On a diffuse interface model of tumour growth

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We consider a diffuse interface model of tumour growth proposed by A. Hawkins-Daruud
et al. ((2013) *J. Math. Biol.* **67** 1457–1485). This model consists of the Cahn–Hilliard equation for the tumour cell fraction φ nonlinearly coupled with a reaction–diffusion equation for ψ, which represents the nutrient-rich extracellular water volume fraction. The coupling is expressed through a suitable proliferation function $p(\varphi)$ multiplied by the differences of the chemical potentials for φ and ψ. The system is equipped with no-flux boundary conditions which give the conservation of the total mass, that is, the spatial average of $\varphi + \psi$. Here, we prove the existence of a weak solution to the associated Cauchy problem, provided that the potential F and p satisfy sufficiently general conditions. Then we show that the weak solution is unique and continuously depends on the initial data, provided that p satisfies slightly stronger growth restrictions. Also, we demonstrate the existence of a strong solution and that any weak solution regularizes in finite time. Finally, we prove the existence of the global attractor in a phase space characterized by an *a priori* bounded energy.

1 Introduction

Modelling tumour growth dynamic has recently become a major issue in applied mathematics (see, for instance, [11,25], cf. also [2,29]). The models can be divided into two broad categories: continuum models and discrete or cellular automata models (however, see, e.g., [11, Chap.7] for hybrid continuum-discrete models). Concerning the former ones, the necessity of dealing with multiple interacting constituents has led to the consideration of

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diffuse-interface models based on continuum mixture theory (see, for instance, [10, 28, 34] and references therein, cf. also [7, 13, 20]). Such models generally consist of Cahn–Hilliard equations with transport and reaction terms which govern various types of cell concentrations. The reaction terms depend on the nutrient concentration (e.g., oxygen) which obeys an advection-reaction-diffusion equation coupled with the Cahn–Hilliard equations. The cell velocities satisfy a generalized Darcy's (or Brinkman's) law where, besides the pressure gradient, there is also the so-called Korteweg force due to the cell concentration. Numerical simulations of diffuse-interface model for tumour growth have been carried out in several papers (see, for instance, [11, Chap. 8] and references therein). Nonetheless, a rigorous mathematical analysis of the resulting systems of differential equations is still in its infancy. In particular, to the best of our knowledge, the first related papers are concerned with the so-called Cahn–Hilliard–Hele–Shaw system (see [24], cf. also [5, 32, 33]) in which the nutrient is neglected. Moreover, a very recent contribution (see [8]) is devoted to analyze an approximation of a model recently proposed in [21] (see also [22, 35]). In this model, velocities are set to zero and the state variables are reduced to the tumour cell fraction φ and the nutrient-rich extracellular water fraction ψ. The corresponding PDE system is given by

$$\varphi_t = \Delta\mu + p(\varphi)(\psi - \mu), \tag{1.1}$$

$$\mu = -\Delta\varphi + F'(\varphi), \tag{1.2}$$

$$\psi_t = \Delta\psi - p(\varphi)(\psi - \mu), \tag{1.3}$$

in $\Omega \times (0, \infty)$, where $\Omega \subset \mathbb{R}^3$ is a bounded smooth domain. Here F is the typical double-well associated with the Ginzburg–Landau free-energy functional, while p is a proliferation function which must be nonnegative and may have, for instance, the form $p(s) = p_0(1 - s^2)\chi_{[-1,1]}(s)$ for $s \in \mathbb{R}$, $p_0 > 0$. Here, $\chi_{[-1,1]}$ represents the indicator function of $[-1, 1]$. However, in this paper we suppose p to be, at least, Lipschitz continuous, but we allow it to satisfy a suitable growth condition (cf. (3.4)). Also, it is worth observing that more general potentials F, possibly depending on ψ as well, might be taken into account since they are relevant from the modelling viewpoint (cf. [21] and references therein). This could be the subject of a future work.

System (1.1)–(1.3) is equipped with the no-flux boundary conditions

$$\partial_n\varphi = \partial_n\mu = \partial_n\psi = 0 \quad \text{on } \partial\Omega \times (0, \infty), \tag{1.4}$$

and initial conditions

$$\varphi(0) = \varphi_0, \qquad \psi(0) = \psi_0 \quad \text{in } \Omega. \tag{1.5}$$

In [8], the authors consider a relaxed model in which the chemical potential μ contains a viscous term $\alpha\varphi_t$, $\alpha > 0$ and equation (1.1) has an additional term $\alpha\mu_t$ which requires a further initial condition. For this model, existence and uniqueness of a variational solution is proven under very general conditions on F, while p is taken to be globally bounded and Lipschitz continuous. Then, imposing substantial restrictions on F (e.g., polynomial growth of order 4), the authors prove the existence of a sequence $\{\alpha_n\}$ and a sequence of solutions which converges to a weak solution to problem (1.1)–(1.5) as α_n goes to 0.

Such a solution is more regular than the weak solution to (1.1)–(1.5) and it is also unique, provided that φ_0 is smooth enough.

Here, we want to analyse problem (1.1)–(1.5) without any regularizing term. More precisely, it is not difficult to check that system (1.1)–(1.3) with (1.4) is characterized by the total energy balance law (see [21, (10)])

$$\frac{d}{dt}\mathscr{E}(\varphi,\psi) + \|\nabla\mu\|^2 + \|\nabla\psi\|^2 + \int_\Omega p(\varphi)(\mu - \psi)^2 = 0, \qquad (1.6)$$

where the energy $\mathscr{E}$ is given by

$$\mathscr{E}(\varphi,\psi) := \frac{1}{2}\|\nabla\varphi\|^2 + \frac{1}{2}\|\psi\|^2 + \int_\Omega F(\varphi). \qquad (1.7)$$

Therefore, it seems natural to find a solution assuming that the initial data have just finite energy. This is our first result, namely, existence of a weak solution of finite energy. The assumptions on F and p are more general than the ones in [8] for the case $\alpha = 0$. In particular, in the present contribution p can have a polynomially controlled growth. Concerning F, we can take any C^2 and λ_1-convex potential satisfying $|F'| \leqslant \lambda_2 F + \lambda_3$ for some non-negative constants $\lambda_1, \lambda_2, \lambda_3$. For instance, $F(s) = \exp(s)$ or F with arbitrary polynomial growth. Also, with a further restriction on the growth of p' and assuming F to have a polynomially controlled growth, we can establish the continuous dependence on the initial data (and so the uniqueness of weak solutions).

The proof is obtained by suitably approximating the potential F with a coercive quadratic potential F_m and finding an approximating solution of such a problem through a Faedo–Galerkin scheme. The crucial point is then to obtain appropriate *a priori* estimates to pass to the limit via compactness results with respect to m. In particular, a bootstrap argument is used in order to derive the optimal regularity estimate for φ, which is necessary in order to prove the continuous dependence estimate as well as for the analysis of the global longtime behaviour. For similar double approximation techniques the reader is referred to, e.g., [12, 15].

Then, we prove a regularity result which helps us to investigate the global longtime behaviour of the solutions. Concerning this issue, observe that conditions (1.4) imply the conservation of the *total* mass

$$\int_\Omega \big(\varphi(t) + \psi(t)\big) = \int_\Omega (\varphi_0 + \psi_0), \quad \forall t \geqslant 0. \qquad (1.8)$$

However, we are not able to obtain independent global bounds for the spatial averages of $\varphi(t)$ and $\psi(t)$. On account of this fact, we can show that (1.1)–(1.4) generates a dynamical system taking as phase space a bounded set in the finite energy space with a constraint on the total mass. We can thus prove that such a system has a global attractor.

This is just a preliminary step towards the theoretical analysis of more refined models. For instance, one may include the fluid velocity either given as a datum or satisfying a generalized Darcy's (or Brinkman's) law. Also, one should take a logarithmic potential F, which is physically more relevant, and non-constant (possibly degenerate) mobility in the Cahn–Hilliard equation. On the other hand, the free energy functional may contain a non-local spatial interaction in place of the usual term $|\nabla\varphi|^2$ giving rise to a convolution

operator acting on φ in place of $\Delta\varphi$ in (1.2) (see, for instance, [34], cf. also [17, 18]). These are just some examples of challenging extensions of the simplified model expressed by (1.1)–(1.3).

1.1 Plan of the paper

In Section 2, we define the notation and we recall a useful inequality. In Section 3, we prove that Problem (1.1)–(1.5) admits a unique weak solution (which continuously depends on the data) under proper assumptions on the nonlinearities F and p. In Section 4, we establish a regularity result for Problem (1.1)–(1.5) that holds under the same condition on p which ensures uniqueness. This result turns out to be crucial in order to eventually prove the existence of the global attractor.

2 Notation and preliminaries

Let Ω be a sufficiently regular, bounded domain in $\mathbb{R}^3$, let $T > 0$ and set $Q = \Omega \times (0, T)$. Then, we define $H := L^2(\Omega)$ and $V := H^1(\Omega)$ and denote by $\|\cdot\|$, $(\cdot,\cdot)$ the norm and the scalar product in H, respectively. If X is a (real) Banach space, the notation $\langle\cdot,\cdot\rangle$ will be used to denote the duality pairing between X and its dual X', while $(\cdot,\cdot)_X$ will denote the scalar product in X. For every $f \in V'$, $\overline{f}$ will stand for the average of f over Ω, i.e., $\overline{f} := |\Omega|^{-1}\langle f, 1\rangle$. Here, $|\Omega|$ is the Lebesgue measure of Ω.

Since it is convenient to rewrite the equations (1.1) and (1.3) as abstract equations in the framework of the Hilbert triplet (V, H, V'), we introduce the Riesz isomorphism $A : V \to V'$ associated to the standard scalar product of V, that is,

$$\langle Au, v\rangle := (u, v)_V = \int_\Omega (\nabla u \cdot \nabla v + uv) \quad \forall u, v \in V. \tag{2.1}$$

We notice that $Au = -\Delta u + u$ if $u \in D(A) := \{\varphi \in H^2(\Omega) : \partial_{\mathbf{n}}\varphi = 0 \text{ on } \partial\Omega\}$ and that the restriction of A to $D(A)$ is an isomorphism from $D(A)$ onto H. We also remark that

$$\langle Au, A^{-1}v^*\rangle = \langle v^*, u\rangle, \quad \forall u \in V \text{ and } v^* \in V',$$
$$\langle u^*, A^{-1}v^*\rangle = (u^*, v^*)_{V'}, \quad \forall u^*, v^* \in V'$$

where $(\cdot,\cdot)_{V'}$ is the dual scalar product in V' associated to the standard one in V, and recall that $\langle v^*, u\rangle = \int_\Omega v^* u$ if $v^* \in H$ and we have

$$\frac{d}{dt}\|v^*\|_{V'}^2 = 2\langle\partial_t v^*, A^{-1}v^*\rangle, \quad \forall v^* \in H^1(0, T; V').$$

Moreover, by a classical spectral theorem there exist a sequence of eigenvalues λ_j with $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots$ and $\lambda_j \to \infty$, and a family of eigenfunctions $w_j \in D(A)$ such that $Aw_j = \lambda_j w_j$. The family of w_j is an orthonormal basis in H and it is also orthogonal in V and $D(A)$.

We shall repeatedly use the following Gagliardo–Nirenberg inequality in dimension 3 (see, e.g., [4, 14, 16, 27] for more details)

Lemma 1 *Let $1 \leqslant p_1, p_2 \leqslant \infty$, $0 \leqslant r < l$ $(r, l \in \mathbb{N})$ and assume that*

$$\theta := \frac{3/m - 3/p_1 - r}{3/p_2 - 3/p_1 - l} \in [r/l, 1).$$

Then

$$\|u\|_{W^{r,m}(\Omega)} \leqslant c\|u\|_{L^{p_1}(\Omega)}^{1-\theta}\|u\|_{W^{l,p_2}(\Omega)}^{\theta}, \quad \forall u \in W^{l,p_2} \cap L^{p_1}(\Omega). \tag{2.2}$$

3 Existence and uniqueness of weak solutions

In this section, we prove that Problem (1.1)–(1.5) admits a weak solution, provided that F and p have polynomial growth with given orders ρ and q, respectively. The upper bounds on ρ and q in Theorem 1 ensure the existence of a weak solution with optimal regularity for φ, i.e., $\varphi \in L^2(0, T; H^3(\Omega))$. Such assumptions can be relaxed if only existence of the weak solution is required (cf. Corollary 1). An additional restriction on the proliferation function p allows us to prove uniqueness as well as a continuous dependence estimate on the initial data for weak solutions. In any case, our assumptions on F and p are more general than those made in [21] (cf. also [8] when $\alpha = 0$).

Let us begin with the existence result, which will be proven, for the case where the growth ρ of F is greater than 4, by means of a double approximation procedure, namely by first exploiting the Faedo–Galerkin scheme to prove existence for $\rho \leqslant 4$ and then by approximating F with a sequence of potentials having growth which is at most 4.

The assumptions we need for the existence are the following

(F) $F \in C^2(\mathbb{R})$ can be written as

$$F(s) = F_0(s) + \lambda(s), \tag{3.1}$$

where $F_0 \in C^2(\mathbb{R})$ and $\lambda \in C^2(\mathbb{R})$ satisfies $|\lambda''(s)| \leqslant \alpha$, for all $s \in \mathbb{R}$, and for some constant $\alpha \geqslant 0$. Moreover, we assume

$$c_1(1 + |s|^{\rho-2}) \leqslant F_0''(s) \leqslant c_2(1 + |s|^{\rho-2}), \tag{3.2}$$

$$F(s) \geqslant c_3|s| - c_4, \tag{3.3}$$

for all $s \in \mathbb{R}$, with $c_1, c_2, c_3 > 0$, $c_4 \in \mathbb{R}$ and with $\rho \in [2, 6)$.

(P) $p \in C_{loc}^{0,1}(\mathbb{R})$ satisfies

$$0 \leqslant p(s) \leqslant c_5(1 + |s|^q), \tag{3.4}$$

for all $s \in \mathbb{R}$, with $c_5 > 0$ and with $q \in [1, 9)$.

Before stating the existence result, let us introduce the definition of weak solution to Problem (1.1)–(1.5).

Definition 1 *Let $\varphi_0 \in V$, $\psi_0 \in H$ and $0 < T < \infty$ be given. Then, a pair $[\varphi, \psi]$ is a weak solution to (1.1)–(1.5) on $[0, T]$ if*

$$\varphi \in L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)), \quad \varphi_t \in L^r(0, T; D(A^{-1})), \tag{3.5}$$

$$\mu := -\Delta\varphi + F'(\varphi) \in L^2(0, T; V), \tag{3.6}$$

$$\psi \in L^\infty(0, T; H) \cap L^2(0, T; V), \quad \psi_t \in L^r(0, T; D(A^{-1})), \tag{3.7}$$

for some $r > 1$, and the following identities are satisfied

$$\langle \varphi_t, \chi \rangle + (\nabla\mu, \nabla\chi) = \big(p(\varphi)(\psi - \mu), \chi\big), \tag{3.8}$$

$$\langle \psi_t, \xi \rangle + (\nabla\psi, \nabla\xi) = -\big(p(\varphi)(\psi - \mu), \xi\big), \tag{3.9}$$

for all $\chi, \xi \in D(A)$ and for almost all $t \in (0, T)$, together with the initial conditions (1.5).

Remark 1 Notice that the regularity properties of weak solution imply that

$$\varphi \in C_w([0, T]; V), \quad \psi \in C_w([0, T]; H).$$

Hence, the initial conditions (1.5) make sense. Moreover, we point out that the required regularity for $\partial\Omega$ in order to prove our theorems is at least $C^{2,1}$. For instance, we need some regularity for the eigenfunctions (see proof of Theorem 1) as well as when we deduce that $\varphi \in L^2(0, T; H^2(\Omega))$ (cf. (3.5)).

Theorem 1 *Assume that (F) and (P) are satisfied. Let $\varphi_0 \in V$ and $\psi_0 \in H$. Then, for every $T > 0$, Problem (1.1)–(1.5) admits a weak solution on $[0, T]$ such that*

$$\varphi \in L^2(0, T; H^3(\Omega)), \tag{3.10}$$

$$F(\varphi) \in L^\infty\big(0, T; L^1(\Omega)\big), \quad \sqrt{p(\varphi)}(\mu - \psi) \in L^2(0, T; H), \tag{3.11}$$

which satisfies the following energy inequality

$$\mathscr{E}(\varphi, \psi) + \int_0^t \big(\|\nabla\mu\|^2 + \|\nabla\psi\|^2\big)d\tau + \int_0^t \int_\Omega p(\varphi)(\mu - \psi)^2 \leqslant \mathscr{E}(\varphi_0, \psi_0), \quad \forall t > 0, \tag{3.12}$$

where $\mathscr{E}$ is given by (1.7). Furthermore, if $q \leqslant 4$, then we have

$$\varphi_t, \psi_t \in L^2(0, T; V'), \tag{3.13}$$

and (3.12) holds with the equality sign. Moreover, in this case the weak formulation (3.8), (3.9) is satisfied also for all $\chi, \xi \in V$.

Remark 2 The bound $\rho < 6$ is required only to gain the optimal regularity $\varphi \in L^2(0, T; H^3(\Omega))$. Actually, we should only require $\rho \leqslant 6$. However, due to technical reasons, we are not able to perform our bootstrap technique in the case $\rho = 6$ (cf. Step II in the proof of Theorem 1). Nevertheless, the existence of a weak solution (without this optimal regularity) can be proven under more general assumptions on F (together with

a slight restriction on q), in particular, for F with polynomial growth of arbitrary order (see Corollary 1).

The following lemma will turn out to be useful in the proof of Theorem 1 (cf. Step II). Indeed, it allows us to suitably approximate a regular potential having general ρ−growth (in particular in the case of $\rho > 4$) and satisfying conditions (3.2), (3.3) with a sequence of regular potentials having quadratic growth.

Lemma 2 *Assume that F satisfies (F) with $\rho > 2$. Then there exists a sequence of $F_m \in C^2(\mathbb{R})$ satisfying*

$$|F_m(s)| \leqslant \alpha_m(1 + s^2), \qquad \forall s \in \mathbb{R},$$

for some constant $\alpha_m \geqslant 0$, such that $F_m(s) \to F(s)$ pointwise for all $s \in \mathbb{R}$ as $m \to +\infty$ and fulfilling, for every $m \in \mathbb{N}$, the bounds

$$|F_m(s)| \leqslant k_0|F(s)|, \quad |F_m'(s)| \leqslant k_1|F'(s)|, \quad |F_m''(s)| \leqslant k_2|F''(s)|, \quad \forall s \in \mathbb{R}, \tag{3.14}$$

and the equi-coercivity conditions

$$F_m(s) \geqslant k_3 s^2 - k_4, \qquad F_m''(s) \geqslant -k_5, \quad \forall s \in \mathbb{R}, \tag{3.15}$$

where k_i, $i = 0, \ldots, 5$ are some positive constants depending on F and ρ only.

Proof Without loss of generality, we set $F_0(0) = F_0'(0) = 0$ (this condition can always be assumed by redefining the function λ). Set $H_0 = F_0'$ and let H_{0m} be the Yosida regularization of H_0 defined by (cf., e.g., [6, p. 28])

$$H_{0m}(s) = m(s - J_m(s)), \qquad J_m(s) = \left(I + \frac{1}{m}H_0\right)^{-1}(s), \qquad \forall s \in \mathbb{R}.$$

Introduce now

$$F_{0m}(s) = \int_0^s H_{0m}(\sigma)d\sigma,$$

for all $s \in \mathbb{R}$, and set

$$F_m(s) = F_{0m}(s) + \lambda(s).$$

Let us check that the sequence of F_m satisfies all the stated conditions. We shall use standard results from the theory of maximal monotone operators (applied to the single-valued monotone function H_0 defined on the whole of $\mathbb{R}$).

First, notice that H_{0m} is Lipschitz continuous with Lipschitz constant equal to m, and then $|H_{0m}(s)| \leqslant m|s|$, for all $s \in \mathbb{R}$, where we have used the fact that $H_{0m}(0) = J_m(0) = 0$, since $F_0'(0) = 0$. Therefore,

$$|F_{0m}(s)| \leqslant \frac{1}{2}ms^2, \qquad \forall s \in \mathbb{R},$$

which implies that F_m has at most quadratic growth for each m.

Moreover, we have $|H_{0m}(s)| \leqslant |H_0(s)| = |F_0'(s)|$ and also $H_{0m}(s) \to H_0(s) = F_0'(s)$, for all $s \in \mathbb{R}$ as $m \to +\infty$. Therefore, by the Lebesgue theorem, we deduce

$$F_{0m}(s) \to \int_0^s F_0'(\sigma)d\sigma = F_0(s), \quad \text{as } m \to +\infty$$

and this implies that $F_m(s) \to F(s)$ for all $s \in \mathbb{R}$ as $m \to +\infty$.

Next, the bound $(3.14)_1$ is immediate, since, for all $s \in \mathbb{R}$ we have

$$|F_{0m}(s)| \leqslant \left| \int_0^s |H_{0m}(\sigma)|d\sigma \right| \leqslant \left| \int_0^s |F_0'(\sigma)|d\sigma \right| = F_0(s),$$

$$|F_{0m}'(s)| = |H_{0m}(s)| \leqslant |H_0(s)| = |F_0'(s)|.$$

Also, we can take the growth condition (3.2) into account. As far as $(3.14)_3$ is concerned, notice first that we have $F_{0m}''(s) = H_{0m}'(s) = m(1 - J_m'(s))$. Moreover, from

$$r + \frac{1}{m}H_0(r) = s \iff r = J_m(s), \tag{3.16}$$

we deduce

$$J_m'(s) = \frac{1}{1 + \frac{1}{m}F_0''(J_m(s))}.$$

Hence, we have

$$F_{0m}''(s) = \frac{F_0''(J_m(s))}{1 + \frac{1}{m}F_0''(J_m(s))} \leqslant F_0''(J_m(s)) \leqslant c_2(1 + |J_m(s)|^{\rho-2})$$

$$\leqslant c_2(1 + |s|^{\rho-2}) \leqslant \frac{c_2}{c_1}F_0''(s), \qquad \forall s \in \mathbb{R}.$$

Bound $(3.14)_3$ then follows from this last estimate and (3.2).

Furthermore, we have

$$F_{0m}''(s) \geqslant \frac{F_0''(J_m(s))}{1 + F_0''(J_m(s))} \geqslant \frac{c_1}{1 + c_1}, \qquad \forall s \in \mathbb{R}, \quad \forall m,$$

and this, together with the assumption on λ, yields $(3.15)_2$. Let us finally check that also $(3.15)_1$ holds. To this end we first recall the following property: let $\beta > 0$ and $\gamma \in \mathbb{R}$ be two constants such that

$$F_0(s) \geqslant \beta s^2 - \gamma, \qquad \forall s \in \mathbb{R}.$$

Then, we have

$$F_{0m}(s) \geqslant \frac{\beta}{2}s^2 - \gamma, \qquad \forall s \in \mathbb{R},$$

and for all $m \geqslant m_0(\beta)$. We give the proof for the reader's convenience. Observe that

$$F_{0m}(s) = \frac{1}{2m}H_{0m}^2(s) + F_0(J_m(s)) \geqslant \frac{1}{2m}H_{0m}^2(s) + \beta J_m^2(s) - \gamma$$

$$= \frac{1}{2m}H_{0m}^2(s) + \beta\left(s - \frac{1}{m}H_{0m}(s)\right)^2 - \gamma \geqslant \frac{1}{4m}H_{0m}^2(s) + \beta\left(1 - \frac{4\beta}{m}\right)s^2 - \gamma$$

$$\geqslant \frac{\beta}{2}s^2 - \gamma,$$

provided we choose $m \geqslant m_0(\beta) := 8\beta$.

Using now (3.2) and the fact that $\rho > 2$, we can write

$$F_0(s) \geqslant \hat{c}_1(|s|^\rho + s^2) \geqslant \frac{\hat{c}_1}{\delta} s^2 - C_\delta,$$

where $\delta > 0$ will be fixed later. By employing the property recalled above and the fact that we have $\lambda(s) \geqslant -\alpha s^2 - \widetilde{\alpha}$, we deduce

$$F_m(s) = F_{0m}(s) + \lambda(s) \geqslant \left(\frac{\hat{c}_1}{2\delta} - \alpha\right)s^2 - C_\delta,$$

which holds for all $s \in \mathbb{R}$ and for all $m \geqslant 8\hat{c}_1/\delta$. Let us choose, e.g., $\delta = \hat{c}_1/2(1+\alpha)$. Therefore, we have

$$F_m(s) \geqslant s^2 - C, \qquad \forall s \in \mathbb{R}, \qquad \forall m \geqslant m_0,$$

where $m_0 = 16(1+\alpha)$. Hence, $(3.15)_1$ is proven and the proof of the lemma is complete. $\square$

Proof of Theorem 1. Step I (case $\rho \leqslant 4$).

Let us first prove the existence of a weak solution with optimal regularity (3.10) under the assumption that F has growth of power 4 at most. We shall use a Faedo–Galerkin approximation method. Let us then take the family $\{w_j\}_{j\geqslant 1}$ of the eigenfunctions of A as a Galerkin basis in V, and let P_n be the orthogonal projectors in H onto the n-dimensional subspace $\mathscr{W}_n := \langle w_1, \ldots w_n \rangle$ spanned by the first n eigenfunctions. For $n \in \mathbb{N}$ fixed, we look for three functions of the form

$$\varphi_n(t) := \sum_{k=1}^n a_k^n(t)w_k, \quad \psi_n(t) := \sum_{k=1}^n b_k^n(t)w_k, \quad \mu_n(t) := \sum_{k=1}^n c_k^n(t)w_k$$

that solve the following approximating problem

$$(\varphi_n', w_j) + (\nabla\mu_n, \nabla w_j) = \big(p(\varphi_n)(\psi_n - \mu_n), w_j\big), \tag{3.17}$$

$$(\mu_n, w_j) = (\nabla\varphi_n, \nabla w_j) + \big(F'(\varphi_n), w_j\big), \tag{3.18}$$

$$(\psi_n', w_j) + (\nabla\psi_n, \nabla w_j) = -\big(p(\varphi_n)(\psi_n - \mu_n), w_j\big), \tag{3.19}$$

$$\varphi_n(0) = \varphi_{0n}, \quad \psi_n(0) = \psi_{0n}, \tag{3.20}$$

for $j = 1, \ldots, n$, where $\varphi_{0n} := P_n\varphi_0$ and $\psi_{0n} := P_n\psi_0$ (prime denotes the derivative with respect to time).

It is easy to see that solving the approximate problem (3.17)–(3.20) is equivalent to solving a Cauchy problem for a system of $2n$ ordinary differential equations in the $2n$ unknowns a_j^n, b_j^n. Since $F' \in C^1$ and $p \in C_{loc}^{0,1}$, the Cauchy–Lipschitz theorem ensures that there exists $T_n^* \in (0, \infty]$ such that this system has a unique maximal solution $\mathbf{a}^n := (a_1^n, \ldots, a_n^n)$, $\mathbf{b}^n := (b_1^n, \ldots, b_n^n)$ on $[0, T_n^*)$ with $\mathbf{a}^n, \mathbf{b}^n \in C^1([0, T_n^*); \mathbb{R}^n)$. Hence, the approximate problem (3.17)–(3.20) admits a unique solution $\varphi_n, \psi_n, \mu_n \in C^1([0, T_n^*); \mathscr{W}_n)$.

We now deduce the basic estimates on the sequence of approximating solutions. In particular, these estimates will guarantee that $T_n^* = \infty$ for every $n \in \mathbb{N}$.

Multiply then (3.17) by c_j^n, (3.18) by $a_j^{n\prime}$, (3.19) by b_j^n and sum the resulting identities over $j = 1, \ldots, n$. We get the following energy identity satisfied by the solution of the approximate problem

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla\varphi_n\|^2 + \frac{1}{2}\|\psi_n\|^2 + \int_\Omega F(\varphi_n)\right) + \|\nabla\mu_n\|^2 + \|\nabla\psi_n\|^2 + \int_\Omega p(\varphi_n)(\mu_n - \psi_n)^2 = 0.$$
(3.21)

By integrating (3.21) in time between 0 and t, using (F), (P) and the assumptions on the initial data we immediately deduce the following estimates

$$\|\varphi_n\|_{L^\infty(0,T;V)} \leqslant C, \quad \|\psi_n\|_{L^\infty(0,T;H)\cap L^2(0,T;V)} \leqslant C,$$
(3.22)

$$\|\nabla\mu_n\|_{L^2(0,T;H)} \leqslant C, \quad \|\sqrt{p(\varphi_n)}(\psi_n - \mu_n)\|_{L^2(0,T;H)} \leqslant C,$$
(3.23)

$$\|F(\varphi_n)\|_{L^\infty(0,T;L^1(\Omega))} \leqslant C.$$
(3.24)

where henceforth $C = C(\|\varphi_0\|_V, \|\psi_0\|)$ denotes a non-negative constant depending on the norms of the initial data (and on F, Ω).

Let us now control the sequence of the averages of μ_n. From (3.18) we get

$$|(\mu_n, 1)| = |(F'(\varphi_n), 1)| \leqslant c_6(F(\varphi_n), 1) + c_7 \leqslant C,$$
(3.25)

where c_6, c_7 are two non-negative constants depending only on F, Ω and where we have used assumption (F) and (3.24). Therefore, the sequence of $\overline{\mu}_n$ is bounded in $L^\infty(0,T)$ and this bound, together with the first of (3.23), yields

$$\|\mu_n\|_{L^2(0,T;V)} \leqslant C.$$
(3.26)

We now prove that the sequence of φ_n is controlled in $L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))$. Indeed, notice first that (3.18) can be written as

$$\mu_n = -\Delta\varphi_n + P_n F'(\varphi_n).$$
(3.27)

Observe now that $\|P_n F'(\varphi_n)\| \leqslant \|F'(\varphi_n)\|$. Thus, the sequence of φ_n is bounded in $L^\infty(0,T;L^6(\Omega))$, we deduce from (3.2) the bound

$$\|F'(\varphi_n)\|_{L^\infty(0,T;H)} \leqslant C.$$
(3.28)

Hence, (3.27) and (3.26) give that the sequence of $-\Delta\varphi_n + \varphi_n$ is bounded in $L^2(0,T;H)$ and, on account of the homogeneous Neumann boundary condition for φ_n, a classical elliptic regularity result implies

$$\|\varphi_n\|_{L^\infty(0,T;V)\cap L^2(0,T;H^2(\Omega))} \leqslant C.$$
(3.29)

By using inequality (2.2), we deduce from (3.29) that the sequence of φ_n is bounded in $L^{10}(Q)$ and moreover the sequence of $\nabla\varphi_n$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;V) \hookrightarrow L^{10/3}(Q)$. On the other hand, note that $\|A^{1/2}u\|^2 = (Au, u) = \|\nabla u\|^2 + \|u\|^2$, for all $u \in D(A)$. Hence $\|A^{1/2}u\| \geqslant \|\nabla u\|$ (which holds, by density, also for all $u \in V = D(A^{1/2})$). Therefore,

we have

$$\|\nabla\big(P_n F'(\varphi_n)\big)\| \leqslant \|A^{1/2} P_n F'(\varphi_n)\| = \|P_n A^{1/2} F'(\varphi_n)\| \leqslant \|\nabla F'(\varphi_n)\| + \|F'(\varphi_n)\|,$$

and hence (3.2) together with (3.28) and (3.22)$_1$ give

$$
\begin{aligned}
\|P_n F'(\varphi_n)\|_{L^2(0,T;V)} &\leqslant \|F''(\varphi_n)\nabla\varphi_n\|_{L^2(Q)} + \|F'(\varphi_n)\|_{L^2(Q)} \\
&\leqslant \|F''(\varphi_n)\|_{L^5(Q)} \|\nabla\varphi_n\|_{L^{10/3}(Q)} + \|F'(\varphi_n)\|_{L^2(Q)} \\
&\leqslant c\big(1 + \|\varphi_n\|_{L^{10}(Q)}^2\big) \|\nabla\varphi_n\|_{L^{10/3}(Q)} + \|F'(\varphi_n)\|_{L^2(Q)} \\
&\leqslant C.
\end{aligned}
$$

Using (3.26) and the elliptic regularity result again, we infer from (3.27) that

$$\|\varphi_n\|_{L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))} \leqslant C. \tag{3.30}$$

We now deduce the estimates for the sequences of time derivatives φ_n' and ψ_n'. Take $\chi \in D(A) \hookrightarrow L^\infty(\Omega)$ and write it as $\chi = \chi_1 + \chi_2$, where $\chi_1 = P_n\chi \in \mathscr{W}_n$ and $\chi_2 \in (I - P_n)\chi \in \mathscr{W}_n^\perp$ (recall that χ_1, χ_2 are orthogonal in H, V and $D(A)$). Then, from (3.17) we have

$$\langle \varphi_n', \chi \rangle = \langle \varphi_n', \chi_1 \rangle = -(\nabla\mu_n, \nabla\chi_1) + \big(p(\varphi_n)(\psi_n - \mu_n), \chi_1\big), \tag{3.31}$$

and a similar identity follows from (3.19). Observe that

$$
\begin{aligned}
|\big(p(\varphi_n)(\psi_n - \mu_n), \chi_1\big)| &\leqslant \|p(\varphi_n)\|_{L^{6/5}(\Omega)} \|\psi_n - \mu_n\|_{L^6(\Omega)} \|\chi_1\|_{L^\infty(\Omega)} \\
&\leqslant c\|p(\varphi_n)\|_{L^{6/5}(\Omega)} \|\psi_n - \mu_n\|_{L^6(\Omega)} \|\chi\|_{D(A)}.
\end{aligned}
$$

The term $(\psi_n - \mu_n)$ is controlled in $L^2(0,T;L^6(\Omega))$, then we need to control the sequence of $p(\varphi_n)$ in $L^\sigma(0,T;L^{6/5}(\Omega))$ with some $\sigma > 2$ in order to get the control of the sequences of φ_n', ψ_n' in $L^r(0,T;D(A^{-1}))$ with some $r > 1$. To this aim notice that from assumption (P) it follows that

$$\|p(\varphi_n)\|_{L^\sigma(0,T;L^{6/5+\epsilon}(\Omega))} \leqslant c(1 + \|\varphi_n\|_{L^{\sigma q}(0,T;L^{6q/5+\epsilon q}(\Omega))}^q), \tag{3.32}$$

where $\sigma > 2$ and $\epsilon > 0$. On the other hand, we know that the sequence of φ_n is bounded in $L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))$ (cf. (3.30)), and, thanks to inequality (2.2), we have the following embedding

$$L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \hookrightarrow L^{8\theta/(\theta-6)}(0,T;L^\theta(\Omega)), \quad \text{for } 6 \leqslant \theta \leqslant \infty. \tag{3.33}$$

Hence, choosing $\theta = 54/5$, we obtain

$$\|\varphi_n\|_{L^{18}(0,T;L^{54/5}(\Omega))} \leqslant C. \tag{3.34}$$

Recalling that $q < 9$, we can then fix $\sigma > 2$ and $\epsilon > 0$ such that $\sigma q \leqslant 18$ and $6q/5 + \epsilon q \leqslant 54/5$ (both σ and ϵ depending on q). Thus, we have $L^{18}(0,T;L^{54/5}(\Omega)) \hookrightarrow L^{\sigma q}(0,T;L^{6q/5+\epsilon q}(\Omega))$. Therefore, on account of (3.32) and (3.34), we get the desired control of $p(\varphi_n)$ in $L^\sigma(0,T;L^{6/5}(\Omega))$ with some $\sigma > 2$. Summing up, we have proven the

following bounds

$$\|\varphi_n'\|_{L^r(0,T;D(A^{-1}))} \leqslant C, \quad \|\psi_n'\|_{L^r(0,T;D(A^{-1}))} \leqslant C, \quad \text{for some } r > 1, \tag{3.35}$$

where we have used (3.32) and (3.23)$_1$ in (3.31) to get the first bound and (3.32) and (3.22)$_2$ to obtain the second bound.

We now deduce from estimates (3.22), (3.26), (3.30) and (3.35) the existence of three functions $\varphi \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))$, $\psi \in L^\infty(0,T;H) \cap L^2(0,T;V)$ and $\mu \in L^2(0,T;V)$, with $\varphi_t, \psi_t \in L^r(0,T;D(A^{-1}))$ which are the (weak) limits (up to subsequences) of φ_n, ψ_n, μ_n and φ_n', ψ_n', respectively. In order to pass to the limit in the approximate problem, we first observe that thanks to the compact embedding

$$L^\infty(0,T;V) \cap W^{1,r}(0,T;D(A^{-1})) \hookrightarrow\hookrightarrow C([0,T];L^\kappa(\Omega)), \quad 2 \leqslant \kappa < 6$$

given by the Aubin–Lions lemma (see, e.g., [23]), we deduce that, up to a subsequence, $\varphi_n \to \varphi$ pointwise almost everywhere in $Q = \Omega \times (0,T)$. Then, since $(\psi_n - \mu_n)$ converges weakly to $(\psi - \mu)$ in $L^2(0,T;L^6(\Omega))$, in order to pass to the limit in $\big(p(\varphi_n)(\psi_n - \mu_n), w_j\big)$ on the right-hand side of (3.17) and (3.19) it is enough that $p(\varphi_n)$ converges strongly to $p(\varphi)$ in $L^2(0,T;L^{6/5}(\Omega))$ (up to a subsequence). But we know that $p(\varphi_n) \to p(\varphi)$ pointwise almost everywhere in Q and furthermore, from (3.32), (3.34) and the from embedding $L^{18}(0,T;L^{54/5}(\Omega)) \hookrightarrow L^{\sigma q}(0,T;L^{6q/5+\epsilon q}(\Omega))$ (with $\sigma > 2$ and $\epsilon > 0$ fixed as above), we have $p(\varphi_n) \rightharpoonup p(\varphi)$ weakly in $L^\sigma(0,T;L^{6/5+\epsilon}(\Omega))$. Hence we deduce

$$p(\varphi_n) \to p(\varphi), \quad \text{strongly in } L^2(0,T;L^{6/5}(\Omega)). \tag{3.36}$$

This convergence, combined with the weak convergence $(\mu_n - \psi_n) \rightharpoonup (\mu - \psi)$ in $L^2(0,T;L^6(\Omega))$, allows us to pass to the limit in the non-linear term on the right-hand side of (3.17) and (3.19) (recall that $w_j \in C^1(\overline{\Omega})$, assuming that $\partial\Omega$ is smooth enough, e.g., $C^{2,1}$). By means of the convergences deduced above we can therefore pass to the limit in the approximate problem (3.17)–(3.20) and deduce that φ, ψ, μ satisfy (3.8)–(3.9). The argument is standard and the details are left to the reader.

The energy inequality (3.12) can be proven by integrating in time (3.21) between 0 and t and passing to the limit as $n \to \infty$ in the resulting identity. The only non-trivial point is the following inequality

$$\int_0^t \int_\Omega p(\varphi)(\mu - \psi)^2 \leqslant \liminf_{n\to\infty} \int_0^t \int_\Omega p(\varphi_n)(\mu_n - \psi_n)^2. \tag{3.37}$$

We know from (3.33) written for $\theta = 14$, that the sequence of φ_n is bounded in $L^{14}(Q)$ and hence, on account of (P), the sequence of $\sqrt{p(\varphi_n)}$ is bounded in $L^{28/q}(Q)$. Since $\varphi_n \to \varphi$ also pointwise almost everywhere in Q, then we have $\sqrt{p(\varphi_n)} \to \sqrt{p(\varphi)}$ strongly in $L^\gamma(Q)$, for every $\gamma < 28/q$. In particular, we have $\sqrt{p(\varphi_n)} \to \sqrt{p(\varphi)}$ strongly in $L^3(Q)$. Therefore, we have

$$\sqrt{p(\varphi_n)}(\mu_n - \psi_n) \rightharpoonup \sqrt{p(\varphi)}(\mu - \psi), \quad \text{in } L^{6/5}(Q),$$

and, due (3.23)$_2$, this last weak convergence is also in $L^2(Q)$. Hence, (3.37) follows.

Moreover, if $q \leqslant 4$ we can easily deduce the regularity $\varphi_t, \psi_t \in L^2(0, T; V')$ by comparison in the variational formulation of (1.1) and (1.3). Indeed, estimating the term $p(\varphi)(\psi - \mu)$ in V', we get

$$\|p(\varphi)(\psi - \mu)\|_{V'} \leqslant c\|p(\varphi)\|_{L^{3/2}(\Omega)}\|\psi - \mu\|_{L^6(\Omega)}. \tag{3.38}$$

But, since $q \leqslant 4$ and $\varphi \in L^\infty(0, T; L^6(\Omega))$, then assumption (P) implies that we have $p(\varphi) \in L^\infty(0, T; L^{3/2}(\Omega))$ and therefore, on account of (3.6) and of $(3.7)_1$, (3.38) gives

$$p(\varphi)(\psi - \mu) \in L^2(0, T; V').$$

Hence, (3.13) follows immediately.

Finally, let us take $\chi = \mu$ and $\xi = \psi$ in the variational formulation (3.8), (3.9) of (1.1) and (1.3) (with test functions χ, ξ now in V), respectively, and sum the resulting identities. The choices for χ and ξ are allowed since we have $\mu, \psi \in L^2(0, T; V)$. Next, let us recall (3.13) for φ_t, ψ_t, (3.10) and (3.7) for φ, ψ, and the chain rule applied to the product $\langle \varphi_t, F'(\varphi) \rangle$, noting that $F'(\varphi) \in L^2(0, T; V)$, to write the identities

$$\langle \varphi_t, \mu \rangle = \frac{1}{2}\frac{d}{dt}\|\nabla\varphi\|^2 + \frac{d}{dt}\int_\Omega F(\varphi), \qquad \langle \psi_t, \psi \rangle = \frac{1}{2}\frac{d}{dt}\|\psi\|^2. \tag{3.39}$$

Here, we have used [9, Proposition 4.2] and the fact that (3.2) ensures that F is a quadratic perturbation of a convex function. Observe that the first term on the right-hand side of $(3.39)_1$ can be justified by means of a regularization argument which employs the time convolution of φ by a family of mollifiers (see, e.g., proof of [31, Lemma 4.1]). Summing up, we obtain

$$\frac{d}{dt}\left(\frac{1}{2}\|\nabla\varphi\|^2 + \frac{1}{2}\|\psi\|^2 + \int_\Omega F(\varphi)\right) + \|\nabla\mu\|^2 + \|\nabla\psi\|^2 + \int_\Omega p(\varphi)(\mu - \psi)^2 = 0. \tag{3.40}$$

By integrating the energy identity (3.40) in time between 0 and t we deduce (3.12) with the equal sign for all $t > 0$. This completes the proof of the theorem for the case $\rho \leqslant 4$.

Step II (case $4 < \rho < 6$).

In this case, we first approximate the potential F with a sequence of potentials $F_m \in C^2(\mathbb{R})$ satisfying the conditions stated in Lemma 2.

Let us now consider problem (1.1)–(1.5) with F replaced by F_m and call it Problem P_m. Since F_m satisfies condition (F) with $\rho \leqslant 4$ (each F_m has quadratic growth on $\mathbb{R}$) then, for each $m \in \mathbb{N}$, Step I ensures the existence of a weak solution $[\varphi_m, \psi_m]$ to Problem P_m such that $\varphi_m \in L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega))$, $\psi_m \in L^\infty(0, T; H) \cap L^2(0, T; V)$, $\mu_m \in L^2(0, T; V)$ and satisfying the energy inequality (3.12).

Due to (3.12) (written for each solution φ_m, ψ_m with F_m in place of F), assumptions (F) and (P), $(3.14)_1$ and (3.15), we can argue as for the Faedo–Galerkin approximating solutions $[\varphi_n, \psi_n]$ (cf. Step I) and we can still recover the basic estimates (3.22), (3.26) for the sequences of φ_m and ψ_m (notice that in Problem P_m the initial conditions are not approximated).

We now show that the sequence φ_m is still controlled in $L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega))$. This bound will be achieved through an iteration argument.

Notice first that the sequence φ_m is bounded in $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$. Indeed, by multiplying the identity $\mu_m = -\Delta\varphi_m + F'_m(\varphi_m)$ by $\Delta\varphi_m$ we obtain

$$\|\Delta\varphi_m\|^2 = -(\mu_m, \Delta\varphi_m) + (F'_m(\varphi_m), \Delta\varphi_m)$$
$$\leqslant \frac{1}{2}\|\mu_m\|^2 + \frac{1}{2}\|\Delta\varphi_m\|^2 - \int_\Omega F''_m(\varphi_m)|\nabla\varphi_m|^2.$$

By using $(3.15)_2$, this last estimate yields

$$\|\Delta\varphi_m\|^2 \leqslant \|\mu_m\|^2 + 2k_5\|\nabla\varphi_m\|^2. \tag{3.41}$$

The desired bound of φ_m in $L^\infty(0,T;V) \cap L^2(0,T;H^2(\Omega))$ then follows from (3.41) by applying the basic estimates (3.22), (3.26) and elliptic regularity.

Using the obtained bound and interpolation (cf. (3.47) below), we see that the sequence of φ_m is bounded in $L^{2(\rho-1)}(0,T;L^{6(\rho-1)/(\rho-3)}(\Omega))$ as well. Hence, (3.2) together with $(3.14)_2$ imply that the sequence of $F'_m(\varphi_m)$ is bounded in $L^2(0,T;L^{6/(\rho-3)}(\Omega))$. Therefore, from (1.2) and (3.26) we infer that the sequence of $-\Delta\varphi_m + \varphi_m$ is bounded in $L^2(0,T;L^{6/(\rho-3)}(\Omega))$. Then, by using elliptic regularity theory (see, e.g., [1, 19, 26]) we get

$$\|\varphi_m\|_{L^\infty(0,T;V) \cap L^2(0,T;W^{2,\frac{6}{\rho-3}}(\Omega))} \leqslant C. \tag{3.42}$$

Thanks to inequality (2.2), we deduce from (3.42) that the sequence of φ_m is bounded in $L^{2(11-\rho)}(Q)$. Moreover, $\nabla\varphi_m$ is bounded in $L^\infty(0,T;H) \cap L^2(0,T;W^{1,6/(\rho-3)}(\Omega)) \hookrightarrow L^{2(11-\rho)/3}(Q)$. Therefore, using $(3.14)_3$ and (3.2) we get

$$\|\nabla F'_m(\varphi_m)\|_{L^{s_0}(Q)} \leqslant k_2 \|F''(\varphi_m)\nabla\varphi_m\|_{L^{s_0}(Q)}$$
$$\leqslant k_2 \|F''(\varphi_m)\|_{L^{2(11-\rho)/(\rho-2)}(Q)} \|\nabla\varphi_m\|_{L^{2(11-\rho)/3}(Q)}$$
$$\leqslant c\left(1 + \|\varphi_m\|_{L^{2(11-\rho)}(Q)}^{\rho-2}\right) \|\nabla\varphi_m\|_{L^{2(11-\rho)/3}(Q)}$$
$$\leqslant C, \quad s_0 = \frac{2(11-\rho)}{\rho+1}.$$

In addition, we know that the sequence of $F'_m(\varphi_m)$ is bounded in $L^2(0,T;L^{6/(\rho-3)}(\Omega))$. Let us now first consider the case $4 < \rho \leqslant 5$. In this case we have $s_0 \in [2, 14/5)$ and since $F'_m(\varphi_m)$ is bounded in $L^2(0,T;L^2(\Omega))$, we obtain

$$\|F'_m(\varphi_m)\|_{L^2(0,T;V)} \leqslant C.$$

By comparison in (1.2) and using (3.26) and elliptic regularity again, we deduce the desired bound

$$\|\varphi_m\|_{L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))} \leqslant C. \tag{3.43}$$

On the other hand, if $5 < \rho < 6$, then $s_0 \in (10/7, 2)$. In this case, the sequence of $F'_m(\varphi_m)$ is still bounded in $L^2(0,T;L^2(\Omega))$, but we have

$$\|F'_m(\varphi_m)\|_{L^{s_0}(0,T;W^{1,s_0}(\Omega))} \leqslant C.$$

By comparison in (1.2) and using (3.26) and elliptic regularity again, we now deduce

$$\|\varphi_m\|_{L^\infty(0,T;V)\cap L^{s_0}(0,T;W^{3,s_0}(\Omega))} \leqslant C. \tag{3.44}$$

In this case, we can repeat the argument above and improve the estimates for the sequence of φ_m by means of a bootstrap procedure performed for a finite number of steps. Indeed, observe first that, thanks to (2.2), we have (for any $s \in (1,2]$)

$$\mathbb{X}_s := L^\infty(0,T;V) \cap L^s(0,T;W^{3,s}(\Omega)) \hookrightarrow L^{7s}(Q), \tag{3.45}$$

$$\mathbb{Y}_s := L^\infty(0,T;H) \cap L^s(0,T;W^{2,s}(\Omega)) \hookrightarrow L^{\frac{7}{3}s}(Q). \tag{3.46}$$

Taking (3.44)–(3.46) into account, the sequences of φ_m and $\nabla\varphi_m$ are bounded in $L^{7s_0}(Q)$ and in $L^{7s_0/3}(Q)$, respectively. Hence, by means of (3.2) and $(3.14)_3$, we have

$$\|\nabla F'_m(\varphi_m)\|_{L^{7s_0/(\rho+1)}(Q)} \leqslant k_2 \|F''(\varphi_m)\nabla\varphi_m\|_{L^{7s_0/(\rho+1)}(Q)} \leqslant C.$$

On the other hand, $F'_m(\varphi_m)$ is bounded in $L^2(0,T;L^2(\Omega))$ and hence also in $L^{s_1}(Q)$, where $s_1 = \min\{2, 7s_0/(\rho+1)\}$. We therefore deduce that

$$\|F'_m(\varphi_m)\|_{L^{s_1}(0,T;W^{1,s_1}(\Omega))} \leqslant C, \quad s_1 := \min\left\{2, \frac{7}{\rho+1}s_0\right\}.$$

If $s_1 = 2$, then by comparison in (1.2) and using (3.26) and elliptic regularity, we get the desired bound for the sequence of φ_m in $L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))$. If $s_1 < 2$ then, by comparison in (1.2) and using (3.26) and elliptic regularity, we infer

$$\|\varphi_m\|_{L^\infty(0,T;V)\cap L^{s_1}(0,T;W^{3,s_1}(\Omega))} \leqslant C.$$

Repeating the argument, we now have the sequences of φ_m and $\nabla\varphi_m$ bounded in $\mathbb{X}_{s_1}$ and in $\mathbb{Y}_{s_1}$, respectively, and hence $\|\nabla F'_m(\varphi_m)\|_{L^{7s_1/(\rho+1)}(Q)} \leqslant C$. Moreover, we know that the sequence of $F'_m(\varphi_m)$ is bounded in $L^{s_2}(Q)$, where $s_2 = \min\{2, 7s_1/(\rho+1)\}$. This implies

$$\|F'_m(\varphi_m)\|_{L^{s_2}(0,T;W^{1,s_2}(\Omega))} \leqslant C, \quad s_2 := \min\left\{2, \left(\frac{7}{\rho+1}\right)^2 s_0\right\}.$$

Again, if $s_2 = 2$ we get the desired claim; otherwise, by using elliptic regularity we infer that the sequence of φ_m is bounded in $\mathbb{X}_{s_2}$ and we repeat the previous argument. By iterating the procedure k times we get

$$\|F'_m(\varphi_m)\|_{L^{s_k}(0,T;W^{1,s_k}(\Omega))} \leqslant C, \quad s_k := \min\left\{2, \left(\frac{7}{\rho+1}\right)^k s_0\right\}.$$

Since $\rho < 6$, after a finite number of steps, as soon as we get $s_k = 2$, the bootstrap procedure ends yielding the bound of the sequence of φ_m in $L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega))$ (which cannot be improved since the regularity of φ_m is related through (1.2) to $\mu_m \in L^2(0,T;V)$).

As far as the estimates for the sequences of time derivatives φ'_m, ψ'_m are concerned, the argument is exactly the same as for the sequences of time derivatives φ'_n, ψ'_n of the Faedo–Galerkin approximating solutions (cf. Step I). Hence, (3.35) still holds for φ'_m, ψ'_m. Finally, the passage to the limit in Problem P_m (notice that $F'_m(\varphi_m) \to F'(\varphi)$ pointwise

almost everywhere in Q), the proof of the energy inequality (3.12) for $q \in [1, 9)$, the proofs of (3.13) and of the energy identity for $q \leqslant 4$ can be carried out along as done at the end of Step I.   □

The existence of a weak solution without the optimal regularity $\varphi \in L^2(0, T; H^3(\Omega))$ can still be ensured under a more general assumption on F, provided we impose a slight restriction (i.e., $q < 7$) on the growth of p. More precisely, we have the following

Corollary 1 *Assume that $F \in C^2(\mathbb{R})$ satisfies*

(F)$_1$ $F''(s) \geqslant -\lambda_1$,

(F)$_2$ $|F'(s)| \leqslant \lambda_2 F(s) + \lambda_3$,

for all $s \in \mathbb{R}$, where $\lambda_1, \lambda_2, \lambda_3$ are some non-negative constants. Moreover, assume that $p \in C_{loc}^{0,1}(\mathbb{R})$ satisfies (3.4) with $q \in [1, 7)$. Let $\varphi_0 \in V$ and $\psi_0 \in H$. Then, for every $T > 0$, Problem (1.1)–(1.5) admits a weak solution on $[0, T]$ satisfying (3.5)–(3.7), (3.11) and the energy inequality (3.12). Finally, if $q \leqslant 4$, then we have (3.13) and (3.12) holds with the equality sign.

Proof   We can follow the Faedo–Galerkin approximation procedure in Step I of the proof of Theorem 1, assuming first that $\varphi_0 \in D(A)$ in order to control the sequence of $\int_\Omega F(\varphi_{0n})$ in the identity obtained by integrating (3.21) in time. Existence of weak solution in the case $\varphi_0 \in V$ can then be recovered by means of a density argument. The basic estimates (3.22)–(3.24) still hold, as well as the controls (3.25), ensured by (F)$_2$, and (3.26). As far as estimate (3.29) is concerned, this can now be recovered by using (F)$_1$. Indeed, multiplying (3.27) by $\Delta\varphi_n$ in H we get

$$\|\Delta\varphi_n\|^2 = -(\mu_n, \Delta\varphi_n) + \big(P_n F'(\varphi_n), \Delta\varphi_n\big)$$
$$= -(\mu_n, \Delta\varphi_n) - \int_\Omega F''(\varphi_n)|\nabla\varphi_n|^2,$$

which yields

$$\|\Delta\varphi_n\|^2 \leqslant \|\mu_n\|^2 + 2\lambda_1\|\nabla\varphi_n\|^2.$$

Estimate (3.29) then follows from this last inequality by using (3.26), the first of (3.22) and elliptic regularity.

Next, in order to get the control of the sequences of time derivatives φ_n', ψ_n' in the space $L^r(0, T; D(A^{-1}))$, for some $r > 1$, and in order to pass to the limit in the approximate problem (3.17)–(3.20) we can still argue as in Step I of the proof of Theorem 1, with the difference that now we can only rely on the control given by (3.29), together with the following embedding

$$L^\infty(0, T; V) \cap L^2(0, T; H^2(\Omega)) \hookrightarrow L^{4\eta/(\eta-6)}(0, T; L^\eta(\Omega)), \quad \text{for } 6 \leqslant \eta \leqslant \infty. \qquad (3.47)$$

Indeed, by using (3.47) with $\eta = 42/5$ we can easily see that, since $q \in [1, 7)$, estimates (3.35) and the strong convergence (3.36) still hold.

As far as the energy inequality (3.12) is concerned, let us observe that the sequence of φ_n is now bounded in $L^{10}(Q)$ (cf. (3.29) and (3.47) with $\eta = 10$). Hence, on account of (3.4) and of pointwise convergence we have $\sqrt{p(\varphi_n)} \to \sqrt{p(\varphi)}$ strongly in $L^\delta(Q)$, for every $\delta < 20/q$. In particular we have $\sqrt{p(\varphi_n)} \to \sqrt{p(\varphi)}$ strongly in $L^{5/2}(Q)$, which implies that

$$\sqrt{p(\varphi_n)}(\mu_n - \psi_n) \rightharpoonup \sqrt{p(\varphi)}(\mu - \psi), \quad \text{in } L^{10/9}(Q).$$

Due to $(3.23)_3$, this weak convergence also holds in $L^2(Q)$ and still yields (3.37) and then (3.12) as well.

Finally, assume that $q \leqslant 4$. By arguing as in Step I of the proof of Theorem 1 we again deduce (3.13). In order to prove that (3.12) holds with the equality sign, let us first observe that from assumption (F) we have $F''(s) \geqslant -c_*$, for some $c_* \in \mathbb{R}$, and therefore we can write F as

$$F(s) = G_0(s) - c_* \frac{s^2}{2},$$

where $G_0 \in C^2(\mathbb{R})$ is convex. Introduce now the functional $\mathscr{G}_0 : H \to \mathbb{R} \cup \{+\infty\}$ given by

$$\mathscr{G}_0(\varphi) = \int_\Omega \left(\frac{1}{2} |\nabla \varphi|^2 + G_0(\varphi) \right), \quad \text{if } \varphi \in V \text{ and } G_0(\varphi) \in L^1(\Omega),$$

and $\mathscr{G}_0(\varphi) = +\infty$ otherwise. Then, $\mathscr{G}_0$ is convex and lower semicontinuous on H and we have (see, e.g., [3, Proposition 2.8])

$$\partial \mathscr{G}_0(\varphi) = -\Delta \varphi + G_0'(\varphi), \quad \forall \varphi \in D(\partial \mathscr{G}_0) = D(A).$$

Since $\partial \mathscr{G}_0(\varphi) = -\Delta \varphi + G_0'(\varphi) = \mu + c_* \varphi \in L^2(0, T; V)$, then we can apply [9, Proposition 4.2] and write

$$\langle \varphi_t, \mu \rangle = \langle \varphi_t, \partial \mathscr{G}_0(\varphi) - c_* \varphi \rangle = \frac{d}{dt} \mathscr{G}_0(\varphi) - \frac{c_*}{2} \frac{d}{dt} \|\varphi\|^2 = \frac{d}{dt} \int_\Omega \left(\frac{1}{2} |\nabla \varphi|^2 + F(\varphi) \right).$$

This identity allows us to recover (3.40), and hence (3.12) with the equality sign, by arguing exactly as at the end of Step I of the proof of Theorem 1. $\qquad \square$

The next result is concerned with the uniqueness of weak solutions and their continuous dependence with respect to the initial data. In order to prove this result, assumption (F) still suffices, but we need to strengthen (P) as follows

(P1) Let $p \in C^{0,1}_{loc}(\mathbb{R})$ be such that $p \geqslant 0$ and

$$|p'(s)| \leqslant c_5 (1 + |s|^{q-1}),$$

for almost any $s \in \mathbb{R}$, with $1 \leqslant q \leqslant 4$.

Then we have

Theorem 2 *Assume that (F) and (P1) are satisfied. Let $\varphi_0 \in V$ and $\psi_0 \in H$. Then, for every $T > 0$ the weak solution to Problem (1.1)–(1.5) on $[0, T]$ given by Theorem 1 is*

unique. Moreover, let $[\varphi_{0i}, \psi_{0i}] \in V \times H$ *be two initial data and* $[\varphi_i, \psi_i]$, $i = 1, 2$ *be the corresponding weak solutions. Then, the following continuous dependence estimate holds*

$$\|\varphi_2(t) - \varphi_1(t)\|_{V'} + \|\psi_2(t) - \psi_1(t)\|_{V'} + \|\varphi_2 - \varphi_1\|_{L^2(0,t;V)} + \|\psi_2 - \psi_1\|_{L^2(0,t;H)}$$
$$\leqslant \Lambda(t)\big(\|\varphi_{02} - \varphi_{01}\|_{V'} + \|\psi_{02} - \psi_{01}\|_{V'}\big), \quad \forall t \in [0, T],$$

where Λ *is a continuous positive function which depends on the norms of the initial data and on* F, p, Ω *and* T.

Remark 3 Notice that the restriction $1 \leqslant q \leqslant 4$ on the growth of p which is needed to establish the uniqueness is exactly the same condition which ensures the validity of the energy identity (3.40) which is proven in Theorem 1.

Proof Let us rewrite the chemical potential μ and (3.8)–(3.9) in the following form

$$\langle \varphi_t, \chi \rangle + (\nabla\mu, \nabla\chi) + (\mu, \chi) = \big(p(\varphi)\psi - (p(\varphi) - 1)\mu, \chi\big), \tag{3.48}$$

$$\mu = A\varphi + G'(\varphi), \tag{3.49}$$

$$\langle \psi_t, \xi \rangle + (\nabla\psi, \nabla\xi) + (\psi, \xi) = -\big((p(\varphi) - 1)\psi + p(\varphi)\mu, \xi\big), \tag{3.50}$$

for all $\chi, \xi \in V$, where $G(s) := F(s) - \frac{1}{2}s^2$.

We now write system (3.48)–(3.50) for two weak solutions $[\varphi_i, \psi_i]$, $i = 1, 2$, and take the difference of each equation. Setting $\varphi := \varphi_2 - \varphi_1$, $\psi := \psi_2 - \psi_1$ and $\mu := \mu_2 - \mu_1$, we have

$$\langle \varphi_t, \chi \rangle + (\nabla\mu, \nabla\chi) + (\mu, \chi)$$
$$= \big((p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2) + p(\varphi_1)\psi - (p(\varphi_1) - 1)\mu, \chi\big), \tag{3.51}$$

$$\mu = A\varphi + G'(\varphi_2) - G'(\varphi_1), \tag{3.52}$$

$$\langle \psi_t, \xi \rangle + (\nabla\psi, \nabla\xi) + (\psi, \xi)$$
$$= -\big((p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2) - (p(\varphi_1) - 1)\psi + p(\varphi_1)\mu, \xi\big), \tag{3.53}$$

for all $\chi, \xi \in V$. Let us take $\chi = A^{-1}\varphi$ in (3.51) and $\xi = A^{-1}\psi$ in (3.53) and sum the resulting identities. Taking also (3.52) into account, we get

$$\frac{1}{2}\frac{d}{dt}\|\varphi\|_{V'}^2 + \|\varphi\|_V^2 + \big(G'(\varphi_2) - G'(\varphi_1), \varphi\big) + \frac{1}{2}\frac{d}{dt}\|\psi\|_{V'}^2 + \|\psi\|^2$$
$$= \Big((p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2) + p(\varphi_1)\psi - (p(\varphi_1) - 1)\mu, A^{-1}\varphi\Big)$$
$$+ \Big(-(p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2) - (p(\varphi_1) - 1)\psi + p(\varphi_1)\mu, A^{-1}\psi\Big). \tag{3.54}$$

We now need to estimate the terms on the right-hand side. Observe first that

$$\big(p(\varphi_1)\psi - (p(\varphi_1) - 1)\mu, A^{-1}\varphi\big) \leqslant \big(\|p(\varphi_1)(\psi - \mu)\|_{V'} + \|\mu\|_{V'}\big)\|\varphi\|_{V'}. \tag{3.55}$$

We have to estimate in V' the term $p(\varphi_1)(\psi - \mu)$. Let us first estimate $p(\varphi_1)\chi$ in V. By using assumption (P1) we get

$$\|p(\varphi_1)\nabla\chi\| \leqslant c\big(1 + \|\varphi_1\|_{L^\infty(\Omega)}^q\big)\|\nabla\chi\|. \tag{3.56}$$

Moreover, we have

$$\|p'(\varphi_1)\nabla\varphi_1\chi\| \leqslant \|p'(\varphi_1)\nabla\varphi_1\|_{L^3(\Omega)}\|\chi\|_{L^6(\Omega)} \leqslant \|p'(\varphi_1)\nabla\varphi_1\|_{L^3(\Omega)}\|\chi\|_V. \qquad (3.57)$$

However, $\nabla\varphi_1 \in L^\infty(0,T;H) \cap L^2(0,T;H^2(\Omega)) \hookrightarrow L^8(0,T;L^3(\Omega))$. On the other hand, $\varphi_1 \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \hookrightarrow L^8(0,T;L^\infty(\Omega))$ (cf. (3.33) with $\theta = \infty$). Thus, thanks to assumption (P1), we also have $p'(\varphi_1) \in L^{8/(q-1)}(0,T;L^\infty(\Omega))$. Hence, we find

$$p'(\varphi_1)\nabla\varphi_1 \in L^{8/q}(0,T;L^3(\Omega)). \qquad (3.58)$$

Moreover, observe that

$$\|p(\varphi_1)\chi\| \leqslant c\|p(\varphi_1)\|_{L^3(\Omega)}\|\chi\|_V, \qquad (3.59)$$

and

$$\|p(\varphi_1)\|_{L^3(\Omega)} \leqslant c\big(1 + \|\varphi_1\|_{L^{3q}(\Omega)}^q\big). \qquad (3.60)$$

Observing that $\varphi_1 \in L^\infty(0,T;V) \cap L^2(0,T;H^3(\Omega)) \hookrightarrow L^{8q/(q-2)}(0,T;L^{3q}(\Omega))$ (cf. (3.33)), then we have

$$p(\varphi_1) \in L^{8/(q-2)}(0,T;L^3(\Omega)). \qquad (3.61)$$

By collecting (3.56)–(3.61) we get

$$\|p(\varphi_1)\chi\|_V \leqslant \alpha_1(t)\|\chi\|_V,$$

where the function α_1 is given by

$$\alpha_1(t) := c\big(\|p(\varphi_1(t))\|_{L^3(\Omega)} + \|\varphi_1(t)\|_{L^\infty(\Omega)}^q + \|p'(\varphi_1(t))\nabla\varphi_1(t)\|_{L^3(\Omega)} + 1\big),$$

and, since $q \leqslant 4$, we have $\alpha_1 \in L^2(0,T)$. Therefore, we obtain

$$|\langle p(\varphi_1)(\psi - \mu), \chi\rangle| \leqslant |\big((\psi - \mu), p(\varphi_1)\chi\big)| \leqslant \alpha_1(t)\|\psi - \mu\|_{V'}\|\chi\|_V,$$

which yields

$$\|p(\varphi_1)(\psi - \mu)\|_{V'} \leqslant \alpha_1(t)\|\psi - \mu\|_{V'}. \qquad (3.62)$$

By combining (3.55) with (3.62) we deduce

$$\big(p(\varphi_1)\psi - \big(p(\varphi_1) - 1\big)\mu, A^{-1}\varphi\big) \leqslant \alpha_1(t)\big(\|\psi\|_{V'} + \|\mu\|_{V'}\big)\|\varphi\|_{V'}. \qquad (3.63)$$

For the estimate of μ in V', by means of assumption (F) and using the continuous embedding $L^{6/5}(\Omega) \hookrightarrow V'$, it is easy to see that

$$\|\mu\|_{V'} \leqslant \|\varphi\|_V + \|G'(\varphi_2) - G'(\varphi_1)\|_{V'}$$
$$\leqslant \|\varphi\|_V + c\big(1 + \|\varphi_1\|_{L^{3(\rho-2)/2}(\Omega)}^{\rho-2} + \|\varphi_2\|_{L^{3(\rho-2)/2}(\Omega)}^{\rho-2}\big)\|\varphi\|_{L^6(\Omega)}$$

$$\leqslant c\big(1 + \|\varphi_1\|_V^{\rho-2} + \|\varphi_2\|_V^{\rho-2}\big)\|\varphi\|_V \leqslant \Gamma\|\varphi\|_V, \tag{3.64}$$

since $3(\rho - 2)/2 \leqslant 6$, because $\rho < 6$. In the last inequality, we have used (3.5)$_1$. In (3.64) and also in the estimates below, Γ denotes a positive constant that depends on the norms of the initial data of the two solutions, i.e., $\Gamma = \Gamma\big(\|\varphi_{01}\|_V, \|\varphi_{02}\|_V, \|\psi_{01}\|, \|\psi_{02}\|\big)$ (of course, Γ depends also on F and Ω). The value of Γ may change even within the same line. From (3.63) and (3.64) we get

$$|(p(\varphi_1)\psi - (p(\varphi_1) - 1)\mu, A^{-1}\varphi)| \leqslant \alpha_1(t)\Gamma\big(\|\psi\|_{V'} + \|\varphi\|_V\big)\|\varphi\|_{V'}$$
$$\leqslant \frac{1}{10}\|\varphi\|_V^2 + \Gamma\alpha_1^2(t)\big(\|\psi\|_{V'}^2 + \|\varphi\|_{V'}^2\big). \tag{3.65}$$

The next term on the right-hand side of (3.54) to be estimated is the following

$$|\big((p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2), A^{-1}\varphi\big)| \leqslant \|(p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2)\|_{V'}\|\varphi\|_{V'}. \tag{3.66}$$

Let us first control the term $\big((p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2)$ in V'. We have, for every $\chi \in V$,

$$|\langle\big((p(\varphi_1) - p(\varphi_2))(\psi_2 - \mu_2), \chi\big)\rangle| \leqslant \|p(\varphi_1) - p(\varphi_2)\|\,\|\psi_2 - \mu_2\|_{L^3}\|\chi\|_{L^6}$$
$$\leqslant c\|p(\varphi_1) - p(\varphi_2)\|\,\|\psi_2 - \mu_2\|_{L^3}\|\chi\|_V. \tag{3.67}$$

On the other hand, thanks to (P1), we obtain

$$\|p(\varphi_2) - p(\varphi_1)\| \leqslant c\big(1 + \|\varphi_1\|_{L^\infty(\Omega)}^{q-1} + \|\varphi_2\|_{L^\infty(\Omega)}^{q-1}\big)\|\varphi\|$$
$$\leqslant c\big(1 + \|\varphi_1\|_{L^\infty(\Omega)}^{q-1} + \|\varphi_2\|_{L^\infty(\Omega)}^{q-1}\big)\|\varphi\|_{V'}^{1/2}\|\varphi\|_V^{1/2}. \tag{3.68}$$

Moreover, by using (2.2) and the interpolation inequality $\|\mu_2\| \leqslant \|\mu_2\|_{V'}^{1/2}\|\mu_2\|_V^{1/2}$, we get

$$\|\mu_2\|_{L^3(\Omega)} \leqslant c\|\mu_2\|^{1/2}\|\mu_2\|_V^{1/2} \leqslant c\|\mu_2\|_{V'}^{1/4}\|\mu_2\|_V^{3/4} \leqslant \Gamma\|\mu_2\|_V^{3/4}, \tag{3.69}$$

where in the last estimate we have exploited the inequality $\|\mu_2\|_{V'} \leqslant \Gamma(1 + \|\varphi_2\|_V) \leqslant \Gamma$ which can be deduced by arguing as in (3.64). Hence, from (3.67)–(3.69) we infer

$$\|(p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2)\|_{V'} \leqslant \alpha_2(t)\|\varphi\|_{V'}^{1/2}\|\varphi\|_V^{1/2}, \tag{3.70}$$

where

$$\alpha_2(t) := c\big(1 + \|\varphi_1(t)\|_{L^\infty(\Omega)}^{q-1} + \|\varphi_2(t)\|_{L^\infty(\Omega)}^{q-1}\big)\big(\|\psi_2(t)\|_{L^3(\Omega)} + \Gamma\|\mu_2(t)\|_V^{3/4}\big). \tag{3.71}$$

Observe that $\alpha_2 \in L^{4/3}(0, T)$ since $q \leqslant 4$. Indeed, both factors in (3.71) are in $L^{8/3}(0, T)$, recalling that $\varphi_1, \varphi_2 \in L^\infty(0, T; V) \cap L^2(0, T; H^3(\Omega)) \hookrightarrow L^8(0, T; L^\infty(\Omega))$ and properties (3.5)–(3.7) (in particular we have $\psi_2 \in L^{10/3}(Q)$). Hence, from (3.66) we get

$$|\big((p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2), A^{-1}\varphi\big)| \leqslant \alpha_2(t)\|\varphi\|_V^{1/2}\|\varphi\|_{V'}^{3/2}$$
$$\leqslant \frac{1}{10}\|\varphi\|_V^2 + \alpha_2^{4/3}(t)\|\varphi\|_{V'}^2. \tag{3.72}$$

We now estimate the following term (cf. the right-hand side of (3.54))

$$
\begin{aligned}
|(-(p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2), A^{-1}\psi)| &\leqslant \|(p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2)\|_{V'}\|A^{-1}\psi\|_V \\
&= \|(p(\varphi_2) - p(\varphi_1))(\psi_2 - \mu_2)\|_{V'}\|\psi\|_{V'} \\
&\leqslant \alpha_2(t)\|\varphi\|_{V'}^{1/2}\|\varphi\|_V^{1/2}\|\psi\|_{V'} \\
&\leqslant \frac{1}{10}\|\varphi\|_V^2 + \alpha_2^{4/3}(t)\|\varphi\|_{V'}^{2/3}\|\psi\|_{V'}^{4/3} \\
&\leqslant \frac{1}{10}\|\varphi\|_V^2 + \alpha_2^{4/3}(t)(\|\varphi\|_{V'}^2 + \|\psi\|_{V'}^2), \qquad (3.73)
\end{aligned}
$$

where, in the third inequality, (3.70) has been used. We now estimate the last term on the right-hand side of (3.54)

$$
\begin{aligned}
|(-(p(\varphi_1) - 1)\psi + p(\varphi_1)\mu, A^{-1}\psi)| &\leqslant (\|p(\varphi_1)(\psi - \mu)\|_{V'} + \|\psi\|_{V'})\|\psi\|_{V'} \\
&\leqslant (\alpha_1\|\psi - \mu\|_{V'} + \|\psi\|_{V'})\|\psi\|_{V'} \\
&\leqslant (1 + \alpha_1)\|\psi\|_{V'}^2 + \alpha_1\Gamma\|\varphi\|_V\|\psi\|_{V'} \\
&\leqslant \frac{1}{10}\|\varphi\|_V^2 + \Gamma(1 + \alpha_1^2)\|\psi\|_{V'}^2, \qquad (3.74)
\end{aligned}
$$

where we have used (3.62) in the second inequality and (3.64) in the third inequality.

Moreover, setting $\hat{\beta} := \alpha + 1 - c_1$, we have

$$
(G'(\varphi_2) - G'(\varphi_1), \varphi) \geqslant -\hat{\beta}\|\varphi\|^2 \geqslant -\frac{1}{10}\|\varphi\|_V^2 - c\|\varphi\|_{V'}^2. \qquad (3.75)
$$

Finally, plugging estimates (3.65) and (3.72)–(3.75) into (3.54) yields the following differential inequality

$$
\frac{d}{dt}\left(\|\varphi\|_{V'}^2 + \|\psi\|_{V'}^2\right) + \|\varphi\|_V^2 + \|\psi\|^2 \leqslant \hat{\gamma}\left(\|\varphi\|_{V'}^2 + \|\psi\|_{V'}^2\right), \qquad (3.76)
$$

where

$$
\hat{\gamma} := \Gamma\left(\alpha_1^2 + \alpha_2^{4/3} + 1\right) \in L^1(0, T).
$$

An application of Gronwall's inequality to (3.76) ends the proof. □

4 Strong solutions and the global attractor

Here, we establish a regularity result for Problem (1.1)–(1.5) that holds under the same condition on p which ensures uniqueness (cf. (P1)). This result will be used to deduce some uniform in time higher-order estimates which will be crucial in order to prove the existence of the global attractor.

Theorem 3 *Suppose (F) and (P1) hold. Let $\varphi_0 \in H^3(\Omega)$ and $\psi_0 \in V$. Then, for every $T > 0$, the solution $[\varphi, \psi]$ to Problem (1.1)–(1.5) on $[0, T]$ given by Theorem 1 satisfies*

$$\varphi \in L^\infty(0, T; H^3(\Omega)), \quad \varphi_t \in L^2(0, T; V),$$
$$\mu \in L^\infty(0, T; V),$$
$$\psi \in L^\infty(0, T; V), \quad \psi_t \in L^2(0, T; H).$$

Proof The proof is carried out by deducing formally some higher-order identities and estimates which can be justified rigorously by means of a suitable approximation procedure (see the proof of Theorem 1).

Testing (1.1) by μ_t in H and using (1.2), we find

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \|\nabla\varphi_t\|^2 + \int_\Omega F''(\varphi)\varphi_t^2 = \big(p(\varphi)(\psi - \mu), \mu_t\big),$$

whence

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \|\nabla\varphi_t\|^2 + \int_\Omega F''(\varphi)\varphi_t^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega p(\varphi)\mu^2 = \frac{1}{2}\int_\Omega p'(\varphi)\varphi_t\mu^2 + \big(p(\varphi)\psi, \mu_t\big).$$
(4.1)

Test now (1.3) by ψ_t in H to get

$$\|\psi_t\|^2 = -\frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2 - \frac{1}{2}\frac{d}{dt}\int_\Omega p(\varphi)\psi^2 + \frac{1}{2}\int_\Omega p'(\varphi)\varphi_t\psi^2 + \big(p(\varphi)\mu, \psi_t\big). \quad (4.2)$$

By summing (4.1) with (4.2) we obtain

$$\frac{1}{2}\frac{d}{dt}\|\nabla\mu\|^2 + \|\nabla\varphi_t\|^2 + \int_\Omega F''(\varphi)\varphi_t^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega p(\varphi)\mu^2$$
$$+ \|\psi_t\|^2 + \frac{1}{2}\frac{d}{dt}\|\nabla\psi\|^2 + \frac{1}{2}\frac{d}{dt}\int_\Omega p(\varphi)\psi^2$$
$$= \frac{1}{2}\int_\Omega p'(\varphi)\varphi_t\mu^2 + \frac{d}{dt}\int_\Omega p(\varphi)\psi\mu - \int_\Omega p'(\varphi)\varphi_t\psi\mu + \frac{1}{2}\int_\Omega p'(\varphi)\varphi_t\psi^2,$$

so that

$$\frac{1}{2}\frac{d}{dt}\Big(\|\nabla\mu\|^2 + \|\nabla\psi\|^2 + \int_\Omega p(\varphi)(\mu - \psi)^2\Big) + \|\nabla\varphi_t\|^2 + \int_\Omega F''(\varphi)\varphi_t^2 + \|\psi_t\|^2$$
$$= \frac{1}{2}\int_\Omega p'(\varphi)\varphi_t(\mu - \psi)^2. \quad (4.3)$$

Observe now that

$$\left|\frac{1}{2}\int_\Omega p'(\varphi)\varphi_t(\mu - \psi)^2\right| \leqslant \frac{1}{2}\|p'(\varphi)\|\|\varphi_t\|_{L^6(\Omega)}\|\mu - \psi\|_{L^6(\Omega)}^2 \leqslant c\|p'(\varphi)\|\|\varphi_t\|_V\|\mu - \psi\|_V^2. \quad (4.4)$$

Moreover, we have (using (1.1) and (1.4))

$$\|\varphi_t\|_V \leqslant (1 + c_\Omega)\|\nabla\varphi_t\| + |\Omega|^{1/2}|\overline{\varphi}_t|$$
$$\leqslant (1 + c_\Omega)\|\nabla\varphi_t\| + \frac{1}{|\Omega|^{1/2}}\left|\int_\Omega p(\varphi)(\mu - \psi)\right|$$

$$\leq (1 + c_\Omega)\|\nabla\varphi_t\| + \frac{1}{|\Omega|^{1/2}}\|p(\varphi)\|_{L^{6/5}(\Omega)}\|\mu - \psi\|_{L^6(\Omega)}, \tag{4.5}$$

where c_Ω is the constant appearing in the Poincaré–Wirtinger inequality. Hence, by combining (4.4) with (4.5), we get, applying, in particular, the Young inequality with exponents 4 and 4/3 in the last line,

$$\left|\frac{1}{2}\int_\Omega p'(\varphi)\varphi_t(\mu - \psi)^2\right| \leq c\|p'(\varphi)\|\big(\|\nabla\varphi_t\| + \|p(\varphi)\|_{L^{6/5}(\Omega)}\|\mu - \psi\|_V\big)\big(\|\mu\|_V^2 + \|\psi\|_V^2\big)$$

$$\leq \frac{1}{2}\|\nabla\varphi_t\|^2 + c\|p'(\varphi)\|^2\big(\|\mu\|_V^4 + \|\psi\|_V^4\big) + c\|p'(\varphi)\|\|p(\varphi)\|_{L^{6/5}(\Omega)}\big(\|\mu\|_V^3 + \|\psi\|_V^3\big)$$

$$\leq \frac{1}{2}\|\nabla\varphi_t\|^2 + c\big(1 + \|p'(\varphi)\|^2\big)\big(\|\mu\|_V^4 + \|\psi\|_V^4\big) + c\|p'(\varphi)\|^4\|p(\varphi)\|_{L^{6/5}(\Omega)}^4. \tag{4.6}$$

Thanks to (P1) and to $(3.5)_1$ we can see that $p'(\varphi)$ is controlled in $L^\infty(0, T; H)$. Moreover, we know that φ is bounded in $L^{18}(0, T; L^{54/5}(\Omega))$ (cf. (3.34)) and φ is also bounded in $L^{4q}(0, T; L^{6q/5}(\Omega))$ since $q \leq 4$, Thanks to this bound, assumption (P1) entails that $p(\varphi)$ is controlled in $L^4(0, T; L^{6/5}(\Omega))$. Thus, we have

$$\|p'(\varphi)\|_{L^\infty(0,T;H)} \leq \Gamma, \qquad \|p(\varphi)\|_{L^4(0,T;L^{6/5}(\Omega))} \leq \Gamma, \tag{4.7}$$

where henceforth $\Gamma = \Gamma\big(\|\varphi_0\|_V, \|\psi_0\|\big)$ will denote a positive constant that depends on the norms of the initial data (and on F, p, Ω).

Furthermore, we have

$$\|\mu\|_V \leq (1 + c_\Omega)\|\nabla\mu\| + |\Omega|^{1/2}|\overline{\mu}| \leq (1 + c_\Omega)\|\nabla\mu\| + \Gamma, \tag{4.8}$$

$$\|\psi\|_V \leq \|\nabla\psi\| + \Gamma. \tag{4.9}$$

By plugging estimate (4.6) into (4.3) and using (4.7)–(4.9) and (3.3), we get

$$\frac{1}{2}\frac{d}{dt}\left(\|\nabla\mu\|^2 + \|\nabla\psi\|^2 + \int_\Omega p(\varphi)(\mu - \psi)^2\right) + \frac{1}{2}\|\nabla\varphi_t\|^2 + \|\psi_t\|^2 \leq c_3\|\varphi_t\|^2$$

$$+ \Gamma\big(\|\mu\|_V^2\|\nabla\mu\|^2 + \|\psi\|_V^2\|\nabla\psi\|^2\big) + \Gamma\big(\|\mu\|_V^2 + \|\psi\|_V^2 + \|p(\varphi)\|_{L^{6/5}(\Omega)}^4\big). \tag{4.10}$$

We now need an estimate for the L^2-norm of φ_t in (4.10). This can be obtained by testing (1.1) by φ_t in H, integrating by parts in Ω and using (1.2). This yields

$$\|\varphi_t\|^2 = (\mu, \Delta\varphi_t) + \big(p(\varphi)(\psi - \mu), \varphi_t\big)$$

$$= -\frac{1}{2}\frac{d}{dt}\|\Delta\varphi\|^2 - \int_\Omega F''(\varphi)\nabla\varphi \cdot \nabla\varphi_t + \big(p(\varphi)(\psi - \mu), \varphi_t\big).$$

Hence, we have

$$\frac{1}{2}\frac{d}{dt}\|\Delta\varphi\|^2 + \frac{1}{2}\|\varphi_t\|^2 \leq \left|\int_\Omega F''(\varphi)\nabla\varphi \cdot \nabla\varphi_t\right| + \frac{1}{2}\|p(\varphi)\|_{L^3(\Omega)}^2\|\mu - \psi\|_{L^6(\Omega)}^2$$

$$\leq \|F''(\varphi)\|_{L^{7/2}(\Omega)}\|\nabla\varphi\|_{L^{14/3}(\Omega)}\|\nabla\varphi_t\| + c\|p(\varphi)\|_{L^3(\Omega)}^2\big(\|\mu\|_V^2 + \|\psi\|_V^2\big)$$

$$\leq \frac{1}{8c_3}\|\nabla\varphi_t\|^2 + c\|F''(\varphi)\|_{L^{7/2}(\Omega)}^2\|\nabla\varphi\|_{L^{14/3}(\Omega)}^2 + c\|p(\varphi)\|_{L^3(\Omega)}^2\big(\|\mu\|_V^2 + \|\psi\|_V^2\big). \tag{4.11}$$

Recalling that φ is bounded in $L^{14}(Q)$ (cf. (3.30) and (3.33) with $\theta = 14$), (F) implies that $F''(\varphi)$ is bounded in $L^{7/2}(Q)$ (note that $\rho < 6$). Moreover, $\nabla\varphi$ is bounded in $L^{14/3}(Q)$ (cf. (3.30) and (3.46) with $s = 2$). Therefore the second term on the right-hand side of the last inequality in (4.11) is bounded in $L^1(0, T)$.

Furthermore, φ is also bounded in $L^8(0, T; L^\infty(\Omega))$ (cf. (3.30) and (3.33) with $\theta = \infty$) and, for $q \leqslant 4$, (P1) implies that $p(\varphi)$ is bounded in $L^2(0, T; L^3(\Omega))$.

By combining (4.10) with (4.11), also on account of (4.8) and (4.9), we obtain the following differential inequality

$$
\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(\|\nabla\mu\|^2 + \|\nabla\psi\|^2 + 2c_3\|\Delta\varphi\|^2 + \int_\Omega p(\varphi)(\mu - \psi)^2\right) + \frac{1}{4}\|\nabla\varphi_t\|^2 + \|\psi_t\|^2 \\
&\leqslant \sigma_1\left(\|\nabla\mu\|^2 + \|\nabla\psi\|^2\right) + \sigma_2,
\end{aligned}
\tag{4.12}
$$

where

$$
\sigma_1 := c\|p(\varphi)\|_{L^3(\Omega)}^2, \quad \sigma_2 := c\|F''(\varphi)\|_{L^{7/2}(\Omega)}^2\|\nabla\varphi\|_{L^{14/3}(\Omega)}^2 + \Gamma\,\|p(\varphi)\|_{L^3(\Omega)}^2.
\tag{4.13}
$$

Notice that

$$
\|\sigma_1\|_{L^1(0,T)} \leqslant \Gamma, \quad \|\sigma_2\|_{L^1(0,T)} \leqslant \Gamma.
$$

Using Gronwall's lemma and recalling the assumptions on the initial data (in particular, $\varphi_0 \in H^3(\Omega)$ implies that $\mu(0) \in V$) from (4.12) we get that $\nabla\mu$ and $\Delta\varphi$ belong to $L^\infty(0, T; H)$, $\psi \in L^\infty(0, T; V)$, $\nabla\varphi_t$ and ψ_t belong to $L^2(0, T; H)$. Also, thanks to (F), we have that $F'(\varphi) \in L^\infty(0, T; H)$. Therefore, $\mu \in L^\infty(0, T; H)$ so that

$$
\mu \in L^\infty(0, T; V).
\tag{4.14}
$$

Moreover, due to elliptic regularity result for homogeneous Neumann problems, we deduce $\varphi \in L^\infty(0, T; H^2(\Omega))$. From this property and (4.14) we infer we have also

$$
\varphi \in L^\infty(0, T; H^3(\Omega)).
\tag{4.15}
$$

Indeed, since $\varphi \in L^\infty(0, T; H^2(\Omega))$, we have $F'(\varphi) \in L^\infty(0, T; V)$. From (4.14) we then get $\Delta\varphi \in L^\infty(0, T; V)$ and (4.15) follows by standard elliptic regularity.

Finally, as far as φ_t is concerned, by integrating (4.11) in time between 0 and t we get $\varphi_t \in L^2(0, T; H)$ and this bound together with the bound for $\nabla\varphi_t$ deduced above imply $\varphi_t \in L^2(0, T; V)$.   □

We now show that (1.1)–(1.4) define a dynamical system on a suitable phase space. Let $M > 0$ be given. Set

$$
\mathscr{W}_M := \{w = [\varphi, \psi] \in V \times H : \mathscr{E}(w) \leqslant M\}.
$$

and endow $\mathscr{W}_M$ with the metric

$$
\mathbf{d}_{\mathscr{W}_M}(w_2, w_1) := \|\varphi_2 - \varphi_1\|_V + \|\psi_2 - \psi_1\|, \quad \forall w_i := [\varphi_i, \psi_i] \in \mathscr{W}_M, \quad i = 1, 2,
$$

so that it is a complete metric space. As a consequence of Theorems 1 and 2, assuming that (F) and (P1) are satisfied, we can define a semigroup $\{S_M(t)\}_{t \geqslant 0}$ of *closed* operators on $\mathscr{W}_M$ (cf. [30]) by setting

$$[\varphi(t), \psi(t)] = S_M(t)[\varphi_0, \psi_0], \quad \forall t \geqslant 0,$$

where $[\varphi, \psi]$ is the unique (weak) solution to Problem (1.1)–(1.5).

Notice that we have the total mass constraint

$$|\overline{\varphi(t)} + \overline{\psi(t)}| = |\overline{\varphi}_0 + \overline{\psi}_0| \leqslant Q(M), \quad \forall t \geqslant 0,$$

where henceforth by $Q = Q(M)$ we denote a non-negative continuous monotone increasing function of M (which may also depend on F, p and Ω). Such a function may change even within the same line.

Theorem 4 *Let (F) and (P1) be satisfied. Then the dynamical system $(\mathscr{W}_M, \{S_M(t)\}_{t \geqslant 0})$ possesses the global attractor.*

Proof The proof is carried out by showing the existence of a compact (in $\mathscr{W}_M$) absorbing set $\mathscr{B}_M$ for the semigroup $\{S_M(t)\}_{t \geqslant 0}$. This fact will allow us to apply a general result on the existence of global attractors for semigroup of closed operators proven in [30].

Let us first write (4.12) in the form

$$\frac{d\Phi}{dt} + \frac{1}{4}\|\nabla \varphi_t\|^2 + \|\psi_t\|^2 \leqslant \sigma_1 \Phi + \sigma_2, \tag{4.16}$$

where

$$\Phi := \frac{1}{2}\|\nabla \mu\|^2 + \frac{1}{2}\|\nabla \psi\|^2 + c_3\|\Delta \varphi\|^2 + \frac{1}{2}\int_\Omega p(\varphi)(\mu - \psi)^2,$$

and σ_1 and σ_2 are defined as in (4.13). Notice that, since $\Gamma = \Gamma\left(\|\varphi_0\|_V, \|\psi_0\|\right)$ and since $[\varphi_0, \psi_0] \in \mathscr{W}_M$, then the constant Γ that bounds the L^1-norm of σ_2 will depend only on M.

Integrating the energy identity (3.40) between t and $t + 1$, we get, for all $t \geqslant 0$,

$$\int_t^{t+1} \|\nabla \mu\|^2 d\tau \leqslant M, \quad \int_t^{t+1} \|\nabla \psi\|^2 d\tau \leqslant M, \quad \int_t^{t+1} \int_\Omega p(\varphi)(\mu - \psi)^2 \leqslant M. \tag{4.17}$$

Recalling that $q \leqslant 4$, we deduce from (P1) that

$$\int_t^{t+1} \sigma_1(\tau) d\tau \leqslant c\left(1 + \|\varphi\|_{L^{2q}(t,t+1;L^{3q}(\Omega))}^{2q}\right)$$
$$\leqslant c\left(1 + \|\varphi\|_{L^\infty(t,t+1;V)}^{2q} + \|\varphi\|_{L^2(t,t+1;H^3(\Omega))}^{2q}\right) \leqslant Q(M). \tag{4.18}$$

Moreover, on account of (F) and (P1), we obtain

$$
\begin{aligned}
\int_t^{t+1} \sigma_2(\tau)d\tau &\leqslant c\|F''(\varphi)\|_{L^{7/2}(t,t+1;L^{7/2}(\Omega))}^2 \|\nabla\varphi\|_{L^{14/3}(t,t+1;L^{14/3}(\Omega))}^2 + Q(M) \\
&\leqslant c\big(1+\|\varphi\|_{L^{7(\rho-2)/2}(t,t+1;L^{7(\rho-2)/2}(\Omega))}^{2(\rho-2)}\big)\|\nabla\varphi\|_{L^{14/3}(t,t+1;L^{14/3}(\Omega))}^2 + Q(M) \\
&\leqslant c\big(1+\|\varphi\|_{L^{\infty}(t,t+1;V)}^{2(\rho-2)} + \|\varphi\|_{L^2(t,t+1;H^3(\Omega))}^{2(\rho-2)}\big)\big(\|\nabla\varphi\|_{L^{\infty}(t,t+1;H)}^2 + \|\nabla\varphi\|_{L^2(t,t+1;H^2(\Omega))}^2\big) \\
&\quad + Q(M) \leqslant c\big(1+\|\varphi\|_{L^{\infty}(t,t+1;V)}^{2(\rho-1)} + \|\varphi\|_{L^2(t,t+1;H^3(\Omega))}^{2(\rho-1)}\big) + Q(M) \leqslant Q(M). \quad (4.19)
\end{aligned}
$$

In (4.18) and (4.19), we have used the fact that the $L^2(t,t+1;H^3(\Omega))$−norm of φ can be controlled, uniformly in time, in terms of $\|\varphi_0\|_V, \|\psi_0\|$ and hence of M, when $4 < \rho < 6$. Indeed, we can use the iteration argument outlined in the proof of Theorem 1 (cf. Step II; if $\rho = 4$ no iteration is needed).

Therefore, we have (see (4.17))

$$
\int_t^{t+1} \Phi(\tau)d\tau \leqslant \frac{3M}{2} + c_3 \int_t^{t+1} \|\Delta\varphi(\tau)\|^2 d\tau \leqslant Q(M). \quad (4.20)
$$

Thanks to (4.18)–(4.20) we can now apply the uniform Gronwall's lemma to (4.16) and obtain

$$
\Phi(t) \leqslant Q(M), \quad \forall t \geqslant 1. \quad (4.21)
$$

On the other hand, the definition of the phase space $\mathscr{W}_M$ and (3.3) yield

$$
\|\varphi(t)\|_{L^\rho(\Omega)} \leqslant Q(M), \qquad \|\psi(t)\| \leqslant Q(M), \quad \forall t \geqslant 0. \quad (4.22)
$$

Hence, we deduce from (4.21) and (4.22) that

$$
\|\varphi(t)\|_{H^2(\Omega)} \leqslant Q(M), \quad \forall t \geqslant 1. \quad (4.23)
$$

Moreover, (4.21) and (4.23) give

$$
\|\mu(t)\|_V \leqslant Q(M), \quad \forall t \geqslant 1.
$$

Also, using (4.23) once more, we have

$$
\|\nabla F'(\varphi(t))\| \leqslant \|F''(\varphi(t))\nabla\varphi(t)\| \leqslant Q(M), \quad \forall t \geqslant 1.
$$

The last two bounds, (1.2) and elliptic regularity imply

$$
\|\varphi(t)\|_{H^3(\Omega)} \leqslant Q(M), \quad \forall t \geqslant 1. \quad (4.24)
$$

Finally, from (4.21) and $(4.22)_2$, we get

$$
\|\psi(t)\|_V \leqslant Q(M), \quad \forall t \geqslant 1. \quad (4.25)
$$

Thanks to (4.24) and (4.25), we have thus proven that there exists $\Lambda = \Lambda(M) > 0$ such that

$$\mathscr{B}_M := \{w := [\varphi, \psi] \in H^3(\Omega) \times H^1(\Omega) : \ \|\varphi\|_{H^3(\Omega)} \leqslant \Lambda, \ \|\psi\|_{H^1(\Omega)} \leqslant \Lambda, \ \mathscr{E}(w) \leqslant M\}$$

is an absorbing set for the semigroup $\{S_M(t)\}_{t \geqslant 0}$ in $\mathscr{W}_M$. Since $\mathscr{B}_M$ is also compact in $\mathscr{W}_M$, the conclusion follows from [30, Theorem 2]. □

5 Conclusions

We provided some theoretical results for a thermodynamically consistent phase field model of tumour growth proposed in [20]. The same contribution contains a numerical scheme as well as some numerical simulations. A formal asymptotic analysis is carried out in [22] to identify the corresponding sharp interface problem. The first theoretical results concerning existence and uniqueness of solutions are proven in [8]. Our first main result is a well-posedness theorem for weak solutions which are just energy bounded. The growth assumptions on the potential F and the proliferation function are more general than the ones in [8]. We point out that the existence of a solution is obtained through an approximation scheme which might be helpful also for numerical approximations. A further main result is based on the dissipative nature of the model, namely, the fact that the total energy decreases in time. More precisely, we demonstrate the existence of a global attractor in a suitable phase space with bounded total energy. This requires, in particular, to show some regularity property of the solution itself. We think that establishing the well-posedness of this model is a step towards its validation. Moreover, our rigorous results on the dissipative longterm dynamics can play a role in designing more appropriate numerical schemes and/or in understanding numerical simulations.

Acknowledgements

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