{m-1} but not F_m.*

Proof. Choose any $\chi : H_n \twoheadrightarrow \mathbb{Z}$ with $m = \min\{m(\chi), m(-\chi)\}$; for example, take $\chi = \chi_1 + \cdots + \chi_{m(\chi)-1} + 2\chi_m$, so $m(\chi) = n-1$ and $m(-\chi) = m$, and then Corollary 2.4 gives the result. □

We can also conclude the following result about arbitrary normal subgroups.

Corollary 2.6. *A non-trivial normal subgroup of H_n is of type F_{n-1} if and only if it has finite index in H_n.*

Proof. The thing to prove is that if N is a non-trivial normal subgroup of H_n with infinite index, then N is not of type F_{n-1}. As explained at the end of [20], N must contain the second derived subgroup of H_n, which is the finite-support alternating group on $[n] \times \mathbb{N}$. Up to replacing N with a finite index supergroup, we can assume it contains the derived subgroup of H_n, which is the finite-support symmetric group on $[n] \times \mathbb{N}$. Being an infinite index subgroup of H_n containing the commutator subgroup, N lies in the kernel of a character $\chi : H_n \twoheadrightarrow \mathbb{Z}$ and, since $[\chi] \notin \Sigma^{n-1}(H_n)$ (as the computation shows $\Sigma^{n-1}(H_n) = \emptyset$), Citation 1.3 says that N is not of type F_{n-1}. □

Another consequence of our computation is that the Houghton groups all have the Bieri Σ-property (Definition 1.4).

Corollary 2.7. *Every H_n has the Bieri Σ-property.*

Proof. We know that $\Sigma^1(H_n)^c = \{[-\chi_1], \ldots, [-\chi_n]\}$ (this was already known in [6, Proposition 8.3]). Hence, thanks to standard forms, $\mathrm{conv}_{\leq m}\, \Sigma^1(H_n)^c$ equals the set of $[\chi]$ with $m(\chi) \leq m$. By Theorems 2.2 and 2.3, it is also the case that $[\chi] \in \Sigma^m(H_n)^c$ if and only if $m(\chi) \leq m$. □

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From the computation it is also clear that we can triangulate $\Sigma(H_n)$ into the boundary of an $(n-1)$-simplex in such a way that $\Sigma^m(H_n)^c$ is precisely the $(m-1)$-skeleton of this simplex. This is an example of the 'polyhedral' behaviour we indicated before defining the Bieri Σ-property.

3. Complexes

In this section we recall the CAT(0) cube complex on which H_n acts, and define an important family of CAT(0) subcomplexes called blankets (Definition 3.1).

3.1. Cube complex

There is a natural cube complex X_n on which H_n acts, which we recall now. Everything in this subsection is taken from [**20**]. First, define M_n to be the monoid of injections $\phi : [n] \times \mathbb{N} \to [n] \times \mathbb{N}$ that are eventual translations (i.e. satisfy the same condition as elements of H_n except they need not be surjective), so H_n consists precisely of the bijective elements of M_n. The 0-skeleton of X_n is defined to be M_n.

To define the 1-skeleton of X_n we need to recall some important elements, t_i of M_n. For each $1 \le i \le n$, define $t_i \in M_n$ to be

$$t_i : (j, x) \mapsto \begin{cases} (j, x) & \text{if } j \neq i, \\ (j, x+1) & \text{if } j = i. \end{cases}$$

Now declare that two 0-cubes ϕ, ψ in X_n (i.e. elements of M_n) span a 1-cube whenever $\phi = t_i \circ \psi$ or $\psi = t_i \circ \phi$ for some $1 \le i \le n$. Such a 1-cube is *labelled by* t_i. (Since our maps act from the right, $t_i \circ \phi$ means *pre*compose with t_i.) We define the higher-dimensional cubes of X_n by declaring that for every $\phi \in M_n$ and every $K \subseteq [n]$ there is a $|K|$-cube spanned by

$$\left\{ \left(\prod_{i \in I} t_i \right) \circ \phi \,\middle|\, I \subseteq K \right\}.$$

Note that the t_i all commute with each other, so specifying an order in the product is unnecessary. Since more subscripts and superscripts will soon appear, we will now write X for X_n, and there should be no risk of ambiguity.

It is known that X is a CAT(0) cube complex. The group H_n acts on M_n from the right via $(\phi)\eta := \phi \circ \eta$, and this extends to an action of H_n on X. Each cube stabilizer is finite. There is an H_n-invariant Morse function $f : X \to \mathbb{R}$ defined on $X^{(0)}$ by

$$f(\phi) := |([n] \times \mathbb{N}) \setminus \operatorname{image}(\phi)|.$$

Each sublevel set $X^{f \le q}$ is H_n-cocompact. Note that $X^{f \le 0}$ (that is, $X^{f=0}$) is precisely H_n.

Since elements of M_n are eventual translations just like elements of H_n, any character $\chi : H_n \to \mathbb{R}$ naturally extends to a monoid homomorphism $\chi : M_n \to \mathbb{R}$ given by the same definition as on H_n. Then, viewing M_n as $X^{(0)}$, any χ extends to a continuous map $\chi : X \to \mathbb{R}$. The lexicographically ordered function (χ, f) is a Morse function in the sense of Definition 1.6 on any $X^{f \le q}$.

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3.2. Blankets

Blankets are certain subcomplexes of X that we will use later to cover the complex $X^{0\leq\chi}$. The definition is as follows.

Definition 3.1 (blanket). For $K \subseteq [n]$ consider the subcomplex $\bigcap_{i\in K} X^{\chi_i\leq 0}$ of X. We will call any connected component of such a subcomplex a K-*blanket*, and generally refer to K-blankets for arbitrary K as *blankets*.

Recall that a subcomplex Z of a CAT(0) cube complex Y is *locally combinatorially convex* if every link in Z of a 0-cube $z \in Z$ is a full subcomplex of the link of z in Y, and *combinatorially convex* if it is connected and locally combinatorially convex. It is well known (for example, see [**9**, Lemma 2.12]) that combinatorially convex subcomplexes are themselves CAT(0), and hence contractible. In particular, each connected component of a locally combinatorially convex subcomplex is contractible.

Lemma 3.2 (blankets are CAT(0)). *For any K, $\bigcap_{i\in K} X^{\chi_i\leq 0}$ is locally combinatorially convex. In particular, blankets are combinatorially convex in X, and hence* CAT(0) *and contractible.*

Proof. It is enough to show that each $X^{\chi_i\leq 0}$ is locally combinatorially convex. Note that if ϕ, ψ are adjacent 0-cubes in X, say with $\psi = t_j \circ \phi$, then $\chi_i(\psi) - \chi_i(\phi)$ is 0 if $i \neq j$ and 1 if $i = j$. Thus if we have a cube C containing ϕ, and $\psi_1, \ldots, \psi_r$ are the 0-faces of C adjacent to ϕ, then the maximum and minimum values of χ_i on C lie in $\{\chi_i(\phi), \chi_i(\psi_1), \ldots, \chi_i(\psi_r)\}$. In particular, if $\phi \in X^{\chi_i\leq 0}$ and these ψ_i lie in the link of ϕ in $X^{\chi_i\leq 0}$, then all of C lies in $X^{\chi_i\leq 0}$. This shows that the link of ϕ in $X^{\chi_i\leq 0}$ is a full subcomplex of the link of ϕ in X. □

Corollary 3.3 (intersections of blankets are blankets). *Let $Z_1, \ldots, Z_r$ be blankets, say with Z_i a K_i-blanket. Then if $Z_1 \cap \cdots \cap Z_r$ is non-empty it is contractible and in fact is a $(K_1 \cup \cdots \cup K_r)$-blanket.*

Proof. Since each Z_i is combinatorially convex, any non-empty $Z_1 \cap \cdots \cap Z_r$ is combinatorially convex, and hence contractible. Moreover, as a (contractible hence) connected subcomplex of $\bigcap_{i\in K_1\cup\cdots\cup K_r} X^{\chi_i\leq 0}$ we know it lies in a $(K_1 \cup \cdots \cup K_r)$-blanket. It also contains a $(K_1 \cup \cdots \cup K_r)$-blanket for trivial (general topological) reasons, and hence it must equal a $(K_1 \cup \cdots \cup K_r)$-blanket. □

4. The proof

In this section we will use blankets and the Morse function (χ, f) to prove our main result, Theorem 2.3. Without loss of generality, $\chi = a_1\chi_1 + \cdots + a_n\chi_n$ is in ascending standard form, so $a_1 \leq \cdots \leq a_n = 0$. Let us write $X_{f\leq k}$ for $X^{f\leq k}$ and $X^{t\leq\chi}_{f\leq k}$ for $X_{f\leq k} \cap X^{t\leq\chi}$. As a remark, in what follows we may occasionally implicitly assume $n \geq 2$; the only character of H_1 is 0, so while our main results are (vacuously) true for $n = 1$, some of the arguments used in this section may not literally be true for $n = 1$.

Lemma 4.1. *If $X^{0\le\chi}_{f\le 3n-3}$ is not $(m(\chi)-1)$-connected then $[\chi]\in\Sigma^{m(\chi)}(H_n)^c$.*

Proof. We proceed by contrapositive. If $[\chi]\in\Sigma^{m(\chi)}(H_n)$, then since $X_{f\le 3n-3}$ is H_n-cocompact we know that the filtration $(X^{t\le\chi}_{f\le 3n-3})_{t\in\mathbb{R}}$ is essentially $(m(\chi)-1)$-connected. By [20, Proposition 6.6], every ascending link with respect to (χ, f) of a 0-cube in $X_{f\le 3n-3}$ is $(m(\chi)-2)$-connected, so by Lemma 1.7 the inclusion $X^{t\le\chi}_{f\le 3n-3}\to X^{s\le\chi}_{f\le 3n-3}$ (for any $s\le t$) induces an isomorphism in π_k for all $k\le m(\chi)-2$ and a surjection in $\pi_{m(\chi)-1}$. We are assuming that for all t there exists $s\le t$ such that this inclusion induces the trivial map in π_k for all $k\le m(\chi)-1$, so in fact for such s the complex $X^{s\le\chi}_{f\le 3n-3}$ is $(m(\chi)-1)$-connected. Without loss of generality, $s\in\chi(H_n)$ and so after translating by an element of H_n we get $X^{s\le\chi}_{f\le 3n-3}\cong X^{0\le\chi}_{f\le 3n-3}$, so $X^{0\le\chi}_{f\le 3n-3}$ is $(m(\chi)-1)$-connected. □

Our goal now is to prove that $X^{0\le\chi}_{f\le 3n-3}$ is not $(m(\chi)-1)$-connected. We will cover it with its intersection with certain blankets and apply the Strong Nerve Lemma. For each $1\le i\le n$, let $(Z^\alpha_i)_{\alpha\in I_i}$ be the collection of $\{i\}$-blankets in X (so the connected components of $X^{\chi_i\le 0}$) and set

$$Y^\alpha_i := Z^\alpha_i\cap X^{0\le\chi}_{f\le 3n-3}.$$

Here, I_i is just an appropriate indexing set.

Lemma 4.2. *The Y^α_i for $1\le i\le m(\chi)$ cover $X^{0\le\chi}_{f\le 3n-3}$.*

Proof. It suffices to show that

$$X^{0\le\chi}\subseteq\bigcup_{i=1}^{m(\chi)} X^{\chi_i\le 0}.$$

First, note that since $\chi=a_1\chi_1+\cdots+a_{m(\chi)}\chi_{m(\chi)}$ with $a_i<0$ for all i, any 0-cube v in X satisfying $\chi(v)\ge 0$ must satisfy $\chi_i(v)\le 0$ for some i. This proves that the inclusion is true on the 0-skeleton. Now take an arbitrary cube in $X^{0\le\chi}$ and let v be its 0-face at which f is maximized. Then all 0-faces w of the cube satisfy $\chi_i(w)\le\chi_i(v)$, and hence as soon as v lies in $X^{\chi_i\le 0}$ so does the cube. □

Lemma 4.3. *Any non-empty intersection of any number of subcomplexes of the form Y^α_i (with $1\le i\le m(\chi)$) is $(m(\chi)-2)$-connected.*

Proof. For such an intersection to be non-empty, it must feature at most one term of the form Y^α_i for each i, so without loss of generality it is $Y^{\alpha_1}_{i_1}\cap\cdots\cap Y^{\alpha_r}_{i_r}$, with the i_j pairwise distinct (here $\alpha_j\in I_{i_j}$). Call this intersection Y, and let Z be $Z^{\alpha_1}_{i_1}\cap\cdots\cap Z^{\alpha_r}_{i_r}$, so $Y=Z\cap X^{0\le\chi}_{f\le 3n-3}$. To understand Y we will now apply Morse theoretic techniques to Z, similar to those applied to X in [20]. The first step is to get from Z to $Z_{f\le 3n-3}$. By Corollary 3.3, Z is contractible. If ϕ and ψ are 0-cubes of X with $\phi=t_i\circ\psi$ and $\phi\in Z$ then $\psi\in Z$. Hence, for any 0-cube ϕ in Z, the f-descending link of ϕ in X lies in Z. Since this is $(n-2)$-connected for $f(\phi)>2n-1$ [20, Citation 6.4], [14, Lemma 3.52], we see that $Z_{f\le 3n-3}$ is $(n-2)$-connected, and hence $(m(\chi)-2)$-connected (here we are

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just using f as a standard Morse function as in [**1**]). Now we need to get from $Z_{f\leq 3n-3}$ to $Y = Z^{0\leq\chi}_{f\leq 3n-3}$, and to do this we will use the Morse function (in the sense of Definition 1.6) (χ, f). For ϕ a 0-cube in $Z_{f\leq 3n-3}$, as in [**20**], the (χ, f)-ascending link of ϕ in $X_{f\leq 3n-3}$ is the join of an f-ascending part and an f-descending part. The f-descending part lies in Z for the same reasons as above. The f-ascending part lies in Z since it consists of directions labelled by t_k for $m(\chi) < k \leq n$, and each χ_{i_j} is constant in those directions. Hence the ascending link of ϕ in $Z_{f\leq 3n-3}$ equals the ascending link of ϕ in $X_{f\leq 3n-3}$, and this is $(m(\chi)-2)$-connected by [**20**, Proposition 6.6]. Now the Morse Lemma 1.7 tells us that $Y = Z^{0\leq\chi}_{f\leq 3n-3}$ is $(m(\chi)-2)$-connected. □

Let L be the nerve of the covering of $X^{0\leq\chi}_{f\leq 3n-3}$ by the Y_i^α. Since $[\chi] \in \Sigma^{m(\chi)-1}(H_n)$ by Theorem 2.2, we know from Lemma 4.1 that $X^{0\leq\chi}_{f\leq 3n-3}$ is $(m(\chi)-2)$-connected, so the Strong Nerve Lemma (Citation 1.8), which applies by Lemma 4.3, says that L is also $(m(\chi)-2)$-connected. To prove Theorem 2.3, the last result we need is that L is not $(m(\chi)-1)$-acyclic.

Lemma 4.4. *The nerve L is not $(m(\chi)-1)$-acyclic.*

Proof. Each vertex of L has a *type* in $[m(\chi)]$, given by declaring that the vertex corresponding to Y_i^α has type i. Vertices of the same type cannot be adjacent, so L is $(m(\chi)-1)$ dimensional. Thus it suffices to exhibit a non-trivial $(m(\chi)-1)$-cycle. For this we will find, for each $1 \leq i \leq m(\chi)$, a pair of distinct vertices Y_i^1 and Y_i^2 of type i, such that $Y_1^{\epsilon_1} \cap \cdots \cap Y_{m(\chi)}^{\epsilon_{m(\chi)}} \neq \emptyset$ for every choice of $\epsilon_j \in \{1,2\}$. This will then yield an embedded $(m(\chi)-1)$-sphere in L, which must be homologically non-trivial since L is $(m(\chi)-1)$ dimensional. For each i take Y_i^1 to be the component Y_i^α containing the identity element of H_n, and take Y_i^2 to be the Y_i^α containing the transposition τ_i in H_n that swaps $(i,1)$ and $(i,2)$. By construction, $Y_1^{\epsilon_1} \cap \cdots \cap Y_{m(\chi)}^{\epsilon_{m(\chi)}} \neq \emptyset$ for every choice of $\epsilon_j \in \{1,2\}$, for instance, this intersection contains the element of H_n that is the product of those τ_i with $\epsilon_i = 2$. It remains to show that for each i we have $Y_i^1 \neq Y_i^2$. It is enough to show that $Z_i^1 \neq Z_i^2$. If $Z_i^1 = Z_i^2$ (call it Z) then we can connect the identity to τ_i via an edge path in Z, and since Z is combinatorially convex, without loss of generality this edge path consists of a path along which f strictly increases followed by a path along which f strictly decreases (see [**14**, Figure 3.8] for some intuition). Since the path lies in Z, χ_i is non-positive on the whole path. Since $\chi_i(\mathrm{id}) = \chi_i(\tau_i) = 0$, none of the edges in the path can be labelled by t_i. In particular, adjacent vertices in the path must restrict to identical permutations on $\{i\} \times \mathbb{N}$, and hence all vertices on the path must restrict to the same permutation on $\{i\} \times \mathbb{N}$. Since id and τ_i do not, we have a contradiction. □

Proof of Theorem 2.3. By Lemma 4.3, the Strong Nerve Lemma (Citation 1.8) applies. The Strong Nerve Lemma together with Lemma 4.4 says that $X^{0\leq\chi}_{f\leq 3n-3}$ is not $(m(\chi)-1)$-acyclic. Now Lemma 4.1 implies $[\chi] \in \Sigma^{m(\chi)}(H_n)^c$. □

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References

1. M. Bestvina and N. Brady, Morse theory and finiteness properties of groups, *Invent. Math.* **129**(3) (1997), 445–470.
2. R. Bieri and B. Renz, Valuations on free resolutions and higher geometric invariants of groups, *Comment. Math. Helv.* **63**(3) (1988), 464–497.
3. R. Bieri and H. Sach, Groups of piecewise isometric permutations of lattice points (arXiv:1606.07728, 2016).
4. R. Bieri, W. D. Neumann and R. Strebel, A geometric invariant of discrete groups, *Invent. Math.* **90**(3) (1987), 451–477.
5. R. Bieri, R. Geoghegan and D. H. Kochloukova, The sigma invariants of Thompson's group F, *Groups Geom. Dyn.* **4**(2) (2010), 263–273.
6. K. S. Brown, Finiteness properties of groups, in *Proceedings of the Northwestern conference on cohomology of groups (Evanston, IL, 1985)*, *J. Pure Appl. Algebra* **44** (1987), 45–75.
7. K.-U. Bux, Finiteness properties of soluble arithmetic groups over global function fields, *Geom. Topol.* **8** (2004), 611–644.
8. K.-U. Bux and C. Gonzalez, The Bestvina–Brady construction revisited: geometric computation of Σ-invariants for right-angled Artin groups, *J. Lond. Math. Soc. (2)* **60**(3) (1999), 793–801.
9. F. Haglund, Finite index subgroups of graph products, *Geom. Dedicata* **135** (2008), 167–209.
10. J. Harlander and D. H. Kochloukova, The Σ^2-conjecture for metabelian groups: the general case, *J. Algebra* **273**(2) (2004), 435–454.
11. C. H. Houghton, The first cohomology of a group with permutation module coefficients, *Arch. Math. (Basel)* **31**(3) (1978/79), 254–258.
12. D. H. Kochloukova, Subgroups of constructible nilpotent-by-abelian groups and a generalization of a result of Bieri, Neumann and Strebel, *J. Group Theory* **5**(2) (2002), 219–231.
13. D. H. Kochloukova, On the Σ^2-invariants of the generalised R. Thompson groups of type F, *J. Algebra* **371** (2012), 430–456.
14. S. R. Lee, Geometry of Houghton's groups (arXiv:1212.0257, 2012).
15. J. Meier, H. Meinert and L. VanWyk, Higher generation subgroup sets and the Σ-invariants of graph groups, *Comment. Math. Helv.* **73**(1) (1998), 22–44.
16. H. Meinert, The homological invariants for metabelian groups of finite Prüfer rank: a proof of the Σ^m-conjecture, *Proc. Lond. Math. Soc. (3)* **72**(2) (1996), 385–424.
17. H. Sach, FPn properties of generalized Houghton groups (arXiv:1608.00933, 2016).
18. S. Witzel and M. C. B. Zaremsky, The Σ-invariants of Thompson's group F via Morse theory, in *Topological methods in group theory* (eds. N. Broaddus, M. Davis, J.-F. Lafont and I. J. Ortiz), London Mathematical Society Lecture Note Series, Volume 451, pp. 173–193 (Cambridge University Press, Cambridge, 2018).
19. S. Witzel and M. C. B. Zaremsky, The Basilica Thompson group is not finitely presented, *Groups Geom. Dyn.*, in press.
20. M. C. B. Zaremsky, On the Σ-invariants of generalized Thompson groups and Houghton groups, *Int. Math. Res. Not. IMRN* **2017**(19) (2017), 5861–5896.
21. M. C. B. Zaremsky, Geometric structures related to the braided Thompson groups (arXiv:1803.0271, 2018).