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Bounded complexity, mean equicontinuity and discrete spectrum

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Abstract. We study dynamical systems that have bounded complexity with respect to three kinds metrics: the Bowen metric d_n, the max-mean metric $\hat{d}_n$ and the mean metric $\bar{d}_n$, both in topological dynamics and ergodic theory. It is shown that a topological dynamical system (X, T) has bounded complexity with respect to d_n (respectively $\hat{d}_n$) if and only if it is equicontinuous (respectively equicontinuous in the mean). However, we construct minimal systems that have bounded complexity with respect to $\bar{d}_n$ but that are not equicontinuous in the mean. It turns out that an invariant measure μ on (X, T) has bounded complexity with respect to d_n if and only if (X, T) is μ-equicontinuous. Meanwhile, it is shown that μ has bounded complexity with respect to $\hat{d}_n$ if and only if μ has bounded complexity with respect to $\bar{d}_n$, if and only if (X, T) is μ-mean equicontinuous and if and only if it has discrete spectrum.

Key words: bounded complexity, equicontinuity, mean equicontinuity, discrete spectrum
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1. *Introduction*

Throughout this paper, by a topological dynamical system (t.d.s.) we mean a pair (X, T), where X is a compact metric space with a metric d and T is a continuous map from X to itself. Let $\mathcal{B}_X$ be the Borel σ-algebra on X and μ be a probability measure on $(X, \mathcal{B}_X)$. We say that μ is an invariant measure for T if, for every $B \in \mathcal{B}_X$, $\mu(T^{-1}B) = \mu(B)$.

Entropy is a very useful invariant to describe the complexity of a dynamical system that measures the rate of the exponential growth of the orbits. For some simple systems (for example dynamical systems with zero entropy), it is useful to consider the complexity function itself. This kind of consideration can be traced back to the work by Morse and Hedlund, who studied the complexity function of a subshift and proved that the boundedness of the function is equivalent to the eventual periodicity of the system (for progress on the high dimensional analogue, see [**3**]). In [**10**], Ferenczi studied the measure-theoretic complexity of ergodic systems, using the α-names of a partition and the Hamming distance. He proved that, when the measure is ergodic, the complexity function is bounded if and only if the system has a discrete spectrum (for the result dealing with the non-ergodic case, see [**35**]). In [**21**], Katok introduced a notion using the modified notion of spanning sets with respect to an invariant measure μ and an error ε, which be used to define the complexity function. In [**2**], Blanchard, Host and Maass studied topological complexity via the complexity function of an open cover and showed that the complexity function is bounded for any open cover if and only if the system is equicontinuous.

Recently, in the investigation of the Sarnak's conjecture, Huang, Wang and Ye [**18**] introduced the measure complexity of an invariant measure μ, similar to the one introduced by Katok [**21**], by using the mean metric instead of the Bowen metric (for discussion and results related to mean metric, see also [**27**, **34**]). They showed that if an invariant measure has discrete spectrum, then the measure complexity, with respect to this invariant measure, is bounded. An open question was posed as to whether the converse statement holds. Motivated by this open question, and inspired by the discussions in [**2**, **10**, **12**, **13**, **16**, **17**, **21**, **25**], in this paper, we study topological and measure-theoretic complexity via a sequence of metrics, induced by a metric d, namely the metrics d_n, $\hat{d}_n$ and $\bar{d}_n$.

To be precise, for $n \in \mathbb{N}$, we define three metrics on X as follows. For $x, y \in X$, let

$$d_n(x, y) = \max\{d(T^i x, T^i y) : 0 \le i \le n-1\}, \quad \bar{d}_n(x, y) = \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y)$$

and

$$\hat{d}_n(x, y) = \max\{\bar{d}_k(x, y) : 1 \le k \le n\}.$$

It is clear that, for all $x, y \in X$,

$$d_n(x, y) \ge \hat{d}_n(x, y) \ge \bar{d}_n(x, y).$$

For $x \in X$, $\varepsilon > 0$ and a metric ρ on X, let $B_\rho(x, \varepsilon) = \{y \in X : \rho(x, y) < \varepsilon\}$. We say that a dynamical system (X, T) has bounded topological complexity with respect to a sequence of metrics $\{\rho_n\}$ if, for every $\varepsilon > 0$, there exists a positive integer C such that, for each $n \in \mathbb{N}$, there are points $x_1, x_2, \ldots, x_m \in X$ with $m \le C$ satisfying $X = \bigcup_{i=1}^m B_{\rho_n}(x_i, \varepsilon)$. In this paper, we will focus on the situation where $\rho_n = d_n$, $\hat{d}_n$ and $\bar{d}_n$.

We also study the measure-theoretic complexity of invariant measures. That is, for a given $\varepsilon > 0$ and an invariant measure μ, we consider the measure complexity with respect to $\{\rho_n\}$ with $\rho_n = d_n$, $\hat{d}_n$ and $\bar{d}_n$, defined by

$$\min\bigg\{m \in \mathbb{Z}_+ : \exists x_1, \ldots, x_m \in X,\ \mu\bigg(\bigcup_{i=1}^{m} B_{\rho_n}(x_i, \varepsilon)\bigg) > 1 - \varepsilon\bigg\}.$$

As expected, the bounded complexity of a topological dynamical system, or a measure preserving system, is related to various notions of equicontinuity.

It is shown that (see Theorems 3.1 and 3.5) a topological dynamical system (X, T) has bounded complexity with respect to d_n (respectively $\hat{d}_n$) if and only if it is equicontinuous (respectively equicontinuous in the mean). At the same time, we construct minimal systems that have bounded complexity, with respect to $\bar{d}_n$ but not equicontinuous in the mean, and that are uniquely ergodic or not (see Propositions 3.8 and 3.9).

It turns out that an invariant measure μ on (X, T) has bounded complexity with respect to d_n if and only if (X, T) is μ-equicontinuous (see Theorem 4.1). Meanwhile, it is shown that μ has bounded complexity with respect to $\hat{d}_n$ if and only if μ has bounded complexity with respect to $\bar{d}_n$, if and only if (X, T) is μ-mean equicontinuous, if and only if (X, T) is μ-equicontinuous in the mean, if and only if it has discrete spectrum (see Theorems 4.3, 4.4 and 4.7).

The structure of the paper is as follows. In §2, we recall some basic notions that we will use in the paper. In §3, we prove the topological results for systems with bounded complexity, with respect to three kinds of metrics. In §4, we consider the corresponding results in the measure-theoretical setting. In Appendix A, we give some examples.

2. *Preliminaries*

In this section we recall some notions and aspects of dynamical systems that will be used later.

2.1. *General notions.*

In the article, the sets of integers, non-negative integers and natural numbers are denoted by $\mathbb{Z}$, $\mathbb{Z}_+$ and $\mathbb{N}$, respectively. We use $\#(A)$ to denote the number of elements of a finite set A.

A t.d.s. (X, T) is *transitive* if, for each pair of non-empty open subsets U and V, $N(U, V) = \{n \in \mathbb{Z}_+ : U \cap T^{-n}V \neq \emptyset\}$ is infinite; it is *totally transitive* if (X, T^n) is transitive for each $n \in \mathbb{N}$; and it is *weakly mixing* if $(X \times X, T \times T)$ is transitive. We say that $x \in X$ is a *transitive point* if its orbit $\mathrm{Orb}(x, T) = \{x, Tx, T^2x, \ldots\}$ is dense in X. The set of transitive points is denoted by $\mathrm{Trans}(X, T)$. It is well known that if (X, T) is transitive, then $\mathrm{Trans}(X, T)$ is a dense G_δ subset of X.

A t.d.s. (X, T) is *minimal* if $\mathrm{Trans}(X, T) = X$, i.e. it contains no proper subsystems. A point $x \in X$ is called a *minimal point* or *almost periodic point* if $(\overline{\mathrm{Orb}(x, T)}, T)$ is a minimal subsystem of (X, T).

2.2. *Equicontinuity and mean equicontinuity.*

A t.d.s. (X, T) is called *equicontinuous* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that whenever $x, y \in X$ with $d(x, y) < \delta$, $d(T^n x, T^n y) < \varepsilon$ for $n = 0, 1, 2, \ldots$. It is well known that a t.d.s. (X, T) with T being surjective is equicontinuous if and only if there exists a compatible metric ρ on X such that T acts on X as an isometry, i.e. $\rho(Tx, Ty) = \rho(x, y)$ for any $x, y \in X$. Moreover, a transitive equicontinuous system is conjugate to a minimal rotation on a compact abelian metric group, and (X, T, μ) has discrete spectrum, where μ is the unique normalized Haar measure on X.

When studying dynamical systems with discrete spectrum, Fomin [**11**] introduced a notion called *stable in the mean in the sense of Lyapunov* or simply *mean-L-stable*. A t.d.s. (X, T) is *mean-L-stable* if, for every $\varepsilon > 0$, there is a $\delta > 0$ such that $d(x, y) < \delta$. This implies that $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ except for a set of upper density less than ε. Fomin proved that if a minimal system is mean-L-stable, then it is uniquely ergodic. Mean-L-stable systems are also discussed briefly by Oxtoby in [**29**], and he proved that each transitive mean-L-stable system is uniquely ergodic. Auslander in [**1**] systematically studied mean-L-stable systems, and provided new examples. See Scarpellini [**30**] for a related work. It was an open question whether every ergodic invariant measure on a mean-L-stable system had discrete spectrum [**30**]. This question was answered affirmatively by Li, Tu and Ye in [**25**].

A t.d.s. (X, T) is called *mean equicontinuous* (respectively *equicontinuous in the mean*) if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that, whenever $x, y \in X$ with $d(x, y) < \delta$, $\limsup_{n\to\infty} \bar{d}_n(x, y) < \varepsilon$ (respectively $\bar{d}_n(x, y) < \varepsilon$ for each $n \in \mathbb{N}$). It is not hard to show that a dynamical system is mean equicontinuous if and only if it is mean-L-stable. For works related to mean equicontinuity, we refer to [**7**, **13**, **14**, **25**, **26**]. We remark that by the result in [**7**], a minimal null or tame system is mean equicontinuous. We will show in this paper that a minimal system is mean equicontinuous if and only if it is equicontinuous in the mean (for the proof for the general case, see [**31**]).

2.3. *μ-equicontinuity and μ-mean equicontinuity.* When studying the chaotic behaviors of dynamical systems, Huang, Lu and Ye [**17**] introduced a notion that reflects the equicontinuity with respect to a subset or a measure.

Following [**17**], for a t.d.s. (X, T), we say that a subset K of X is *equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ and all $x, y \in K$ with $d(x, y) < \delta$. For an invariant measure μ on (X, T), we say that T is *μ-equicontinuous* if, for any $\tau > 0$, there exists a T-equicontinuous measurable subset K of X with $\mu(K) > 1 - \tau$. It was shown in [**17**] that if (X, T) is μ-equicontinuous and μ is ergodic, then μ has a discrete spectrum. We note that the μ-equicontinuity was studied further in [**12**].

In the process to study mean equicontinuity, the above notions were generalized to mean equicontinuity with respect to an invariant measure by García-Ramos in [**13**]. Particularly, he proved that, for an ergodic invariant measure μ, (X, T) is μ-mean equicontinuous if and only if μ has discrete spectrum. For a different approach, see [**24**].

2.4. *Hausdorff metric.* Let $K(X)$ be the hyperspace on X, i.e. the space of non-empty closed subsets of X equipped with the Hausdorff metric d_H defined by

$$d_H(A, B) = \max\Big\{\max_{x\in A}\min_{y\in B} d(x, y), \max_{y\in B}\min_{x\in A} d(x, y)\Big\} \quad \text{for } A, B \in K(X).$$

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As (X, d) is compact, $(K(X), d_H)$ is also compact. For $n \in \mathbb{N}$, it is easy to see that the map $X^n \to K(X)$, $(x_1, \ldots, x_n) \mapsto \{x_1, \ldots, x_n\}$, is continuous. Then, $\{A \in K(X) : \#(A) \leq n\}$ is a closed subset of $K(X)$.

2.5. *Discrete spectrum.* Let (X, T) be an invertible t.d.s., that is, T is a homeomorphism on X. Let μ be an invariant measure on (X, T) and let $L^2(\mu) = L^2(X, \mathcal{B}_X, \mu)$ for short. An *eigenfunction* for μ is some non-zero function $f \in L^2(\mu)$ such that $Uf := f \circ T = \lambda f$ for some $\lambda \in \mathbb{C}$. In this case, λ is called the *eigenvalue* corresponding to f. It is easy to see that every eigenvalue has norm one, that is $|\lambda| = 1$. If $f \in L^2(\mu)$ is an eigenfunction, then $\{U^n f : n \in \mathbb{Z}\}$ is precompact in $L^2(\mu)$, that is the closure of $\{U^n f : n \in \mathbb{Z}\}$ is compact in $L^2(\mu)$. Generally, we say that f is *almost periodic* if $\{U^n f : n \in \mathbb{Z}\}$ is precompact in $L^2(\mu)$. It is well known that the set of all bounded almost periodic functions forms a U-invariant and conjugation-invariant subalgebra of $L^2(\mu)$ (denoted by A_c). The set of all almost periodic functions is just the closure of A_c (denoted by H_c), and is also spanned by the set of eigenfunctions. The invariant measure μ is said to have *discrete spectrum* if $L^2(\mu)$ is spanned by the set of eigenfunctions, that is $H_c = L^2(\mu)$. We remark that when μ is not ergodic, the structure of a system (X, T, μ) with discrete spectrum can be very complicated, we refer to [**8**, **23**] and the example we provide at the end of §4 for details.

3. *Topological dynamical systems with bounded topological complexity*

In this section, we will study the topological complexity of dynamical systems with respect to three kinds of metrics.

3.1. *Topological complexity with respect to $\{d_n\}$.* Let (X, T) be a t.d.s. For $n \in \mathbb{N}$ and $x, y \in X$, define

$$d_n(x, y) = \max\{d(T^i x, T^i y) : i = 0, 1, \ldots, n-1\}.$$

It is easy to see that, for each $n \in \mathbb{N}$, d_n is a metric on X which is topologically equivalent to the metric d. Let $x \in X$ and $\varepsilon > 0$. The open ball of centre x and radius ε in the metric d_n is

$$B_{d_n}(x, \varepsilon) = \{y \in X : d_n(x, y) < \varepsilon\} = \bigcap_{i=0}^{n-1} T^{-i} B(T^i x, \varepsilon).$$

Let K be a subset of X, $n \in \mathbb{N}$ and $\varepsilon > 0$. A subset F of K is said to *(n, ε)-span K with respect to T* if, for every $x \in K$, there exists $y \in F$ with $d_n(x, y) < \varepsilon$, that is

$$K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon).$$

Let $\mathrm{span}_K(n, \varepsilon)$ denote the smallest cardinality of any (n, ε)-spanning set for K with respect to K, that is

$$\mathrm{span}_K(n, \varepsilon) = \min\left\{\#(F) : F \subset K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon)\right\}.$$

We say that a subset K of X has *bounded topological complexity with respect to $\{d_n\}$* if, for every $\varepsilon > 0$, there exists a positive integer $C = C(\varepsilon)$ such that $\mathrm{span}_K(n, \varepsilon) \leq C$ for all

$n \geq 1$. If the whole set X has bounded topological complexity with respect to $\{d_n\}$, we will say that the dynamical system (X, T) has the property.

We first show that a subset with bounded topological complexity with respect to $\{d_n\}$ is equivalent to the equicontinuity property.

THEOREM 3.1. *Let (X, T) be a t.d.s. and $K \subset X$ be a compact set. Then K has bounded topological complexity with respect to $\{d_n\}$ if and only if it is equicontinuous.*

Proof. ($\Leftarrow$) Fix $\varepsilon > 0$. By the definition of equicontinuity, there exists $\delta > 0$ such that $d(T^n x, T^n y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ and all $x, y \in K$ with $d(x, y) < \delta$. By the compactness of K, there exists a finite subset F of K such that $K \subset \bigcup_{x \in F} B(x, \delta)$. Then, $K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon)$ for all $n \geq 1$. So, K has bounded topological complexity with respect to $\{d_n\}$.

($\Rightarrow$) Assume the contrary that K is not equicontinuous. There exists $\varepsilon > 0$ such that, for any $k \geq 1$, there are $x_k, y_k \in K$ and $m_k \in \mathbb{N}$ such that $d(x_k, y_k) < 1/k$ and $d(T^{m_k} x_k, T^{m_k} y_k) \geq \varepsilon$. Without loss of generality, we may assume that $x_k \to x_0$ as $k \to \infty$. Then, we have $x_0 \in K$ and $y_k \to x_0$ as $k \to \infty$. For any $k \in \mathbb{N}$, by the triangle inequality, either $d(T^{m_k} x_k, T^{m_k} x_0) \geq \varepsilon/2$ or $d(T^{m_k} y_k, T^{m_k} x_0) \geq \varepsilon/2$. Without loss of generality, we always have $d(T^{m_k} x_k, T^{m_k} x_0) \geq \varepsilon/2$ for all $k \in \mathbb{N}$. Then, $d_{m_k+1}(x_0, x_k) \geq \varepsilon/2$ for all $k \in \mathbb{N}$.

As K has bounded topological complexity with respect to $\{d_n\}$, for the constant $\varepsilon/6$, there exists $C > 0$ such that, for every $n \geq 1$, there exists a subset F_n of K with $\#(F_n) \leq C$ such that $K \subset \bigcup_{x \in F_n} B_{d_n}(x, \varepsilon/6)$. We view $\{F_n\}$ as a sequence in the hyperspace $K(X)$. By the compactness of $K(X)$, there is a subsequence $F_{n_i} \to F$ as $i \to \infty$ in the Hausdorff metric d_H. As $F_n \subset K$ and K is compact, we have $F \subset K$. By the fact $\{A \in K(X) : \#(A) \leq C\}$ is closed, we have $\#(F) \leq C$. For any $i \in \mathbb{N}$ and any $x \in K$, there exists $z_{n_i} \in F_{n_i}$ such that $d_{n_i}(x, z_{n_i}) < \varepsilon/6$. Without loss of generality, assume that $z_{n_i} \to z$ as $i \to \infty$. Then $z \in F$. As the sequence $\{d_n\}$ of metrics is increasing, that is $d_n(u, v) \leq d_{n+1}(u, v)$ for all $u, v \in X$ and $n \in \mathbb{N}$, we have $d_{n_i}(x, z_{n_j}) \leq d_{n_j}(x, z_{n_j}) < \varepsilon$ for all $j \geq i$. Letting j go to infinity, we get $d_{n_i}(x, z) \leq \varepsilon/6$. This implies that

$$K \subset \bigcup_{z \in F} \{x \in K : d_{n_i}(x, z) \leq \varepsilon/6\}$$

for all n_i. By the monotonicity of $\{d_n\}$, we have

$$K \subset \bigcup_{z \in F} \{x \in K : d_n(x, z) \leq \varepsilon/6\}$$

for all $n \in \mathbb{N}$. Enumerate F as $\{z_1, \ldots, z_m\}$ and let

$$K_j = \bigcap_{n=1}^{\infty} \{x \in K : d_n(x, z_j) \leq \varepsilon/6\}$$

for $j = 1, \ldots, m$. Then each K_i is a closed set. By the monotonicity of $\{d_n\}$, we have $K = \bigcup_{j=1}^{m} K_j$.

For the sequence $\{x_k\}$ in K, passing to a subsequence if necessary, we assume that the sequence $\{x_k\}$ is in the same K_j. As K_j is closed, x_0 is also in K_j. Note that, for any $u, v \in K_j$ and any $n \geq 1$, $d_n(u, v) \leq d_n(u, z_j) + d_n(z_j, v) \leq \varepsilon/3$. Particularly, we have $d_{m_k+1}(x_0, x_k) \leq \varepsilon/3$ for any $k \in \mathbb{N}$, which is a contradiction. □

Remark 3.2. In the definition of (n, ε)-spanning set F of K, we require F to be a subset of K. In fact, we can define

$$\mathrm{span}'_K(n, \varepsilon) = \min\left\{\#(F) : F \subset X \text{ and } K \subset \bigcup_{x \in F} B_{d_n}(x, \varepsilon)\right\}.$$

It is clear that $\mathrm{span}_K(n, 2\varepsilon) \leq \mathrm{span}'_K(n, \varepsilon) \leq \mathrm{span}_K(n, \varepsilon)$. So Proposition 3.1 still holds if, in the definition of topological complexity with respect to $\{d_n\}$, we replace $\mathrm{span}_K(n, \varepsilon)$ by $\mathrm{span}'_K(n, \varepsilon)$.

COROLLARY 3.3. *A dynamical system* (X, T) *is equicontinuous if and only if for every* $\varepsilon > 0$ *there exists a positive integer* C *such that* $\mathrm{span}_X(n, \varepsilon) \leq C$ *for all* $n \geq 1$.

Remark 3.4. It is shown in [**2**] that the complexity defined by using the open covers is bounded if and only if the system is equicontinuous. In fact, we can prove Corollary 3.3 by using this result and the fact that [**33**, Theorem 7.7] if α is an open cover of X with Lebesgue number δ, then

$$N\left(\bigvee_{i=0}^{n-1} T^{-i}\alpha\right) \leq \mathrm{span}_X(n, \delta/2).$$

3.2. *Topological complexity with respect to* $\{\hat{d}_n\}$. For $n \in \mathbb{N}$ and $x, y \in X$, define

$$\hat{d}_n(x, y) = \max\left\{\frac{1}{k}\sum_{i=0}^{k-1} d(T^i x, T^i y) : k = 1, 2, \ldots, n\right\}.$$

It is easy to see that, for each $n \in \mathbb{N}$, $\hat{d}_n$ is a metric on X that is topologically equivalent to the metric d. For $x \in X$ and $\varepsilon > 0$, let $B_{\hat{d}_n}(x, \varepsilon) = \{y \in X : \hat{d}_n(x, y) < \varepsilon\}$. Let K be a subset of X. For $n \in \mathbb{N}$ and $\varepsilon > 0$, define

$$\widehat{\mathrm{span}}_K(n, \varepsilon) = \min\left\{\#(F) : F \subset K \subset \bigcup_{x \in F} B_{\hat{d}_n}(x, \varepsilon)\right\}.$$

We say that a subset K of X has *bounded topological complexity with respect to* $\{\hat{d}_n\}$ if, for every $\varepsilon > 0$, there exists a positive integer $C = C(\varepsilon)$ such that $\widehat{\mathrm{span}}_K(n, \varepsilon) \leq C$ for all $n \geq 1$.

As $\hat{d}_n(x, y) \leq d_n(x, y)$ for all $n \in \mathbb{N}$ and $x, y \in X$, if K has bounded topological complexity with respect to $\{d_n\}$, then it is also bounded topological complexity with respect to $\{\hat{d}_n\}$. We say that a subset K of X is *equicontinuous in the mean* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that $\hat{d}_n(x, y) < \varepsilon$ for all $n \in \mathbb{Z}_+$ and all $x, y \in K$ with $d(x, y) < \delta$.

The following result follows the same lines in Theorem 3.1, just replace the distance d_n by $\hat{d}_n$, as the sequence $\{\hat{d}_n\}$ of metrics is also increasing.

THEOREM 3.5. *Let* (X, T) *be a t.d.s. and* K *be a compact subset of* X. *Then,* K *has bounded topological complexity with respect to* $\hat{d}_n$ *if and only if it is equicontinuous in the mean.*

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We say that a subset K of X is *mean equicontinuous* if, for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon$$

for all $x, y \in K$ with $d(x, y) < \delta$. If X is mean equicontinuous, then we say that (X, T) is *mean equicontinuous*. It is clear that if K is equicontinuous in the mean, then it is mean equicontinuous. We can show that for minimal systems they are equivalent.

PROPOSITION 3.6. *Let (X, T) be a minimal t.d.s. Then (X, T) is mean equicontinuous if and only if it is equicontinuous in the mean.*

Proof. It is clear that equicontinuity in the mean implies mean equicontinuity.

Assume that (X, T) is mean equicontinuous. For each $\varepsilon > 0$ there is $\delta_1 > 0$ such that if $d(x, y) < \delta_1$, then

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{\varepsilon}{8}.$$

Fix $z \in X$. For each $N \in \mathbb{N}$, let

$$A_N = \left\{x \in \overline{B(z, \delta_1/2)} : \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i z) \le \frac{\varepsilon}{4}, n = N, N+1, \ldots\right\}.$$

Then, A_N is closed and $\overline{B(z, \delta_1/2)} = \bigcup_{N=1}^{\infty} A_N$. By the Baire Category theorem, there is $N_1 \in \mathbb{N}$ such that A_{N_1} contains an open subset U of X. By the minimality, we know that there is $N_2 \in \mathbb{N}$ with $\bigcup_{i=0}^{N_2-1} T^{-i}U = X$. Let δ_2 be the Lebesgue number of the open cover $\{T^{-i}U : 0 \le i \le N_2 - 1\}$ of X. Let $N = \max\{N_1, 2N_2\}$. By the continuity of T, there exists $\delta_3 > 0$ such that $d(x, y) < \delta_3$ implies that $d(T^i x, T^i y) < \varepsilon/4$ for any $0 \le i \le N$. Put $\delta = \min\{\delta_2, \delta_3\}$. Let $x, y \in X$ with $d(x, y) < \delta$ and $n \in \mathbb{N}$. If $n \le N$, then

$$\frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) \le \frac{1}{n}\cdot n \cdot \frac{\varepsilon}{4} < \varepsilon.$$

If $n > N$, there exists $0 \le i_0 \le N_2 - 1$ such that $x, y \in T^{-i_0}U$, i.e. $T^{i_0}x, T^{i_0}y \in U$, and then

$$\begin{aligned}
\frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) &\le \frac{1}{n}\sum_{i=0}^{i_0-1} d(T^i x, T^i y) + \frac{1}{n}\sum_{i=0}^{n-1} d(T^i T^{i_0} x, T^i T^{i_0} y) \\
&\le \frac{\varepsilon}{4} + \frac{1}{n}\sum_{i=0}^{n-1} d(T^i T^{i_0} x, T^i z) + \frac{1}{n}\sum_{i=0}^{n-1} d(T^i T^{i_0} x, T^i z) \\
&\le \frac{\varepsilon}{4} + \frac{\varepsilon}{4} + \frac{\varepsilon}{4} < \varepsilon.
\end{aligned}$$

Therefore, $\hat{d}_n(x, y) < \varepsilon$ for all $n \in \mathbb{Z}_+$. This implies that (X, T) is equicontinuous in the mean. □

Remark 3.7. When this paper was finished, we became aware of the work of [**31**] that Qiu and Zhao can show that in general a t.d.s. is mean equicontinuous if and only if it is equicontinuous in the mean.

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3.3. *Topological complexity with respect to* $\{\bar{d}_n\}$. For $n \in \mathbb{N}$ and $x, y \in X$, define

$$\bar{d}_n(x, y) = \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y).$$

It is easy to see that, for each $n \in \mathbb{N}$, $\bar{d}_n$ is a metric on X that is topologically equivalent to the metric d. For $x \in X$ and $\varepsilon > 0$, let $B_{\bar{d}_n}(x, \varepsilon) = \{y \in X : \bar{d}_n(x, y) < \varepsilon\}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, define

$$\overline{\mathrm{span}}_K(n, \varepsilon) = \min\bigg\{\#(F) : F \subset K \subset \bigcup_{x \in F} B_{\bar{d}_n}(x, \varepsilon)\bigg\}.$$

We say that a subset K of X has *bounded topological complexity with respect to* $\{\bar{d}_n\}$ if, for every $\varepsilon > 0$, there exists a positive integer $C = C(\varepsilon)$ such that $\overline{\mathrm{span}}_K(n, \varepsilon) \le C$ for all $n \ge 1$.

As $\bar{d}_n(x, y) \le \hat{d}_n(x, y)$, for all $n \in \mathbb{N}$ and $x, y \in X$, if K has bounded topological complexity with respect to $\{\hat{d}_n\}$, then it also has bounded topological complexity with respect to $\{\bar{d}_n\}$. Intuitively, dynamical systems with bounded topological complexity with respect to $\{\bar{d}_n\}$ should have similar properties to those with respect to $\{\hat{d}_n\}$ or $\{d_n\}$. But we will see that this is far from being true. The key point is that the sequence $\{\bar{d}_n\}$ of metrics might be not monotonous. If a dynamical system has bounded topological complexity with respect to $\{\bar{d}_n\}$, then by Theorem 4.7 in the next section every invariant measure has a discrete spectrum. So it is simple in the measure-theoretic sense. But, we have the following proposition that is a surprise in some sense. Since the construction is somewhat long and complicated, we move it to Appendix A.

PROPOSITION 3.8. *There is a distal, non-equicontinuous, non-uniquely ergodic, minimal system that has bounded topological complexity with respect to* $\{\bar{d}_n\}$.

We can modify the example in Proposition 3.8 to be uniquely ergodic and we also present the construction in Appendix A.

PROPOSITION 3.9. *There is a distal, non-equicontinuous, uniquely ergodic, minimal system that has bounded topological complexity with respect to* $\{\bar{d}_n\}$.

Remark 3.10. As each distal mean equicontinuous minimal system is equicontinuous, the systems constructed in Propositions 3.8 and 3.9 are not mean equicontinuous.

We have a natural question.

Question 1. Is there a non-trivial weakly mixing, or even strongly mixing minimal system, with bounded topological complexity with respect to $\{\bar{d}_n\}$?

We are informed by Huang and Xu [**19**] that the above question has an affirmative answer for weakly mixing minimal systems. The question of whether there is a non-trivial strongly mixing minimal system with bounded topological complexity with respect to $\{\bar{d}_n\}$ is still open.

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4. *Invariant measures with bounded measure-theoretic complexity*

In this section, we will study the measure-theoretic complexity of invariant (Borel probability) measures with respect to three kinds of metrics.

4.1. *Measure-theoretic complexity with respect to $\{d_n\}$.*

Let (X, T) be a t.d.s. and μ be an invariant measure on (X, T). For $n \in \mathbb{N}$ and $\varepsilon > 0$, let

$$\operatorname{span}_\mu(n, \varepsilon) = \min\bigg\{\#(F) : F \subset X \text{ and } \mu\bigg(\bigcup_{x\in F} B_{d_n}(x, \varepsilon)\bigg) > 1 - \varepsilon\bigg\}.$$

Recall that this is the same notion that is defined in [**21**] by Katok. We say that μ has *bounded complexity with respect to* $\{d_n\}$ if, for every $\varepsilon > 0$, there exists a positive integer $C = C(\varepsilon)$ such that $\operatorname{span}_\mu(n, \varepsilon) \le C$ for all $n \ge 1$.

We will show that an invariant measure with bounded complexity with respect to $\{d_n\}$ is equivalent to the μ-equicontinuity property.

THEOREM 4.1. *Let (X, T) be a t.d.s. and μ be an invariant measure on (X, T). Then, μ has bounded complexity with respect to $\{d_n\}$ if and only if T is μ-equicontinuous.*

Proof. ($\Leftarrow$) First assume that (X, T) is μ-equicontinuous. Fix $\varepsilon > 0$. There exists a T-equicontinuous measurable subset K of X with $\mu(K) > 1 - \varepsilon$. As the measure μ is regular, we can require the set K to be compact. Now, the result follows from Theorem 3.1, as $\operatorname{span}_\mu(n, \varepsilon) \le \operatorname{span}_K(n, \varepsilon)$.

($\Rightarrow$) For any $\tau > 0$, we need to find a T-equicontinuous set K with $\mu(K) > 1 - \tau$. Now, fix $\tau > 0$. As μ has bounded complexity with respect to $\{d_n\}$, for any $M > 0$, there exists $C = C_M > 0$ such that, for every $n \ge 1$, there exists a subset F_n of X with $\#(F_n) \le C$ such that

$$\mu\bigg(\bigcup_{x\in F_n} B_{d_n}\bigg(x, \frac{1}{M}\bigg)\bigg) > 1 - \frac{\tau}{2^{M+2}}.$$

As the measure μ is regular, pick a compact subset K_n of $\bigcup_{x\in F_n} B_{d_n}(x, 1/M)$ with $\mu(K_n) > 1 - \tau/2^{M+2}$. Without loss of generality, assume that $F_n \to F_M$, $K_n \to K_M$ as $n \to \infty$ in the Hausdorff metric. Then, $\#(F_M) \le C$. As K_n is closed,

$$\mu(K_M) \ge \limsup_{n\to\infty} \mu(K_n) \ge 1 - \frac{\tau}{2^{M+2}}.$$

For any $x \in K_M$ and $n \in \mathbb{N}$, there exists an $N > 0$ such that, for any $k > N$, there exists $x_k \in K_k$ and $y_k \in F_k$ such that $d_n(x, x_k) < 1/M$ and $d_k(x_k, y_k) < 1/M$. Without loss of generality, assume that $y_k \to y$ as $k \to \infty$. Then, $y \in F_M$. By the monotonicity of $\{d_n\}$, we have

$$d_n(x, y_k) \le d_n(x, x_k) + d_n(x_k, y_k) \le d_n(x, x_k) + d_k(x_k, y_k) \le \frac{2}{M}.$$

Letting k go to infinity, we have $d_n(x, y) \le 2/M$. Then, $K_M \subset \bigcup_{x\in F_M} B_{d_n}(x, 3/M)$ and $\operatorname{span}_{K_M}(n, 3/M) \le \#(F_M) \le C_M$.

Let $K = \bigcap_{M=1}^{\infty} K_M$. Then, $\mu(K) > 1 - \tau$ and for any $M \ge 1$,

$$\operatorname{span}_K\bigg(n, \frac{3}{M}\bigg) \le \operatorname{span}_{K_M}\bigg(n, \frac{3}{M}\bigg) \le C_M$$

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for all $n \geq 1$. Now, by Theorem 3.1, K is T-equicontinuous. This proves that (X, T) is μ-equicontinuous. □

Remark 4.2. Similar to the observation in Remark 3.4, the open cover version of Theorem 4.1 was proved in [**17**, Proposition 3.3].

4.2. *Measure-theoretic complexity with respect to* $\{\hat{d}_n\}$. For $n \in \mathbb{N}$ and $\varepsilon > 0$, let

$$\widehat{\mathrm{span}}_\mu(n, \varepsilon) = \min\bigg\{\#(F) : F \subset X \text{ and } \mu\bigg(\bigcup_{x\in F} B_{\hat{d}_n}(x, \varepsilon)\bigg) > 1 - \varepsilon\bigg\}.$$

We say that μ has *bounded complexity with respect to* $\{\hat{d}_n\}$ if, for every $\varepsilon > 0$, there exists a positive integer $C = C(\varepsilon)$ such that $\widehat{\mathrm{span}}_\mu(n, \varepsilon) \leq C$ for all $n \geq 1$.

We will show that an invariant measure with bounded complexity with respect to $\{\hat{d}_n\}$ is equivalent to the following two kinds of measure-theoretic equicontinuity. We say that T is *μ-equicontinuous in the mean* if, for any $\tau > 0$, there exists a measurable subset K of X with $\mu(K) > 1 - \tau$ that is equicontinuous in the mean, and *μ-mean equicontinuous* if, for any $\tau > 0$, there exists a measurable subset K of X with $\mu(K) > 1 - \tau$ that is mean equicontinuous.

THEOREM 4.3. *Let (X, T) be a t.d.s. and μ be an invariant measure on (X, T). Then, the following statements are equivalent:*
(1) *μ has bounded complexity with respect to $\hat{d}_n$;*
(2) *T is μ-equicontinuous in the mean;*
(3) *T is μ-mean equicontinuous.*

Proof. (1) ⇒ (2) Following the proof of Theorem 4.1, we know that, for a given $\tau > 0$, there is a compact subset K such that $\mu(K) \geq 1 - \tau$ and, for any $M \geq 1$, $\widehat{\mathrm{span}}_K(n, 6/M) \leq C_M$ for all $n \geq 1$. By Theorem 3.5, K is equicontinuous in the mean. This proves that (X, T) is μ-equicontinuous in the mean.

(2) ⇒ (3) is obvious.

(3) ⇒ (1) Now assume that (X, T) is μ-mean equicontinuous. Fix $\varepsilon > 0$. Then, there is a compact $K = K(\varepsilon) \subset X$ such that $\mu(K) > 1 - 2\varepsilon$ and K is mean equicontinuous. There exists a $\delta > 0$ such that

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \varepsilon/4$$

for all $x, y \in K$ with $d(x, y) < \delta$. As K is compact, there exists a finite subset F of K such that $K \subset \bigcup_{x\in F} B(x, \delta)$. Enumerate F as $\{x_1, x_2, \ldots, x_m\}$. For $j = 1, \ldots, m$ and $N \in \mathbb{N}$, let

$$A_N(x_j) = \bigg\{y \in B(x_j, \delta) \cap K : \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x_j, T^i y) < \varepsilon/2, n = N, N+1, \ldots\bigg\}.$$

It is easy to see that, for each $j = 1, \ldots, m$, $\{A_N(x_j)\}_{N=1}^\infty$ is an increasing sequence and $B(x_j, \delta) \cap K = \bigcup_{N=1}^\infty A_N(x_j)$. Choose $N_1 \in \mathbb{N}$ and a compact subset K_1 of $A_{N_1}(x_1)$

such that $\mu(K_j) > \mu(B(x_j, \delta) \cap K) - \varepsilon/2m$. Choose $N_2 \in \mathbb{N}$ and a compact subset K_2 of $A_{N_2}(x_2)$ such that $K_1 \cap K_2 = \emptyset$ and $\mu(K_1 \cup K_2) > \mu((B(x_1, \delta) \cup B(x_2, \delta)) \cap K) - 2\varepsilon/2m$. By induction, we can choose compact subsets K_j of $A_{N_j}(x_j)$, for $j = 1, \ldots, m$, with $\mu(\bigcup_{j=1}^m K_j) > \mu(K) - \varepsilon/2 > 1 - \varepsilon$ and $K_i \cap K_j = \emptyset$, for $1 \le i < j \le m$.

Let $K_0 = \bigcup_{j=1}^m K_j$ and $N_0 = \max\{N_j : j = 1, 2, \ldots, m\}$. There exists $\delta_1 > 0$ such that for every $x, y \in K$ with $d(x, y) < \delta_1$, there exists $j \in \{1, 2, \ldots, m\}$ with $x, y \in K_j$. By the continuity of T, there exists $\delta_2 > 0$ such that $d_N(x, y) < \varepsilon$ for every $x, y \in X$ with $d(x, y) < \delta_2$. Let $\delta_3 = \min\{\delta_1, \delta_2\}$. By the compactness of K_0, there exists a finite subset H of K_0 such that $H \subset \bigcup_{x\in H} B(x, \delta_3)$. Fix $n \ge 1$ and $y \in K_0$. There exists $x \in H$ with $d(x, y) < \delta_3$. If $n < N_0$, then $\hat{d}_n(x, y) \le d_{N_0}(x, y) < \varepsilon$. If $n \ge N_0$, there exists $j \in \{1, 2, \ldots, m\}$ with $x, y \in K_j \subset A_{N_j}(x_j)$. By the construction of $A_{N_j}(x_j)$ and $n \ge N_j$,

$$\frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i y) \le \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x, T^i x_j) + \frac{1}{n}\sum_{i=0}^{n-1} d(T^i x_j, T^i y) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

For any $n \ge 1$, we have $\hat{d}_n(x, y) < \varepsilon$. Then,

$$K_0 \subset \bigcup_{x\in H} B_{\hat{d}_n}(x, \varepsilon)$$

and

$$\mu\bigg(\bigcup_{x\in H} B_{\hat{d}_n}(x, \varepsilon)\bigg) \ge \mu(K_0) > 1 - \varepsilon.$$

This implies that $\widehat{\text{span}}_\mu(n, \varepsilon) \le \#(H)$ for all $n \ge 1$. Then, μ has bounded complexity with respect to $\{\hat{d}_n\}$. □

4.3. *Measure-theoretic complexity with respect to* $\{\bar{d}_n\}$. For $n \in \mathbb{N}$, $\varepsilon > 0$, let

$$\overline{\text{span}}_\mu(n, \varepsilon) = \min\bigg\{\#(F) : F \subset X \text{ and } \mu\bigg(\bigcup_{x\in F} B_{\bar{d}_n}(x, \varepsilon)\bigg) > 1 - \varepsilon\bigg\}.$$

We say that μ has *bounded complexity with respect to* $\{\bar{d}_n\}$ if, for every $\varepsilon > 0$, there exists a positive integer $C = C(\varepsilon)$ such that $\overline{\text{span}}_\mu(n, \varepsilon) \le C$ for all $n \ge 1$.

Unlike the topological case, we can prove that bounded measure-theoretic complexity with respect to $\{\bar{d}_n\}$ and $\{\hat{d}_n\}$ are equivalent.

THEOREM 4.4. *Let (X, T) be a t.d.s. and μ be an invariant measure on (X, T). Then, μ has bounded complexity with respect to $\{\bar{d}_n\}$ if and only if it has bounded complexity with respect to $\{\hat{d}_n\}$.*

Proof. It is clear that if μ has bounded complexity with respect to $\{\hat{d}_n\}$, then by definition it also has bounded complexity with respect to $\{\bar{d}_n\}$.

Now assume that μ has bounded complexity with respect to $\{\bar{d}_n\}$. Let $\varepsilon > 0$. There is $C = C(\varepsilon)$ such that, for any $n \in \mathbb{N}$, there is $F_n \in X$ with $\#(F_n) \le C$ such that

$$\mu\bigg(\bigcup_{x\in F_n} B_{\bar{d}_n}(x, \varepsilon/8)\bigg) > 1 - \varepsilon/8.$$

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By the Birkhoff pointwise ergodic theorem for $\mu \times \mu$ almost every (a.e.) $(x, y) \in X^2$

$$\bar{d}_N(x, y) = \frac{1}{N} \sum_{i=0}^{N-1} d(T^i x, T^i y) \to d^*(x, y).$$

So, for a given $0 < r < \min\{1, \varepsilon/2C\}$, by Egorov's theorem there are $R \subset X^2$ with $\mu \times \mu(R) > 1 - r^2$ and $N_0 \in \mathbb{N}$ such that if $(x, y) \in R$, then

$$|\bar{d}_n(x, y) - \bar{d}_{N_0}(x, y)| < r \quad \text{for } n \geq N_0.$$

By Fubini's theorem, there is $A \subset X$ such that $\mu(A) > 1 - r$ and for any $x \in A$, $\mu(R_x) > 1 - r$, where

$$R_x = \{y \in X : (x, y) \in R\}.$$

Enumerate $F_{N_0} = \{x_1, x_2, \ldots, x_m\}$. Then, $m \leq C$. Let

$$I = \{1 \leq i \leq m : A \cap B_{\bar{d}_{N_0}}(x_i, \varepsilon/8) \neq \emptyset\}.$$

Denote $\#(I) = m'$. Then, $1 \leq m' \leq m$. For each $i \in I$, pick $y_i \in A \cap B_{\bar{d}_{N_0}}(x_i, \varepsilon/8)$. Then, we have $B_{\bar{d}_{N_0}}(x_i, \varepsilon/8) \subset B_{\bar{d}_{N_0}}(y_i, \varepsilon/4)$ for all $i \in I$. As

$$\mu\bigg(A \cap \bigcap_{i \in I} R_{y_i} \cap \bigcup_{x \in F_{N_0}} B_{\bar{d}_{N_0}}(x, \varepsilon/8)\bigg) \geq 1 - r - m'r - \varepsilon/8 > 1 - \varepsilon,$$

choose a compact subset

$$K \subset A \cap \bigcap_{i \in I} R_{y_i} \cap \bigcup_{x \in F_{N_0}} B_{\bar{d}_{N_0}}(x, \varepsilon/8)$$

with $\mu(K) > 1 - \varepsilon$. If $x \in K$, there exists $i \in I$ such that $x \in R_{y_i} \cap B_{\bar{d}_{N_0}}(y_i, \varepsilon/4)$. Then, $(y_i, x) \in R$. By the construction of R, for any $n \geq N_0$,

$$\bar{d}_n(x, y_i) = \bar{d}_n(y_i, x) \leq \bar{d}_{N_0}(y_i, x) + r < \varepsilon/4 + r < \varepsilon/2.$$

Let $\delta_1 > 0$ be a Lebesgue number of the open cover of K by

$$\{K \cap B_{\bar{d}_{N_0}}(y_i, \varepsilon/4) : i \in I\}.$$

By the continuity of T, there exists $0 < \delta < \delta_1$ such that if $d(x_1, x_2) < \delta$, then $d_{N_0}(x_1, x_2) < \varepsilon$. Let $x_1, x_2 \in K$ with $d(x_1, x_2) < \delta$. There is $i \in I$ such that $x_1, x_2 \in A_{y_i} \cap B_{\bar{d}_{N_0}}(y_i, \varepsilon/4)$. Fix $n \geq 1$. If $n < N_0$, $\bar{d}_n(x_1, x_2) \leq d_{N_0}(x_1, x_2) < \varepsilon$. If $n \geq N_0$,

$$\bar{d}_n(x_1, x_2) \leq \bar{d}_n(x_1, y_i) + \bar{d}_n(x_2, y_i) < \varepsilon/2 + 2r < \varepsilon.$$

Then, $\hat{d}_n(x_1, x_2) < \varepsilon$ for all $n \geq 1$. By the compactness of K, there exists a finite subset H of K such that $K \subset \bigcup_{x \in H} B(x, \delta)$. For any $n \geq 1$, we have

$$K \subset \bigcup_{x \in H} B_{\hat{d}_n}(x, \varepsilon)$$

and then,

$$\mu\bigg(\bigcup_{x \in H} B_{\hat{d}_n}(x, \varepsilon)\bigg) \geq \mu(K) > 1 - \varepsilon.$$

This implies that $\widehat{\text{span}}_\mu(n, \varepsilon) \leq \#(H)$ for all $n \geq 1$. Then, μ has bounded complexity with respect to $\{\hat{d}_n\}$. $\square$

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We can restate [**18**, Proposition 4.1] as follows.

PROPOSITION 4.5. *Let (X, T) be an invertible t.d.s. and μ be an invariant measure on (X, T). If μ has discrete spectrum, then it has bounded complexity with respect to $\{\bar{d}_n\}$.*

It is conjectured in [**18**] that the converse of Proposition 4.5 is also true. If μ is ergodic, by [**13**, Corollary 39], we know that μ has discrete spectrum if and only if μ is mean equicontinuous. So, by Theorem 4.3, if an ergodic measure μ has bounded complexity with respect to $\{\bar{d}_n\}$, then it has discrete spectrum. We will show in Theorem 4.7 that, in general, the converse of Proposition 4.5 is also true.

The following result was proved in [**24**, Theorem 2.7], see also [**13**, Corollary 39]. Here, we provide a different direct proof.

PROPOSITION 4.6. *Let (X, T) be a t.d.s. and μ be an ergodic invariant measure on (X, T). If μ does not have discrete spectrum, then there exists $\alpha > 0$ such that, for $\mu \times \mu$-a.e. pair $(x, y) \in X \times X$,*

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \alpha.$$

Proof. Let $\mathcal{B}_\mu$ be the completion of the Borel σ-algebra $\mathcal{B}_X$ of X with respect to μ. Corresponding to the discrete part of the spectrum of the action of T, there exists a compact metric abelian group $(G, +)$ with Haar measure ν, an element τ of G such that $(G, \mathcal{B}_\nu, \nu, S)$ is the Kronecker factor of $(X, \mathcal{B}_\mu, \mu, T)$ with an associated factor map $\pi : X \to G$, where $\mathcal{B}_\nu$ be the completion of the Borel σ-algebra of G with respect to ν and S is the translation by τ on G.

Let $\mu = \int_G \mu_z \, d\nu(z)$ be the disintegration of the measure μ over ν. For $s \in G$, let

$$\lambda_s = \int_G \mu_z \times \mu_{z+s} \, d\nu(z).$$

It is a classical result (see e.g. [**15**, §4.3.1 Theorem 18]) that there is $G_0 \subset G$ with $\nu(G_0) = 1$ such that, for every $s \in G_0$, the system $(X \times X, \lambda_s, T \times T)$ is ergodic and

$$\mu \times \mu = \int_G \lambda_s \, d\nu(s)$$

is the ergodic decomposition $\mu \times \mu$ under $T \times T$.

By the Birkhoff ergodic theorem, the limit

$$\lim_{n\to+\infty} \bar{d}_n(x, y)$$

exists and is equal to

$$\int_{X\times X} d(x_1, x_2) \, d\lambda_s(x_1, x_2)$$

for some $s = s(x, y) \in G_0$ for $\mu \times \mu$-a.e. $(x, y) \in X^2$.

Now, it is sufficient to show that if $(X, \mathcal{B}_\mu, \mu, T)$ does not have a discrete spectrum, then there exists $\alpha > 0$ such that $\int_{X\times X} d(x_1, x_2) \, d\lambda_s(x_1, x_2) \geq \alpha$ for all $s \in G_0$.

As X is compact, pick a countable dense subset $\{y_n : n \in \mathbb{N}\}$ in X. For $z \in G$,

$$c(z) := \inf_{n\in\mathbb{N}} \int_X d(x, y_n)\, d\mu_z(x).$$

It is clear that $c(z) > 0$ if and only if μ_z is not a Dirac measure. Moreover, $c(\cdot)$ is a non-negative measurable function on G. Put

$$\alpha := \int_G c(z)\, d\nu(z).$$

Since $(X, \mathcal{B}_\mu, \mu, T)$ is ergodic and does not have discrete spectrum, by Rohlin's theorem μ_z is not a Dirac measure for ν-a.e. $z \in G$. This means that $c(z) > 0$ for ν-a.e. $z \in G$ and thus $\alpha > 0$. For each $y \in X$, there exists a subsequence $\{n_i\}$ such that $y_{n_i} \to y$ as $i \to \infty$. Then, for each $x \in X$, $d(x, y_{n_i}) \to d(x, y)$ as $i \to \infty$. By the Lebesgue dominated convergence theorem, for each $z \in G_0$,

$$\int_X d(x, y)\, d\mu_z(x) = \lim_{i\to\infty} \int_X d(x, y_{n_i})\, d\mu_z(x) \geq c(z).$$

Thus, for each $s \in G_0$,

$$\begin{aligned}
\int_{X\times X} d(x_1, x_2)\, d\lambda_s(x_1, x_2) &= \int_G \bigg(\int_{X\times X} d(x, y)\, d\mu_z \times \mu_{z+s}(x, y)\bigg)\, d\nu(z) \\
&= \int_G \bigg(\int_X \bigg(\int_X d(x, y)\, d\mu_z(x)\bigg)\, d\mu_{z+s}(y)\bigg)\, d\nu(z) \\
&\geq \int_G \int_X c(z)\, d\mu_{z+s}(y)\, d\nu(z) \\
&= \int_G c(z)\, d\nu(z) = \alpha > 0.
\end{aligned}$$

This finishes the proof. □

Now we are able to show the converse of Proposition 4.5.

THEOREM 4.7. *Let (X, T) be an invertible t.d.s. and μ be an invariant measure on (X, T). If μ has bounded complexity with respect to $\{\bar{d}_n\}$, then it has discrete spectrum.*

Proof. Let A be the collection of points $z \in X$ that are generic to some ergodic measure, that is, for each $z \in A$, $(1/n) \sum_{i=0}^{n-1} \delta_{T^i z} \to \mu_z$ as $n \to \infty$ and μ_z is ergodic. Then, A is measurable and $\mu(A) = 1$. We first prove the following claim.

CLAIM 1. *μ_z has discrete spectrum for μ-a.e. $z \in A$.*

Proof of the Claim 1. Let $A_1 = \{z \in A : \mu_z$ does not has discrete spectrum$\}$. We need to prove that A_1 is measurable and has zero μ-measure. The ergodic decomposition of μ can be expressed as $\mu = \int_A \mu_z\, d\mu(z)$ (see e.g. [**28**, Theorem 6.4]). For $k \in \mathbb{N}$ and $z \in A$, put

$$F_k(z) = \mu_z \times \mu_z\bigg(\bigg\{(x, y) \in X \times X : \liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \frac{1}{k}\bigg\}\bigg).$$

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As $\int_A \mu_z \times \mu_z \, d\mu(z)$ is an invariant measure on $(X \times X, T \times T)$, for each $k \in \mathbb{N}$, F_k is a measurable function on A. By Theorem 4.6, we know that μ_z does not have discrete spectrum if and only if there exists $k \in \mathbb{N}$ such that $F_k(z) = 1$. Then,

$$A_1 = \bigcup_{k \in \mathbb{N}} \{z \in A : F_k(z) = 1\}$$

and it is measurable. Now it is sufficient to prove $\mu(G_1) = 0$. If not, then $\mu(A_1) > 0$ and there exists $k \in \mathbb{N}$ such that $\mu(\{z \in A : F_k(z) = 1\}) > 0$. Let $A_2 = \{z \in A : F_k(z) = 1\}$ and put $r = \mu(A_2)$. Then, for every $z \in A_2$ and for $\mu_z \times \mu_z$-a.e. $(x, y) \in X \times X$,

$$\liminf_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) > \frac{1}{k}. \tag{1}$$

By Theorems 4.3 and 4.4, (X, T) is μ-mean equicontinuous. Then, there exists $M \subset X$ with $\mu(M) > 1 - r^2/4$ such that M is mean equicontinuous. By regularity of μ, we can assume that M is compact and $M \subset A$. Let $A_3 = \{z \in A : \mu_z(M) > 1 - r/2\}$. Then, A_3 is measurable, as $\mu = \int_A \mu_z \, d\mu(z)$ is the ergodic decomposition of μ. We have

$$\begin{aligned} 1 - \frac{r^2}{4} < \mu(M) = \int_A \mu_z(M)\, d\mu(z) &\le \int_{A_3} \mu_z(M)\, d\mu(z) + \int_{A \setminus A_3} \mu_z(M)\, d\mu(z) \\ &\le \mu(A_3) + (1 - \mu(A_3))\left(1 - \frac{r}{2}\right), \end{aligned}$$

which implies that $\mu(A_3) > 1 - r/2$. Then, $\mu(A_2 \cap A_3) > r + (1 - r/2) - 1 = r/2 > 0$. Pick $z \in A_2 \cap A_3$. As M is mean equicontinuous, there exists a $\delta > 0$ such that, for any $x, y \in M$, with $d(x, y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{1}{k}.$$

As M is compact, there exists a finite open cover $\{U_1, U_2, \ldots, U_m\}$ of M, with diameter less than δ. Since $z \in A_3$, $\mu_z(M) > 1 - r/2$. Then, there exists $i \in \{1, \ldots, m\}$ such that $\mu_z(U_i) > 0$ and also $\mu_z \times \mu_z(U_i) > 0$. Note that the diameter of U_i is less than δ, so for any $x, y \in U_i$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} d(T^i x, T^i y) < \frac{1}{k},$$

which contradicts (1). This ends the proof of Claim 1. □

Let

$$A_0 = \{z \in A : \mu_z \text{ has discrete spectrum}\}.$$

By Claim 1, we have $\mu(A_0) = 1$. Let $f \in C(X)$ be a Lipschitz continuous function on X. Then, there exists $C > 0$ such that $|f(x) - f(y)| \le C d(x, y)$ for all $x, y \in X$.

Recall that the associated operator $U : L^2(\mu) \to L^2(\mu)$ is defined by $Uf = f \circ T$ for all $f \in L^2(\mu)$. Inspired by the idea of [**30**, Theorem 1], we have the following claim.

CLAIM 2. *For any $\tau > 0$, there exists $M^* \in \mathcal{B}$ with $\mu(M^*) > 1 - \tau$ such that $f \cdot \mathbf{1}_{M^*}$ is almost periodic, i.e. $\{U^n(f \cdot \mathbf{1}_{M^*}) : n \in \mathbb{Z}\}$ is precompact in $L^2(\mu)$.*

Proof of the Claim 2. By Theorems 4.3 and 4.4, (X, T) is μ-mean equicontinuous. Fix a constant $\tau > 0$. Then, there exists $M \subset X$ with $\mu(M) > 1 - \tau$ such that M is mean equicontinuous. Let $M^* = \bigcup_{n\in\mathbb{Z}} T^{-n}M$. To show that $f \cdot \mathbf{1}_{M^*}$ is almost periodic, we only need to prove for any sequence $\{t_n\}$ in $\mathbb{Z}$ that there exists a subsequence $\{s_n\}$ of $\{t_n\}$ such that $\{U^{s_n}(f \cdot \mathbf{1}_{M^*})\}$ is a Cauchy sequence in $L^2(\mu)$.

By regularity of μ, we can assume that M is compact and $M \subset A_0$. Choose a countable dense subset $\{z_m\}$ in M. As μ_{z_1} has discrete spectrum, there exists a subsequence $\{t_{n,1}\}$ of $\{t_n\}$ such that $\{U^{t_{n,1}} f : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\mu_{z_1})$. Inductively, assume that for each $i \le m-1$ we have defined $\{t_{n,i}\}$ (which is a subsequence of $\{t_{n,i-1}\}$) such that $\{U^{t_{n,i}} f : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\mu_{z_i})$. As μ_{z_m} has discrete spectrum, there exists a subsequence $\{t_{n,m}\}$ of $\{t_{n,m-1}\}$ such that $\{U^{t_{n,m}} f : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\mu_{z_m})$. Let $s_n = t_{n,n}$ for $n \ge 1$. By the usual diagonal procedure, $\{U^{s_n} f : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\mu_{z_m})$ for all $m \ge 1$.

Fix $\varepsilon > 0$. As M is mean equicontinuous in (X, T), there exists $\delta > 0$ such that, for any $x, y \in M$, with $d(x, y) < \delta$,

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^i x, T^i y))^2 < \varepsilon.$$

Fix $z \in M$. There exists $m \in \mathbb{N}$ such that $d(z, z_m) < \delta$. For any $j \ne k \in \mathbb{N}$,

$$\begin{aligned}
\|U^{s_j} f - U^{s_k} f\|^2_{L^2(\mu_z)} &= \int_X |U^{s_j} f - U^{s_k} f|^2 \, d\mu_z \\
&= \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} |f(T^{s_j+i}z) - f(T^{s_k+i}z)|^2 \\
&\le C^2 \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^{s_j+i}z, T^{s_k+i}z))^2 \\
&\le C^2 \bigg(\limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^{s_j+i}z, T^{s_j+i}z_m))^2 \\
&\quad + \limsup_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^{s_k+i}z, T^{s_k+i}z_m))^2 \\
&\quad + \lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} (d(T^{s_j+i}z_m, T^{s_j+i}z_m))^2 \bigg) \\
&\le C^2 (2\varepsilon + \|U^{s_j} f - U^{s_k} f\|^2_{L^2(\mu_{z_m})}).
\end{aligned}$$

As $\{U^{s_n} f : n \in \mathbb{N}\}$ is a Cauchy sequence in $L^2(\mu_{z_m})$ for all $m \ge 1$, $\{U^{s_n} f : n \in \mathbb{N}\}$ is also a Cauchy sequence in $L^2(\mu_z)$. Then, for each $z \in M$,

$$\lim_{N\to\infty} \sup_{j,k\ge N} \int_X |U^{s_j} f - U^{s_k} f|^2 \, d\mu_z = 0.$$

For each $y \in M^*$, there exists $n \in \mathbb{Z}$ and $z \in M$ such that $T^n z = y$. Then, $\mu_z = \mu_y$. For $z \in M^*$, put

$$f_N(z) = \sup_{j,k\ge N} \int_X |U^{s_j} f - U^{s_k} f|^2 \, d\mu_z.$$

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By the dominated convergence theorem,

$$\lim_{N\to\infty}\int_{M^*} f_N(z)\,d\mu(z)=\int_{M^*}\lim_{N\to\infty} f_N(x)\,d\mu(z)=0. \tag{2}$$

It is easy to see that

$$\begin{aligned}\sup_{j,k\geq N}\int_{M^*}\int_X |U^{s_j}f-U^{s_k}f|^2\,d\mu_z\,d\mu(z)\\ &\leq\int_{M^*}\bigg(\sup_{j,k\geq N}\int_X |U^{s_j}f-U^{s_k}f|^2\,d\mu_z\bigg)\,d\mu(z)\\ &=\int_{M^*} f_N(z)\,d\mu(z),\end{aligned}$$

from which we deduce

$$\lim_{N\to\infty}\bigg(\sup_{j,k\geq N}\int_{M^*}\int_X |U^{s_j}f-U^{s_k}f|^2\,d\mu_z\,d\mu(z)\bigg)=0\quad\text{by (2)}.$$

As $\int_{M^*} g\,d\mu=\int_{M^*}(\int g\,d\mu_z)\,d\mu(z)$, for any $g\in L^2(\mu)$, we have

$$\lim_{N\to\infty}\bigg(\sup_{j,k\geq N}\int_{M^*}|U^{s_j}f-U^{s_k}f|^2\,d\mu(z)\bigg)=0.$$

Note that $T(M^*)=M^*$, so

$$\int_{M^*}|U^{s_j}f-U^{s_k}f|^2\,d\mu(z)=\int|U^{s_j}(f\cdot\mathbf{1}_{M^*})-U^{s_k}(f\cdot\mathbf{1}_{M^*})|^2\,d\mu.$$

Thus, $\{U^{s_n}(f\cdot\mathbf{1}_{M^*})\}$ is a Cauchy sequence in $L^2(\mu)$. This ends the proof of Claim 2. □

Note that the collection of almost periodic functions g is closed in $L^2(\mu)$. As the measure of M^* in Claim 2 can be arbitrarily close to 1, f is also an almost periodic function in $L^2(\mu)$. As the collection of Lipschitz continuous functions is dense in $C(X)$ (see e.g. [**5**, Theorem 11.2.4.]) and $C(X)$ is dense in $L^2(\mu)$, then for every function $g\in L^2(\mu)$ is almost periodic in $L^2(\mu)$, that is μ has discrete spectrum. □

Remark 4.8. After we had finished this paper, Nhan-Phu Chung informed us that Theorem 4.7 was also proved in [**32**, Theorem 3.2] by a different method. Note that an invariant measure μ has bounded complexity with respect to $\{\bar{d}_n\}$ in our sense if and only if every $\varepsilon>0$ and the scaling sequences with respect to μ and d are bounded as in [**32**, Definition 3.1]. It should be noticed that in the introduction of [**32**] it requires a mild condition that the standard (Lebesgue) space (X,μ) is non-atomic.

In Theorem 4.7, we show that if an invariant measure μ of a t.d.s. (X,T) has bounded complexity with respect to $\{\bar{d}_n\}$, then almost all of the ergodic components in the ergodic decomposition of μ have discrete spectrum. In the following remark we provide an example which shows that it may happen there are uncountably many pairwise non-isomorphic ergodic components in the ergodic decomposition, and the set of unions of all eigenvalues of the ergodic components are countable.

Remark 4.9. The space X is the product $\{0, 1\}^{\mathbb{N}} \times (S^1)^{\mathbb{N}}$. Let $\{\tau_i : i \in \mathbb{N}\}$ be a family of irrational numbers independent over the rational numbers. The measure μ is the product of the Bernoulli measure $(\frac{1}{2}, \frac{1}{2})$ on $\{0, 1\}^{\mathbb{N}}$ and the product measure $\lambda_{\mathbb{N}}$ on $(S^1)^{\mathbb{N}}$, where each coordinate is equipped with the Lebesgue measure λ.

The transformation $T : X \to X$ is defined in the following way: let $\omega = (\omega_i)_{i\geq 1} \in \{0, 1\}^{\mathbb{N}}$ and $w = (w)_{i\geq 1} \in (S^1)^{\mathbb{N}}$. Define $T(\omega, w) = (\omega, w')$, where $(w')_i = w_i$ if $\omega_i = 0$ and $(w')_i = T_i w_i$ if $\omega_i = 1$, where T_i is the translation by τ_i on $(S^1)_i$. It is easy to see that $\{\omega\} \times (S^1)^{\mathbb{N}}$ is T-invariant for any $\omega \in \{0, 1\}^{\mathbb{N}}$.

Let the distance on X be the sum of the distances

$$d_1(\omega, \omega') = \sum_{i\geq 1} \frac{1}{2^i} \mid \omega_i - \omega_i' \mid \quad \text{and} \quad d_2(s, s') = \sum_{i\geq 1} \frac{1}{2^i} d'(s_i, s_i'),$$

where d' is the distance on the circle S^1, so that $d((\omega, s), (\omega', s')) = d_1(\omega, \omega') + d_2(s, s')$.

It is not difficult to see that T has bounded complexity with respect to $\{\bar{d}_n\}$. Note that the ergodic components are $\{\omega\} \times (w', \Pi_{i|\omega_i=1}(S^1)_i)$, where $w' \in \Pi_{i|\omega_i=0}(S^1)_i$.

Remark 4.10. Assume that (X, T) is a minimal system with bounded complexity with respect to $\{\bar{d}_n\}$ for an invariant measure μ. It is interesting to know whether almost all of the ergodic measures in the ergodic decomposition of μ are isomorphic. After the first draft version of this paper was finished, Cyr and Kra informed us in [**4**] that there exists an example that does not satisfy the condition, see Proposition B.1 in Appendix B.

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A. *Appendix. Two examples*

The aim of this appendix is to construct the two examples announced in Section 3. We remark that the measure complexity for a minimal distal system can be very complicated. For an example see example [**20**].

A.1. *The construction of the system in Proposition 3.8.* We view the unit circle $\mathbb{T}$ as $\mathbb{R}/\mathbb{Z}$ and also as $[0, 1)$ (mod 1). For $a \in \mathbb{R}$, we let $\|a\| = \min\{|a - z| : z \in \mathbb{Z}\}$, which induces a distance on $\mathbb{T}$. Let $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ be an irrational number and $R_\alpha : \mathbb{T} \to \mathbb{T}$, $x \to x + \alpha$ the rotation on $\mathbb{T}$ by α. In this subsection, we will construct a skew product map $T : \mathbb{T}^2 \to \mathbb{T}^2$ with $T(x, y) = (x + \alpha, y + h(x))$ for any $x, y \in \mathbb{T}$, where $h : \mathbb{T} \to \mathbb{R}$ is continuous and will be defined below.

The general idea to construct h is as follows. First, we choose some disjoint intervals (see (A.1)) in $\mathbb{T}$ and define $h_1 : \mathbb{T} \to \mathbb{R}$ such that h_1 takes positive values in some intervals, negative values in the rest intervals and zero at other points (see (A.8) below). This results in $\sum_{i=0}^{n-1} h_1(R_\alpha^i x)$ being small under certain conditions (see Lemmas A.1 and A.2). Then, we choose smaller disjoint intervals (see (A.5)) in $\mathbb{T}$ and define $h_2 : \mathbb{T} \to \mathbb{R}$ such that h_2 takes smaller positive values in some intervals, smaller negative values in the rest intervals and zero at other points. We do this for each h_k and, finally, we put $h = \sum_{k=1}^{\infty} h_k$. Note that for $n \in \mathbb{N}$ and $x, y \in \mathbb{T}$, $T^n(x, y) = (R_\alpha^n x, y + \sum_{i=0}^{n-1} h(R_\alpha^i x))$. When calculating $\sum_{i=0}^{n-1} h(R_\alpha^i x)$, we only need to take care that $\sum_{i=0}^{n-1}(\sum_{j=1}^{k} h_j)(R_\alpha^i x)$ when k is large enough (see (A.13)). By carefully choosing h_k, we can show the minimality, non-equicontinuity and non-unique ergodicity. Now let us begin the construction.

Let $\eta = \frac{1}{100}$, $M_1 = 10$ and $N_1 = 10M_1$. As α is irrational, the two-side orbit $\{n\alpha : n \in \mathbb{Z}\}$ of 0 under the rotation R_α is pairwise distinct. Choose $\delta_1 > 0$ small enough such that the intervals

$$[i\alpha - \delta_1, i\alpha + \delta_1], \quad i = -1, 0, 1, \ldots, 2N_1$$

are pairwise disjoint on $\mathbb{T}$. Put

$$E_1 = \bigcup_{i=0}^{2N_1-1} [i\alpha - \delta_1, i\alpha + \delta_1], \tag{A.1}$$

and

$$F_1 = \{i\alpha - \delta_1, i\alpha + \delta_1 : i = 0, 1, \ldots, 2N_1 - 1\}.$$

The total length of intervals in E_1 is $4N_1\delta_1$. Shrinking δ_1 if necessary, we can require $4N_1\delta_1 < \eta/2$. Put

$$l_1 = \min\{\|x - y\| : x, y \in F_1, x \neq y\}$$

and $\gamma_0 = 2l_1$.

For $k = 2, 3, 4, \ldots,$ we will define $M_k, N_k, \delta_k, E_k, F_k, l_k$ and γ_{k-1} by induction. Assume that $M_{k-1}, N_{k-1}, \delta_{k-1}, E_{k-1}, F_{k-1}, l_{k-1}$ and γ_{k-2} have been defined such that the total length of intervals in E_{k-1} is less than $\eta/2^{k-1}$. As R_α is uniquely ergodic on $\mathbb{T}$, choose $M_k > N_{k-1}$ large enough such that, for any $x, y \in \mathbb{T}$, we have

$$\{0 \leq i \leq M_k - 1 : R_\alpha^i x \in (y, y + l_{k-1})\} \neq \emptyset. \tag{A.2}$$

and for any $n \geq M_k$ and any $x \in \mathbb{T}$,

$$\frac{1}{n}\#(\{0 \leq i \leq n - 1 : R_\alpha^i x \in E_{k-1}\}) < \frac{\eta}{2^{k-1}}. \tag{A.3}$$

Let $N_k = 10^k M_k$. Choose $\delta_k > 0$ small enough such that

$$\{i\alpha \pm \delta_k : i = 0, 1, 2, \ldots, 2N_k - 1\} \cap F_{k-1} = \emptyset,$$

and

$$[i\alpha - \delta_k, i\alpha + \delta_k], \quad i = -1, 0, 1, \ldots, 2N_k$$

are pairwise disjoint intervals on $\mathbb{T}$. Choose $0 < \gamma_{k-1} < \delta_{k-1}$ small enough such that

$$[i\alpha - \gamma_{k-1}, i\alpha + \gamma_{k-1}], \quad -2N_k \leq i \leq 2N_k + 2N_{k-1} \tag{A.4}$$

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are pairwise disjoint intervals on $\mathbb{T}$. Put

$$E_k = \bigcup_{i=0}^{2N_k-1} [i\alpha - \delta_k, i\alpha + \delta_k] \tag{A.5}$$

and

$$F_k = F_{k-1} \cup \{i\alpha + \delta_k, i\alpha - \delta_k : i = 0, 1, \ldots, 2N_k - 1\}. \tag{A.6}$$

The total length of the intervals in E_k is $4N_k\delta_k$. Shrinking δ_k, if necessary, we can require $4N_k\delta_k < \eta/2^k$. Let

$$l_k = \min\bigg(\{\|x - y\| : x, y \in F_k, x \neq y\} \cup \bigg\{\frac{\gamma_i}{2k^2} : i = 1, 2, \ldots, k - 1\bigg\}\bigg). \tag{A.7}$$

This completes the induction.

For each $k \in \mathbb{N}$, define $h_k^*, h_k : \mathbb{R} \to [-1/2, 1/2)$ such that

$$h_k^*(x) = \begin{cases} \dfrac{1}{N_k}\bigg(1 - \bigg|\dfrac{x - m}{\gamma_k}\bigg|\bigg) & \text{for } x \in [m - \gamma_k, m + \gamma_k] \text{ with } m \in \mathbb{Z}, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$h_k(x) = \sum_{i=0}^{N_k-1} h_k^*(x - i\alpha) - \sum_{i=N_k}^{2N_k-1} h_k^*(x - i\alpha).$$

As the intervals in E_k are pairwise disjoint and $\gamma_k < \delta_k$, it is easy to check that, for any $x \in \mathbb{R}$,

$$h_k(x) = \begin{cases} h_k^*(x - i\alpha) & \text{if } x \in [i\alpha - \gamma_k, i\alpha + \gamma_k] \pmod 1,\ i = 0, 1, 2, \ldots, N_k - 1, \\ -h_k^*(x - i\alpha) & \text{if } x \in [i\alpha - \gamma_k, i\alpha + \gamma_k] \pmod 1,\ i = N_k, \ldots, 2N_k - 1, \\ 0 & \text{otherwise.} \end{cases} \tag{A.8}$$

In particular, $h_k(x) = 0$ for $x \notin E_k \pmod 1$,

$$h_k(i\alpha) = \begin{cases} \dfrac{1}{N_k} & \text{for } i = 0, 1, \ldots, N_k - 1, \\ -\dfrac{1}{N_k} & \text{for } i = N_k, N_k + 1, \ldots, 2N_k - 1, \end{cases}$$

and for any $z \in [-\delta_x, \delta_x]$,

$$\sum_{s=0}^{2N_k-1} h_k(R_\alpha^s z) = 0. \tag{A.9}$$

It is also easy to see that, for any $x \in \mathbb{R}$,

$$|h_k(x)| \leq \frac{1}{N_k} = \frac{1}{10^k M_k} < \frac{1}{10^k}, \tag{A.10}$$

and h_k is Lipschitz continuous with a Lipschitz constant $1/N_k\gamma_k$, that is, for any $x, y \in \mathbb{R}$,

$$|h_k(x) - h_k(y)| \leq \frac{1}{N_k\gamma_k}|x - y|. \tag{A.11}$$

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For any $x \in \mathbb{R}$, we have $h_k(x+1) = h_k(x)$, so we can regard h_k as a function from $\mathbb{T}$ to $\mathbb{R}$. Now, define $h : \mathbb{T} \to \mathbb{R}$ as for each $x \in \mathbb{T}$

$$h(x) = \sum_{k=1}^{\infty} h_k(x).$$

It is easy to see that h is continuous, since

$$\sum_{k=1}^{\infty} |h_k(x)| \leq \sum_{k=1}^{\infty} \frac{1}{10^k} < \frac{1}{2}.$$

For $k \geq 1$, we set

$$h_{1,k}(x) = \sum_{i=1}^{k} h_i(x) \quad \text{and} \quad h_{k,\infty}(x) = \sum_{i=k}^{\infty} h_i(x). \tag{A.12}$$

Then,

$$h(x) = h_{1,k}(x) + h_{k+1,\infty}(x)$$

and

$$\|h_{k,\infty}(x)\| \leq \sum_{i=k}^{\infty} \|h_i(x)\| \leq \sum_{i=k}^{\infty} \frac{1}{10^i M_i} \leq \frac{1}{M_k} \sum_{i=k}^{\infty} \frac{1}{10^i} = \frac{1}{M_k} \frac{1}{9 \cdot 10^{k-1}} < \frac{1}{9 \cdot 10^{k-1}}. \tag{A.13}$$

Finally, we define a skew product map as follows:

$$T : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + h(x)).$$

It is clear that T is continuous. We will show that the system $(\mathbb{T}^2, T)$ is as required. By the definition, it is clear that $(\mathbb{T}^2, T)$ is distal.

For any real function g on $\mathbb{T}$ and $x \in \mathbb{T}$, we set $H_0^g \equiv 0$ and

$$H_n^g(x) := \sum_{i=0}^{n-1} g(R_\alpha^i x)$$

for $n \geq 1$. Recall that $h(x) = \sum_{k=1}^{\infty} h_k(x)$, so

$$H_n^h(x) = \sum_{i=1}^{\infty} H_n^{h_i}(x) = H_n^{h_{1,k}}(x) + H_n^{h_{k+1,\infty}}(x).$$

We choose a compatible metric d on $\mathbb{T}^2$ by

$$d((x_1, y_1), (x_2, y_2)) := \|x_1 - x_2\| + \|y_1 - y_2\|,$$

for any $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$. We remark that, for $n \in \mathbb{N}$ and $x, y \in \mathbb{T}$,

$$T^n(x, y) = \left(R_\alpha^n x, y + \sum_{i=0}^{n-1} h(R_\alpha^i x)\right) = (R_\alpha^n x, y + H_n^h(x)).$$

To estimate the orbit of (x, y), the key point is to control $H_n^h(x)$. The following two lemmas will be useful in the estimation.

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LEMMA A.1. *Assume $x \in \mathbb{T}$, $i \in \mathbb{N}$ and $m \in \mathbb{N}$. If x, $R_\alpha^m x \in E_i^c \cup [-\delta_i, \delta_i]$, then we have*

$$H_m^{h_i}(x) = 0.$$

Proof. Let $J = \{0 \le j \le m-1 : R_\alpha^j x \in [-\delta_i, \delta_i]\}$. We first claim that

$$\{0 \le k \le m-1 : R_\alpha^k x \in E_i\} = \bigcup_{j \in J} \{j + l : 0 \le l \le 2N_i - 1\}.$$

To see this equality, firstly we note that if $j \in J$, then $R_\alpha^j x \in [-\delta_i, \delta_i]$. This implies that $R_\alpha^{j+l} x \in [l\alpha - \delta_i, l\alpha + \delta_i] \subset E_i$ for $0 \le l \le 2N_i - 1$. Since $j \le m-1$ and $R_\alpha^m x \in E_i^c \cup [\delta_i, \delta_i]$, we have $j + 2N_i - 1 \le m$ and then $\{j + l : 0 \le l \le 2N_i - 1\} \subset \{0, 1, 2, \ldots, m-1\}$. Thus,

$$\{0 \le k \le m-1 : R_\alpha^k x \in E_i\} \supset \bigcup_{j \in J} \{j + l : 0 \le l \le 2N_i - 1\}.$$

Conversely, if $k \in \{0, 1, 2, \ldots, m-1\}$ with $R_\alpha^k x \in E_i$. This means that

$$R_\alpha^k x \in [s\alpha - \delta_i, s\alpha + \delta_i] \quad \text{for some } 0 \le s \le 2N_i - 1.$$

If $k < s$, then $x \in [(s-k)\alpha - \delta_i, (s-k)\alpha + \delta_i]$, which contradicts the assumption $x \in E_i^c \cup [-\delta_i, \delta_i]$. This implies that $k \ge s$. Hence, we have $k - s \in J$ and

$$k \in \{(k-s) + l : 0 \le l \le 2N_i - 1\}.$$

Thus, we get $\{0 \le k \le m-1 : R_\alpha^k x \in E_i\} \subset \cup_{j \in J} \{j + l : 0 \le l \le 2N_i - 1\}$. This proves the claim.

By the claim, we have

$$\begin{aligned} H_m^{h_i}(x) &= \sum_{\substack{0 \le k \le m-1 \\ R_\alpha^k x \notin E_i}} h_i(R_\alpha^k x) + \sum_{\substack{0 \le k \le m-1 \\ R_\alpha^k x \in E_i}} h_i(R_\alpha^k x) \\ &= 0 + \sum_{j \in J} \sum_{l=0}^{2N_i - 1} h_i(R_\alpha^l (R_\alpha^j x)) = 0. \quad \text{by (A.9)} \end{aligned}$$

This finishes the proof of Lemma A.1. □

LEMMA A.2. *Assume $x \in \mathbb{T}$, $m, k \in \mathbb{N}$ and $1 \le j \le k-1$. If $\|m\alpha\| < l_k$ and x, $R_\alpha^m x \in [i\alpha - \delta_j, i\alpha + \delta_j]$ for some $0 \le i \le 2N_j - 1$, then we have*

$$\|H_m^{h_j}(x)\| < \frac{1}{k^2}.$$

Proof. First, by (A.7), we have

$$\begin{aligned} l_k &\le \min\{\|(i\alpha + \delta_k) - (j\alpha + \delta_k)\| : 0 \le i < j \le 2N_k - 1\} \\ &= \min_{0 \le r \le 2N_k - 1} \|r\alpha\|. \end{aligned}$$

Thus, $m \ge 2N_k$ since $\|m\alpha\| < l_k$. Next, by the construction of E_j, we have

$$R_\alpha^{2N_j - i} x,\ R_\alpha^{m-i} x = R_\alpha^{m - 2N_j}(R_\alpha^{2N_j - i} x) \in E_j^c \cup [-\delta_j, \delta_j]$$

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since $x,\ R_\alpha^m x \in [i\alpha - \delta_j, i\alpha + \delta_j]$. By Lemma A.1, we have $H^{h_j}_{m-2N_j}(R_\alpha^{2N_j - i}x) = 0$ and

$$\begin{aligned}
H^{h_j}_m(x) &= H^{h_j}_{2N_j - i}(x) + H^{h_j}_{m-2N_j}(R_\alpha^{2N_j-i}x) + H^{h_j}_i(R_\alpha^{m-i}x)\\
&= H^{h_j}_{2N_j-i}(x) + H^{h_j}_i(R_\alpha^{m-i}x)\\
&= (H^{h_j}_{2N_j-i}(x) + H^{h_j}_i(R_\alpha^{-i}x)) + (H^{h_j}_i(R_\alpha^{m-i}x) - H^{h_j}_i(R_\alpha^{-i}x))\\
&= H^{h_j}_{2N_j}(R_\alpha^{-i}x) + (H^{h_j}_i(R_\alpha^{m-i}x) - H^{h_j}_i(R_\alpha^{-i}x)).
\end{aligned}$$

Notice that $R_\alpha^{-i}x \in [-\delta_j, \delta_j]$. By (A.9), we have

$$H^{h_j}_{2N_j}(R_\alpha^{-i}x) = \sum_{s=0}^{2N_j-1} h_j(R_\alpha^s(R_\alpha^{-i}x)) = 0.$$

This implies that

$$\begin{aligned}
\|H^{h_j}_m(x)\| &\le \|H^{h_j}_i(R_\alpha^{m-i}x) - H^{h_j}_i(R_\alpha^{-i}x)\|\\
&\le \sum_{s=0}^{i-1} \|h_j(R_\alpha^{m-i+s}x) - h_j(R_\alpha^{-i+s}x)\|\\
&\le i \cdot l_k \cdot \frac{1}{N_j\gamma_j},
\end{aligned}$$

where the last inequality follows from (A.11) and $\|R_\alpha^{-i+s}x - R_\alpha^{m-i+s}x\| = \|m\alpha\| < l_k$ for $s = 0, 1, 2, \ldots, i-1$. Finally, by (A.7),

$$\|H^{h_j}_m(x)\| \le 2N_j \cdot \frac{\gamma_j}{2k^2} \cdot \frac{1}{N_j\gamma_j} = \frac{1}{k^2}.$$

This finishes the proof of Lemma A.2. □

PROPOSITION A.3. $(\mathbb{T}^2, T)$ *is minimal.*

Proof. We need to show every point (x, y) has a dense orbit. Fix $(x, y) \in \mathbb{T}^2$, $0 < \epsilon < 1$ and $k \in \mathbb{N}$. There exists $n_1 \in \mathbb{N}$ such that $R_\alpha^{n_1}x \in [-\epsilon\gamma_k, \epsilon\gamma_k]$. Let $(x_1, y_1) = T^{n_1}(x, y)$. Then $x_1 = R_\alpha^{n_1}x$ and $\|x_1\| \le \epsilon\gamma_k$.

Now fix $(x', y') \in \mathbb{T}^2$. Note that points in F_{k-1} divide the unit circle into open arcs, with length not less than l_{k-1}. The collection of these arcs is denoted by $\mathcal{F}_{k-1}$. There exists $(a_1, a_2) \in \mathcal{F}_{k-1}$ such that $x' \in [a_1, a_2]$. As $(a_1, a_2) \cap F_{k-1} = \emptyset$, either $(a_1, a_2) \subset E_{k-1}$ or $(a_1, a_2) \subset E^c_{k-1}$. If $(a_1, a_2) \subset E_{k-1}$, then $[a_1, a_2) \subset [j\alpha - \delta_{k-1}, j\alpha + \delta_{k-1}]$ for some $0 \le j \le 2N_{k-1} - 1$, and we take $a = j\alpha + \delta_{k-1}$. If $(a_1, a_2) \subset E^c_{k-1}$, we take $a \in [a_1, a_2)$ such that $x' \in [a, a + l_{k-1}) \subset [a_1, a_2)$, since the length of $[a_1, a_2]$ is not less than l_{k-1}. Note that in any case $(a, a + l_{k-1})$ is a subset of some $(b_1, b_2) \in \mathcal{F}_{k-1}$. For any $1 \le i \le k-2$, as $F_i \subset F_{k-1}$, $(b_1, b_2) \cap E_i = \emptyset$. Then (b_1, b_2) is either a subset of $[j\alpha - \delta_i, j\alpha + \delta_i]$ for some $0 \le j \le 2N_i - 1$ or a subset of E^c_i. Summing up the above arguments, we have:

(i) $(a, a + l_{k-1}) \subset E^c_{k-1}$ and $\min\{\|x' - x''\| : x'' \in (a, a + l_{k-1})\} \le 2\delta_{k-1}$;

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(ii) for all $1 \leq i \leq k-2$, either $(a, a+l_{k-1}) \subset [j\alpha - \delta_i, j\alpha + \delta_i]$ for some $0 \leq j \leq 2N_i - 1$ or $(a, a+l_{k-1}) \subset E_i^c$.

By (A.2), there exists an integer $n_2 \in [0, M_k)$ such that $R_\alpha^{n_2}(x_1) \in (a, a+l_{k-1})$. Suppose

$$y' - (y_1 + H_{n_2}^h(x_1)) = b \ (\mathrm{mod}\ 1).$$

Then, $b \in [0, 1)$. By (A.2), there exists an integer $n_3 \in [[10^k b]M_k, ([10^k b]+1)M_k)$ such that $n_3 \geq n_2$ and $R_\alpha^{n_3}(x_1) \in (a, a+l_{k-1})$. Note that $n_3 < 10^k M_k = N_k$ and

$$b - \frac{2}{10^k} \leq \frac{([10^k b]-1)M_k}{N_k} \leq \frac{n_3 - n_2}{N_k} \leq \frac{([10^k b]+1)M_k}{N_k} \leq b + \frac{2}{10^k}.$$

By (i) and Lemma A.1, we have $H_{n_3-n_2}^{h_{k-1}}(R_\alpha^{n_2} x_1) = 0$. By (ii) and Lemmas A.1 and A.2, we have

$$\|H_{n_3-n_2}^{h_i}(R_\alpha^{n_2} x_1)\| < \frac{1}{(k-1)^2}$$

for $1 \leq i \leq k-2$. Thus, we have

$$\begin{aligned}
\|y' - (y_1 + H_{n_3}^h(x_1))\| &= \|y' - (y_1 + H_{n_2}^h(x_1)) - H_{n_3-n_2}^h(R_\alpha^{n_2} x_1)\| \\
&= \left\| b - \sum_{i=k}^{\infty} H_{n_3-n_2}^{h_i}(R_\alpha^{n_2} x_1) - \sum_{i=1}^{k-2} H_{n_3-n_2}^{h_i}(R_\alpha^{n_2} x_1) \right\| \\
&\leq \|b - H_{n_3-n_2}^{h_k}(R_\alpha^{n_2} x_1)\| \\
&\quad + \sum_{i=k+1}^{\infty} \|H_{n_3-n_2}^{h_i}(R_\alpha^{n_2} x_1)\| + \sum_{i=1}^{k-2} \|H_{n_3-n_2}^{h_i}(R_\alpha^{n_2} x_1)\| \\
&\leq \|b - H_{n_3-n_2}^{h_k}(n_2\alpha)\| + \|H_{n_3-n_2}^{h_k}(n_2\alpha) - H_{n_3-n_2}^{h_k}(n_2\alpha + x_1)\| \\
&\quad + \sum_{i=k+1}^{\infty} \frac{n_3 - n_2}{N_i} + (k-2) \cdot \frac{1}{(k-1)^2} \\
&\leq \left\| b - (n_3 - n_2)\frac{1}{N_k} \right\| + \epsilon \frac{n_3 - n_2}{N_k} + \sum_{i=k+1}^{\infty} \frac{1}{10^i} + \frac{1}{k-1} \\
&\leq \frac{3}{10^k} + 2\epsilon + \frac{1}{k-1}.
\end{aligned}$$

We deduce that

$$\begin{aligned}
d((x', y'), T^{n_3+n_1}(x, y)) &= d((x', y'), T^{n_3}(x_1, y_1)) \\
&= \|x' - R_\alpha^{n_3} x_1\| + \|y' - (y_1 + H_{n_3}^h(x_1))\| \\
&\leq l_{k-1} + 2\delta_{k-1} + \frac{3}{10^k} + 2\epsilon + \frac{1}{k-1}.
\end{aligned}$$

This implies that $(x', y') \in \overline{\mathrm{Orb}((x, y), T)}$ if we let $k \to +\infty$ and $\epsilon \to 0$. Hence $(\mathbb{T}^2, T)$ is minimal. □

For $1 \leq j \leq k$, we let

$$E_{j,k} = \bigcup_{i=j}^{k} E_i.$$

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By (A.3), for any $n \geq M_{k+1}$ and $x \in \mathbb{T}$,

$$\frac{1}{n}\#(\{0 \leq i \leq n-1 : R_\alpha^i x \in E_{j,k}\}) < \sum_{i=j}^{k} \frac{\eta}{2^i} < \frac{\eta}{2^{j-1}} < \frac{1}{2}. \tag{A.14}$$

PROPOSITION A.4. *$(\mathbb{T}^2, T)$ is not equicontinuous.*

Proof. To show that $(\mathbb{T}^2, T)$ is not equicontinuous, it is sufficient to show that, for any $\epsilon > 0$, there exist $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$ and $n \in \mathbb{N}$ such that $d((x_1, y_1), (x_2, y_2)) \leq \epsilon$ and $d(T^n(x_1, y_1), T^n(x_2, y_2)) \geq \frac{1}{9}$.

Fix $\epsilon > 0$. There exists $k \in \mathbb{N}$ such that $l_k + \delta_k < \epsilon$. Put $x' = \delta_k + \frac{1}{2}l_k$. We have $R_\alpha^i x' \in E_k^c$ and $h_k(R_\alpha^i x') = 0$ for $i = 0, 1, \ldots, N_k - 1$. By (A.14), we can choose integers $n_1 \in [0, M_k - 1]$ and $n_2 \in [\frac{1}{2}N_k - M_k, \frac{1}{2}N_k - 1]$ such that

$$R_\alpha^{n_1}0,\ R_\alpha^{n_1}x',\ R_\alpha^{n_2}0,\ R_\alpha^{n_2}x' \in E_{1,k-1}^c.$$

By using Lemma A.1 and the fact $R_\alpha^{n_1}x', R_\alpha^{n_2}x' \in E_k^c$, we have

$$\begin{aligned}
H_{n_2-n_1}^h(R_\alpha^{n_1}0) &= H_{n_2-n_1}^{h_{1,k-1}}(R_\alpha^{n_1}0) + H_{n_2-n_1}^{h_k}(R_\alpha^{n_1}0) + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}0)\\
&= H_{n_2-n_1}^{h_k}(R_\alpha^{n_1}0) + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}0)\\
&= (n_2-n_1)\frac{1}{N_k} + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}0)
\end{aligned}$$

and

$$\begin{aligned}
H_{n_2-n_1}^h(R_\alpha^{n_1}x') &= H_{n_2-n_1}^{h_{1,k-1}}(R_\alpha^{n_1}x') + H_{n_2-n_1}^{h_k}(R_\alpha^{n_1}x') + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}x')\\
&= H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}x').
\end{aligned}$$

Note that $\frac{1}{2}N_k - 2M_k \leq n_2 - n_1 \leq \frac{1}{2}N_k$ and $N_k = 10^k M_k$, so

$$\frac{2}{5} \leq (n_2-n_1)\frac{1}{N_k} \leq \frac{1}{2}.$$

By (A.10), we have

$$\begin{aligned}
\|H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}0) - H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1}x')\| &\leq \sum_{i=k+1}^{\infty}(\|H_{n_2-n_1}^{h_i}(R_\alpha^{n_1}0)\| + \|H_{n_2-n_1}^{h_i}(R_\alpha^{n_1}x')\|)\\
&\leq \sum_{i=k+1}^{\infty} 2(n_2-n_1)\frac{1}{N_i} \leq \sum_{i=k+1}^{\infty}\frac{2N_k}{N_i}\\
&\leq 2\sum_{i=k+1}^{\infty}\frac{1}{10^{i-k}} = \frac{2}{9}.
\end{aligned}$$

Thus,

$$\tfrac{16}{90} \leq \|H_{n_2-n_1}^h(R_\alpha^{n_1}0) - H_{n_2-n_1}^h(R_\alpha^{n_1}x')\| \leq \tfrac{65}{90},$$

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and

$$\begin{aligned} d(T^{n_2-n_1}(R_\alpha^{n_1}0, 0), T^{n_2-n_1}(R_\alpha^{n_1}x', 0)) \\ = d((R_\alpha^{n_2}0, H_{n_2-n_1}^h(R_\alpha^{n_1}0))), (R_\alpha^{n_2}x', H_{n_2-n_1}^h(R_\alpha^{n_1}x')) \\ \geq \|H_{n_2-n_1}^h(R_\alpha^{n_1}0) - H_{n_2-n_1}^h(R_\alpha^{n_1}x')\| \geq \tfrac{16}{90} \geq \tfrac{1}{9} \end{aligned}$$

with

$$d((R_\alpha^{n_1}0, 0), (R_\alpha^{n_1}x', 0)) = \|R_\alpha^{n_1}0 - R_\alpha^{n_1}x'\| = \|x'\| = \delta_k + \tfrac{1}{2}\ell_k \leq \epsilon.$$

This implies that $(\mathbb{T}^2, T)$ is not equicontinuous. □

PROPOSITION A.5. *$(\mathbb{T}^2, T)$ is not uniquely ergodic.*

Proof. Let $m_{\mathbb{T}^2}$ be the unique normalized Haar measure on $\mathbb{T}^2$. For any $m_{\mathbb{T}^2}$-integrable function $f(x, y)$, by the Fubini's theorem, we have

$$\begin{aligned} \int_{\mathbb{T}^2} f \circ T(x, y)\, dm_{\mathbb{T}^2} &= \int_{\mathbb{T}} \int_{\mathbb{T}} f(R_\alpha x, y + h(x))\, dm_{\mathbb{T}}(y)\, dm_{\mathbb{T}}(x) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} f(R_\alpha x, y)\, dm_{\mathbb{T}}(y)\, dm_{\mathbb{T}}(x) \\ &= \int_{\mathbb{T}} \int_{\mathbb{T}} f(x, y)\, dm_{\mathbb{T}}(y)\, dm_{\mathbb{T}}(x) \\ &= \int_{\mathbb{T}^2} f(x, y)\, dm_{\mathbb{T}^2}. \end{aligned}$$

Therefore $m_{\mathbb{T}^2}$ is T-invariant.

If $(\mathbb{T}^2, T)$ is uniquely ergodic, then $m_{\mathbb{T}^2}$ is the unique invariant measure. We take a measurable function

$$f(x, y) = \mathbf{1}_{\mathbb{T}\times[0,1/2)}(x, y) - \mathbf{1}_{\mathbb{T}\times[1/2,1)}(x, y).$$

Note that the boundary of $\mathbb{T} \times [0, \frac{1}{2})$ and $\mathbb{T} \times [\frac{1}{2}, 1)$ have zero $m_{\mathbb{T}^2}$-measure. By unique ergodicity, we have

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=0}^{n-1} f(T^i(x, y)) = \int_{\mathbb{T}^2} f\, dm_{\mathbb{T}^2} = 0 \tag{A.15}$$

for each $(x, y) \in \mathbb{T}^2$.

For $k \geq 1$, put

$$A_k = \left\{ s \in \left\{ \frac{1}{10}N_k, \frac{1}{10}N_k + 1, \ldots, \frac{4}{10}N_k \right\} : R_\alpha^s 0 = s\alpha \in \bigcap_{j=1}^{k-1} E_j^c \right\}.$$

For $i \in A_k$, it is clear that $0, R_\alpha^i 0 \in \bigcap_{j=1}^{k-1} E_j^c \cup [-\delta_j, \delta_j]$, so we have that by Lemma A.1

$$H_i^h(0) - \frac{i}{N_k} = \sum_{j=1}^{\infty} H_i^{h_j}(0) - \frac{i}{N_k} = \sum_{j=k}^{\infty} H_i^{h_j}(0) - \frac{i}{N_k} = \sum_{j=k+1}^{\infty} H_i^{h_j}(0)$$

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where in the last equality we use the fact that $H_i^{h_k}(0) = \sum_{l=0}^{i-1} h_k(R_\alpha^l 0) = i/N_k$ by (A.8). Notice that

$$\|H_i^{h_j}(0)\| \le \sum_{l=0}^{i-1} \|h_j(R_\alpha^l 0)\| \le \frac{i}{N_j}.$$

Therefore

$$\left\|H_i^h(0) - \frac{i}{N_k}\right\| = \left\|\sum_{j=k+1}^{\infty} H_i^{h_j}(0)\right\| \le \sum_{j=k+1}^{\infty} \|H_i^{h_j}(0)\| \le \sum_{j=k+1}^{\infty} \frac{i}{N_j}$$
$$\le \sum_{j=k+1}^{\infty} \frac{i}{10^j M_j} \le \sum_{j=k+1}^{\infty} \frac{1}{10^j} < \frac{1}{10}.$$

It is clear that $\frac{1}{10} \le i/N_k \le \frac{4}{10}$. So $H_i^h(0) \in [0, \frac{1}{2})$ and

$$f(T^i(0,0)) = f((i\alpha, H_i^h(0))) = 1. \tag{A.16}$$

Put $S_k = \{0, 1, \ldots, \frac{1}{2}N_k - 1\}$ and $B_k = \{s \in S_k : R_\alpha^s 0 \in E_{1,k-1}\}$, and by the construction (A.3)

$$\frac{\#(B_k)}{(1/2)N_k} \le \sum_{j=1}^{k-1} \frac{\eta}{2^j} < \eta = \frac{1}{100}.$$

Hence

$$\frac{\#(A_k)}{(1/2)N_k} \ge \frac{(3/10)N_k - \#B_k}{(1/2)N_k} > \frac{59}{100}.$$

Since $f(T^i(0,0)) = 1$ for $i \in A_k$ and $f(T^i(0,0)) \in \{-1, 1\}$ for $i \in S_k \setminus A_k$, we have

$$\frac{1}{(1/2)N_k} \sum_{i=0}^{(1/2)N_k - 1} f(T^i(0,0)) = \frac{1}{(1/2)N_k}\left(\sum_{i \in A_k} f(T^i(0,0)) + \sum_{i \in S_k \setminus A_k} f(T^i(0,0))\right)$$
$$\ge \frac{1}{(1/2)N_k}(\#(A_k) - \#(S_k \setminus A_k))$$
$$\ge \frac{59}{100} - \frac{41}{100} = \frac{18}{100}.$$

Thus

$$\limsup_{k\to\infty} \frac{1}{(1/2)N_k} \sum_{i=0}^{(1/2)N_k - 1} f(T^i(0,0)) \ge \frac{18}{100} > 0,$$

which contradicts (A.15). Therefore $(\mathbb{T}^2, T)$ is not uniquely ergodic. This completes the proof. □

For any real function g on $\mathbb{T}$, $n \in \mathbb{N}$ and $x, y \in X$, we set

$$\bar{d}_n^g(x, y) := \frac{1}{n} \sum_{m=0}^{n-1} \|H_m^g(x) - H_m^g(y)\|.$$

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Then for any $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$, we have

$$\bar{d}_n((x_1, y_1), (x_2, y_2)) = \frac{1}{n} \sum_{m=0}^{n-1} d((R_\alpha^m x_1, y_1 + H_m^h(x_1)), (R_\alpha^m x_2, y_1 + H_m^h(x_2)))$$
$$\leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}_n^h(x_1, x_2).$$

The main result of this subsection is as follows.

PROPOSITION A.6. *$(\mathbb{T}^2, T)$ has bounded topological complexity with respect to $\{\bar{d}_n\}$.*

Proof. It is sufficient to show that, for any $\epsilon \in (0, \frac{1}{100})$, there exist two constants $C(\epsilon) > 0$ and $K(\epsilon) \in \mathbb{N}$ such that $\overline{\text{span}}(n, 17\epsilon) \leq C(\epsilon)$ for any $n > K(\epsilon)$.

First, we choose an integer $q \in \mathbb{N}$ such that

$$\sum_{i=q+1}^{\infty} \frac{\eta}{2^i} < \epsilon \quad \text{and} \quad \frac{1}{10^q} < \epsilon. \tag{A.17}$$

Then there exists $\delta(\epsilon) > 0$ such that

$$\sum_{i=1}^{q} \|H_s^{h_i}(x) - H_s^{h_i}(y)\| < \epsilon, \tag{A.18}$$

for any $0 \leq s \leq M_{q+1} - 1$ and any $x, y \in \mathbb{T}$ with $\|x - y\| < \delta(\epsilon)$.

Put $c_\epsilon = \lceil 1/\epsilon \rceil$ and $c_\delta = \lceil 1/\delta(\epsilon) \rceil$. Let

$$C(\epsilon) = 100 c_\epsilon^{11} c_\delta \quad \text{and} \quad K(\epsilon) = 2N_{q+2}.$$

In the following, we are going to show that, for any $n > K(\epsilon)$, there exists a cover $\mathcal{T}$ of $\mathbb{T}^2$ (that depends on n) such that

$$\#(\mathcal{T}) \leq C(\epsilon) \quad \text{and} \quad \bar{d}_n((x_1, y_1), (x_2, y_2)) \leq 17\epsilon$$

for any $(x_1, y_1), (x_2, y_2) \in W \in \mathcal{T}$. This will imply $\overline{\text{span}}(n, 17\epsilon) \leq C(\epsilon)$ for any $n > K(\epsilon)$.

Now fix an integer $n > K(\epsilon)$. There exists a unique integer $k \geq q + 2$ such that

$$2N_k < n \leq 2N_{k+1}.$$

Recall that

$$h = h_{1,k-1} + h_k + h_{k+1} + h_{k+2,\infty},$$
$$\bar{d}_n((x_1, y_1), (x_2, y_2)) \leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}_n^h(x_1, x_2),$$

and

$$\bar{d}_n^h(x_1, x_2) \leq \bar{d}_n^{h_{1,k-1}}(x_1, x_2) + \bar{d}_n^{h_k}(x_1, x_2) + \bar{d}_n^{h_{k+1}}(x_1, x_2) + \bar{d}_n^{h_{k+2,\infty}}(x_1, x_2).$$

We divide the remaining proof into four steps, bounding each term of the sum above.

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Step 1. We will construct a finite cover $\mathcal{P}$ of $\mathbb{T}$ such that

$$\#(\mathcal{P}) \le c_\delta c_\epsilon^2 \quad \textit{and} \quad \bar{d}_n^{h_{1,k-1}}(x, y) < 6\epsilon$$

for $x, y \in P \in \mathcal{P}$.

Firstly, for any $x \in \mathbb{T}$ and $\ell \ge 2$, we define

$$n_\ell^*(x) = \min\{i \ge 0 : R_\alpha^i x \in E_{1,\ell-1}^c\} \quad \text{and} \quad x_\ell^* = R_\alpha^{n_\ell^*(x)} x.$$

Clearly, $n_\ell^*(x) \le M_\ell - 1$ by (A.2). By Lemma A.1, if $R_\alpha^m x \in E_i^c$ for some $1 \le i \le \ell - 1$ and $m \ge M_\ell$, we have $H_{m-n_\ell^*(x)}^{h_i}(x_\ell^*) = 0$ and then

$$H_m^{h_i}(x) = H_{n_\ell^*(x)}^{h_i}(x) + H_{m-n_\ell^*(x)}^{h_i}(x_\ell^*) = H_{n_\ell^*(x)}^{h_i}(x). \tag{A.19}$$

Next, let

$$\mathcal{P}_1 = \left\{\left[\frac{j}{c_\delta}, \frac{j+1}{c_\delta}\right) : 0 \le j \le c_\delta - 1\right\},$$

$$\mathcal{P}_2 = \left\{\left\{x \in \mathbb{T} : \sum_{i=1}^{q} H_{n_{q+1}^*(x)}^{h_i}(x) \in \left[\frac{j}{c_\epsilon}, \frac{j+1}{c_\epsilon}\right)\right\} : 0 \le j \le c_\epsilon - 1\right\},$$

$$\mathcal{P}_3 = \left\{\left\{x \in \mathbb{T} : \sum_{i=q+1}^{k-1} H_{n_k^*(x)}^{h_i}(x) \in \left[\frac{j}{c_\epsilon}, \frac{j+1}{c_\epsilon}\right)\right\} : 0 \le j \le c_\epsilon - 1\right\}.$$

Put $\mathcal{P} = \mathcal{P}_1 \vee \mathcal{P}_2 \vee \mathcal{P}_3$. It is clear that $\mathcal{P}$ is a partition of $\mathbb{T}$ and $\#(\mathcal{P}) \le c_\delta c_\epsilon^2$.

Fix two points x, y which are in the same atom of $\mathcal{P}$. If there exists $m \ge M_k$ with $R_\alpha^m x, R_\alpha^m y \in E_{q+1,k-1}^c$, then by (A.19) we have for $q + 1 \le i \le k - 1$,

$$H_m^{h_i}(x) = H_{n_k^*(x)}^{h_i}(x) \quad \text{and} \quad H_m^{h_i}(y) = H_{n_k^*(y)}^{h_i}(y).$$

Thus,

$$\left\| \sum_{i=q+1}^{k-1} (H_m^{h_i}(x) - H_m^{h_i}(y)) \right\| = \left\| \sum_{i=q+1}^{k-1} (H_{n_k^*(x)}^{h_i}(x) - H_{n_k^*(y)}^{h_i}(y)) \right\| \le \frac{1}{c_\epsilon} \le \epsilon,$$

as x, y are in the same atom in $\mathcal{P}_3$.

By (A.3) for any $z \in \mathbb{T}$,

$$\frac{1}{M_{q+1}} \#\{0 \le i \le M_{q+1} - 1 : R_\alpha^i z \in E_{1,q}^c\} \ge 1 - \sum_{i=1}^{\infty} \frac{\eta}{2^i} > \frac{1}{2}.$$

If there exists $m \ge M_k$ with $R_\alpha^m x, R_\alpha^m y \in E_{q+1,k-1}^c$, then we can find an integer $M \in [m - M_{q+1}, m - 1]$ such that $R_\alpha^M x \in E_{1,q}^c$ and $R_\alpha^M y \in E_{1,q}^c$. Note that $M \ge m - M_{q+1} > M_{q+1}$. By (A.19), for $1 \le i \le q$,

$$H_M^{h_i}(x) = H_{n_{q+1}^*(x)}^{h_i}(x) \quad \text{and} \quad H_M^{h_i}(y) = H_{n_{q+1}^*(y)}^{h_i}(y).$$

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Then

$$\left\|\sum_{i=1}^{q}(H_M^{h_i}(x)-H_M^{h_i}(y))\right\| = \left\|\sum_{i=1}^{q}(H_{n^*_{q+1}(x)}^{h_i}(x)-H_{n^*_{q+1}(y)}^{h_i}(y))\right\| \le \frac{1}{c_\epsilon} \le \epsilon,$$

as x, y are in the same atom in $\mathcal{P}_2$.

As x, y are in the same atom in $\mathcal{P}_1$, $\|R_\alpha^M x - R_\alpha^M y\| = \|x-y\| \le 1/c_\delta \le \delta(\epsilon)$. Note that $m - M \le M_{q+1} - 1$. By (A.18) we have

$$\left\|\sum_{i=1}^{q} H_{m-M}^{h_i}(R_\alpha^M x) - \sum_{i=1}^{q} H_{m-M}^{h_i}(R_\alpha^M y)\right\| < \epsilon.$$

Hence, if there exists $m \ge M_k$ with $R_\alpha^m x,\ R_\alpha^m y \in E_{q+1,k-1}^c$, then we have

$$\begin{aligned}
\|H_m^{h_{1,k-1}}(x) - H_m^{h_{1,k-1}}(y)\| &\le \left\|\sum_{i=1}^{q}(H_m^{h_i}(x)-H_m^{h_i}(y))\right\| + \left\|\sum_{i=q+1}^{k-1}(H_m^{h_i}(x)-H_m^{h_i}(y))\right\| \\
&\le \left\|\sum_{i=1}^{q}(H_M^{h_i}(x)-H_M^{h_i}(y))\right\| \\
&\quad + \left\|\sum_{i=1}^{q} H_{m-M}^{h_i}(R_\alpha^M x) - \sum_{i=1}^{q} H_{m-M}^{h_i}(R_\alpha^M y)\right\| \\
&\quad + \left\|\sum_{i=q+1}^{k-1}(H_m^{h_i}(x)-H_m^{h_i}(y))\right\| \\
&\le 3\epsilon.
\end{aligned}$$

Finally,

$$\begin{aligned}
\bar{d}_n^{h_{1,k-1}}(x,y) &= \frac{1}{n}\sum_{j=0}^{n-1}\|H_j^{h_{1,k-1}}(x) - H_j^{h_{1,k-1}}(y)\| \\
&\le \frac{1}{n}\Bigg(\sum_{\substack{M_k\le j\le n-1\\ R_\alpha^j x,\, R_\alpha^j y\in E_{q+1,k-1}^c}} \|H_j^{h_{1,k-1}}(x) - H_j^{h_{1,k-1}}(y)\| \\
&\quad + \sum_{\substack{M_k\le j\le n-1\\ R_\alpha^j x\in E_{q+1,k-1}}} 1 + \sum_{\substack{M_k\le j\le n-1\\ R_\alpha^j y\in E_{q+1,k-1}}} 1 + \sum_{0\le j\le M_k-1} 1\Bigg) \\
&\le 3\epsilon + \frac{1}{n}\#(\{0\le j\le n-1 : R_\alpha^j x \in E_{q+1,k-1}\}) \\
&\quad + \frac{1}{n-1}\#(\{0\le j\le n-1 : R_\alpha^j y \in E_{q+1,k-1}\}) + \frac{M_k}{n-1} \\
&< 6\epsilon,
\end{aligned}$$

where the last inequality follows from (A.14) and (A.17).

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Step 2. We will construct a finite cover $\mathcal{Q}$ of $\mathbb{T}$ such that

$$\#(\mathcal{Q}) \leq 10c_\epsilon^4 \quad \text{and} \quad \bar{d}_n^{h_k}(x, y) \leq 4\epsilon$$

for any $x, y \in Q \in \mathcal{Q}$.

There are two cases. The first case is $n \leq 2c_\epsilon N_k$. In this case, we put

$$Q_0 = \mathbb{T} \Big\backslash \bigg(\bigcup_{-2c_\epsilon N_k \leq i < (2+2c_\epsilon)N_k} [i\alpha - \gamma_k, i\alpha + \gamma_k] \bigg),$$

and

$$Q_{r,s} = \bigcup_{rN_k/c_\epsilon \leq i < (r+1)N_k/\epsilon} \bigg[i\alpha + \frac{\gamma_k s}{c_\epsilon^2}, i\alpha + \frac{\gamma_k(s+1)}{c_\epsilon^2} \bigg].$$

Let

$$\mathcal{Q} = \{Q_0\} \cup \{Q_{r,s} : -2c_\epsilon^2 \leq r \leq (2+2c_\epsilon)c_\epsilon - 1, -c_\epsilon^2 \leq s \leq c_\epsilon^2 - 1\}.$$

It is clear that $\mathcal{Q}$ is a cover of $\mathbb{T}$ and $\#(\mathcal{Q}) \leq 2c_\epsilon^2 \cdot 5c_\epsilon^2 = 10c_\epsilon^3/\epsilon \leq 10c_\epsilon^4$. For $x, y \in Q_0$, we have $\bar{d}_n^{h_k}(x, y) = 0$ by (A.8).

Now assume that $x, y \in Q_{r,s}$ for some r and s. There exist integers $m_1, m_2 \in [rN_k/c_\epsilon, (r+1)N_k/c_\epsilon]$ and $x_1, y_1 \in [\gamma_k s/c_\epsilon^2, \gamma_k(s+1)/c_\epsilon^2]$ such that

$$x = R_\alpha^{m_1} x_1 \quad \text{and} \quad y = R_\alpha^{m_2} y_1.$$

Without loss of generality, we can assume that $m_1 \leq m_2$. For any $1 \leq m \leq n$, we have

$$\begin{aligned}
\|H_m^{h_k}(x) - H_m^{h_k}(y)\| &= \bigg\| \sum_{i=0}^{m-1} (h_k(R_\alpha^i x) - h_k(R_\alpha^i y)) \bigg\| \\
&\leq \bigg\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i y_1)) \bigg\| \\
&\leq \bigg\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i x_1) \bigg\| \\
&\quad + \bigg\| \sum_{i=m_2}^{m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \bigg\| \\
&\leq \sum_{i=m_1}^{m_2-1} \|h_k(R_\alpha^i x_1)\| + \sum_{i=m_1+m}^{m_2+m-1} \|h_k(R_\alpha^i x_1)\| + m \cdot \frac{\gamma_k}{c_\epsilon^2} \cdot \frac{1}{N_k \gamma_k} \\
&\leq 2(m_2 - m_1)\frac{1}{N_k} + m \cdot \frac{\gamma_k}{c_\epsilon^2} \cdot \frac{1}{N_k \gamma_k} \quad \text{by (A.10) and (A.11)} \\
&\leq 4\epsilon.
\end{aligned}$$

Hence, summing up, we obtain

$$\bar{d}_n^{h_k}(x, y) \leq 4\epsilon \quad \text{for } x, y \in Q \in \mathcal{Q}.$$

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The second case is $n > 2c_\epsilon N_k$. In this case, we put

$$Q_0 = \mathbb{T} \Big\backslash \Big(\bigcup_{0 \le i < 2N_k} [i\alpha - \gamma_k, i\alpha + \gamma_k] \Big),$$

and

$$Q_{r,s} = \bigcup_{rN_k/c_\epsilon \le i < (r+1)N_k/c_\epsilon} \bigg[i\alpha + \frac{\gamma_k s}{c_\epsilon^2}, i\alpha + \frac{\gamma_k(s+1)}{c_\epsilon^2} \bigg].$$

Let

$$\mathcal{Q} = \{Q_0\} \bigcup \{Q_{r,s} : 0 \le r \le 2c_\epsilon - 1, -c_\epsilon^2 \le s \le c_\epsilon^2 - 1\}.$$

It is clear that $\mathcal{Q}$ is a cover of $\mathbb{T}$ and $\#\mathcal{Q} \le 10c_\epsilon^4$. Given $x, y \in Q_0$, by (A.4) and (A.8) we have

$$\#\{0 \le m \le n-1 : H_m^{h_k}(x) \ne 0\} \le 2N_k$$

and

$$\#\{0 \le m \le n-1 : H_m^{h_k}(y) \ne 0\} \le 2N_k.$$

Then by (A.10)

$$\bar{d}_n^{h_k}(x, y) \le \frac{1}{n} \cdot 4N_k \le 2\epsilon.$$

Now assume that $x, y \in Q_{r,s}$ for some r and s. There exist integers $m_1, m_2 \in [rN_k/c_\epsilon, (r+1)N_k/c_\epsilon]$ and $x_1, y_1 \in [\gamma_k s/c_\epsilon^2, \gamma_k(s+1)/c_\epsilon^2]$, such that

$$x = R_\alpha^{m_1} x_1 \quad \text{and} \quad y = R_\alpha^{m_2} y_1.$$

Without loss of generality, we can assume that $m_1 \le m_2$. Recall that $2N_k < n \le 2N_{k+1}$. By (A.4) and (A.8), we have

$$h_k(R_\alpha^i x_1) = h_k(R_\alpha^i y_1) = 0, \tag{A.20}$$

for any $2N_k < i \le 2N_k + n \le 2N_k + 2N_{k+1}$. For any $1 \le m \le n$, we have

$$\begin{aligned}
\|H_m^{h_k}(x) - H_m^{h_k}(y)\| &= \bigg\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i y_1) \bigg\| \\
&\le \bigg\| \sum_{i=m_1}^{m_1+m-1} h_k(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_k(R_\alpha^i x_1) \bigg\| \\
&\quad + \bigg\| \sum_{i=m_2}^{m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \bigg\| \\
&\le \sum_{i=m_1}^{m_2-1} \|h_k(R_\alpha^i x_1)\| + \sum_{i=m_1+m}^{m_2+m-1} \|h_k(R_\alpha^i x_1)\| \\
&\quad + \bigg\| \sum_{m_2 \le i \le m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \bigg\| \\
&= 2(m_2 - m_1)\frac{1}{N_k}
\end{aligned}$$

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$$
\begin{aligned}
&+\Big\| \sum_{m_2 \le i \le m_2+m-1} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \Big\| \quad \text{by (A.10)} \\
&= 2(m_2 - m_1)\frac{1}{N_k} \\
&+\Big\| \sum_{\substack{m_2 \le i \le m_2+m-1 \\ i \le 2N_k}} (h_k(R_\alpha^i x_1) - h_k(R_\alpha^i y_1)) \Big\| \quad \text{by (A.20)} \\
&\le 2(m_2 - m_1)\frac{1}{N_k} + 2N_k \cdot \frac{\gamma_k}{c_\epsilon^2} \cdot \frac{1}{N_k \gamma_k} \quad \text{by (A.11)} \\
&\le 4\epsilon.
\end{aligned}
$$

Hence, summing up we get

$$\bar{d}_n^{h_k}(x, y) \le 4\epsilon \quad \text{for } x, y \in Q \in \mathcal{Q}.$$

Step 3. We will construct a finite cover $\mathcal{I}$ of $\mathbb{T}$ such that

$$\#(\mathcal{I}) \le 10c_\epsilon^3 \quad \textit{and} \quad \bar{d}_n^{h_{k+1}}(x, y) \le 4\epsilon,$$

for any $x, y \in I \in \mathcal{I}$.

Put

$$I_0 = \mathbb{T} \Big\backslash \Big(\bigcup_{i=-2N_{k+1}}^{2N_{k+1}} [i\alpha - \gamma_{k+1}, i\alpha + \gamma_{k+1}] \Big),$$

and

$$I_{r,s} = \bigcup_{rN_{k+1}/c_\epsilon \le i < (r+1)N_{k+1}/c_\epsilon} \Big[i\alpha + \frac{\gamma_{k+1} s}{c_\epsilon^2}, i\alpha + \frac{\gamma_{k+1}(s+1)}{c_\epsilon^2} \Big].$$

Put

$$\mathcal{I} = \{I_0\} \bigcup \{I_{r,s} : -2c_\epsilon \le r \le 2c_\epsilon - 1, -c_\epsilon^2 \le s \le c_\epsilon^2 - 1\}.$$

It is clear that $\mathcal{I}$ is a cover of $\mathbb{T}$ and $\#(\mathcal{I}) \le 10c_\epsilon^3$. Given $x, y \in I_0$, we have $\bar{d}_n^{h_{k+1}}(x, y) = 0$ by (A.8).

Now assume that $x, y \in I_{r,s}$, for some r and s. There exist integers $m_1, m_2 \in [rN_k/c_\epsilon, (r+1)N_k/c_\epsilon]$ and $x_1, y_1 \in [\gamma_{k+1}s/c_\epsilon^2, \gamma_{k+1}(s+1)/c_\epsilon^2]$ such that

$$x = R_\alpha^{m_1} x_1 \quad \text{and} \quad y = R_\alpha^{m_2} y_1.$$

Without loss of generality, we can assume that $m_1 \le m_2$. For any $1 \le m \le n$, we have

$$
\begin{aligned}
\|H_m^{h_{k+1}}(x) - H_m^{h_{k+1}}(y)\| &= \Big\| \sum_{i=0}^{m-1} (h_{k+1}(R_\alpha^i x) - h_{k+1}(R_\alpha^i y)) \Big\| \\
&\le \Big\| \sum_{i=m_1}^{m_1+m-1} h_{k+1}(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_{k+1}(R_\alpha^i y_1) \Big\| \\
&\le \Big\| \sum_{i=m_1}^{m_1+m-1} h_{k+1}(R_\alpha^i x_1) - \sum_{i=m_2}^{m_2+m-1} h_{k+1}(R_\alpha^i x_1) \Big\|
\end{aligned}
$$

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$$
\begin{aligned}
&+\left\|\sum_{i=m_2}^{m_2+m-1}(h_k(R_\alpha^i x_1)-h_k(R_\alpha^i y_1))\right\| \\
&\le 2(m_2-m_1)\frac{1}{N_{k+1}}+m\cdot\frac{\gamma_{k+1}}{c_\epsilon^2}\cdot\frac{1}{N_{k+1}\gamma_{k+1}} \quad \text{by (A.10) and (A.11)} \\
&\le 4\epsilon.
\end{aligned}
$$

Hence, summing up, we have

$$\bar{d}_n^{h_{k+1}}(x, y)\le 4\epsilon \quad \text{for } x, y\in Q\in\mathcal{I}.$$

Step 4. We will construct a finite cover $\mathcal{T}$ of $\mathbb{T}^2$ such that

$$\#(\mathcal{T})\le 100c_\epsilon^{11}c_\delta \quad \textit{and} \quad \bar{d}_n((x_1, y_1), (x_2, y_2))\le 17\epsilon$$

for any $(x_1, y_1), (x_2, y_2)\in W\in\mathcal{T}$.

Note that, for any $x\in\mathbb{T}$,

$$\|h_{k+2,\infty}(x)\|\le\sum_{i=k+2}^{\infty}\frac{1}{N_i}\le\frac{2}{N_{k+2}}.$$

For any $x, y\in\mathbb{T}$ and $1\le m\le n$, by (A.17) and $2N_k<n\le 2N_{k+1}$, we have

$$
\begin{aligned}
\|H_m^{h_{k+2,\infty}}(x)-H_m^{h_{k+2,\infty}}(y)\| &= \left\|\sum_{i=0}^{m-1}(h_{k+2,\infty}(R_\alpha^i x)-h_{k+2,\infty}(R_\alpha^i y))\right\| \\
&\le\frac{m}{N_{k+2}}\le\frac{4N_{k+1}}{N_{k+2}}<\epsilon.
\end{aligned}
$$

Hence,

$$\bar{d}_n^{h_{k+2,\infty}}(x, y)<\epsilon. \tag{A.21}$$

Finally, let $\mathcal{S}=\{[j/c_\epsilon, j+1/c_\epsilon) : j=0, 1, \ldots, c_\epsilon-1\}$ and put

$$\mathcal{T}=(\mathcal{S}\vee\mathcal{P}\vee\mathcal{Q}\vee\mathcal{I})\times\mathcal{S}.$$

It is clear that $\mathcal{T}$ is a finite cover of $\mathbb{T}^2$ with

$$\#(\mathcal{T})\le c_\epsilon\cdot c_\delta c_\epsilon^2\cdot 10c_\epsilon^4\cdot 10c_\epsilon^3\cdot c_\epsilon=100c_\epsilon^{11}c_\delta=C(\epsilon).$$

Hence, for $(x_1, y_1), (x_2, y_2)\in W\in\mathcal{T}$, by Steps 1, 2, 3 and (A.21), we have

$$
\begin{aligned}
\bar{d}_n^h(x_1, x_2) &\le \bar{d}_n^{h_{1,k-1}}(x_1, x_2)+\bar{d}_n^{h_k}(x_1, x_2)+\bar{d}_n^{h_{k+1}}(x_1, x_2)+\bar{d}_n^{h_{k+2,\infty}}(x_1, x_2) \\
&< 6\epsilon+4\epsilon+4\epsilon+\epsilon=15\epsilon.
\end{aligned}
$$

We deduce that

$$\bar{d}_n((x_1, y_1), (x_2, y_2))\le\|x_1-x_2\|+\|y_1-y_2\|+\bar{d}_n^h(x_1, x_2)<17\epsilon.$$

This implies $\overline{\text{span}}(n, 17\epsilon)\le C(\epsilon)$, for all $n>K(\epsilon)$, which ends the proof. □

A.2. *The construction of the system in Proposition 3.9.* First, we need the following Furstenberg's dichotomy result.

PROPOSITION A.7. [**9**] *Suppose (Ω_0, μ_0, T_0) is a uniquely ergodic topological dynamical system, with μ_0 being the unique ergodic measure, and $h : \Omega_0 \to \mathbb{T}$ is a continuous function. Let $T : \Omega_0 \times \mathbb{T}$ be defined by $T(x, y) = (T_0(x), y + h(x))$. Then, exactly one of the following is true:*

(1) *T is uniquely ergodic and $\mu_0 \times m_{\mathbb{T}}$ is the unique invariant measure;*

(2) *there exists a measurable map $g : \Omega_0 \to \mathbb{T}$ and a non-zero integer s, such that $s \cdot h(x) = g(T_0(x)) - g(x)$, for μ_0-a.e. $x \in \Omega_0$.*

Now we modify the example $(\mathbb{T}^2, T)$ in the previous subsection, to be uniquely ergodic. As $(\mathbb{T}^2, T)$ is not uniquely ergodic, by Furstenberg's dichotomy result there is an $m_{\mathbb{T}}$-measurable function $g(x)$ and a non-zero integer s such that

$$s \cdot h(x) = g(x + \alpha) - g(x) \tag{A.22}$$

for $m_{\mathbb{T}}$-a.e. $x \in \mathbb{T}$. We define

$$\phi : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (x, s \cdot y)$$

and

$$\widetilde{T} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + s \cdot h(x)).$$

Then, $\widetilde{T} \circ \phi = \phi \circ T$, in other words, the following diagram commutes.

$$\begin{array}{ccc} \mathbb{T}^2 & \xrightarrow{T} & \mathbb{T}^2 \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ \mathbb{T}^2 & \xrightarrow{\widetilde{T}} & \mathbb{T}^2 \end{array}$$

Take an irrational number $\beta \in \mathbb{R}$ such that α and β are rationally independent. Then, the system defined by

$$T_{\alpha,\beta} : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \to (x + \alpha, y + \beta)$$

is uniquely ergodic and $m_{\mathbb{T}^2}$ is the unique invariant measure. Finally, we define

$$\widetilde{T}_\beta : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (x + \alpha, y + s \cdot h(x) + \beta).$$

We will show that the system $(\mathbb{T}^2, \widetilde{T}_\beta)$ is the one we need. It is clear that $(\mathbb{T}^2, \widetilde{T}_\beta)$ is distal.

PROPOSITION A.8. *$(\mathbb{T}^2, \widetilde{T}_\beta)$ is uniquely ergodic and minimal.*

Proof. Let $K = \{x \in \mathbb{T} : s \cdot h(x) = g(x + \alpha) - g(x)\}$ and

$$\pi : \mathbb{T}^2 \to \mathbb{T}^2, \quad (x, y) \mapsto (x, y - g(x)).$$

By (A.22), we have $m_{\mathbb{T}}(K) = 1$. It is easy to see that $\pi : K \times \mathbb{T} \to K \times \mathbb{T}$ is an invertible map with $\pi \circ \widetilde{T}_\beta = T_{\alpha,\beta} \circ \pi$. For each $\widetilde{T}_\beta$-invariant measure μ, we have $\mu(K \times \mathbb{T}) = 1$ and $\mu \circ \pi^{-1}$ is $T_{\alpha,\beta}$-invariant. We have $\mu \circ \pi^{-1} = m_{\mathbb{T}^2}$, since $m_{\mathbb{T}^2}$ is the unique invariant probability measure of $T_{\alpha,\beta}$. Thus, $\mu = m_{\mathbb{T}^2} \circ \pi$. This implies that $m_{\mathbb{T}^2} \circ \pi$ is the only invariant measure for $(\mathbb{T}^2, \widetilde{T}_\beta)$. Moreover, $(\mathbb{T}^2, \widetilde{T}_\beta)$ is minimal, since the only invariant measure $m_{\mathbb{T}^2} \circ \pi$ is of full support. □

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PROPOSITION A.9. *$(\mathbb{T}^2, \widetilde{T}_\beta)$ is not equicontinuous.*

Proof. It is sufficient to show for any $\epsilon > 0$, there exist $(x_1, y_1), (x_2, y_2) \in \mathbb{T}^2$ and a positive integer n such that $d((x_1, y_1), (x_2, y_2)) \le \epsilon$ and $d(\widetilde{T}_\beta^n(x_1, y_1), \widetilde{T}_\beta^n(x_2, y_2)) \ge \frac{1}{200}$.

Assume that $10^p \le |s| < 10^{p+1}$, for some non-negative integer p. Given $\epsilon > 0$, there exists $k \in \mathbb{N}$ such that $k > p + 10$ and $l_k + \delta_k < \epsilon$. Put $x' = \delta_k + \frac{1}{2} l_k$. We have $R_\alpha^i x' \in E_k^c$ and $h_k(R_\alpha^i x') = 0$ for $i = 0, 1, 2, \ldots, N_k - 1$. By (A.3), for any $x \in \mathbb{T}$,

$$\frac{1}{M_k}\#\{0 \le i \le M_k - 1 : R_\alpha^i x \in E_{1,k-1}^c\} \ge 1 - \sum_{i=1}^{\infty} \frac{\eta}{2^i} > \frac{1}{2}.$$

Then there are integers $n_1 \in [0, M_k - 1]$ and

$$n_2 \in [10^{k-p-2} M_k - M_k, 10^{k-p-2} M_k - 1]$$

such that $R_\alpha^{n_1} 0, R_\alpha^{n_1} x', R_\alpha^{n_2} 0, R_\alpha^{n_2} x' \in E_{1,k-1}^c$. By using Lemma A.1 and the fact $R_\alpha^{n_1} x', R_\alpha^{n_2} x' \in E_k^c$, we have

$$\begin{aligned}
H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x') &= H_{n_2-n_1}^{h_{1,k-1}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{1,k-1}}(R_\alpha^{n_1} x') \\
&\quad + H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} x') \\
&\quad + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} x') \\
&= H_{n_2-n_1}^{h_k}(R_\alpha^{n_1} 0) + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} x') \\
&= (n_2 - n_1)\frac{1}{N_k} + H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} x').
\end{aligned}$$

Moreover, we have

$$\begin{aligned}
\|H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} 0) - H_{n_2-n_1}^{h_{k+1,\infty}}(R_\alpha^{n_1} x')\| &\le \sum_{i=k+1}^{\infty} (|H_{n_2-n_1}^{h_i}(R_\alpha^{n_1} 0)| + |H_{n_2-n_1}^{h_i}(R_\alpha^{n_1} x')|) \\
&\le \sum_{i=k+1}^{\infty} 2(n_2 - n_1)\frac{1}{10^i M_i} \le \sum_{i=k+1}^{\infty} \frac{2 \cdot 10^{k-p-2} M_k}{10^i M_i} \\
&\le 2 \sum_{i=k+1}^{\infty} \frac{10^{-p-2}}{10^{i-k}} = \frac{2}{9} \cdot 10^{-p-2}.
\end{aligned}$$

Note that $10^{k-p-2} M_k - 2M_k \le n_2 - n_1 \le 10^{k-p-2} M_k$. We have

$$|s| \cdot \|H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x')\| \le |s| \cdot (10^{-p-2} + \tfrac{2}{9} \cdot 10^{-p-2}) \le \tfrac{2}{10}$$

and

$$|s| \cdot \|H_{n_2-n_1}^h(R_\alpha^{n_1} 0) - H_{n_2-n_1}^h(R_\alpha^{n_1} x')\| \ge |s| \cdot \left(10^{-p-2} - \frac{2}{10^k} - \frac{2}{9} \cdot 10^{-p-2}\right) \ge \frac{1}{200}.$$

Let $x_1 = R_\alpha^{n_1} 0$, $x_2 = R_\alpha^{n_1} x'$, $y_1 = y_2 = 0$ and $n = n_1 - n_2$. Then,

$$d((x_1, y_1), (x_2, y_2)) = \|R_\alpha^{n_1} 0 - R_\alpha^{n_1} x'\| = \|x'\| = \delta_k + \frac{1}{2}\ell_k < \epsilon$$

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and

$$\begin{aligned}
&d(\widetilde{T}^n_\beta(x_1, y_1), \widetilde{T}^n_\beta(x_2, y_2)) \\
&\quad = d((R^{n_2}_\alpha 0, s \cdot H^h_{n_2-n_1}(R^{n_1}_\alpha 0) + (n_2 - n_1)\beta), \\
&\qquad (R^{n_2}_\alpha x', s \cdot H^h_{n_2-n_1}(R^{n_1}_\alpha x') + (n_2 - n_1)\beta)) \\
&\quad \geq \|s \cdot (H^h_{n_2-n_1}(R^{n_1}_\alpha 0) - H^h_{n_2-n_1}(R^{n_1}_\alpha x'))\| \geq \tfrac{1}{200}.
\end{aligned}$$

This implies that $(\mathbb{T}^2, \widetilde{T}_\beta)$ is not equicontinuous. □

PROPOSITION A.10. *$(\mathbb{T}^2, \widetilde{T}_\beta)$ has bounded topological complexity with respect to $\{\bar{d}_n\}$.*

Proof. For $\epsilon > 0$, let $\mathscr{T}$, c_ϵ and c_δ be defined in Proposition A.6. Then, for $(x_1, y_1), (x_2, y_2) \in W \in \mathscr{T}$, we have

$$\begin{aligned}
\bar{d}^{s\cdot h+\beta}_n(x_1, x_2) &= \frac{1}{n}\sum_{m=0}^{n-1} \|H^{sh+\beta}_m(x) - H^{sh+\beta}_m(y)\| \\
&= \frac{1}{n}\sum_{m=0}^{n-1} \|sH^h_m(x) - sH^h_m(y)\| \\
&\leq \frac{1}{n}\sum_{m=0}^{n-1} |s| \cdot \|H^h_m(x) - H^h_m(y)\| \\
&\leq |s| \cdot \bar{d}^h_n(x_1, x_2) \leq 15|s|\epsilon
\end{aligned}$$

and

$$\bar{d}_n((x_1, y_1), (x_2, y_2)) \leq \|x_1 - x_2\| + \|y_1 - y_2\| + \bar{d}^h_n(x_1, x_2) \leq (15|s| + 2)\epsilon.$$

Hence $\overline{\mathrm{span}}(n, (15|s| + 2)\epsilon) \leq 100c^{11}_\epsilon c_\delta$. Thus, $(\mathbb{T}^2, \widetilde{T}_\beta)$ has bounded topological complexity with respect to $\{\bar{d}_n\}$. □

B. *Appendix. An example by Cyr and Kra*

We first introduce some concepts. Following [**6**], by an *assignment*, we mean a function Ψ, defined on an abstract metrizable Choquet simplex $\mathcal{P}$, whose 'values' are measure-theoretic dynamical systems, i.e. for $p \in \mathcal{P}$, $\Psi(p)$ has the form $(X_p, \mathcal{B}_p, \mu_p, T_p)$. Two assignments, Ψ on a simplex $\mathcal{P}$ and Ψ' on a simplex $\mathcal{P}'$, are said to be *equivalent* if there exists an affine homeomorphism $\pi : \mathcal{P} \to \mathcal{P}'$ of Choquet simplexes such that for every $p \in \mathcal{P}$ the systems $\Psi(p)$ and $\Psi'(\pi(p))$ are isomorphic as measure-theoretic dynamical systems. A topological dynamical system (X, T) determines an assignment on the simplex of T-invariant probability measures by the rule $\mu \to (X, \mathcal{B}_X, \mu, T)$, where $\mathcal{B}_X$ denotes the collection of Borel sets on X. By [**6**, Theorem 1] or [**22**, Theorem 1], we know that if Y is zero dimensional and (Y, S) has no periodic points, then the assignment determined by (Y, S) is equivalent to an assignment determined by some minimal system (X, T). If (Y, S) is invertible, then we can require that (X, T) is also invertible [**22**]. Applying [**6**, Theorem 1] or [**22**, Theorem 1], there is a minimal system (X, T) whose assignment is equivalent to that of (Y, S).

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PROPOSITION B.1. *There exists a minimal system with bounded complexity with respect to $\{\bar{d}_n\}$, for an invariant measure μ, for which there exist two non-isomorphic ergodic measures in the ergodic decomposition.*

Proof. Pick two Sturmian shifts, (Y_1, σ) and (Y_2, σ), in the full shifts $(\{0, 1\}^{\mathbb{Z}}, \sigma)$ and $(\{2, 3\}^{\mathbb{Z}}, \sigma)$, respectively. Then, (Y_1, σ) and (Y_2, σ) are minimal and uniquely ergodic. Let ν_1 and ν_2 be the unique invariant measures of (Y_1, σ) and (Y_2, σ), respectively. Then both ν_1 and ν_2 have discrete spectrum. We can require that the spectra of ν_1 and ν_2 are different and then ν_1 and ν_2 are not isomorphic. Let $Y = Y_1 \cup Y_2 \subset \{0, 1, 2, 3\}^{\mathbb{Z}}$. It is clear that Y is zero dimensional and (Y, σ) has no periodic points.

By [**6**, Theorem 1] or [**22**, Theorem 1], there is a minimal system (Y, S) whose assignment is equivalent to that of (Y, σ). This means that (X, T) carries exactly two ergodic measures, μ_1 and μ_2, and $(X, \mathcal{B}_X, \mu_i, T)$ is isomorphic to $(Y_i, \mathcal{B}_{Y_i}, \nu_i, \sigma)$. Let $\mu = \frac{1}{2}\mu_1 + \frac{1}{2}\mu_2$. As both μ_1 and μ_2 have discrete spectrum, so is μ. By Proposition 4.5, μ has bounded complexity with respect to $\{\bar{d}_n\}$. But the ergodic measures in the ergodic decomposition of μ are μ_1 and μ_2, which are not isomorphic. □

Remark B.2. It should be noted that the same idea of construction in Proposition B.1 can be used to provide countably many non-isomorphic ergodic measures in the ergodic decomposition, but the uncountably many case is still not clear.

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