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Ball-avoiding theorems

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Dedicated to Jacob Palis on the occasion of his 60th birthday

Abstract. Consider a nice hyperbolic dynamical system (singularities not excluded). Statements about the topological smallness of the subset of orbits, which avoid an open subset of the phase space (for every moment of time, or just for a not too small subset of times), play a key role in showing hyperbolicity or ergodicity of semi-dispersive billiards, especially, of hard-ball systems. As well as surveying the characteristic results, called ball-avoiding theorems, and giving an idea of the methods of their proofs, their applications are also illustrated. Furthermore, we also discuss analogous questions (which had arisen, for instance, in number theory), when the Hausdorff dimension is taken instead of the topological one. The answers strongly depend on the notion of dimension which is used. Finally, ball-avoiding subsets are naturally related to repellers extensively studied by physicists. For the interested reader we also sketch some analytical and rigorous results about repellers and escape times.

1. *Introduction*

The seminal work of Chernov and Sinai [**SCh(1987)**] not only established the K-property of dispersive billiards in the general, multidimensional case, but—through their theorem on local ergodicity for semi-dispersive billiards —also opened the possibility of showing the K-property of semi-dispersive billiards. Indeed, in 1989, by using this fundamental tool, Krámli *et al* [**KSSz(1989)**] first showed the K-property of a billiard, which was semi-dispersive but not dispersive. Our method, which has been further developed in a series of works (for a survey of the results see [**Sz(1996)**]), consists of three essential parts using dynamical–topological, geometric–algebraic, and, finally, dynamical–measure-theoretic tools. The dynamical–topological methods of these proofs are distilled in so-called *ball-avoiding theorems*, whose content we are going to formulate here.

Assume $(\mathbf{M}, \mathcal{F}, S^{\mathbb{R}_+}, \mu)$ is a semigroup of endomorphisms (or $(\mathbf{M}, \mathcal{F}, S^{\mathbb{R}}, \mu)$ is a group of automorphisms) of a probability space $(\mathbf{M}, \mathcal{F}, \mu)$. For formulating topological statements, we will, in general, assume that $\mathbf{M}$ is a Riemannian manifold with or without

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boundary. Most of our methods will use some hyperbolicity and/or mixing properties of the dynamics involved. Fix an arbitrary subset H of $\mathbb{R}_+$ (or of $\mathbb{R}$) and a subset $B \subset \mathbf{M}$. For B and H given in this way, the ball-avoiding subset $A_H(B) \subset \mathbf{M}$ is defined as follows:

$$A_H(B) = \{x \in \mathbf{M} : \ S^H x \cap B = \emptyset\}.$$

In words, it consists of phase points whose orbits avoid the subset B in prescribed moments of time (B, in general, need not be a ball, but often it is, and the term ball-avoiding already has traditional usage). If B is not too small, e.g. it is open, then $A_H(B)$, as a collection of non-typical trajectories, is expected to be small. *Ball-avoiding theorems* claim that, by assuming that B is not very small and H is unbounded (or semi-unbounded, at least), $A_H(B)$ is small in a well-defined sense (i.e. its topological codimension is at least one or two, and, moreover, $\mu\{A_H(B)\} = 0$), under weaker or stronger assumptions on the hyperbolic and/or ergodic behaviour of the dynamics. It is worth stressing that, although some general results have only been formulated for semi-dispersive billiards, their validity is wider: they are true for a class of 'hyperbolic' systems with singularities possessing a smooth invariant probability measure.

If $\mathbf{M}$ is a separable and metrizable space, then let $\{B_i : i = 1, 2, \dots\}$ be a basis of the topology in $\mathbf{M}$ (then each $A_H(B_i)$ is closed provided that the group $S^{\mathbb{R}}$ is continuous). Denote

$$ND := \{x \in \mathbf{M} : S^{\mathbb{R}} x \text{ is not everywhere dense in } \mathbf{M}\}. \tag{1.1}$$

Plainly, $ND = \cup_i (A_{\mathbb{R}_-}(B_i) \cap A_{\mathbb{R}_+}(B_i))$. If one shows that each $A_{\mathbb{R}_-}(B_i) \cap A_{\mathbb{R}_+}(B_i)$ is a zero-measure subset of codimension two, then ND will necessarily be *slim* (for the definition see §3), i.e. topologically small.

Although the question ball-avoiding theorems answer is natural, in this form they seem to have not been treated before [**KSSz(1989)**]. In the particular case $H = \mathbb{R}$ the set $A_{\mathbb{R}}(B)$ is an invariant subset. These sets were used by Smale (cf. [**H(1970)**]) and later by others to analyse possible dimensions of compact, proper invariant subsets of a hyperbolic diffeomorphism. The difference between their treatment of the problem and between ours reflects the very difference between smooth (Anosov) systems and those with singularities. On the other hand, there is a very active and interesting direction of research investigating, in particular, the same subsets ND from a different point of view. These results generalize a classical theorem of Jarnik [**J(1929)**] and of Besicovitch [**B(1934)**] claiming that the set of badly approximable (or Diophantine) numbers in the interval $[0, 1]$ has Hausdorff dimension one. The typical result then claims that the *Hausdorff dimension* of the subset ND is maximal, i.e. agrees with $\dim \mathbf{M}$. In other words, despite the fact that these orbits are non-typical, nevertheless the Hausdorff dimension does not sense this atypicality.

Also, in the last few years physicists have become interested in open systems, e.g. in open billiards, which actually live on a ball-avoiding subset of the phase space of a closed billiard. As a consequence, these systems have also been investigated from the mathematical point of view. Since the interest of their authors was different from ours (cf. [**ChMT(2000)**]) we will be satisfied only to give a brief account of their main characteristic results.

This work is partitioned into three parts. In the first, consisting of §§2–5, the simplest ball-avoiding theorems are presented: a weak one in §2 and, after a brief summary of some

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useful notions from topological dimension theory given in §3, and some strong ones in §§4 and 5. In the second part, consisting of §§6–9, first the relevance of ball-avoiding theorems for hard ball systems is explained. Then various forms of them are surveyed. Our additional aim is to present the different methods used in their proofs, or at least to hint at them, and to also collect the most interesting open problems. Finally, in the third part some related directions mentioned above are reviewed.

I. Weak and strong ball-avoiding theorems

2. *An (abstract) weak ball-avoiding lemma*

Let $(\mathbf{M}, \mathcal{F}, S^{\mathbb{R}_+}, \mu)$ be a semigroup of endomorphisms of a probability space $(\mathbf{M}, \mathcal{F}, \mu)$. Fix an arbitrary subset H of $\mathbb{R}_+$ satisfying $\sup H = +\infty$.

LEMMA 2.1. [**KSSz(1989)**] *If the semigroup $\mathcal{S}^{\mathbb{R}_+}$ is mixing, then, for any $B \in \mathcal{F}$ with $\mu\{B\} > 0$, one has*

$$\mu\{A_H(B)\} = 0.$$

Since the proof is extremely simple, it will be presented below.

Proof. Denote

$$A_H^\tau(B) := \{x \in \mathbf{M} : \; S^{H\cap[0,\tau]}x \cap B = \emptyset\}.$$

Then, on the one hand,

$$\mu\{A_H^\tau(B)\} \searrow \mu\{A_H(B)\} \tag{2.2}$$

if $\tau \to \infty$. On the other hand, for every $t \in H$, we have

$$\mu\{A_H^\tau(B) \cap \{S^t x \notin B\}\} \geq \mu\{A_H(B)\} \tag{2.3}$$

Then, by mixing and (2.2), (2.3) leads to

$$\lim_{\tau\to\infty} \lim_{t\to\infty, t\in H} \mu\{A_H^\tau(B) \cap \{S^t x \notin B\}\} = \mu\{A_H(B)\}\mu\{B^c\} \geq \mu\{A_H(B)\}$$

implying $\mu\{A_H(B)\} = 0$ for $\mu\{B\} > 0$. □

Remark. Any irrational rotation of $\mathbb{R}/\mathbb{Z}$ serves as an example of an ergodic automorphism for which the claim of the lemma is not valid. Different is the situation if $H = \mathbb{R}$, since then ergodicity is, of course, sufficient to imply $\mu\{A_H(B)\} = 0$.

Remark. The proof of the lemma immediately implies that its analogue for discrete time semigroups $T^{\mathbb{Z}_+}$ is also true.

3. *Simple facts from topological dimension theory*

Here we briefly summarize some necessary notions and facts from topological dimension theory (for details see [**E(1978)**] or [**HW(1941)**]).

Assume first, in general, that X is a separable metric space. We will denote by $\dim X$ the small inductive topological dimension of X whose recursive definition will just be recovered.

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Definition 3.1.

(i) $\dim X = -1$ if and only if $X = \emptyset$;

(ii) $\dim X \le n$ if and only if there exists a basis $\mathcal{U}$ of open neighbourhoods for X such that for every $U \in \mathcal{U}$ one has $\dim \partial U \le n - 1$ ($n = 0, 1, 2, \dots$);

(iii) $\dim X = n$ if and only if $\dim X \le n$ and it is not true that $\dim X \le n - 1$.

Definition 3.2. If $A \subset X$, and for some natural number k one has $\dim A \le \dim X - k$, then we say that the topological codimension of A in X is at least k (or often we briefly say that the topological codimension is k).

From now on we assume that $\mathbf{M}$ is a connected, smooth manifold (boundary permitted) and μ is a smooth measure on $\mathbf{M}$.

PROPOSITION 3.3. *For any* $A \subset \mathbf{M}$, $\dim A \le \dim \mathbf{M} - 1$ *(in other words, the topological codimension of* A *in* $\mathbf{M}$ *is at least 1) if and only if* $\operatorname{int} A = \emptyset$.

PROPOSITION 3.4. *If* $F \subset \mathbf{M}$ *is closed, then the following statements are equivalent:*

(i) $\operatorname{codim}_{\mathbf{M}} F \ge 2$*;*

(ii) $F \ne \mathbf{M}$ *and, for every open connected set* $G \subset \mathbf{M}$*, the difference set* $G \setminus F$ *is also connected;*

(iii) $\operatorname{int} F = \emptyset$ *and for every point* $x \in \mathbf{M}$ *and for any neighbourhood* V *of* x *in* $\mathbf{M}$ *there exists a smaller neighborhood* $W \subset V$ *of the point* x *such that, for every pair of points* $y, z \in W \setminus F$*, there is a continuous curve* γ *in the set* $V \setminus F$ *connecting the points* y *and* z*.*

For the main applications of strong ball-avoiding theorems we need another concept of topological smallness closely related to being of codimension two (this will be clear from the content of §5).

Definition 3.5. [**KSSz(1989)**] We say that $A \subset \mathbf{M}$ is a *slim* subset if and only if it is the subset of an F_σ zero-set of codimension at least two (A is a zero-set if $\mu\{A\} = 0$).

By their definition, slim subsets of $\mathbf{M}$ form a σ-ideal. The key property of slim subsets is expressed by the following

PROPOSITION 3.6. [**KSSz(1989)**] *If* $\mathbf{M}$ *is connected, and* A *is slim, then* $\mathbf{M} \setminus A$ *contains an arcwise connected* G_δ*-set of full measure.*

In applications, in particular in the inductive arguments, the following integrability property of codimension two subsets is often very useful.

PROPOSITION 3.7. [**KSSz(1989)**] *If* $\mathbf{M} = \mathbf{N}_1 \times \mathbf{N}_2$*, where* $\mathbf{N}_1$ *and* $\mathbf{N}_2$ *are connected smooth manifolds, and* $F \subset \mathbf{M}$ *is a closed subset such that, for every* $w \in \mathbf{N}_1$*, the (closed) section* $F_w := \{p \in \mathbf{N}_2 : (w, p) \in F\}$ *obeys*

$$\operatorname{codim}_{\mathbf{N}_2} F_w \ge 2,$$

then

$$\operatorname{codim}_{\mathbf{M}} F \ge 2.$$

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4. *The Smale–Williams theorem for Anosov diffeomorphisms*

Assume $\mathbf{M}$ is a smooth Riemannian manifold and $T : \mathbf{M} \to \mathbf{M}$ is an Anosov C^1-diffeomorphism. Smale and Williams (see [**H(1970)**]) proved the following nice theorem.

THEOREM 4.1. *Assume that the set of periodic points of T is dense in $\mathbf{M}$. If F is a compact invariant subset of $\mathbf{M}$ satisfying* $\operatorname{codim}_{\mathbf{M}} F \geq 1$, *then* $\operatorname{codim}_{\mathbf{M}} F \geq 2$.

The combination of Theorem 4.1 with our weak Lemma 2.1 provides a strong ball-avoiding statement for smooth systems.

Corollary. For any $B \neq \emptyset \subset \mathbf{M}$ open, $A_{\mathbb{Z}}(B)$ is a closed set of topological codimension of at least two.

Proof of the corollary. Topological transitivity, the invariance of $A_{\mathbb{Z}}(B)$, and the openness of B imply that $\operatorname{int} A_{\mathbb{Z}}(B) \neq \emptyset$. Proposition 3.3 then proves the claim. □

Proof of Theorem 4.1. Throughout the whole paper we will denote by $\{\gamma^s\}$ and $\{\gamma^u\}$ the invariant foliations defined by the dynamics in question, and by $\gamma^s_\varepsilon(x)$ and $\gamma^u_\varepsilon(x)$ the local invariant manifolds of size ε through the point x.

Denote by $\mathcal{P}$ the set of periodic points of $\mathbf{M} \setminus F$. We use the following simple statements.

CLAIM 1. *$\mathcal{P}$ is dense in $\mathbf{M}$.*

CLAIM 2. *If $x \in \mathcal{P}$, then $\gamma^u(x) \cap F = \gamma^s(x) \cap F = \emptyset$.*

These claims easily provide the truth of the theorem. Indeed, let $y \in F$ and choose $\varepsilon > 0$ small. The foliations $\{\gamma^u\}$, $\{\gamma^s\}$ define a local product structure and using it we can consider a parallelogram $\gamma^u_\varepsilon(y) \times \gamma^s_\varepsilon(y)$. Moreover, we define $F_0 = F \cap (\gamma^u_\varepsilon(y) \times \gamma^s_\varepsilon(y))$. By Claim 2, for any $x \in \mathcal{P}$, $(\gamma^u(x) \cup \gamma^s(x)) \cap F = \emptyset$, and, consequently, $F_0 \subset (\gamma^u_\varepsilon(y) \setminus \cup_{x\in\mathcal{P}} \gamma^s(x)) \times (\gamma^s_\varepsilon(y) \setminus \cup_{x\in\mathcal{P}} \gamma^u(x))$.

Claim 1 and Proposition 3.3 then say that the factors of the previous product set each have codimension at least one. Hence the Theorem follows by the product theorem (cf. Theorem III.4 of [**HW(1941)**]).

Let us now prove Claim 1. Take an arbitrary open subset G of $\mathbf{M}$. The open set $G \setminus F$ is not empty for otherwise we would have $\dim F = n$. Since $\mathcal{P}$ was dense in $\mathbf{M} \setminus F$, we also have $(G \setminus F) \cap \mathcal{P} \neq \emptyset$.

Turn next to Claim 2. We prove $\gamma^s(x) \cap F = \emptyset$ for an arbitrary $x \in \mathcal{P}$. Assume $T^p x = x$. Select an open neighbourhood G of x disjoint from F. By invariance, $(\cup_{n\in\mathbb{Z}} T^n G) \cap F = \emptyset$. Now for any $y \in \gamma^s(x)$, $\rho(T^{kp} y, x) \to 0$ if $k \to \infty$, and thus, for k sufficiently large, $T^{kp} y \in G$ implying $y \notin F$. □

Remark 4.2. After the aforementioned result, the study of compact invariant subsets was continued, among others by Franks [**F(1977)**], Hancock [**H(1978)**], and Mañé [**M(1978)**]. Since the sets $A_{\mathbb{R}}(B)$ provide natural examples of compact, invariant subsets—in fact, all compact invariant subsets are of this form—this description has been used by several authors. In particular, for every $0 \leq k \leq d - 2$, Przytycki [**P(1980)**] found examples of sets B_k such that $\dim A_{\mathbb{R}}(B_k) = k$.

5. *A strong ball-avoiding theorem for hyperbolic systems (with or without singularities)*
For simplicity, we formulate the theorem for discrete time groups $(\mathbf{M}, \mathcal{F}, T^{\mathbb{Z}}, \mu)$ of hyperbolic systems since the generalization to continuous time is straightforward. Our set-up is that $\mathbf{M}$ is a compact C^∞-manifold and μ is a smooth, invariant probability measure. We also want to permit singularities as is done in [**LW(1995)**], in [**Y(1998)**], or in [**Ch(1998)**]. To save space, we do not list the conditions formulated in these works since we only use some standard consequences of them. Namely, the μ-a.e. existence of the local invariant manifolds and the absolute continuity of the canonical isomorphism between them, and further the simple fact that if in the case of singularities we define trajectory branches as is described, for instance, in [**KSSz(1992)**] or in [**Sim(1992)**], then the dynamics can always be considered continuous on these trajectory branches. On the other hand, the kind of hyperbolicity needed will be implicitly ensured by our assumptions. Start with the corresponding definition.

Definition 5.1. A point $x \in \mathbf{M}$ is called *a zigzag point* if one can find arbitrary small open neighbourhoods U of x such that for every zero-set $A \subset \mathbf{M}$ there exists another zero-set $A' \supset A$ with the following property: for every $y, y' \in U \setminus A'$ there exists a chain (also called a Hopf-chain)

$$\gamma^u_{\text{loc}}(z_0), \gamma^s_{\text{loc}}(z_1), \gamma^u_{\text{loc}}(z_1), \gamma^s_{\text{loc}}(z_2), \ldots, \gamma^u_{\text{loc}}(z_{n-1}), \gamma^s_{\text{loc}}(z_n)$$

(here $z_0 = y, z_n = y'$) of local unstable and stable invariant manifolds inside U such that each intersection

$$\gamma^u_{\text{loc}}(z_i) \cap \gamma^s_{\text{loc}}(z_{i+1}) \quad (i = 0, \ldots, n-1)$$

and

$$\gamma^s_{\text{loc}}(z_i) \cap \gamma^u_{\text{loc}}(z_i) = \{z_i\} \quad (i = 1, \ldots, n-1)$$

consists of exactly one point belonging to $U \setminus A'$.

The following theorem generalizes Lemma 4.3 of [**KSSz(1989)**], and its proof is also based on their ideas.

THEOREM 5.2. *Assume that:*
(i) *the group* $(\mathbf{M}, \mathcal{F}, T^{\mathbb{Z}}, \mu)$ *is mixing;*
(ii) *for the subset* Z *of zigzag points of* $\mathbf{M}$, $\mathbf{M} \setminus Z$ *is slim;*
(iii) $B \neq \emptyset$ $(\subset \mathbf{M})$ *is open; and*
(iv) H $(\subset \mathbb{Z})$ *satisfies* $\sup H = -\inf H = \infty$.
Then $A_H(B) (\subset \mathbf{M})$ *is a closed zero-set of codimension at least two.*

Remark 5.3. For the first glance, condition (ii) of Theorem 5.2, as formulated, might seem too restrictive, but, fortunately, this is not the case. In the case of hyperbolic systems with singularities, with billiards included (cf. [**LW(1995)**]), the singularities, to be denoted by $\mathcal{S}$ (in other words, the set of points where T or T^{-1} is not smooth), form one-codimensional submanifolds of the phase space. Let us denote

$$\Delta_n := \bigcup_{-n \leq k < l \leq n} (T^k \mathcal{S} \cap T^l \mathcal{S})$$

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and, furthermore,

$$\mathbf{M}^* := \mathbf{M} \Big\backslash \bigcup_{n=1}^{\infty} \Delta_n,$$

and

$$\mathbf{M}^0 := \mathbf{M} \Big\backslash \bigcup_{k=-\infty}^{\infty} T^k \mathcal{S}.$$

Then, as one can see, for instance, in [**KSSz(1990)**], for our main model of semi-dispersive billiards, in general, the zigzag property holds not only for sufficient points of $\mathbf{M}_0$, but also for those of $\mathbf{M}^*$. Analogously, in the interesting cases, $\mathbf{M}\backslash\mathbf{M}^*$ has codimension at least two.

Proof. (1) By definition, $A_H(B)$ is closed once B is open.

(2) Denote the inner radius of B by r and choose a ball $\tilde{B} \subset B$ such that $d(\tilde{B}, B^c) \geq r/2$. Define

$$D := \{x \in \mathbf{M} : \rho^{u,s}(x) > 0 \text{ and } \inf\{n \in H : T^n x \in \tilde{B}\} = -\infty \\ \text{and } \sup\{n \in H : T^n x \in \tilde{B}\} = \infty\}.$$

Here $\rho^{u,s}(x)$ denotes the inner radius of the local unstable (stable) invariant manifold $\gamma^u(x)$ $(\gamma^s(x))$ through x. By (ii) and Lemma 2.1 (this presupposes (i)) we have $\mu\{D\} = 1$.

(3) Since, by (ii), non-zigzag points make a slim subset, by Lindelöf's theorem, it is sufficient to check that every zigzag point z has a neighbourhood $U = U(z)$ such that $A_H(B) \cap U$ is slim. This is what we do. Fix z and its neighbourhood according to Definition 5.1 in such a way that $\operatorname{diam} U < r/2$. To $A = U \setminus D$ select $\tilde{A} \supset A$ according to the same definition. We claim that every pair of points $y, y' \in U \setminus \tilde{A}$ can be connected by a curve belonging to $U \setminus A_H(B)$. Since $A_H(B)$ is closed, both y and y' have neighbourhoods in U disjoint of $A_H(B)$. Also, since $U \setminus \tilde{A}$ is dense in U, we can choose $\tilde{y}$ and $\tilde{y}'$ $(\in U \setminus \tilde{A})$ in these neighbourhoods and connect y with $\tilde{y}$ and analogously y' with $\tilde{y}'$ inside these tiny neighbourhoods not intersecting $A_H(B)$.

Connect now $\tilde{y}$ and $\tilde{y}'$ with a Hopf-chain ensured by Definition 5.1. Since $\operatorname{diam} U < r/2$, we know that the outer diameters of all local manifolds figuring in the chain are less than $r/2$. Observe that the property that the intersection points w belong to $U \setminus \tilde{A}$ ensures that they belong to D. This implies that for infinitely many $n \in H \cap \mathbb{Z}_+$ one has $T^n w \in \tilde{B}$. Then for n large enough, $T^n \gamma^s(w) \subset B$ holds, too, implying that $\gamma^s(w) \cap A_H(B) = \emptyset$. Analogously, for the unstable local manifolds figuring in the chain we have $\gamma^u(w) \cap A_H(B) = \emptyset$ and thus the desired connection between $\tilde{y}$ and $\tilde{y}'$ is, indeed, constructed. □

Remark. Compare Theorem 5.2 with the corollary of the Smale–Williams Theorem 4.1. Instead of requiring the density of periodic points we have a smooth, invariant and mixing measure. Furthermore, we also permit singular systems, and our assumption on H is much weaker, for it can even have zero density.

An immediate consequence is the following.

COROLLARY. *Assume that* $(\mathbf{M}, \mathcal{F}, S^{\mathbb{R}}, \mu)$ *is a group of automorphisms satisfying the conditions of Theorem 5.2 suitably modified to the continuous time case (in particular,*

$H \subset \mathbb{R}$, and otherwise satisfies the same assumptions). Then $A_H(B)$ is a closed zero-set of codimension at least two.

By copying the proof given above one finds a simple generalization of Theorem 5.2, which will be applied in §6. Namely, let B_-, B_+ ($\subset \mathbf{M}$), and define

$$A_H(B_-, B_+) := \{x \in \mathbf{M} : S^{H_-}x \cap B_- = S^{H_+}x \cap B_+ = \emptyset\}$$

where $H_- := H \cap \mathbb{R}_-$ and $H_+ := H \cap \mathbb{R}_+$.

THEOREM 5.4. *Assume that besides (i), (ii) and (iv) of Theorem 5.2, the following condition is satisfied:*
(iii)* *$B_- \neq \emptyset$ and $B_+ \neq \emptyset$ ($\subset \mathbf{M}$) are open.*
Then $A_H(B_-, B_+)$ is a closed zero-set of codimension at least two.

Proof. This is the same as that of Theorem 5.2 with the natural modification that now we select $\tilde{B}_- \subset B_-$ and $\tilde{B}_+ \subset B_+$ in such a way that $d(\tilde{B}_-, B_-) \geq r/2$ and $d(\tilde{B}_+, B_+) \geq r/2$, and define

$$D := \{x \in \mathbf{M} : \rho^{u,s}(x) > 0 \text{ and } \inf\{n \in H : T^n x \in \tilde{B}_-\} = -\infty \text{ and } \sup\{n \in H : T^n x \in \tilde{B}_+\} = \infty\}. \quad \square$$

II. Ball-avoiding theorems and hyperbolic properties

6. *Hyperbolic and ergodic properties of hard-ball systems*

6.1. *Isomorphy to semi-dispersive billiards.* The main aim of this section is to provide a motivation and explanation of how ball-avoiding theorems enter into proofs of hyperbolicity and ergodicity of hard-ball systems or, more generally, of semi-dispersive billiards. Consequently, in our exposition the details are surrendered to this goal.

Let us assume, in general, that a system of N (≥ 2) balls of unit mass and of radii $r > 0$ are given on $\mathbf{T}^\nu$, the ν-dimensional unit torus ($\nu \geq 2$). (The assumption that the masses and the radii are identical is not an essential restriction for our purposes.) Denote the phase point of the ith ball by $(q_i, v_i) \in \mathbf{T}^\nu \times \mathbb{R}^\nu$. The configuration space $\tilde{\mathbf{Q}}$ of the N balls is a subset of $\mathbf{T}^{N\cdot\nu}$: from $\mathbf{T}^{N\cdot\nu}$ we cut out $\binom{N}{2}$ cylindric scatterers

$$\tilde{C}_{i,j} = \{Q = (q_1, \dots, q_N) \in \mathbf{T}^{N\cdot\nu} : \|q_i - q_j\| < 2r\}, \tag{6.1}$$

$1 \leq i < j \leq N$. The energy $E = \frac{1}{2}\sum_1^N v_i^2$ and the total momentum $P = \sum_1^N v_i$ are first integrals of the motion. Thus, without loss of generality, we can assume that $E = \frac{1}{2}$ and $P = 0$ and, moreover, that the sum of spatial components $B = \sum_1^N q_i$ is equal to zero (if $P \neq 0$, then the centre of mass has an additional conditionally periodic or periodic motion). For these values of E, P and B, the phase space of the system reduces to $\mathbf{M} := \mathbf{Q} \times S^{d-1}$ where

$$\mathbf{Q} := \left\{Q \in \tilde{\mathbf{Q}} \Big\backslash \bigcup_{1\leq i<j\leq N} \tilde{C}_{i,j} : \sum_1^N q_i = 0\right\}$$

and $d := \dim \mathbf{Q} = N \cdot \nu - \nu$ (here S^k denotes, in general, the k-dimensional unit sphere). It is easy to see the following.

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PROPOSITION 6.2. *The dynamics of the N balls, determined by their uniform motion with elastic collisions, on the one hand, and the billiard flow $\{S^t : t \in \mathbb{R}\}$ in $\mathbf{Q}$ with specular reflections at $\partial\mathbf{Q}$, on the other hand, are isomorphic and they conserve the Liouville measure $d\mu = \text{constant} \cdot dq \cdot dv$. (Thus both dynamics can be denoted by $(\mathbf{M}, S^{\mathbb{R}}, d\mu)$.)*

We recall that a *billiard* is a dynamical system describing the motion of a point particle in a connected, compact domain $\mathbf{Q} \subset \mathbb{R}^d$ or $\mathbf{Q} \subset \mathbb{T}^d = \text{Tor}^d$, $d \geq 2$ with a piecewise C^2-smooth boundary. As usual, the phase space $\mathbf{M}$ of the system is identified with the unit tangent bundle over $\mathbf{Q}$. In other words, the configuration space is $\mathbf{Q}$ while the phase space is $\mathbf{M} = \mathbf{Q} \times S^{d-1}$. The natural projections $\pi : \mathbf{M} \to \mathbf{Q}$ and $p : \mathbf{M} \to \mathcal{S}^{d-1}$ are defined by $\pi(q, v) = q$ and by $p(q, v) = v$, respectively. The billiard dynamical system $(\mathbf{M}, S^{\mathbb{R}}, \mu)$, where μ is the Liouville measure, is called *the standard billiard flow*. If it is isomorphic to a hard ball system in the sense of Proposition 6.2, then it is called *the standard hard ball flow* or *standard billiard ball flow*.

Suppose that $\partial\mathbf{Q} = \cup_1^k \partial\mathbf{Q}_i$ where $\partial\mathbf{Q}_i$ are the smooth components of the boundary. Denote $\partial\mathbf{M} = \partial\mathbf{Q} \times S^{d-1}$ and let $n(q)$ be the unit normal vector of the boundary component $\partial\mathbf{Q}_i$ at $q \in \partial\mathbf{Q}_i$ directed inwards $\mathbf{Q}$. (In billiards, isomorphic to hard ball systems, the scatterers are convex cylinders if $N \geq 3$, and are (strictly convex) balls if $N = 2$.)

Definition 6.3. We say that a billiard is *dispersive* if each $\partial\mathbf{Q}_i$ is strictly convex, and we say it is *semi-dispersive* if each $\partial\mathbf{Q}_i$ is convex.

6.2. *Local ergodicity of semi-dispersive billiards.* Our next aim is to introduce the notion of *sufficiency*, which is basic to the study of semi-dispersive billiards. Assume that $S^{[a,b]}x$ is a finite trajectory segment of a semi-dispersive billiard, which is regular, i.e. it avoids singularities. Let $S^a x = (Q, V) \in M$ and consider the hyperplanar wavefront $\tilde{\Gamma}(S^a x) := \{(Q + dQ, V) : dQ \text{ small} \in \mathbb{R}^d \text{ and } \langle dQ, V\rangle = 0\}$ (indeed, $\pi(\tilde{\Gamma})$ is part of a hyperplane).

Definition 6.4. [**SCh(1987)**] We say that the trajectory segment $S^{[a,b]}x$ is *sufficient* if $\pi(S^b\tilde{\Gamma})$ is strictly convex at $S^b x$. A phase point $x \in \mathbf{M}$ is sufficient if its entire trajectory is sufficient (i.e. it contains a sufficient trajectory segment).

We note that, for semi-dispersive billiards, the tangent vectors of convex orthogonal manifolds (cf. [**KSSz(1990)**]) form an invariant cone field in the tangent space of $\mathbf{M}$ in the sense of [**W(1985)**]. Then the sufficiency of an $x \in \mathbf{M}$ is equivalent to saying that the cone field along the orbit $S^{\mathbb{R}}x$ is eventually strictly invariant in the sense of [**W(1985)**]. Simple geometric considerations (cf. [**KSSz(1990)**]) show that a sufficient trajectory segment generates an expansion rate uniformly larger than one in some neighbourhood of the point $S^a x$.

By using Poincaré recurrence and the ergodic theorem, it is easy to prove the following.

LEMMA 6.5. [**SCh(1987)**] *If $x \in \mathbf{M}$ is sufficient, then there exists an open neighbourhood $U \subset \mathbf{M}$ of x such that the relevant Lyapunov exponents of the system are not zero μ-almost everywhere in U. (In the case of singular orbits, we only consider neighbourhoods in the phase spaces of the corresponding trajectory branches.)*

In other words, in this neighbourhood, the system is hyperbolic. A very deep and delicate result is as follows.

THEOREM 6.6. (Local ergodicity of semi-dispersive billiards [**SCh(1987)**]) *Assume that a semi-dispersive billiard satisfies some geometric conditions and the Chernov–Sinai ansatz, a condition strongly connected with the singularities of the system (the conditions are formulated in detail in* [**KSSz(1990)**]*; also, for a generalization to hyperbolic symplectomorphisms with singularities, see* [**LW(1995)**]*).*

If $x \in M^*$ is a sufficient point, then it has an open neighbourhood U, which belongs to one ergodic component.

If almost every phase point of a semi-dispersive billiard is sufficient, then, of course, it may have at most a countable number of ergodic components. In some cases it is not hard then to derive the *global ergodicity* of the system, i.e. to show that there is just one ergodic component in the phase space. A very important consequence is thus the following.

COROLLARY. [**SCh(1987)**] *Every dispersive billiard is ergodic, and, moreover, is a K-flow. In particular, the system of $N = 2$ balls on the ν-torus is a K-flow if $r < \frac{1}{4}$.*

6.3. *Richness of a symbolic collision sequence.* Consider a semi-dispersive billiard. $\mathbf{M}^*$ will denote the set of phase points whose orbits contain no more than one singular collision, and $\mathbf{M}^{\emptyset}$ the set of phase points with no collision at all. $\mathbf{M}^0 \subset \mathbf{M}^* \setminus \mathbf{M}^{\emptyset}$ will be the subset of regular phase points, and finally we set $\mathbf{M}^1 := \mathbf{M}^* \setminus (\mathbf{M}^0 \cup \mathbf{M}^{\emptyset})$. Moreover, $\mathcal{SR}^+ \subset \partial\mathbf{M}$ will denote the collection of all phase points $x \in \partial\mathbf{M}$ for which the reflection, occurring at x, is singular (tangential or multiple) and, in the case of a multiple collision, x is supplied with the *outgoing* velocity V^+. We remind the reader that a trajectory segment $S^{[a,b]}x$ is called *regular* (or non-singular) if it does not hit singularities ($S^{[a,b]}x \cap \mathcal{SR}^+ = \emptyset$).

Definition 6.7. Consider a non-singular trajectory segment $S^{[a,b]}x$, $-\infty < a < b < \infty$, $x \in \mathbf{M}$. Assume that during the interval $[a, b]$ the orbit hits the boundary $\partial\mathbf{Q}$ in times $a \le t_1, \dots, t_n \le b$ (i.e. for $\forall i : 1 \le i \le n$ $S^{t_i}x \in \partial\mathbf{Q}_{j(i)}$, and if $t \ne t_i$ $(1 \le i \le n)$ but $t \in [a, b]$, then $S^t x \notin \partial\mathbf{Q}$). Then the *symbolic collision sequence* $\Sigma = (\sigma_1, \dots, \sigma_n)$ of the orbit segment is $(j(1), \dots j(n))$. (If the trajectory hits one or several singularities, then, of course, there is a finite number of such sequences since every trajectory branch has its own symbolic collision sequence.)

In applications one usually defines a combinatorial property, called *richness*, for symbolic sequences of orbit segments. The usefulness of such a notion will be clear from Key Lemmas 6.9 and 6.10 and Theorems 6.12 and 6.16, valid for hard-ball systems, where the definition of richness is actually very clear and simple.

Since, as said above, hard-ball systems are isomorphic to billiards where the scatterers are the cylinders (6.1), the symbolic collision sequence of an orbit is, in this case, a sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ of 'colliding pairs', i.e. $\sigma_k = \{i_k, j_k\}$ whenever $Q(t_k) = \pi(S^{t_k}x) \in \partial\tilde{C}_{i_k,j_k}$. The sequence $\Sigma := \Sigma(S^{[a,b]}x) := (\sigma_1, \sigma_2, \dots, \sigma_n)$ is called the *symbolic collision sequence* of the trajectory segment $S^{[a,b]}x$.

Definition 6.8. [**SSz(1995)**] We say that the symbolic collision sequence $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is *connected* if the collision graph of this sequence,

$$\mathcal{G}_\Sigma := (\mathcal{V} = \{1, 2, \ldots, N\}, \mathcal{E}_\Sigma := \{\{i_k, j_k\} : \text{ where } \sigma_k = \{i_k, j_k\}, 1 \le k \le n\}),$$

is connected.

We say that the symbolic collision sequence $\Sigma = (\sigma_1, \ldots, \sigma_n)$ is *C-rich*, with C being a natural number, if it can be decomposed into at least C consecutive, disjoint collision subsequences in such a way that each of them is connected.

6.4. *The role of ball-avoiding theorems in proving hyperbolicity or ergodicity.* The definition of C-richness should be so strong that one could establish the following hypotheses, formulated as 'key lemmas'.

(WEAK) 'KEY LEMMA' 6.9. *Assume that $C \in \mathbb{N}$ is suitably fixed and for a non-singular orbit segment $S^{[a,b]}x$ the symbolic collision sequence $\Sigma(S^{[a,b]}x)$ is C-rich. Then there exists an open neighbourhood U of x and a submanifold $\mathcal{N}$ such that*

(1) *for every $y \in U \setminus \mathcal{N}$ $S^{[a,b]}y$ is sufficient; and*

(2) $\operatorname{codim}_U \mathcal{N} \ge 1$.

(STRONG) 'KEY LEMMA' 6.10. *Assume that $C \in \mathbb{N}$ is suitably fixed and for a non-singular orbit segment $S^{[a,b]}x$ the symbolic collision sequence $\Sigma(S^{[a,b]}x)$ is C-rich. Then there exists an open neighbourhood U of x and a submanifold $\mathcal{N}$ such that:*

(1) *for every $y \in U \setminus \mathcal{N}$, $S^{[a,b]}y$ is sufficient; and*

(2) $\operatorname{codim}_U \mathcal{N} \ge 2$.

An analogous statement also holds for phase points $x \in \mathbf{M}$, where $S^{[a,b]}x$ contains exactly one singularity.

Denote for some $C \in \mathbb{N}$

$$\Pi_C := \{x \in \mathbf{M} : S^{\mathbb{R}_+}x \text{ is not } C\text{-rich}\}. \tag{6.11}$$

Our next theorem shows the role a weak ball-avoiding theorem—actually equation (6.13)—plays in establishing the *hyperbolicity of a hard-ball system.*

THEOREM 6.12. *Assume that for a semi-dispersive billiard, isomorphic to a hard-ball system,*

(1) *the weak 'Key Lemma' 6.9 and*

(2) *the statement*

$$\mu\{\Pi_C\} = 0 \tag{6.13}$$

hold true, where C is the constant from Lemma 6.9.

Then the system is hyperbolic.

A system is said to be *hyperbolic* if all its relevant Lyapunov exponents do not vanish for μ-a.e. phase point.

Proof. By (6.13), for almost every point $x \in \mathbf{M}$, $S^{\mathbb{R}_+}x$ is (non-singular and) C-rich. Now the application of 'Key Lemma' 6.9, Lemma 6.5 and Lindelöf's theorem provide the statement. □

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Let P be a non-trivial, two-class partition of the set $\{1, \ldots, N\}$ of balls and denote

$$F_+ := F_+(P) := \{x \in \mathbf{M} : S^{\mathbb{R}_+} x \text{ is } P\text{-partitioned}\} \tag{6.14}$$

We say that $S^{\mathbb{R}_+} x$ *is* P*-partitioned* if the non-negative semi-trajectory of x does not contain collisions between pairs of balls belonging to different classes of the partition P.

COROLLARY 1. *The claim of Theorem 6.12 holds whenever instead of (6.13) we require, for any non-trivial, two-class partition* P*,*

$$\mu\{F_+\} = 0 \tag{6.15}$$

to be true.

Proof. This follows straightforwardly from the obvious inequality

$$\Pi_C \subset \bigcup_P \bigcup_{t=0}^{\infty} S^{-t} F_+(P) \tag{6.16}$$

where $\cup_P$ runs over all non-trivial two-class partitions P of $\{1, \ldots, N\}$. □

COROLLARY 2. *Under the assumptions of Theorem 6.12, the ergodic components of the system are of positive measure.*

Proof. This is a consequence of Theorem 6.12 and of the Katok–Strelcyn theory (see [**KS(1986)**]). □

Now, complementing Theorem 6.12, we formulate a theorem which illustrates the role a strong ball-avoiding theorem plays when establishing the *ergodicity of a hard-ball system*.

THEOREM 6.17. *Assume that for a semi-dispersive billiard, isomorphic to a hard-ball system, the geometric conditions of Theorem 6.6 hold true. Assume, moreover,*

(1) *the 'Key Lemma' 6.10 is proved; and*

(2) Π_C *is a slim subset where* C *is the constant from 'Key Lemma' 6.10.*

Then the system is ergodic.

Proof. From the geometric conditions it follows that the complement of $\mathbf{M}^*$ is a countable union of codimension-two submanifolds and thus is slim (cf. [**SSz(1995)**]). Consequently, in virtue of condition (2), apart from a slim subset of $\mathbf{M}$, every phase point contains at most one singularity and is C-rich. By applying 'Key Lemma' 6.10, and then Lindelöf's theorem, we obtain that, apart from a slim subset of $\mathbf{M}$, every phase point is sufficient (slim subsets form a σ-ideal!). By referring to Theorem 6.6 and once more to Lindelöf's theorem, the statement of the theorem follows. □

Remark 1. By [**S(1992)**], condition (2) holds whenever $N \geq 3$. In the case $N = 2$, however, the orbits with no collisions at all form a one-codimensional submanifold. Then the argument given above provides that we may have at most a finite number of open ergodic components. To obtain global ergodicity, one can connect these by the dynamics in a straightforward way.

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Remark 2. Among the geometric conditions of Theorem 6.6, an essential one is the Chernov–Sinai ansatz. We note that the methods for settling it use, on the one hand, 'key-lemma'-type statements, and they, on the other hand, are related to ball-avoiding theorems; since the set-up of these methods is more involved and the ideas are much the same as the ones we are discussing, we do not treat them in detail in this paper.

COROLLARY 1. *The statement of Theorem 6.17 holds whenever instead of its condition (2) we require, for any non-trivial, two-class partition P, $\mu\{F_+\} = 0$ and*

$$\operatorname{codim} F_+ \geq 2$$

to be true.

Proof. This follows from Definition 3.5 and the inequality (6.16). □

COROLLARY 2. *Under the assumptions of Theorem 6.17, the system is K-mixing and is, moreover, a B-system.*

The K-property is standard and the B-property is proved in [**ChH(1996)**] and [**OW(1998)**].

7. *Interlude: an instructive example*

Despite its very simplicity the paradigm we are going to study sheds light on two fundamental circumstances:

(1) How do ball-avoiding theorems arise and help in proving hyperbolicity or ergodicity of semi-dispersive billiards?

(2) In which way do their applications possess an inductive character? This also explains the apparent contradiction: on the one hand, ball-avoiding theorems are exploited in proofs of hyperbolicity or ergodicity of the systems, but, on the other hand, the latter properties do occur among the assumptions of most ball-avoiding theorems.

So have a closer look at our example. Its analysis is taken from [**KSSz(1989)**], so consequently our treatment here will be concise. For later use we introduce a very interesting class of semi-dispersive billiards, that of cylindric ones.

In words, *cylindric billiards* are toric billiards where the scatterers are cylinders. In our discussion, the bases of the cylinders will be assumed to be strictly convex, a property which ensures that the scatterers are convex, and thus the arising billiard be semi-dispersive. Because of the simplicity of our model, let us immediately start with a formal definition.

Definition 7.1. (Cylindric billiard) The configuration space of a cylindric billiard is $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \cdots \cup C_k)$, where $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ ($d \geq 2$) is the unit torus. Here the cylindric scatterer C_i ($i = 1, \ldots, k$) is defined as follows.

Let $A_i \subset \mathbb{R}^d$ be a so-called *lattice subspace* of the Euclidean space $\mathbb{R}^d$, which means that the discrete intersection $A_i \cap \mathbb{Z}^d$ has rank $\dim A_i$. In this case the factor $A_i/(A_i \cap \mathbb{Z}^d)$ naturally defines a subtorus of $\mathbb{T}^d$, which will be taken as the generator of the cylinder $C_i \subset \mathbb{T}^d$. Denote by $L_i = A_i^{\perp}$ the orthocomplement of A_i. Under the above conditions, the subspace L_i must also be a lattice subspace. We also assume that $\dim L_i \geq 2$. Let,

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moreover, $D_i \subset L_i$ be a convex, compact domain with a C^2-smooth boundary ∂D_i so that $0 \in \operatorname{int} D_i$. Suppose D_i is strictly convex in the sense that the second fundamental form of its boundary ∂D_i is everywhere positive definite. Furthermore, in order to avoid unnecessary complications, we postulate that the convex domain D_i does not contain any pair of points congruent modulo $\mathbb{Z}^d$. The domain D_i will be taken as the *base* of the cylinder C_i. Finally, suppose that a translation vector $t_i \in \mathbb{R}^d$ is given, which plays an essential role in positioning the cylinder C_i in the ambient torus $\mathbb{T}^d$. Set

$$C_i = \{a + l + t_i \mid a \in A_i, l \in D_i, \}/\mathbb{Z}^d.$$

In order to avoid further unnecessary complications, we also assume that the interior of the configuration space $\mathbf{Q} = \mathbb{T}^d \setminus (C_1 \cup \cdots \cup C_k)$ is connected.

The phase space $\mathbf{M}$ of our billiard will be the unit tangent bundle of $\mathbf{Q}$, i.e. $\mathbf{M} = \mathbf{Q} \times \mathbb{S}^{d-1}$. (Here, as usual, $\mathbb{S}^{d-1}$ is the $(d-1)$-dimensional unit sphere.)

The dynamical system $(\mathbf{M}, S^{\mathbb{R}}, \mu)$ is a *cylindric billiard*.

As we have seen in the first part of §5, hard-ball systems actually belong to the class of cylindric billiards.

The example. Consider a simple cylindric billiard on $\mathbb{T}^3$ with two cylindric scatterers C_1 and C_2, where

$$A_1 := \{(x, y, z) : y = z = 0\}, \quad A_2 := \{(x, y, z) : x = z = 0\},$$
$$D_1 := \{(x, y, z) : x = 0, y^2 + z^2 \le r_1^2\}, \quad D_2 := \{(x, y, z) : y = 0, x^2 + z^2 \le r_2^2\}$$

and r_1 and r_2 are arbitrary. To exclude the possibility of an infinite number of collisions in a finite time interval, we assume that no tangencies occur in the given cylinder configuration.

Our goal here is to demonstrate how the proof of the following theorem can be reduced to the strong ball-avoiding one.

THEOREM 7.2. [**KSSz(1989)**] *The cylindric billiard in* $\mathbf{Q} = \mathbb{T}^3 \setminus (C_1 \cup C_2)$ *defined above is ergodic (and, consequently, a K-mixing flow).*

Proof. The following prerequisites are used:

(i) Lemma 4.15 of [**KSSz(1990)**] claiming that $\mathbf{M} \setminus \mathbf{M}^*$ is a countable union of proper closed submanifolds of codimension two (in our case, the reader can, in fact, directly check the claim);

(ii) the fact that the conditions of the theorem on local ergodicity (Theorem 6.6) are satisfied (as to the non-trivial Chernov–Sinai ansatz see [**KSSz(1989)**]), and Theorem 6.6 itself ensuring that every sufficient point $(Q, V) \in \mathbf{M}^*$ has an open neighbourhood which belongs to one ergodic component;

(iii) and, finally, the fact, that the set of orbits $\mathbf{M}^{\emptyset}$ with no collision at all is the union of the one-codimensional submanifolds $\{v_x = 0\}$ and $\{v_y = 0\}$. Thus the argument to come ensures that the system has at most four ergodic components. These, however, can be connected by using the dynamics providing the desired global ergodicity.

Denote by $\mathbf{M}_{\text{suff}}$ the subset of sufficient points. In view of (i) and (ii) the proof of Theorem 7.2 boils down to showing the following.

LEMMA 7.3. $\mathbf{M}^* \setminus (\mathbf{M}_{\text{suff}} \cup \mathbf{M}^{\emptyset})$ *is a slim subset.*

Since we only have two scatterers, in any symbolic collision sequence $\Sigma = (\sigma_1, \ldots, \sigma_n)$, $n \geq 0$, one has $\sigma_i = 1$ or 2.

Definition 7.4. An orbit segment is *rich* if its symbolic collision sequence contains at least one digit 1 and at least one digit 2.

The following simple observation, whose proof is left for the reader, is basic for our whole approach. In fact, for our model, this lemma does the job of a strong key lemma (cf. 'Key Lemma' 6.10).

LEMMA 7.5. *Every rich orbit segment is sufficient.*

Indeed, denote by $J_i \subset \mathbf{M}^*$ $(i = 1, 2)$ the subset of orbits whose symbolic collision sequence is $(\ldots, i, i, i, \ldots)$. By Lemma 7.5, Lemma 7.3 will follow from the following.

LEMMA 7.6. *The subsets J_i, $i = 1, 2$ are closed zero-sets of codimension two.*

Proof. Consider J_1. Choose an open subset $I \times B \subset C_2$ where I is an open interval of the x-axis, and B is an open ball of the y–z-plane. Fix the x-coordinates (q_x, v_x) of the phase point $(q_x, q_y, q_z; v_x, v_y, v_z)$ $(v_x^2 + v_y^2 + v_z^2 = 1)$. For the fixed (q_x, v_x), consider the dispersive billiard in the y–z-plane with the sole scatterer $D_1' := \{(y, z) : y^2 + z^2 \leq r_1^2\}$. More precisely, assume $0 < |v_x| < 1$. An orbit starting from $(Q, V) = (q_1, q_2, q_3, v_1, v_2, v_3)$ never hits $I \times B$ if and only if

$$\text{for every } t \in \mathbb{R}, \text{ such that } q_x + tv_x \in I,\ \pi_{\{y,z\}} S^t(Q, V) \notin B$$

where $\pi_{\{y,z\}}(q_x, q_y, q_z; v_x, v_y, v_z) := (q_y, q_z; v_y, v_z)$. Let

$$(A_H(B))_{q_x, v_x} := \{(q_y, q_z; v_y, v_z) : \text{for every } t \in \mathbb{R} \text{ such that } q_x + tv_x \in I \text{ one has } \pi_{\{y,z\}} S^t(Q, V) \notin B\}$$

i.e. $(A_H(B))_{q_x, v_x}$ is understood for the projected dynamics with $H := \{t : q_x + tv_x \in I\}$. We note that for the billiard $\pi_{\{y,z\}} S^t(Q, V)$

$$0 < v_y^2 + v_z^2 = 1 - v_x^2 < 1, \tag{7.7}$$

so the system is obtained by a linear time change from a standard dispersive billiard flow. Consequently, by virtue of the corollary to Theorem 6.6, for this projected system, Lemma 2.1 and the corollary of Theorem 5.2 can also be applied whenever (7.7) holds, implying that, on the one hand,

$$\mu\{(A_H(B))_{q_x, v_x}\} = 0$$

and, on the other hand, $(A_H(B))_{q_x, v_x}$ is of codimension two.

Fubini's theorem then first provides $\mu\{A_H(B)\} = 0$, whereas Proposition 3.7 implies that $A_H(B)$ is a codimension two subset. Since $J_1 \subset A_H(B)$, Lemma 7.6 follows. □

Lemma 7.3 easily follows from Lemmas 7.5 and 7.6. Now Theorem 7.2, indeed, follows from Lemma 7.3 by simple arguments left to the reader (or see [**KSSz(1989)**]). □

The reader could certainly observe in the proof of Lemma 7.6 the fundamental role of the ball-avoiding theorems, and further the fact that in proving the hyperbolicity and ergodicity of a higher—in this case three-dimensional—semi-dispersive billiard we used the hyperbolicity and ergodicity of a lower—in this case two-dimensional—billiard.

8. *Results for hard-ball systems*

8.1. *Dynamical method.* As we have seen in the proof of Theorem 7.2, a substantial element has been the strong ball-avoiding Theorem 5.2. The demonstration of Theorem 5.2, in its turn, has used the fundamental tools of the theory of hyperbolic dynamical systems. By more sophisticated versions of the arguments of Theorems 7.2 and 5.2, in particular, it was possible to establish the following.

THEOREM 8.1. [**KSSz(1991)**, **KSSz(1992)**] *Assume $N = 3$ or 4, $\nu \geq 3$. Then the standard hard-ball flow is a K-system.*

The structure of the verification of this theorem is the one given in Theorem 6.17 with the slight difference that richness is not exactly a C-richness for some C, but is introduced according to the following definition.

Definition 8.2. For the models of Theorem 8.1 we say that the symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ is *not rich* if there exist a $n' : 1 \leq n' \leq n$ and two non-trivial two-class partitions P^- and P^+ of the set $\{1, \dots, N\}$ of balls such that $\Sigma^- = (\sigma_1, \dots, \sigma_{n'})$ is partitioned by P^- and $\Sigma^+ = (\sigma_{n'+1}, \dots, \sigma_n)$ is partitioned by P^+, or, in other words, neither $\mathcal{G}_{\Sigma^-}$ nor $\mathcal{G}_{\Sigma^+}$ are connected (cf. Definition 6.8). Otherwise, the symbolic collision sequence is called *rich*.

Remark. The statement of Theorem 8.1 is also shown to hold for $N = 3$, $\nu = 2$ with a more stringent notion of richness.

Denote by $F(P^-, P^+)$ the subset of phase points $x \in \mathbf{M}$ for which $S^{\mathbb{R}_-}x$ is partitioned by P^- and $S^{\mathbb{R}_+}x$ is partitioned by P^+. The role of assumption (2) of Theorem 6.17 is now played by the following strong ball-avoiding theorem.

THEOREM 8.3. [**KSSz(1991)**, **KSSz(1992)**] *If $N = 3$, $\nu \geq 2$ or $N = 4$, $\nu \geq 3$, then for every pair P^- and P^+ of non-trivial two-class partitions of $\{1, \dots, N\}$, the subset $F(P^-, P^+)$ is a closed zero-set of codimension two.*

Remark. As always, the closedness of $F(P^-, P^+)$ follows from the definition of $F(P^-, P^+)$ by considering trajectory branches in the case of singular orbit segments (cf. §2 of [**KSSz(1992)**] or of [**Sim(1992)-I**]).

To substantiate the idea of the proof of Theorem 5.2, the demonstrations of this theorem presented in [**KSSz(1991)**] (case $N = 3$) and [**KSSz(1992)**] (case $N = 4$) use a quite interesting dynamical construction: that of the *pasting of dynamical systems*. We do not give here a detailed argument, we only explain it for the simplest non-trivial case: let $N = 3$, and set $P^- := \{\{2\}, \{1, 3\}\}$ and, further, $P^+ := \{\{1\}, \{2, 3\}\}$. Now $F(P^-, P^+)$ contains orbits where particle 2 has no interaction in the past, whereas particle 1 has not any in the future. In other words, the past trajectory avoids both cylinders $C_{1,2}$ and $C_{2,3}$, whereas the future one avoids $C_{1,2}$ and $C_{1,3}$. Since in the past there is an interaction between 1 and 3, only, and 2 has an almost periodic motion on the torus, we are in a similar situation as in the proof of Theorems 5.2 and 5.4: we have a ball-avoiding problem for the past orbit of the subsystem $\{1, 3\}$, which is actually known to be a K-system. The situation with the future trajectory is similar: particle 1 has an almost periodic motion, and we have a ball-avoiding problem for the future orbit of the K-mixing subsystem $\{2, 3\}$. The difference from the situation of Theorem 5.4 is that the dynamics that should now avoid some balls are different for the past and for the future. This difficulty is resolved by the method of pasting: one uses the unstable invariant manifolds of the subsystem $\{1, 3\}$, and the stable ones of the subsystem $\{2, 3\}$; further on one repeats the idea of the proof of Theorems 5.2 and 5.4 and the arising technical problems can be solved.

Fix a finite symbolic collision sequence Σ and a pair P^- and P^+ of partitions as before. Denote by $F := F(P^-, \Sigma, P^+)$ the subset of phase points $x \in \mathbf{M}$ for which $S^{\mathbb{R}_-}x$ is partitioned by P^-, there exists a $t > 0$ such that $\Sigma(S^{(0,t)}x) = \Sigma$, and, moreover, $S^{t+\mathbb{R}_+}x$ is partitioned by P^+. Encouraged by the success of the previous argument, one is inclined to hope that the method of pasting also permits one to settle the following conjecture.

CONJECTURE 8.4. *For any $N \geq 3, \nu \geq 2$, for an arbitrary symbolic collision sequence Σ and for any pair P^-, P^+ of non-trivial two-class partitions, $F(P^-, \Sigma, P^+)$ is a closed zero-set of codimension two.*

Remark 8.5. (a) The statement of Conjecture 8.4 immediately implies that for any fixed C, Π_C is a slim subset (for the definition of Π_C see (6.11)), which is exactly assumption (2) of Theorem 6.17.

Though we strongly believe that the conjecture is true, nevertheless, the method of pasting in its present form is not strong enough to prove Conjecture 8.4. The reason is, roughly speaking, that one can still consider the unstable manifolds for the subdynamics restricted to the classes of P^- in the time interval $(-\infty, 0]$ and the stable manifolds for the subdynamics restricted to the classes of P^+ in the time interval $[t, \infty)$. It is, however, hard to see why the absolute continuity and transversality statements necessary to formulate the zigzag properties, so basic to repeat the idea of Theorem 5.4, would hold.

(b) In [**KSSz(1991)**], the statement of Theorem 8.1 is also settled for the case $N = 3$, $\nu = 2$. However, to obtain it for this particular case one also had to verify Conjecture 8.4 for the case of a one-element symbolic collision sequence $\Sigma = (\sigma)$. This was actually done in [**KSSz(1991)**] through a concrete analysis of the concrete situation and so far it is not clear how this argument generalizes.

Having seen the limitations of the method of pasting, we will now turn to another method, which we call the mechanical method.

8.2. *Mechanical method.* The mechanical method was elaborated by Simányi in [**Sim(1992)-I**]. It will be presented in the simple case of a weak-type theorem borrowed from [**SSz(1999)**]. A novelty and an essential advantage of the upcoming formulation is that it is absolute, i.e. it is not inductive; afterwards we will also see inductive statements. A non-inductive formulation is needed if one is only able to show hyperbolicity of hard-ball systems, since this is a weak property to permit a possible induction. For the convenience of the reader and brevity of exposition, the set-up is simplified to the case when the masses of the balls are identical, although, in essence, the assumption on the identity of masses is only a minor technical one.

Denote by $R^* = R^*(N, \nu)$ the maximal number R such that for every $r \in (0, R)$ the interior of the configuration domain $\mathbf{Q}$ of the hard-ball system is connected.

THEOREM 8.6. [**SSz(1999)**] *Consider a system of N (≥ 3) particles on the ν-torus $\mathbb{T}^\nu$ ($\nu \geq 2$) satisfying $r < R^*$. Let $P = \{P_1, P_2\}$ be a given, two-class partition of the N particles, where, for simplicity, $P_1 = \{1, \ldots, n\}$ and $P_2 = \{n+1, \ldots, N\}$ ($n < N-1$). Then the closed set*

$$F_+ = \{x \in \mathbf{M} : S^{[0,\infty)}x \text{ is partitioned by } P\}$$

has measure zero.

Proof. The two cases $\min\{n, N-n\} \geq 2$ and $\min\{n, N-n\} = 1$ can be treated similarly, and thus we only consider the first one.

Every point $x \in \mathbf{M}$ can be characterized by the following coordinates in an essentially unique way:

(1) $\pi_{P_1}(x) = x_1 \in \mathbf{M}_1$,

(2) $\pi_{P_2}(x) = x_2 \in \mathbf{M}_2$,

(3) $C_1(x) = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} q_i(x) \in \mathbb{T}^\nu$,

(4) $\dfrac{I_1(x)}{\|I_1(x)\|} \in \mathbb{S}^{\nu-1}$,

(5) $\|I_1(x)\| \in \mathbb{R}_+$,

(6) $E_1(x) = \dfrac{1}{2}\displaystyle\sum_{i=1}^{n} v_i^2(x) \in \mathbb{R}_+$.

where $I_1(x) = [\sum_{i=1}^n v_i(x)]/n$. Here $\pi_{P_i}(x) : \mathbf{M} \to \mathbf{M}_1$, $i = 1, 2$, acts in the following way. First we separate the coordinates of balls belonging to P_i. Since then we lose the normalization conditions formulated at the beginning of §6, π_{P_i} also recovers them by trivial linear rescalings. Non-uniqueness only arises in choosing $C_1(x)$ as an arbitrary representant of $\frac{1}{n}\sum_{i=1}^n q_i(x) \in \mathbb{T}^\nu$. However, it can be uniquely defined locally and this will be satisfactory for our purposes. In what follows the six coordinates corresponding to the characterization given before will, in general, be denoted by $b_1, \ldots, b_6$; thus it will always be assumed that $b_1 \in \mathbf{M}_1$, $b_2 \in \mathbf{M}_2$, $b_3 \in \mathbb{T}^\nu$, $b_4 \in S^{\nu-1}$, $b_5, b_6 \in \mathbb{R}_+$. The relation $\mu\{F_+\} = 0$ will certainly follow if we show that for almost every such choices of the b_i's

$$\mu_{b_1,b_2,b_3,b_5,b_6}\{F_+(b_1, b_2, b_3, b_5, b_6)\} = 0, \tag{8.7}$$

where

$$F_+(b_1, b_2, b_3, b_5, b_6) = \{y_4 \in S^{\nu-1} : (b_1, b_2, b_3, y_4, b_5, b_6) \in F_+\},$$

and $\mu_{b_1,b_2,b_3,b_5,b_6}$ denotes the conditional measure of μ under the conditions corresponding to fixing the values of b_1, b_2, b_3, b_5, b_6. (This conditional measure is equivalent to the Lebesgue measure on its support.)

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The relation $x \in F_+$ is equivalent to saying that for every pair $i \in P_1$, $j \in P_2$ and every $t \geq 0$

$$\varrho(q_i^t(x) - q_j^t(x), 0) \geq 2r, \tag{8.8}$$

where $\varrho(.,.)$ denotes the Euclidean distance. For simplicity, fix $i = 1$ and $j = n + 1$.

Now we will be considering the subdynamics corresponding to our two-class partition P (cf. the 'subsystems, decompositions' part of §2 in the paper [**Sim(1992)-I**]), and will denote them, for simplicity, by S_1 and S_2, respectively (their phase spaces are $\mathbf{M}_1$ and $\mathbf{M}_2$, of course).

Our only task is to represent the time displacement $q_{n+1}^t(x) - q_1^t(x)$ in terms of the coordinates (1)–(6) and the subdynamics S_1, S_2. It is easy to see that for any $t \in \mathbb{R}$ and any $x \in \mathbf{M}$ one has

$$q_{n+1}^t(x) - q_1^t(x) = \tilde{q}_{n+1}^{\alpha t}(x) - \tilde{q}_1^{\beta t}(x) - \frac{N}{N-n} C_1(x) - \tilde{I} \tag{8.9}$$

where for any $t \in \mathbb{R}$ we denote $\tilde{x}_1^t := S_1^t x_1$ and $\tilde{x}_2^t := S_2^t x_2$, and, moreover, $\tilde{I}$ is the relative velocity of the 'baricentres' of the second and first subsystems (this term appears since in $\mathbf{M}_1$ and $\mathbf{M}_2$ the momenta of the subsystems are scaled to be equal to zero); in fact,

$$\tilde{I} = \frac{-N}{N-n} I_1 = \frac{-N}{N-n} \|I_1\| \frac{I_1}{\|I_1\|}, \tag{8.10}$$

and, finally,

$$\alpha = \sqrt{2E_1(x) - n\|I_1(x)\|^2},$$
$$\beta = \sqrt{1 - 2E_1(x) - \frac{n^2}{N-n}\|I_1(x)\|^2}$$

are the corresponding time scalings.

Our task is to show that the event

$$\forall t \geq 0 \quad \varrho\left(\lambda t \frac{I_1}{\|I_1\|} + f(t), \mathbb{Z}^\nu\right) \geq 2r \tag{8.11}$$

has measure zero for every fixed b_1, b_2, b_3, b_5, b_6, where $\lambda = (N-n)^{-1}(-N)\|I_1\|$ and

$$f(t) : \mathbb{R} \to \mathbb{R}^\nu$$

is an arbitrary fixed function such that $f(t) = f(t, x_1, x_2, C_1, \|I_1\|, E_1)$. Actually, by (8.9),

$$f(t) = \tilde{q}_{n+1}^{\alpha t}(x) - \tilde{q}_1^{\beta t}(x) - \frac{N}{N-n} C_1(x).$$

In (8.11), the canonical meaning of zero measure is that in $I_1/\|I_1\|$.

Denote by $\mathcal{L}_{2r}$ the lattice of balls of radius $2r$ centred at points of $\mathbb{Z}^\nu$. Our proof of Theorem 8.6 will be based on the following well-known elementary lemma whose proof can, for instance, be found in [**SSz(1999)**].

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LEMMA 8.12. *Fix a vector $\vec{n} \in S^{\nu-1}$ for which at least one ratio of coordinates is irrational. Consider arbitrary hyperplanes H perpendicular to $\vec{n}$, and denote by $B_R(z)$ the $(\nu-1)$-dimensional ball of radius R in H centred at $z \in H$. Then, for a suitable $\gamma(\vec{n}) > 0$,*

$$\liminf_{R\to\infty} \inf_H \inf_{z\in H} \frac{\text{meas}(B_R(z) \cap \mathcal{L}_{2r})}{\text{meas}(B_R(z))} \geq \gamma(\vec{n}). \tag{8.13}$$

Assume that the statement (8.11) is not true, i.e. the measure of the subset K of $S^{\nu-1}$ described by (8.11) is positive. Then select and fix a Lebesgue density point $\vec{n}$ of K with the property that at least one ratio of the components of $\vec{n}$ is irrational. Denote by $G_\varepsilon \subset S^{\nu-1}$ the ball of radius ε around $\vec{n}$. By (8.13) we can choose R_0 so large that, for $R \geq R_0$,

$$\inf_H \inf_{z\in H} \frac{\text{meas}(B_R(z) \cap \mathcal{L}_{2r})}{\text{meas}(B_R(z))} \geq \frac{\gamma(\vec{n})}{2}.$$

The set $\lambda t G_\varepsilon$ can be arbitrarily well approximated by a ball of radius $\lambda t \varepsilon = R$ (R is fixed, $R \geq R_0$) in the hyperplane orthogonal to $\vec{n}$ through the point $\lambda t \vec{n} + f(t)$ if only t is sufficiently large. Consequently, if $R \geq R_0$, then by choosing t sufficiently large and at the same time putting $\varepsilon = (\lambda t)^{-1} R$, we have

$$\frac{\text{meas}((\lambda t G_\varepsilon + f(t)) \cap \mathcal{L}_{2r})}{\text{meas}(\lambda t G_\varepsilon + f(t))} \geq \frac{\gamma(\vec{n})}{4}.$$

But this inequality contradicts the fact that $\vec{n}$ was chosen as a Lebesgue density point of the subset $K \subset S^{\nu-1}$. Hence Theorem 8.6 follows. □

Remark 8.14. An essential advantage of the formulation, and of the mechanical method as well, is that the definition of F_+ only uses the ball-avoiding property of the non-negative semi-trajectory. For a weak theorem this is not surprising (cf. Lemma 2.1) but for strong theorems this is a great advantage over results like Theorem 5.2. In fact, Simányi used the mechanical method to show the following.

THEOREM 8.15. **[Sim(1992)-I]** *Let $(M, \{S^t\}, \mu)$ be the* standard hard-ball flow *of N (≥ 3) particles on the unit torus $\mathbb{T}^\nu$ $(\nu \geq 2)$. Suppose that $r < R^*$. Assume that for all $n < N$ the n-billiard flow on $\mathbb{T}^\nu$ is a K–flow. Let P be a given, two-class partition of the N particles. Then the set*

$$F_+ = \{x \in M : S^{[0,\infty)}x \textit{ is partitioned by } P\}$$

is a closed, zero set with codimension at least two.

Remark 8.16. By (6.16) and (6.11), the theorem immediately implies that Π_C is a slim subset for an arbitrary $C \in \mathbb{N}$; consequently, it settles a basic assumption (condition (2)) of Theorem 6.17.

Remark 8.17. As one can also convince oneself upon reading the proof of Theorem 8.6, the assumption that the radii be not too large is absolutely essential. Indeed, the mechanical method is based on the conservation of the momenta of the non-interacting subsystems. This, however, does not hold for large radii, more precisely, for radii where the configuration domain consists of more than one connected component. We note that it does not hold either when one considers hard-ball systems in a box. Simányi [**Sim(1998)**], when proving the ergodicity of two hard balls in a box, in fact, used a quite different notion of richness and had to prove a special ball-avoiding theorem directly adapted to that notion. In fact, these (i.e. large radii and systems confined to a box) are *two important open problems of the theory*. The ball-avoiding question, however, is not isolated since it heavily depends on the effective notion of richness.

Theorem 8.15, weak and strong at the same time, is inductive and uses the K-property of smaller systems as its hypothesis. If one is only able or is satisfied to establish weaker properties (hyperbolicity or openness of ergodic components), then one uses a weaker inductive assumption and thus has to strengthen the method. The wisdom of the previous results suggests to us that, from the aspect of an inductive proof, hyperbolicity is a weak notion. So if we want to settle hyperbolicity, then the necessary weak ball-avoiding statement requires a non-inductive proof. On the other hand, as Theorems 8.15 and 8.18 show, the openness of the ergodic components is already a sufficiently strong notion to be used in an inductive argument. Such a theorem, a weak one, was used in [**SSz(1995)**].

THEOREM 8.18. [**SSz(1995)**] *Consider a system of* N (≥ 3) *particles on the unit torus* $\mathbb{T}^\nu$ *with* $r < R^*$. *Assume that, for all* $n < N$, *almost everywhere, none of the relevant Lyapunov exponents of the standard hard-ball system* $(\mathbf{M}, \{S^t\}, \mu)$ *vanishes, the ergodic components of the system are open (and thus of positive measure), and on each of them the flow has the K-property. Let* P *be a given, two-class partition of the* N *particles. Then the set*

$$F_+ = \{x \in \mathbf{M} : S^{[0,\infty)}x \text{ is partitioned by } P\}$$

has measure zero.

9. *Hyperbolic properties of cylindrical billiards*

9.1. *Orthogonal cylindric billiards.* Cylindric billiards, a more general class than hard-ball systems, were defined in Definition 7.1. In [**Sz(1994)**], a special class of cylindric billiards was considered: that of *orthogonal cylindrical billiards*. They are characterized by the additional requirement that the generator subspace of any cylindric scatterer is spanned by some of the coordinate vectors adapted to the orthogonal coordinate system where $\mathbb{T}^d$ is given. In technical terms the scatterers of such a billiard are given by a family $C^j : 1 \leq j \leq J$ of cylinders,

$$C^j := \sigma_{u^j}\left\{(q_1, \ldots, q_d) : \left(\sum_{i \in K^j} q_i^2\right)^{1/2} \leq r^j\right\}$$

on the d-torus where σ_u denotes the translation by a vector $u \in \mathbb{T}^d$.

A basic role in the conditions of ergodicity of orthogonal cylindrical billiards is played by the subsets $K^j \subset \{1, \ldots, d\}$, $|K^j| \geq 2$. These subsets will also be important in defining richness.

Consider the non-singular trajectory segment $S^{[a,b]}x$, $-\infty < a \leq b < \infty$, $x \in M$. Its *symbolic collision sequence* is the list of subsequent cylinders of collisions $(C^{j_1}, \ldots, C^{j_k})$, $k \geq 1$, of the trajectory and can be described by the sequence $(j_1, \ldots, j_k)$, $1 \leq j_l \leq J$,

$1 \leq l \leq k$. (If the trajectory hits one or more singularities, then, as usual, there is a finite number of such sequences for any finite orbit.)

Definition 9.1. We say that the trajectory segment $S^{[a,b]}x$ is *connected* if $\{K^{j_1}, \ldots, K^{j_k}\}$ is a connected cover of the set $\{1, \ldots, d\}$. We say that the trajectory segment $S^{[a,b]}x$ is *rich* if there exists a time $t \in [a, b]$ (with $S^t x \in \partial M$ also permitted) such that both trajectory segments $S^{[a,t]}x$ and $S^{[t,b]}x$ are connected. (If the trajectory segment hits singularities, then the above properties are required for any trajectory branch.)

Finally, the trajectory segment is *poor* if it is not rich.

$\{K^{j_1}, \ldots, K^{j_k}\}$ is said to be a connected cover of the set $\{1, \ldots, d\}$ if it is a cover, and, moreover, no $H \subset \{1, \ldots, d\}$, $H \neq \emptyset$, $H \neq \{1, \ldots, d\}$ exists such that, for every $1 \leq j \leq J$, $K^j \subset H$ or $K^j \subset H^c$ holds.

Denote by M_p^0 the subset of non-rich phase points from M^0. It would be nice to claim that M_p^0 is slim but there may exist some trivial one-codimensional submanifolds of non-sufficient points for our billiard. The trajectories of points lying in these submanifolds are non-sufficient for they (or the corresponding orbits of some auxiliary sub-billiards used in the proof) contain no collisions at all. Therefore, we should exclude from $\mathbf{M}$ a finite union of one-codimensional submanifolds, and as a result we obtain the set $\mathbf{M}^\# \subset \mathbf{M}$. Since the introduction of these submanifolds is a bit lengthy and is not deeply connected to the topic of our survey, we omit their precise description (this is done in detail in the appendix of [**Sz(1994)**]). With a little hand-waving we just repeat that this finite union consists all phase points whose trajectories never collide in at least one non-trivial sub-billiard of our system (such a sub-billiard is obtained by taking a non-empty subset of the cylindric scatterers, and by considering the billiard with these scatterers, only; i.e. we discard the other scatterers). We note that these submanifolds themselves are defined by linear conditions on the velocities.

In [**Sz(1994)**], it is shown that the necessary and sufficient condition of the ergodicity (and the K-property) of orthogonal cylindrical billiards is as follows.

CONDITION 9.2. *$\{K^j : 1 \leq j \leq J\}$ is a connected cover of $\{1, \ldots, d\}$.*

The ball-avoiding theorem used in the proof of sufficiency is the following.

THEOREM 9.3. [**Sz(1994)**] *If Condition 9.2 holds, then $M_p^0 \cap M^\#$ is a slim subset.*

To illustrate the deep analogy with hard-ball systems (cf. in particular, Theorem 8.3) it is worth formulating the basic lemma which immediately provides this theorem. To this end for any pair $\{P^-, P^+\}$ of non-trivial two-class partitions of the *set of coordinates* $\{1, \ldots, d\}$, let us define

$$F := F(P^-, P^+) := \{x \in \mathbf{M}_0 \setminus \partial\mathbf{M} : \text{the } K^j s \text{ corresponding to } \Sigma(S^{(-\infty,0)}x) \text{ and } \Sigma(S^{(0,\infty)}x) \text{ are partitioned by } P^- \text{ and } P^+\}.$$

We note that *a symbolic collision sequence Σ is said to be partitioned by a non-trivial two-class partition P of* $\{1, \ldots, d\}$ if the K^js corresponding to the elements of Σ form a connected cover of $\{1, \ldots, d\}$.

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LEMMA 9.4. *If Condition 9.2 holds, then F is a slim subset.*

This lemma can again be settled by the (dynamical) method of pasting; in fact, for cylindrical billiards the mechanical method does not make sense since the momentum, in general, is not an invariant of motion any more.

9.2. *General case.* According to Remark 8.16, the ball-avoiding type condition of Theorem 6.17 is settled once and for all by Simányi's Theorem 8.15, at least if we have an inductive proof in mind. It would be desirable to arrive at a similar success in cylindrical billiards. One encounters several problems, however. First of all, it is not *a priori* clear what a suitable definition of richness should be so that some analogue of hypothesis (2) of Theorem 6.17 (or that of Theorem 6.12) could be checked. The ideas of the paper [**SSz(2000)**], however, suggest the following definition.

Definition 9.5. In a symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ every $\sigma_i : 1 \le i \le n$, by definition, corresponds to a cylinder with base space L_i. Now we say that Σ is *connected* iff there is no orthogonal splitting $\mathbb{R}^d = B_1 \oplus B_2$ with $\dim B_j > 0$ and with the property that for every $i = 1, \dots, n$ either $L_i \subset B_1$ or $L_i \subset B_2$.

We say that the symbolic collision sequence $\Sigma = (\sigma_1, \dots, \sigma_n)$ is *C-rich*, with C being a natural number, if it can be decomposed into at least C consecutive, disjoint collision subsequences in such a way that each of them is connected.

Remark 9.6. The condition of connectedness is exactly identical to the orthogonal non-splitting property, formulated in [**SSz(2000)**], of the system of subspaces $L_1, \dots, L_n$. Moreover, by Theorem 4.6 and Proposition 4.9 of the same work, in the particular case of hard-ball systems our Definition 9.5 reduces precisely to Definition 6.5 given above.

CONJECTURE 9.7. *For an arbitrary natural number C, the subset of orbits whose symbolic collision sequence is not C-rich, is a slim subset of* **M**.

Finally we formulate a stronger conjecture than the previous one. In principle it is adapted to a possible proof of ergodicity by an induction on to the number of cylinders.

Fix a finite symbolic collision sequence Σ and two cylinders: C_{j^-} and C_{j^+}. Denote by $F(j^-, \Sigma, j^+)$ the subset of phase points $x \in \mathbf{M}$ for which $S^{\mathbb{R}_-}x$ avoids the cylinder C_{j^-}, and there exists a $t > 0$ such that $\Sigma(S^{(0,t)}x) = \Sigma$, and, moreover, $S^{t+\mathbb{R}_+}x$ avoids the cylinder C_{j^+}.

CONJECTURE 9.8. *For an arbitrary symbolic collision sequence Σ and any pair of cylinders C_{j^-}, C_{j^+}, the set $F(j^-, \Sigma, j^+)$ is a closed zero-set of codimension two.*

This conjecture generalizes Conjecture 8.4 and its eventual proof has an analogous difficulty as of that one.

III. Related directions

10. *Replacing topological dimension by the Hausdorff one*

Let $T : \mathbf{M} \to \mathbf{M}$ be a transitive Anosov C^2-diffeomorphism of a compact Riemannian manifold **M**. Our ball-avoiding theorems discussed so far expressed the fact that for an

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orbit to be non-dense is an atypical behaviour, at least as long as the notion of dimension we are considering is the topological one. Surprisingly enough, if we take Hausdorff dimension, then we cannot recover this atypicality as this will be shown by the following selection of theorems.

THEOREM 10.1. [**U(1991)**] *Let G be a non-empty open subset in* **M**. *Then*

$$\mathrm{HD}(G \cap ND) = \dim \mathbf{M}$$

where

$$ND = ND(T) := \{x \in \mathbf{M} : T^{\mathbb{Z}}x \text{ is not everywhere dense in } \mathbf{M}\}$$

and HD *denotes Hausdorff dimension.*

Urbański has also established an analogous statement for Anosov flows.

THEOREM 10.2. [**U(1991)**] *Let G be a non-empty open subset in* **M**. *Then*

$$\mathrm{HD}(G \cap ND) = \dim \mathbf{M}$$

where $ND = ND(S)$ is the set from (1.1).

Dolgopyat has found an interesting strengthening of the question answered by the previous theorems. Note that ND is the set of orbits whose limit points do not fill up the whole space. For $Z \subset \mathbf{M}$ a fixed subset, we can consider the set

$$L_Z := \{x \in \mathbf{M} : \lim T^{\mathbb{Z}}x \cap Z = \emptyset\}$$

where $\lim T^{\mathbb{Z}}x$ denotes the set of limit points of the orbit $\{T^n x : t \in \mathbb{Z}\}$. Dolgopyat's theorem sounds as follows.

THEOREM 10.3. [**D(1997)**] *Assume T is a topologically transitive Anosov C^2-diffeomorphism of $\mathbb{T}^2$, the 2-torus, and denote by $\mathrm{HD}(\mu)$ the Hausdorff dimension of its Sinai–Ruelle–Bowen measure μ. If $Z \subset \mathbb{T}^2$ has Hausdorff dimension less than $\mathrm{HD}(\mu)$, then*

$$\mathrm{HD}(L_Z) = 2.$$

Conversely, for any $p > \mathrm{HD}(\mu)$, one can find a set Z of Hausdorff dimension less than p for which the above statement fails.

The proofs of Theorems 10.1–10.3 all exploit the existence of a finite Markov partition. Furthermore, the verifications of Theorems 10.1–10.2 use a generalization of a result of McMullen [**McM(1987)**], providing a lower bound for the Hausdorff dimension through local densities. On the other hand, to establish Theorem 10.3, Dolgopyat uses formulas of Manning–McCluskey [**MM(1983)**] and Young [**Y(1982)**] which are valid in the two-dimensional setting and this fact explains the dimensional restriction in Theorem 10.3.

For systems with singularities the Markov partition, even if it exists, cannot be finite and the previous methods do not work. Nevertheless it is reasonable to expect the following.

CONJECTURE 10.4. *Theorems 10.1–10.3 are valid for Anosov systems with singularities (for the axioms of these systems see* [**Y(1998)**] *or* [**Ch(1999)**]*).*

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Finally we note that results analogous to the aforementioned theorems have been formulated for certain one-parameter subgroups of some Lie groups but even their listing would go beyond the scope of the present survey. For results and conjectures we refer to [**M(1990)**] and [**K(1998)**] and we just note that in these cases again the method of Markov partitions is not at hand but one can exploit the rich algebraic structure instead.

11. *Ball-avoiding in physics: open systems and repellers*

For a better understanding of the pre-turbulent behaviour of the Lorenz model, in 1979, Pianigiani and Yorke [**PY(1979)**] initiated the study of open dynamical systems. One main model they suggested was a dispersive billiard with a hole. Since it is close to our basic object, let us look at the questions they raise for this model.

Assume we are given a dispersive billiard in **Q** and a small hole is cut in the table. Whenever the billiard particle enters the hole, it gets absorbed with its orbit deleted from the phase space. We select the hole to be an open subset B of the phase space and assume that the initial phase point is given by a measure m_0. Then let

$$p^+(t) := m_0\{S^{[0,t]}x \cap B = \emptyset\}$$

be the probability that the particle stays on the table for at least time t, and let

$$p_A^+(t) := m_0\{S^{[0,t]}x \cap B = \emptyset \text{ and } S^t x \in A\}$$

be the probability that it is in the set A in time t.

Question 1. What is the rate with which $p^+(t)$ converges to zero, when $t \to \infty$?

Question 2. Does the weak limit of the conditional measure

$$\lim_{t\to\infty} \frac{p_A^+(t)}{p^+(t)} = \mu^+\{A\}$$

exist and if it does what is its value?

Question 3. How does μ^+ depend on the initial distribution m_0?

The questions can also be raised in a time-symmetric way. Indeed, denote

$$p(t) := m_0\{S^{[-t,t]}x \cap B = \emptyset\}$$

and

$$p_A(t) := m_0\{S^{[-t,t]}x \cap B = \emptyset \text{ and } S^t x \in A\}$$

$$\mu\{A\} = \lim_{t\to\infty} \frac{p_A(t)}{p(t)}.$$

Then we can pose the same questions for these objects as before.

Pianigiani and Yorke answered Questions 1–3 for expanding maps acting in a domain of $\mathbb{R}^d$. In a recent work of Chernov *et al* [**ChMT(2000)**], the problems are settled for Anosov diffeomorphisms on surfaces with small holes. Their results and previous rigorous results of other authors have been based on analytic calculations obtained originally by physicists.

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Out of these—without aiming at completeness—we only mention the works of Kantz and Grassberger [**KG(1985)**] (related, in particular, to Theorem 11.5 below), Hsu *et al* [**HOG(1988)**] (related, in particular, to Theorem 11.1 below), and Legrand and Sornette [**LS(1990)**] (for an analytic calculation for stadia); for a review we refer to the survey of Tél [**T(1996)**]. Since we only plan to give the flavour of the results of [**ChMT(2000)**], we will omit the very technical formulation of their conditions.

Let $T : \mathbf{M} \to \mathbf{M}$ be a topologically transitive Anosov $C^{1+\alpha}$-diffeomorphism of a compact Riemannian surface and $B \subset \mathbf{M}$ be a nice open subset. Denote $\widetilde{\mathbf{M}} := \mathbf{M} \setminus B$ and let, for every $n \geq 0$,

$$\mathbf{M}_n := \cap_{i=0}^n T^i \widetilde{\mathbf{M}} \quad \text{and} \quad \mathbf{M}_{-n} := \cap_{i=0}^n T^{-i} \widetilde{\mathbf{M}}$$

and, moreover,

$$\mathbf{M}_+ := \cap_{n\geq 1} \mathbf{M}_n, \quad \mathbf{M}_- := \cap_{n \geq 1} \mathbf{M}_{-n}, \quad \Omega := \mathbf{M}_- \cap \mathbf{M}_+.$$

The set Ω is called *the repeller* (in the physics literature, more recently they are often called *chaotic saddles*).

Now for some more notation. For every finite Borel measure m we denote $|m| = m\{\mathbf{M}\}$,

$$(T_* m)\{A\} = m\{T^{-1}(A \cap \mathbf{M}_1)\} \quad (A \subset \widetilde{\mathbf{M}})$$

$$T_+ m := \frac{1}{|T_* m|} T_* m \quad \text{if } |T_* m| \neq 0.$$

We say that the probability measure m on $\widetilde{\mathbf{M}}$ is *conditionally invariant* under T if $T_+ m = m$, or equivalently if there is a $\lambda_+ > 0$ such that $T_* m = \lambda_+ m$. Any conditionally invariant measure m is, of course, supported on $\mathbf{M}_+$, and we also have $\lambda_+ = |T_* m| = m\{\mathbf{M}_{-1} \cap \mathbf{M}_+\} = m\{\mathbf{M}_{-1}\}$. Denote by $\mathcal{M}_n$, $\mathcal{M}_+$, and $\mathcal{M}$ the classes of (SRB-like) probability measures supported on $\mathbf{M}_n$, $\mathbf{M}_+$, and Ω, respectively.

THEOREM 11.1. [**ChMT(2000)**] *There is a unique (SRB-like) conditionally invariant measure $\mu_+ \in \mathcal{M}_+$, i.e. the operator $T_+ : \mathcal{M}_+ \to \mathcal{M}_+$ has a unique fixed point μ_+.*

THEOREM 11.2. [**ChMT(2000)**] *For any measure $m_0 \in \mathcal{M}_0$, the sequence of measures $T_+^n m_0$ converges weakly, as $n \to \infty$, to the conditionally invariant measure μ_+. Moreover, the sequence of measures $\lambda_+^{-n}(T_*^n m_0)$ converges weakly to $\rho(m_0)\mu_+$, where the functions $\rho(m_0)$ and $\rho^{-1}(m_0)$ are uniformly bounded on $\mathcal{M}_0$.*

THEOREM 11.3. [**ChMT(2000)**] *The sequence $T^{-n}\mu_+$ converges weakly, as $n \to \infty$, to a T-invariant probability measure $\hat{\mu}_+ \in \mathcal{M}$. The measure $\hat{\mu}_+$ is ergodic and K-mixing.*

The aforementioned results have their natural duals by changing the signs, and then one obtains μ_-, $\hat{\mu}_-$, λ_-.

THEOREM 11.4. [**ChMT(2000)**] *If for every periodic point $x \in \Omega$, $T^k x = x$ we have $|\det DT^k x| = 1$, then $\hat{\mu}_+ = \hat{\mu}_- = \hat{\mu}$ and $\lambda_+ = \lambda_- = \lambda$. In particular, this happens if the given Anosov diffeomorphism preserves a smooth invariant measure.*

(In [**ChMT(2000)**] it is also conjectured that $\hat{\mu}_+$ is a Bernoulli measure, and has a fast decay of correlations.) The following theorem not only answers Question 1, most

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interesting from the point of view of physical applications, but also proves the *escape rate formula* of [**KG(1985)**]. We call $\gamma_+ := -\log \lambda_+$ the *escape rate* of the system. Denote by λ_+ the positive Lyapunov exponent of the ergodic measure $\hat{\mu}_+$, and by $h(\hat{\mu}_+)$ its Kolmogorov–Sinai entropy.

THEOREM 11.5. [**ChMT(2000)**]

$$\gamma_+ = \lambda_+ - h(\hat{\mu}_+). \tag{11.6}$$

An interesting feature of the escape rate formula (11.6) is that its right-hand side is defined exclusively in terms of the measure $\hat{\mu}_+$ given on the repeller $\mathbf{M}_+$, whereas γ_+ is the rate with which an initial measure given on whole $\mathbf{M}$ gets pulled down to the repeller. It is an interesting task to generalize Theorems 11.1–11.5 to Anosov systems with singularities and subsequently to billiards.

Remark 11.7. For dynamical systems of large linear size L, which actually are appropriate models of transport phenomena, Gaspard and Nicolis [**GN(1990)**] derived a beautiful equation replacing the escape rate formula. For definiteness, let us think of a Lorentz process (i.e. a dispersive, finite-horizon billiard with a periodic configuration of scatterers) in an elongated periodic container of integer length L; the boundary condition in the direction of the y-axis is periodic, whereas those in the direction of the x-axis at $x = 0$ and $x = L$ are open, i.e. $B = (\{x = 0\} \cup \{x = L\}) \times S^1$. This model determines a repeller $\mathbf{M}_+(L)$ with SRB-like invariant measure $\hat{\mu}_+(L)$, for which we denote the positive Lyapunov exponent by λ_L and the K–S entropy by $h(\hat{\mu}_+(L))$. Then, by using the diffusion approximation for the Lorentz process, Gaspard and Nicolis proved analytically that

$$\mathcal{D} = \lim_{L\to\infty} \left(\frac{L}{\pi^2}\right)(\lambda_L - h(\hat{\mu}_+(L)))$$

where $\mathcal{D}$ is the diffusion coefficient of the Lorentz process in the infinite slab (i.e. in the same model with $L = \infty$). Further related formulas and models are beyond the scope of the present survey. As references on this developing direction of research we mention the papers of Gaspard and Dorfman [**GD(1995)**], Tél *et al* [**TVB(1996)**] and Ruelle [**R(1999)**]; for earlier related models of transport see the works of Lebowitz and Spohn [**LS(1978)**] and Krámli *et al* [**KSSz(1987)**].

Notes added in proof. 1. According to a recent observation of P. Bálint, N. Chernov, D. Száz and P. I. Tóth, in Theorem 6.6 (Local ergodicity of semi-dispersive billiards) one should suppose in addition that the boundaries of the scatterers are algebraic. Consequently, according to our present understanding this algebraicity condition should be everywhere assumed where Theorem 6.6 is applied.

2. New developments in the applications of ball-avoiding theorems to ergodicity proofs of billiards can be found in N. Simányi's most recent survey: Hard ball systems and semi-dispersive billiards: hyperbolicity and ergodicity. *Hard Ball Systems and the Lorentz Gas (Encyclopaedia of Mathematical Sciences, vol. 101)*. Ed. D. Szász. Springer.

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