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UNIFORMITIES AND UNIFORMLY CONTINUOUS FUNCTIONS ON LOCALLY CONNECTED GROUPS

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We show that the left and the right uniformities on a locally connected topological group G coincide if and only if every left uniformly continuous real-valued function on G is right uniformly continuous.

A topological group G is said to be a *SIN-group* if the left and the right uniform structures on G coincide. Every precompact topological group and every Abelian topological group is a SIN-group; the linear Lie group $SL_2(\mathbb{R})$ provides the best known specimen of a topological group that is not SIN. (See [**3, 8**].) An obvious corollary of the SIN property is that every left uniformly continuous real-valued function on G is right uniformly continuous (and *vice versa*). Rather surprisingly, it is still unknown if the converse holds true.

OPEN QUESTION. (Itzkowitz, [**5**]) *Is a topological group G SIN whenever every left uniformly continuous real-valued function on G is right uniformly continuous?*

Firstly, Itzkowitz [**4**] has shown that if a locally compact group is either unimodular or metrisable then the answer is "yes". For arbitrary locally compact groups the affirmative answer was obtained independently by Milnes [**6**], Itzkowitz [**5**] and Protasov [**7**]; moreover, Protasov proved that the answer to the problem is in the affirmative for almost metrisable groups in the sense of Pasynkov. (This class includes all locally compact as well as all metrisable groups, see [**8**].) Recently, Hansell and Troallic [**1**] have established an analogous result for a somewhat larger class of q-groups.

In the present note we answer Itzkowitz's question in the affirmative for a class of topological groups satisfying a topological condition of a completely different character.

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Main Theorem. *A locally connected topological group G is a SIN-group if and only if every left uniformly continuous real-valued function on G is right uniformly continuous.*

Before proceeding to the proofs, we find it convenient to introduce the following *ad hoc* concept.

Definition 1: Say that a topological group G is a *functionally SIN-group* (*FSIN*, for short) if every left uniformly continuous real-valued function on G is right uniformly continuous. (Equivalently: if every right uniformly continuous real-valued function on G is left uniformly continuous.)

Now Itzkowitz's question can be reformulated as follows: are the properties SIN and FSIN equivalent for every topological group?

Definition 2: We say that a subset A of a topological group G is *left neutral* in G if for every neighbourhood V of the identity in G there is a neighbourhood U of the identity such that $UA \subseteq AV$. In a similar way we define *right neutral* subsets. A subset that is both left and right neutral is said to be *neutral.*

For example, every left-neutral symmetric subset and every precompact subset of a topological group is neutral. The property of being a neutral subset appears in the paper [**2**] as the '(*)-property,' but we feel that our present terminology is justified, being a straightforward extension of the previously known concept of a *neutral subgroup* [**8**]: a subgroup H of a topological group G is neutral if and only if it forms a neutral subset of G in our sense. Every normal subgroup of a topological group is neutral.

Definition 3: We say that a subset A of a topological group G is *left uniformly discrete* in G if it is uniformly discrete with respect to the left uniform structure, that is, for a suitable neighbourhood V of the identity the left translates aV and bV are disjoint whenever $a, b \in A$ and $a \neq b$. In a similar fashion we define the *right uniformly discrete* subsets.

Theorem 1. *Every left uniformly discrete subset of a FSIN group is left neutral.*

Proof: Let A be a left uniformly discrete subset of a FSIN-group G. Let V be an arbitrary neighbourhood of the identity in G. We shall prove that $UA \subseteq AV$ for a suitable neighbourhood U of the identity.

Using the left uniform discreteness of A and passing to a smaller neighbourhood if necessary, we can assume without loss of generality that V is a symmetric neighbourhood of the identity such that the left translates aV^2 and bV^2 are disjoint whenever $a, b \in A$ and $a \neq b$.

One can choose a left uniformly continuous function $f: G \to \mathbb{R}$ with the properties $f(e_G) = 1$, $f|_{G \setminus V} \equiv 0$, and $0 \leqslant f(x) \leqslant 1$ for all $x \in G$. (One easy way to construct such a function is to choose a left invariant pseudometric ρ on G whose unit ball centred at e_G is contained in V (see [**3**, Theorem 8.2]), and set $f(x) = 1 - \min\{1, \rho(e_G, x)\}$.) For every $a \in A$, the function $f_a: G \to \mathbb{R}$, defined by letting $f_a(x) = f(a^{-1}x)$, is also left uniformly continuous (because $(a^{-1}x)^{-1}(a^{-1}y) = x^{-1}y)$ and has the properties $f_a(a) = 1$, $f_a|_{G \setminus aV} \equiv 0$, and $0 \leqslant f_a(x) \leqslant 1$ for all $x \in G$.

The family of open subsets aV, $a \in A$, is left uniformly discrete in G by the very choice of V. Indeed, if $x \in G$ is arbitrary, then the neighbourhood xV of x can meet no more than one set of the form aV: assuming $av = xw$ and $bu = xz$ with $v, u, w, z \in V$, one obtains $avw^{-1} = x = buz^{-1}$, where $vw^{-1}, uz^{-1} \in V^2$, that is, $aV^2 \cap bV^2 \neq \emptyset$. As a corollary, the function

$$\varphi(x) = \sum_{a \in A} f_a(x): G \to \mathbb{R}$$

is well-defined and is left uniformly continuous. It has the properties $\varphi(a) = 1$ for every $a \in A$, $\varphi|_{G \setminus AV} \equiv 0$, and $0 \leqslant \varphi(x) \leqslant 1$ for all $x \in G$.

By force of the assumed FSIN-property, φ is right uniformly continuous. Therefore, there is a neighbourhood of the identity U in G such that whenever $x, y \in G$ and $xy^{-1} \in U$, one has $|\varphi(x) - \varphi(y)| < 1$.

We claim that $UA \subseteq AV$. Indeed, let $u \in U$ and $a \in A$ be arbitrary. Since $(ua)a^{-1} = u \in U$, one has $|\varphi(ua) - \varphi(a)| < 1$, that is, $|\varphi(ua) - 1| < 1$, and therefore $ua \in AV$. $\square$

Lemma. *Let V be a neighbourhood of the identity in a topological group G. Then there exists a set $A \subseteq G$ such that the left translates aV and bV are disjoint whenever $a, b \in A$ and $a \neq b$, and such that $AVV^{-1} = G$.*

Proof: Let $U = VV^{-1}$ and note that $U = U^{-1}$. Let $\mathcal{A}$ be the set of subsets A of G such that $aU \cap A = \{a\}$ for all $a \in A$. Clearly $\mathcal{A}$ is non-empty and is partially ordered under set inclusion. If $\{A_\iota\}$ is a chain in $\mathcal{A}$, put $A = \bigcup_\iota A_\iota$. We claim that $A \in \mathcal{A}$. For if we fix $a \in A$, then $aU \cap A = aU \cap \bigcup_\iota A_\iota = \bigcup_\iota (aU \cap A_\iota)$. But for each ι, $aU \cap A_\iota$ is empty if $a \notin A_\iota$ and is $\{a\}$ if $a \in A_\iota$, and it is clear that $a \in A_\iota$ for at least one value of ι. Hence $aU \cap A = \{a\}$, and $A \in \mathcal{A}$ as claimed. Therefore, by Zorn's lemma, there is a maximal set $A_0 \in \mathcal{A}$. We claim that $A_0 U = G$. If not, pick $a_0 \in G \setminus A_0 U$. Then $a_0 \notin A_0$, and we have not only $a_0 \notin aU$ for all $a \in A_0$, but also $a \notin a_0 U$ for all $a \in A_0$, by the symmetry of U. Therefore $A_0 \cup \{a_0\} \in \mathcal{A}$, contradicting the maximality of A_0. Therefore $A_0 U = G$, as claimed.

It is now easy to see that $A = A_0$ is the set we require. For the last assertion proved above shows that $AVV^{-1} = G$, and if $a, b \in A$ and $a \neq b$, then $aV \cap bV = \emptyset$, since otherwise $b \in aVV^{-1} = aU$, contradicting the fact that $aU \cap A = \{a\}$. $\square$

Recall that a topological group G is *locally connected* if the connected open subsets form a base for G.

THEOREM 2. *Let G be a locally connected topological group in which every left uniformly discrete subset is left neutral. Then G is SIN.*

PROOF: Let $\mathcal{O}$ be an arbitrary neighbourhood of the identity in G. We wish to find a neighbourhood U of the identity such that $g^{-1}Ug \subseteq \mathcal{O}$ for all $g \in G$.

Let V be a connected symmetric neighbourhood of the identity in G such that $V^5 \subseteq \mathcal{O}$. Choose a subset $A \subseteq G$ as in the Lemma. Since A is a left uniformly discrete subset, it is left neutral by the hypothesis. Therefore, one can find a connected neighbourhood of the identity U with $UA \subseteq AV$.

Let $a \in A$ be arbitrary. We shall show that $a^{-1}Ua \subseteq V$. The connected component of the set AV, containing a, is exactly aV, in view of connectedness of V. The set Ua is contained in AV, is connected, and has a non-empty intersection with aV ($\{a\} \subseteq Ua \cap aV$); therefore, we must have $Ua \subseteq aV$, that is, $a^{-1}Ua \subseteq V$.

Now let $g \in G$ be arbitrary. By the choice of A, there exists a representation $g = avw$, where $a \in A$ and $v, w \in V$. One has:

$$g^{-1}Ug = (avw)^{-1}Uavw = w^{-1}v^{-1}(a^{-1}Ua)vw \subseteq w^{-1}v^{-1}Vvw \subseteq V^5 \subseteq \mathcal{O},$$

as required. ∎

Now the Main Theorem follows from Theorems 1 and 2 as an immediate corollary.

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