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Propagation of electromagnetically generated wake fields in inhomogeneous magnetized plasmas

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Abstract. Generation of wake fields by a short electromagnetic pulse in a plasma with an inhomogeneous background magnetic field and density profile is considered, and a wave equation is derived. Transmission and reflection coefficients are calculated in a medium with sharp discontinuities. Particular attention is focused on examples where the longitudinal part of the electromagnetic field is amplified for the transmitted wave. Furthermore, it is noted that the wake field can propagate out of the plasma and thereby provide information about the electron density profile. A method for reconstructing the background density profile from a measured wake field spectrum is proposed and a numerical example is given.

1. Introduction

As is well known, a short electromagnetic (EM) pulse propagating in an underdense unmagnetized plasma can excite a wake field of plasma oscillations (Tajima and Dawson 1979; Gorbunov and Kirsanov 1987). This has interesting applications to plasma-based particle accelerators (Dawson 1994) and photon acceleration (Wilks et al. 1989; Mironov et al. 1992; Mendonça 2001), and is naturally of importance for the general understanding of the interactions between plasmas and radiation. Effects of the presence of an external magnetic field on the generation of wake fields has been considered before. The regime where the group velocity of the exciting pulse is not approximately equal to the speed of light in vacuum has been considered for electrostatic wake fields in the case of both parallel (with respect to the magnetic field) propagation (Shukla 1993) and perpendicular propagation (Shukla 1999). In the opposite regime, where the group velocity of the exciting pulse approximately coincides with the velocity of light in vacuum, the wake field becomes partially electromagnetic and thereby obtains a nonzero group velocity (Brodin and Lundberg 1998), provided that the pulse propagation is non-parallel to the magnetic field.

In the present paper, we generalize the derivation of Brodin and Lundberg (1998) to an *inhomogeneous* magnetized plasma. A wave equation for the wake field, including arbitrary inhomogeneities in the particle number density and magnetic field, is derived from the cold-electron-fluid equations and the propagation properties are investigated. We address two questions in particular. Firstly, we examine the effect of a strong inhomogeneity on the wake field, by introducing a discontinuity in the background magnetic field and density. The longitudinal part of the electric field of the transmitted wave can be strongly amplified when the ratio between the group velocities of the transmitted and incident wave is small. The amplification factor for the longitudinal electric field is given and analyzed, as well as the transmission and reflection coefficients. Secondly, we consider to what extent the wake field can propagate out of the plasma, and thereby provide information about the background plasma parameters. Since the wake field initially has a frequency equal to the local plasma frequency also in the magnetized case, this provides a way of extracting information about the background electron density profile, i.e. the profile in the absence of the wake field density oscillations. It turns out that even though wave overtaking – for example when a higher-frequency part of the wake field passes a lower-frequency part – may occur, the density profile can still be reconstructed by integrating the ray equations of geometric optics backwards. A numerical example is provided, where the predicted spectrum of the wake field corresponding to an assumed density profile is shown, and a reconstructed profile is calculated.

The paper is organized as follows. After stating the equations governing the wake field in Sec. 2, we derive the wave equation for the longitudinal electric field in Sec. 3. The excitation and propagation in a weakly inhomogeneous medium are examined in Sec. 4. Then, in Sec. 5, the effects of strong inhomogeneities on the wake field, which for example cause field amplification and reflection, are studied. Next, in Sec. 6, the spectral properties of an electromagnetically generated wake field from a nonuniform density profile are investigated. An algorithm for reconstructing the density profile from a measured wake-field spectrum is given and illustrated with a numerical example. Finally, the results are summarized and discussed in Sec. 7.

2. Basic equations

We consider a high-frequency EM pulse with frequency ω_H propagating in a cold, inhomogeneous magnetized plasma. We assume the ordering $\omega_H \gg \omega_p, \omega_c$, where ω_p and $\omega_c \equiv |q\mathbf{B}_0|/m$ are the plasma and electron cyclotron frequencies, q and m are the electron charge and mass, and $\mathbf{B}_0 = B_0\hat{\mathbf{x}}$ is the external magnetic field. We let the EM pulse propagate perpendicularly to the external magnetic field. The ponderomotive force of the EM pulse will generate a 'low-frequency' wake-field mode (which is the low-frequency branch of the extraordinary mode, or plasma oscillations modified by the magnetic field, depending on the choice of terminology) during its path through the plasma. The generation mechanism is most efficient if the pulse has a duration of the order of the inverse plasma frequency or shorter, so that the ion motion can be omitted. In principle, the EM pulse will broaden owing to ordinary dispersion, decrease its energy and frequency owing to the interaction with the wake field, etc. These and other effects have been considered in homogeneous plasmas by, for example, Brodin and Lundberg (1998). We shall focus on the propagation properties of the *wake field*, however, and for this purpose it turns out that we can forget about the details of the EM pulse. Basically, the effect of the EM field is to provide a well-localized ponderomotive source term in the governing equations for the wake field, travelling with almost the speed of light in vacuum.

The wake-field quantities are denoted by an index L. We introduce the corresponding vector and scalar potentials $\mathbf{A}_L(z,t)$ and $\phi_L(z,t)$, using the Coulomb gauge, and the electron density is written $n = n_0(z,t) + n_L(z,t)$, where n_0 is the unperturbed density. Furthermore, the electron fluid velocity $\boldsymbol{v}$ is divided into highand low-frequency parts, and we denote the low-frequency contributions perpendicular and parallel to the direction of propagation by $\mathbf{v}_{L\perp}$ and v_{Lz} respectively. The ponderomotive force of the EM pulse induces longitudinal wake-field motion, which couple to motion in the $\hat{\mathbf{y}}$ direction through the Lorentz force, but there is no wake-field motion in the direction of the external magnetic field, and accordingly we put $\mathbf{A}_L = A_L\hat{\mathbf{y}}$ and $\mathbf{v}_{L\perp} = v_{L\perp}\hat{\mathbf{y}}$. Linearizing in the low-frequency variables, we obtain the following set of equations governing the wake-field generation and propagation:

$$-\mu_0 q n_0 v_{L\perp} + (c^{-2}\partial_t^2 - \partial_z^2)A_L = 0, \tag{1}$$

$$c^{-2}\partial_z\partial_t\phi_L - \mu_0 q n_0\, v_{L\,z} = 0, \tag{2}$$

$$\partial_t v_{L\perp} - \omega_c v_{Lz} + \frac{q}{m}\partial_t A_L = 0, \tag{3}$$

$$\partial_t v_{Lz} + \frac{q}{m}\partial_z\phi_L + \omega_c v_{L\perp} = -\frac{q^2}{2m^2}\partial_z|\mathbf{A}_H|^2, \tag{4}$$

$$\partial_t n_L + \partial_z(n_0 v_{Lz}) = 0. \tag{5}$$

3. Derivation of the wave equation

Before deriving a wave equation for the wake field, it is practical to redefine

$$\frac{q}{m}A_L \to A, \qquad \frac{q}{m}\partial_z\phi \to \psi, \qquad v_\perp \to v, \qquad S = -\frac{q^2}{2m^2}\partial_z|\mathbf{A}_H|^2.$$

Equations (1), (3), and (4) then read

$$-\omega_p^2 v + (\partial_t^2 - c^2\partial_z^2)A = 0, \tag{6}$$

$$\partial_t v - \omega_c\omega_p^{-2}\partial_t\psi + \partial_t A = 0, \tag{7}$$

$$(\omega_p^{-2}\partial_t^2 + 1)\psi + \omega_c v = S. \tag{8}$$

Equations (2) and (5) only give information regarding how v_{Lz} and n_L are related to the other variables, and are therefore omitted at this point. Acting on (7) with $(\partial_t^2 - c^2\partial_z^2)\partial_t^{-1}$ and applying (6) gives

$$0 = (\partial_t^2 - c^2\partial_z^2 + \omega_p^2)v - (\partial_t^2 - c^2\partial_z^2)(\omega_c\psi).$$

Combining this with (8) gives, after some rearrangements,

$$\partial_z^2(\hat{\alpha}\psi) + \hat{\beta}\psi = \hat{\gamma}(\omega_c^{-1}S), \tag{9}$$

where

$$\hat{\alpha} \equiv c^2[\omega_c^{-1}(1 + \omega_p^{-2}\partial_t^2) + \omega_c\omega_p^{-2}],$$

$$\hat{\beta} \equiv -\omega_c^{-1}\{\omega_p^2 + \partial_t^2[2 + \omega_p^{-2}(\omega_c^2 + \partial_t^2)]\},$$

$$\hat{\gamma} \equiv -(\partial_t^2 - c^2\partial_z^2 + \omega_p^2).$$

Equation (9) provides the starting point for analyzing the propagation properties of a wake field generated by an EM pulse under the given circumstances. It is somewhat surprising that the evolution of the wake field is governed by a *single* wave equation, in spite of there being two arbitrary background parameter functions ω_p and ω_c.

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In the case of a static and homogeneous background density and magnetic field, (9) reduces to

$$[\partial_t^4 + (\omega_p^2 + \omega_h^2)\partial_t^2 - c^2\partial_z^2\partial_t^2 - \omega_h^2 c^2\partial_z^2 + \omega_p^4]\psi = \omega_p^2(\partial_t^2 - c^2\partial_z^2 + \omega_p^2)S, \qquad (10)$$

where $\omega_h^2 \equiv \omega_p^2 + \omega_c^2$ is the upper-hybrid frequency. The left-hand side of (10) is the familiar wave operator for the extraordinary electromagnetic mode.

4. Wake-field excitation and propagation

We shall consider wake-field propagation in both strongly and weakly inhomogeneous plasmas. By *weakly* inhomogeneous, we mean that the wavelength of the wake field is much smaller than the characteristic inhomogeneity length scale. Then, to lowest order, derivatives acting on background quantities can be neglected and the wave equation reduces to (10) with a space and possibly a time dependence in ω_p and ω_c. In a *strongly* inhomogeneous plasma, this approximation cannot be applied, and the time evolution of the wake field is given by (9). In this section, only weakly inhomogeneous plasmas will be considered, and it is illustrative to divide our study of wake-field properties into an excitation and a propagation phase. However, we note that the excitation process considered below will also be of relevance for the next section concerning strongly inhomogeneous plasmas, since in that case we shall deal with wake fields *generated* in a weakly inhomogeneous plasma that *propagate* into a strongly inhomogeneous region.

4.1. Excitation

The excitation of one additional wavelength of the wake field takes place during a distance of the order of $2\pi c/\omega_p$, and – as a basic assumption of ours – the variations of n_0 is negligible on this length scale. Thus, as far as the excitation process is concerned, the plasma can essentially be treated as homogeneous. The solution for the wake field can thus be obtained from previous work (Brodin and Lundberg 1998). Changing to comoving coordinates $\xi = z - v_{gH}t$ and $\tau = t$, where v_{gH} is the group velocity of the high-frequency EM pulse, in (10) and neglecting the small derivatives ∂_τ^2 and $\partial_\tau\partial_\xi$ and terms proportional to $v_{gH}^2 - c^2$, it reduces to (reinstating the potentials)

$$(v_{gH}^2\partial_\xi^2 + \omega_p^2)\phi_L = \frac{q\omega_p^2|\mathbf{A}_H|^2}{m}. \qquad (11)$$

This implies

$$\phi_L = \phi_{L0}\sin[k_p(\xi - \xi_0)], \qquad (12)$$

where $k_p = \omega_p/v_{gH}$ is the wake-field wavenumber, ξ_0 is the (constant) position of the (short) EM pulse, and $\phi_{L0} = (q\omega_p/mv_{gH})\int_{-\infty}^{\infty}|\mathbf{A}_H|^2\,d\xi$. The important result here, for our purposes, is the determination of the initial value of the wake-field wavenumber $k_p = \omega_p/v_{gH}$, which corresponds to an initial frequency ω_p (in the laboratory frame). Note that this wake-field frequency will in general vary with the position of generation.

4.2. Propagation

In a weakly inhomogeneous plasma, the wavelength of the wake field is much smaller than the characteristic inhomogeneity length scale. The wake-field properties can thus be considered as locally uniform but globally nonuniform, and therefore we make the ansatz of geometrical optics (Whitham 1974)

$$\psi = \psi_0(z,t)e^{i\theta(z,t)}.$$

The local wavenumber and frequency are defined in terms of the eikonal $\theta(z,t)$ as $k \equiv \partial_z\theta$ and $\omega \equiv -\partial_t\theta$, respectively, and the amplitude $\psi_0(z,t)$ is assumed to vary slowly with z and t. The local dispersion relation then follows, as a lowest-order approximation, from (10),

$$\omega^4 - \omega^2(\omega_h^2 + \omega_p^2 + c^2k^2) + \omega_h^2c^2k^2 + \omega_p^4 = 0. \tag{13}$$

We note that there is a resonance at $\omega^2 = \omega_h^2 \equiv \omega_c^2 + \omega_p^2$ and cutoffs at $\omega_L \equiv \frac{1}{2}\left[-\omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}\right]$ and $\omega_R \equiv \frac{1}{2}[\omega_c + (\omega_c^2 + 4\omega_p^2)^{1/2}]$. The dispersion relation has two positive roots. One branch is valid for $\omega > \omega_R$ and the other for $\omega_L < \omega < \omega_h$. As the wake field is generated with the local plasma frequency ω_p, the wake field must belong to the latter branch. Therefore we write the dispersion relation from now on as

$$\omega = W(k,z,t) \equiv [\chi - (\chi^2 - \omega_h^2c^2k^2 - \omega_p^4)^{1/2}]^{1/2}, \tag{14}$$

where $\chi \equiv \frac{1}{2}(\omega_h^2 + \omega_p^2 + c^2k^2)$. From the dispersion relation, we also derive explicit expressions for the wavenumber and group velocity $v_g \equiv \partial_k\omega$ that we shall use later. They are

$$k = \frac{1}{c}\left[\frac{\omega^2(\omega_h^2 + \omega_p^2) - \omega^4 - \omega_p^4}{\omega_h^2 - \omega^2}\right]^{1/2} \tag{15}$$

and

$$v_g = \frac{\omega^2 - \omega_h^2}{2\omega^2 - \omega_h^2 - \omega_p^2 - c^2k^2}\,\frac{c^2k}{\omega} \tag{16}$$

respectively.

From the geometrical optics approach, there follow useful transport equations for the wavenumber and frequency. Noting the identity $\partial_t k + \partial_z\omega = 0$, it follows that

$$\frac{dk}{dt} = -\partial_z W, \qquad \frac{d\omega}{dt} = \partial_t W, \tag{17a,b}$$

where $d/dt \equiv \partial_t + v_g\partial_z$. Equations (17a,b) are referred to as the *ray equations*. Note that for a time-independent medium, the right-hand side of (17b) is zero, and the wake field propagates with the local wake field group velocity v_g and with unchanged frequency.

Since the group velocity may vary along the rays, the energy carried with the wake field can be compressed as well as attenuated, and from energy conservation one may expect the field amplitude to vary correspondingly; see, for example, Mendonça (2001) for similar effects for ordinary electromagnetic waves in plasmas. The extraordinary mode, however, has several degrees of freedom between which the energy may vary, depending on variations in the background parameters. Therefore the behavior of the amplitudes is not in direct correspondence with the variation of the group velocity. For completeness and for future reference, we state the linear relations between the field variables and ψ that follows from (1)–(5) in the weakly inhomogeneous approximation:

$$\phi = -ik^{-1}\frac{m}{q}\psi, \tag{18}$$

$$v_\perp = \frac{\omega_p^{-2}\omega_c(\omega^2 - c^2k^2)}{\omega^2 - c^2k^2 - \omega_p^2}\psi, \tag{19}$$

$$v_z = -i\omega\omega_p^{-2}\psi, \tag{20}$$

$$A_L = -\frac{m}{q}\frac{\omega_c}{\omega^2 - c^2k^2 - \omega_p^2}\psi, \tag{21}$$

$$\delta n = -ikn\omega_p^{-2}\psi. \tag{22}$$

5. Reflection and transmission properties

We now consider the effect of a strongly inhomogeneous region on the wake field. We assume that the wake field entering this region was generated in a weakly inhomogeneous part of the plasma. Thus the wake field can be taken to be uniform when entering the inhomogeneous region, and the variations in the wake-field frequency ω can be neglected. Under these conditions, we have $\partial_t^2\psi = -\omega^2\psi$ and, away from the exciting electromagnetic pulse, the wave equation (9) reduces to an ordinary differential equation

$$\partial_z^2(\alpha\psi) + \beta\psi = 0, \tag{23}$$

where

$$\alpha \equiv c^2\omega_c^{-1}\omega_p^{-2}(\omega_h^2 - \omega^2),$$
$$\beta \equiv -\omega_c^{-1}\omega_p^{-2}[\omega^4 + \omega_p^4 - \omega^2(\omega_h^2 + \omega_p^2)].$$

Although it is straightforward – at least numerically – to solve (23) for any given background density and magnetic field, we simplify the analysis by treating the inhomogeneity as a discontinuity, in order to clearly illustrate some of the main effects associated with a strong inhomogeneity. We let the discontinuity be located at $z = 0$, and the remaining plasma is assumed to be homogeneous. Thus we can make the following ansatz for the wake field:

$$\left.\begin{array}{ll} \psi_1 = \psi_i e^{i(k_1 z - \omega t)} + \psi_r e^{i(-k_1 z - \omega t)} & (z < 0), \\ \psi_2 = \psi_t e^{i(k_2 z - \omega t)} & (z > 0), \end{array}\right\} \tag{24}$$

where $\omega = \omega_{p1}$. This ansatz does not apply if we are too close to the exciting pulse, or for the fields that were generated in the strongly inhomogeneous region, but both of these parts of the wake field are assumed to be distant from the discontinuity. The subscripts i, r, and t stands for the incident, reflected and transmitted parts, respectively, and the indices 1 and 2 distinguish quantities on the left-hand ($z < 0$) and right-hand ($z > 0$) sides of the discontinuity.

By integrating (23) across $z = 0$, it follows that $\alpha\psi$ and $\partial_z(\alpha\psi)$ are continuous over the discontinuity. We define $r = \psi_r/\psi_i$ and $a = \psi_t/\psi_i$, and we refer to these quantities as the amplification factors for the reflected and transmitted parts of the longitudinal electric field, which essentially are generalized Fresnel coefficients. The continuity conditions and the ansatz (24) imply

$$a = \frac{2k_1}{k_1 + k_2}\frac{\alpha_1}{\alpha_2}, \tag{25}$$

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where α_1 and α_2 are the values of α on the left- and right-hand sides of the discontinuity, respectively, and

$$r = \frac{k_1 - k_2}{k_1 + k_2}. \tag{26}$$

It follows from (23) that

$$S = -\frac{\omega k \omega_c^2}{\left(\omega^2 - c^2k^2 - \omega_p^2\right)^2}|\psi|^2 \tag{27}$$

is – averaged in time and space – a conserved quantity, i.e. $\partial_z S = 0$. (Noting that S is the time- and space-averaged z-component of the Poynting vector $\mathbf{S} \equiv \frac{1}{2}(\mathbf{E} \times \mathbf{B}^* + \mathbf{E}^* \times \mathbf{B})$, this also follows directly from energy conservation. Actually from (23) it first follows that the conserved quantity is equal to $k\alpha^2|\psi|^2$. To see that this is equivalent to (27) requires tedious but straightforward algebra.)

The transmission and reflection coefficients are introduced as $T \equiv S_t/S_i$ and $R \equiv S_r/S_i$. Explicitly, they read

$$T = \frac{4k_1k_2}{(k_1 + k_2)^2} \tag{28}$$

and

$$R = \frac{(k_1 - k_2)^2}{(k_1 + k_2)^2}, \tag{29}$$

and they satisfy the energy conservation law $T + R = 1$.

The quantities a, r, T, and R depends on the four parameters ω_{p1}, ω_{p2}, ω_{c1}, and ω_{c2}. We choose $\omega_{p1} = \omega_{c2} = 1$ (in normalized units) and, rather than presenting complicated surface plots, present one-dimensional graphs of the dependence of a, r, T, and R on ω_{c2} for some given values of ω_{p2}, and vice versa. Figure 1 shows $T(\omega_{c2})$, $R(\omega_{c2})$, and v_{g2} for distinct values of ω_{p2}, and Fig. 2 shows $T(\omega_{p2})$, $R(\omega_{p2})$, and v_{g2} for distinct values of ω_{c2}. As can be seen, for most of the parameter regime, the transmission is close to unity. There are also regions of no propagation in Fig. 2, corresponding to parameter values $\omega_{h2} < \omega < \omega_{R2}$. Close to these regions, the transmission quickly goes to zero and the reflection towards unity, because the wave approaches a resonance or a cutoff. The regions to the left in Figs 1(a) and 2(a) correspond to the branch $\omega > \omega_{R2}$, i.e. the transmitted mode belongs to a different branch of the dispersion relation (13) than the incident mode.

In Fig. 1(a), where $\omega_{p2} = 0.5$, the left part stretches until $\omega_{c2} \approx 0.71$, where the ω_R cutoff prohibits transmission. At $\omega_{c2} \approx 0.87$, T and v_{g2} becomes zero because of the ω_h resonance. Figure 1(b) is the special case for a jump in the magnetic field only, i.e. $\omega_{p2} = 1$. The transmission is everywhere unity, although the group velocity approaches zero with diminishing ω_{c2}. In Fig. 1(c), where $\omega_{p2} = 2$, there is no transmission up to $\omega_{c2} = 3$, which is due to the ω_L cutoff.

In Figs 2(a), (b), and (c), the values of ω_{c2} are 0.5, 1, and 2, respectively. In Fig. 2(a), the ω_R cutoff occurs at $\omega_{p2} \approx 0.71$, the ω_h resonance at $\omega_{p2} \approx 0.87$, and the ω_L cutoff at $\omega_{p2} \approx 1.22$. In both Figs 2(b) and (c), the absence of transmission is due to the ω_L cutoff.

The special cases Figs 1(b) and 2(b) are particularly interesting, representing a jump in the magnetic field only (referred to as case I below) and in the density only (referred to as case II below). The transmission remains unity, although the group velocity goes to zero as $\omega_{c2} \to 0$ (Fig. 1b) and $\omega_{p2} \to 0$ (Fig. 2b). This means that

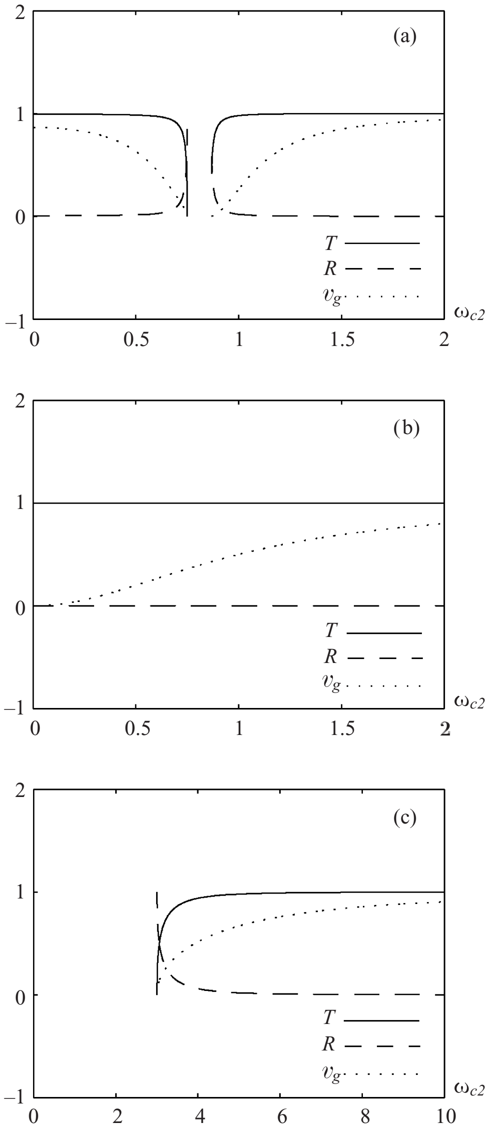

Figure 1. Reflectivity, transmittivity and transmitted group velocity as a functions of the jump in the external magnetic field for $\omega_{p2}/\omega_{p1} = 0.5$ (a), $\omega_{p2}/\omega_{p1} = 1$ (b), and $\omega_{p2}/\omega_{p1} = 2$ (c), with $\omega_{c1}/\omega_{p1} = 1$ in all three parts.

the energy density entering from region 1 will be dramatically amplified in region 2. The question arises as to which field variables this energy will be concentrated.

Case I. For $\omega_{p2} = \omega_{p1} = 1$, the amplification factor for ψ is displayed in Fig. 3(a) together with v_g and v_{ph}. The amplification factor reduces to $a = \omega_{c1}/\omega_{c2}$. It should be emphasized that the wavenumber is preserved over the discontinuity and thus also the phase velocity, which equals c, is preserved. This property is, however, very sensitive to small deviations from exactly constant density. This is illustrated in Fig. 3(b), where $\omega_{p2} = 0.99$, and Fig. 3(c), where $\omega_{p2} = 1.01$. In these cases, the

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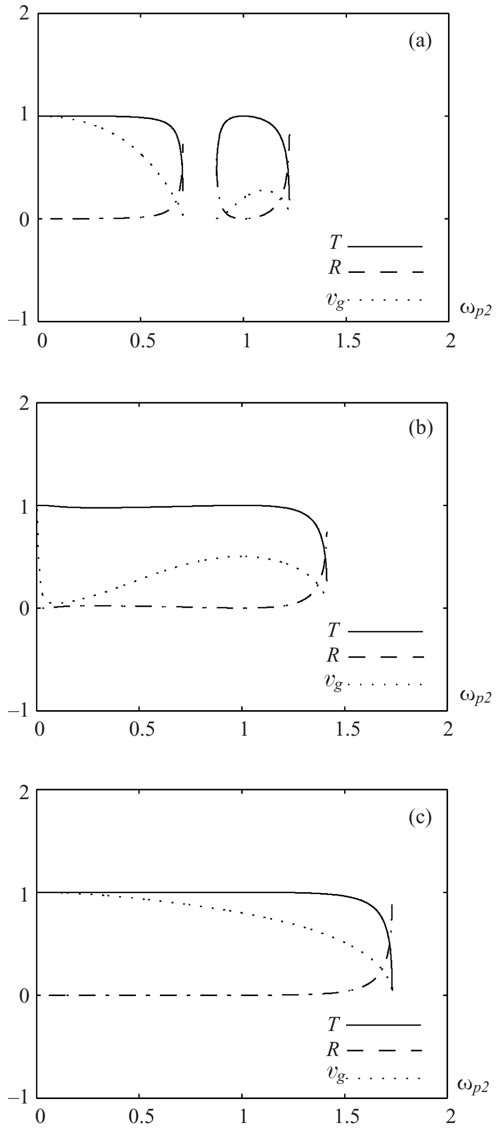

Figure 2. Reflectivity, transmittivity and transmitted group velocity as functions of the jump in the unperturbed density for $\omega_{c2}/\omega_{c1} = 0.5$ (a), $\omega_{c2}/\omega_{c1} = 1$ (b), and $\omega_{c2}/\omega_{c1} = 2$ (c), with $\omega_{c1}/\omega_{p1} = 1$ in all three parts.

longitudinal field can still be amplified – but not by a large factor without also affecting the phase velocity substantially.

Case II. In the case of a uniform magnetic field, $\omega_{c2} = \omega_{c1} = 1$, the field variable that is amplified is the perpendicular electron fluid velocity $v_\perp$. This can be seen by evaluating $v_{\perp 2}/v_{\perp 1}$ using (19) and the amplification factor (25). This case does not share the property with case I of preserved phase velocity over the discontinuity. In Fig. 2(b), we have added a small deviation to ω_c, so that $\omega_{c2} = 1.005$, to illustrate that the group velocity can be made arbitrarily small. However, in the limit $\omega_{p2} \to 0$, the group velocity approaches c, as required in vacuum.

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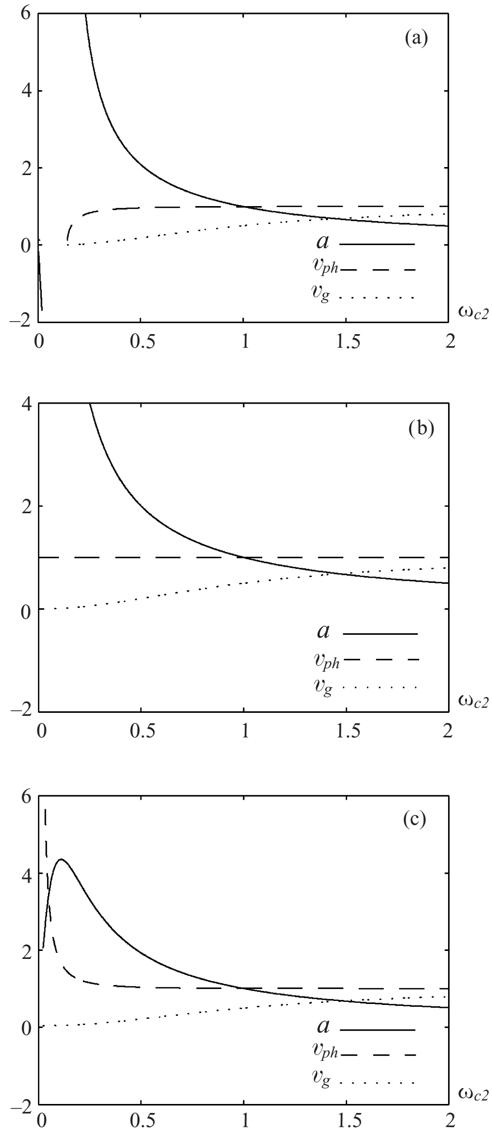

Figure 3. Longitudinal field amplification, phase velocity, and group velocity of the transmitted wave as functions of the jump in external magnetic field for $\omega_{p2}/\omega_{p1} = 0.99$ (a), $\omega_{p2}/\omega_{p1} = 1$ (b), and $\omega_{p2}/\omega_{p1} = 1.01$ (c), with $\omega_{c1}/\omega_{p1} = 1$ in all three parts.

6. Density profile reconstruction

A wake field generated by a short EM pulse in an underdense magnetized plasma has frequency equal to the local plasma frequency, $\omega_p = (q^2 n_0/\epsilon_0 m)^{1/2}$. Owing to the presence of the magnetic field, it has a nonzero group velocity, and, for suitable background parameter profiles, cutoffs and resonances in the plasma are avoided and thus the wake field can propagate out of the plasma. This suggests the possibility of gaining information of the density profile $n_0(z)$ from studying the wake field exiting the plasma.

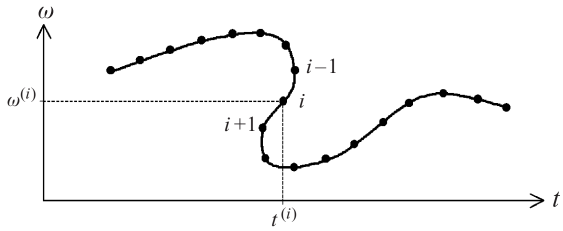

Figure 4. Cartoon of a detected wake-field spectrum as a function of the time of detection. The figure also illustrates the discretization of the curve that can be identified from a wake-field spectrum.

We assume that the plasma is weakly inhomogeneous, so that the results in Secs 4.1 and 4.2 can be applied. Given the ray equations, one may – conceptually speaking – treat the wake field as consisting of particles, 'photons' (dressed photons, that is), created at different times, as the EM pulse plows through the plasma. The 'photons' have distinct frequencies, depending on the position at which they were 'created', and thus also different propagation velocities v_g. The equation of motion for the 'photons' is (16) and the ray equations are (17a,b). The exact evolution of the wake field, on the other hand, is more complicated. This can be understood from the fact that 'photons' created at later times may be overtaken by 'photons' created at earlier times. For our purposes, the particle picture is to be preferred.

We assume that the generated wake-field spectrum is measured immediately outside the plasma boundary. Consistent with the geometrical optics approximation, we shall treat the measured data as a weakly time-dependent spectrum with well-defined sharp (quasimonochromatic) peaks. Because of overtaking 'photons', the data is not necessarily monochromatic at a given time – multiple sharp peaks may occur in the spectrum. Generally, we can express the data as a set of distinct frequencies measured at different times, in which case a sharp curve can be recognized (Fig. 4). Owing to the presence of cutoffs and/or resonances, the curve may be discontinuous.

Given a measured wake field spectrum, as in Fig. 4, the density profile can be reconstructed in the following way. Discretize the frequency curve into N points $\omega^{(0)}, \omega^{(1)}, \ldots, \omega^{(N)}$ with corresponding times of detection $t^{(0)}, t^{(1)}, \ldots, t^{(N)}$. The plasma is discretized into N cells whose positions $z^{(0)}, z^{(1)}, \ldots, z^{(N)}$ and width are yet to be determined. The 'photon' with frequency $\omega^{(N)}$, detected at time $t^{(N)}$, was the last to exit the plasma. Therefore the plasma frequency in the cell at $z^{(N)}$ has the value $\omega^{(N)}$. Next, we retrace the 'photon' $\omega^{(N-1)}$ backwards into the plasma to a position $z^{(N-1)}$ consistent with the time of detection $t^{(N-1)}$ and the equation of motion through the already-reconstructed cell. The plasma frequency in the cell with position $z^{(N-1)}$ is assigned the value $\omega^{(N-1)}$. The third 'photon' $\omega^{(N-2)}$ is retraced through the cells with plasma frequencies $\omega^{(N)}$ and $\omega^{(N-1)}$ to a position $z^{(N-2)}$ and assigns the corresponding cell there the plasma frequency $\omega^{(N-2)}$. This procedure is repeated for all 'photons' along the frequency curve.

In order to demonstrate the method, we calculate a wake-field spectrum numerically from an assumed density profile, using the ray equations (Fig. 5a), and treat this as experimental data from which the density profile can be reconstructed. We consider a plasma magnetized such that $\omega_c = 1.1 \times \omega_{p,\max}$, where $\omega_{p,\max}$ is the

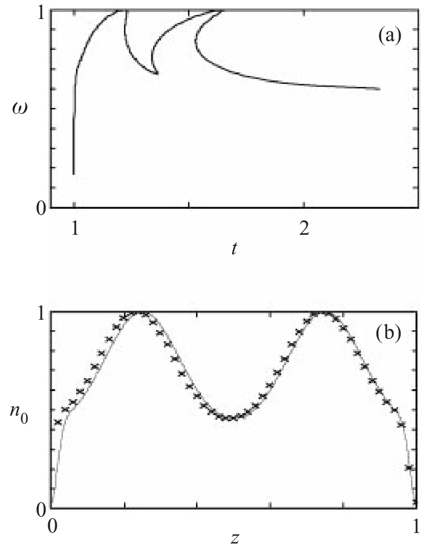

Figure 5. Example of a numerically generated wake field spectrum (a) from an assumed density profile (solid line in (b)). The reconstructed profile is marked with crosses (b).

maximum value of the plasma frequency. For simplicity we normalize such that $L = 1$ and $c = 1$, where L is the length of the plasma, and let the EM pulse enter the plasma at $t = 0$ and exit at $t = 1$.

Retracing the spectrum according to the algorithm presented above results in a density profile that can be compared with the one that we assumed (Fig. 5b). A small numerical error – which can be removed with a finer discretization – can be seen. Note that the entire plasma profile cannot be reconstructed. The leftmost points in Fig. 2(b) are missing. This is because the wake field generated in this region of low density cannot propagate through the plasma, since there is a cutoff prohibiting this. The information of this region is already missing in the wake-field spectrum.

7. Summary and discussion

We have considered the propagation of wake fields generated by a short, high-frequency EM pulse in an inhomogeneous magnetized plasma. A general wave equation, (9), for the wake field driven by the ponderomotive force of the high-frequency pulse has been derived, and the propagation properties have been investigated. If the wake field enters a strongly inhomogeneous region, it may be strongly amplified. The amplification factor for the longitudinal electric field and transmission and reflection coefficients have been derived and analyzed for a discontinuity in the magnetic field and/or the particle number density. In the case of uniform density, the amplification factor becomes $a = \omega_{c1}/\omega_{c2}$. This special case is also characterized by the fact that there is no reflection and the phase velocity of the transmitted wave remains equal to the velocity of light. This result may be of relevance for particle and photon accelerators based on wake fields. For applications such as particle acceleration, it might be desirable to finally let the amplified wake field propagate in an unmagnetized plasma. Simply eliminating the external magnetic field once the wake field is amplified affects the frequency and thus the phase velocity in accordance with the ray equations (17a,b). It seems straightforward, however, to match the discontinuity and the elimination of the magnetic field so that the phase velocity of the resulting field is approximately equal to c, which is the desirable value for particle acceleration purposes.

Furthermore, the spectral properties of a wake field from a plasma with nonuniform density have been investigated, and a method for reconstructing the density profile from a measured wake-field spectrum has been proposed and illustrated with a numerical example. This result shows that wake fields generated by a high-frequency EM pulse can in principle be used as a diagnostic tool in magnetized plasmas. The proposed method is based on a mechanism substantially different from those of existing techniques, such as interferometry and reflectometry (Hutchinson 1987; Hartfuss 1998). It should be possible to extend the method to plasmas where the background density varies in time by using sequential EM pulses. The most interesting case is that of a strongly magnetized plasma (i.e. when the electron cyclotron frequency is larger than the plasma frequency), for which almost all of the wake-field energy – except that generated in a narrow low-density region – may propagate out of the plasma. The requirement that the length scales of inhomogeneities must be larger than the local plasma wavelength $\lambda_p \equiv 2\pi v_{gH}/\omega_p$ for the results to be valid means that it can resolve inhomogeneities of the order $10^8 n^{-1/2}$ m and larger, where n is the electron number density in units of m^{-3}.

References

Brodin, G. and Lundberg, J. 1998 *Phys. Rev.* **E57**, 704.

Dawson, J. M. 1994 *Physica Scripta* **T52**, 7.

Gorbunov, L. M. and Kirsanov, V. I. 1987 *Zh. Eksp. Teor. Fiz.* **93**, 509 [*Sov. Phys. JETP* **66**, 290].

Hartfuss, H. J. 1998 *Plasma Phys. Contr. Fusion* **40**, A231.

Hutchinson, I. H. 1987 *Principles of Plasma Diagnostics*. Cambridge University Press.

Mendonça, J. T. 2001 *Theory of Photon Acceleration*. Bristol: Institute of Physics Publishing.

Mironov, V. A., Sergeev, A. M., Vanin, E. V., Brodin, G. and Lundberg, J. 1992 *Phys. Rev.* **A46**, 6178.

Shukla, P. K. 1993 *Phys. Fluids* **B5**, 3088.

Shukla, P. K. 1999 *Phys. Plasmas* **6**, 1363.

Tajima, T. and Dawson, J. M. 1979 *Phys. Rev. Lett.* **43**, 276.

Whitham, G. B. 1974 *Linear and Nonlinear Waves*. New York: Wiley.

Wilks, S. C., Dawson, J. M., Mori, W. B., Katsouleas, T. and Jones, M. E. 1989 *Phys. Rev. Lett.* **62**, 2600.