Concluding remark

Whereas $QA = QB + QC \geqslant PA + PB + PC$ for any point Q when P is an internal point, areal coordinates may be used to show that $QA^2 + QB^2 + QC^2 \geqslant GA^2 + GB^2 + GC^2$ for the centroid G.

Reference

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106.49 An area problem using barycentric coordinates

In [1], the following question was raised: Given any triangle, show how to construct three Cevians such that the four triangles ANE, BLF, CMD and LMN in Figure 1 have equal areas. Answer: The three Cevians divide each side in the ratio $\Phi : 1$, where Φ is the golden ratio. In [1] and [2], the answer was derived by applying Ceva's theorem and by a Cartesian coordinate argument. In this Note, we give a more natural argument using barycentric coordinates and prove a more general statement (Theorem 1).

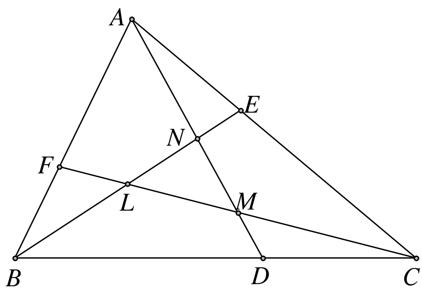

FIGURE 1

In absolute areal (barycentric) coordinates, with respect to the basic triangle ABC, we get $A = (1, 0, 0)$, $B(0, 1, 0)$ and $C = (0, 0, 1)$, and the points D, E and F on the lines BC, CA and AB are $D = (0, d, d')$, $E = (e', 0, e)$ and $F = (f, f', 0)$, where $d + d' = 1$, $e + e' = 1$ and $f + f' = 1$. These points are on the sides of the triangle ABC if, and only if, $0 < d, e, f < 1$. Let us consider the points

$$\begin{aligned} L &= \left(\frac{e'f}{1 - ef'}, \frac{e'f'}{1 - ef'}, \frac{ef}{1 - ef'}\right), \\ M &= \left(\frac{fd}{1 - fd'}, \frac{f'd}{1 - fd'}, \frac{f'd'}{1 - fd'}\right), \qquad (1) \\ N &= \left(\frac{d'e'}{1 - de'}, \frac{de}{1 - de'}, \frac{d'e}{1 - de'}\right). \end{aligned}$$

We have

$$e'f + e'f' + ef = (e + e')(f + f') - ef' = 1 - ef'.$$

and (1) is correct for the point L. This equality can be written as $(1 - ef')L = e'f'B + fE$ and as $(1 - ef')L = efC + e'F$, which proves that the point L lies on the lines BE and CF, so $L = BE \cap CF$. Similarly, equalities $M = CF \cap AD$ and $N = AD \cap BE$ also hold.

For the oriented area $[P_1P_2P_3]$ of the triangle $P_1P_2P_3$, where $P_i = (x_i, y_i, z_i)$ $i = (1, 2, 3)$, the following formula holds (see e.g. [3, Th. 11] or [4]):

$$\frac{1}{\Delta}[P_1P_2P_3] = \begin{vmatrix} x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \end{vmatrix}, \tag{2}$$

where $\Delta = [ABC]$ is the oriented area of the triangle ABC. So, for the triangle LMN we obtain

$$(1 - ef')(1 - fd')(1 - de')\frac{1}{\Delta}[LMN] = (def - d'e'f')^2. \tag{3}$$

From (1), by applying (2) we get

$$(1 - de')\frac{1}{\Delta}[ANE] = \begin{vmatrix} 1 & 0 & 0 \\ d'e' & de & d'e \\ e' & 0 & e \end{vmatrix} = de^2, \tag{4}$$

and analogously we obtain

$$(1 - ef')\frac{1}{\Delta}[BLF] = ef^2, \qquad (1 - fd')\frac{1}{\Delta}[CMD] = fd^2.$$

That is why the equality $[ANE] = [BLF]$ is equivalent to the equality $(1 - de')f^2 = de(1 - ef')$, and because of $e' = 1 - e, f' = 1 - f$, it is equivalent to $(1 - d + de)f^2 = de(1 - e + ef)$. It can be written as the first of the three analogous equalities

$$f^2 - de = f^2d - de^2 + de^2f - def^2,$$

$$d^2 - ef = d^2e - ef^2 + def^2 - d^2ef,$$

$$e^2 - fd = e^2f - fd^2 + d^2ef - de^2f,$$

where the other two are equivalent to $[BLF] = [CMD]$ and $[CMD] = [ANE]$. If these three equalities are added, we get

$$\tfrac{1}{2}\left[(e - f)^2 + (f - d)^2 + (d - e)^2\right] = (f - e)(d - f)(e - d). \tag{5}$$

With the notations $u = |f - e|$, $v = |d - f|$ and $w = |e - d|$, (5) becomes

$$u^2 + v^2 + w^2 = 2uvw, \tag{6}$$

where $0 \leqslant u, v, w < 1$. If one of the quantities u, v, w is equal to zero, then because of (6), all three must be equal to zero. Because of (6), the

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inequality $\sqrt{\frac{1}{3}(u^2 + v^2 + w^2)} \geqslant \sqrt[3]{uvw}$ of the quadratic and geometric means implies $\sqrt{\frac{2}{3}uvw} \geqslant \sqrt[3]{uvw}$, i.e. $uvw \geqslant \frac{27}{8}$, which is impossible because of $u, v, w < 1$. Therefore, it is necessary to have $u = v = w = 0$. We have proved

Theorem 1: With the points $D(0, d, 1 - d)$, $E(1 - e, 0, e)$ and $F(f, 1 - f, 0)$ on the sides of the triangle ABC, the triangles ANE, BLF and CMD have equal oriented areas if, and only if, $d = e = f$.

Now suppose (as in [1] and [2]) that

$$[ANE] = [BLF] = [CMD] = [LMN].$$

Because of the previous theorem, suppose that $d = e = f = 1 - x$, and then $d' = e' = f' = x$. So from (3) and (4) we obtain:

$$(1 - x + x^2)^3 \frac{1}{\Delta}[LMN] = [(1 - x)^3 - x^3]^2 = (1 - 2x)^2(1 - x + x^2)^2,$$

$$(1 - x + x^2)\frac{1}{\Delta}[ANE] = (1 - x)^3,$$

and the equality $[ANE] = [LMN]$ is equivalent to $(1 - x)^3 = (1 - 2x)^2$, i.e. to $x(x^2 + x - 1) = 0$. Because of the condition $0 < x < 1$, it follows finally that $x = \frac{1}{2}(\sqrt{5} - 1)$. It means that, for example, D is $\left(0, \frac{1}{2}(3 - \sqrt{5}), \frac{1}{2}(\sqrt{5} - 1)\right)$, and it is easy to see that this is equivalent to $\overrightarrow{DC} = \frac{1}{2}(\sqrt{5} - 1)\overrightarrow{BD}$. Similar reasoning holds for the points E and F. Let us note that $1 - x = x^2$, and because of $e = f = 1 - x$, $e' = f' = x$, there follows $e'f = (1 - x)x = x^3$, $e'f' = x^2$, $ef = (1 - x)^2 = x^4$, and

$$1 - e'f = 1 - x(1 - x) = 1 - x + x^2 = 2x^2.$$

Therefore, from (1) we get $L\left(\frac{1}{2}x, \frac{1}{2}, \frac{1}{2}x^2\right)$, and besides that, E is $(x, 0, x^2)$. The point L is therefore the midpoint of BE. The points M and N are also the midpoints of CF and AD. All these facts are seen in Figure 1.

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