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Quasiconformal Contactomorphisms and Polynomial Hulls with Convex Fibers

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Abstract. Consider the polynomial hull of a smoothly varying family of strictly convex smooth domains fibered over the unit circle. It is well-known that the boundary of the hull is foliated by graphs of analytic discs. We prove that this foliation is smooth, and we show that it induces a complex flow of contactomorphisms. These mappings are quasiconformal in the sense of Korányi and Reimann. A similar bound on their quasiconformal distortion holds as in the one-dimensional case of holomorphic motions. The special case when the fibers are rotations of a fixed domain in $\mathbf{C}^2$ is studied in details.

1 Preliminaries and Statements of Results

The study of *holomorphic motions* in one complex variable has been largely motivated by complex dynamics. The observation (*cf.* [22]) that the Julia sets of rational maps move holomorphically with respect to the parameter provides a whole class of natural and interesting examples of holomorphic motions. There is also an intimate connection between holomorphic motions and quasiconformal maps, thus holomorphic motions can be used as a tool in proving deep results in the theory of quasiconformal maps [2], [3].

Let us now be more precise: given a subset $Y \subset \overline{\mathbf{C}}$, one can define a holomorphic motion of Y as a mapping $\Psi\colon \Delta \times Y \to \overline{\mathbf{C}}$, such that

(i) $\Psi(0,\cdot) = \Psi_0 = id|_Y$,
(ii) for any fixed $y \in Y$, the map $z \to \Psi(z,y)$ is holomorphic in Δ,
(iii) for any fixed $z \in \Delta$, the map $y \to \Psi(z,y) = \Psi_z(y)$ is an injection.

A remarkable result concerning holomorphic motions is the so called *extended λ-lemma*. This says that a holomorphic motion of the set $Y \subset \overline{\mathbf{C}}$ extends to a motion of the whole $\overline{\mathbf{C}}$. The mappings $\Psi_z(\cdot)$ are quasiconformal with the following precise bound on the quasiconformal distortion:

$$K(\Psi_z) \le \frac{1+|z|}{1-|z|}. \tag{1}$$

Partial results in this direction have been obtained by Bers and Royden [7] and also by Sullivan and Thurston in [30]. The result in the above form was proven by Slodkowski in [26]. The idea of his proof was to embed Y in the boundary $\partial X(0)$ of the 0-fiber $X(0) = \{y \in \mathbf{C} : (0,y) \in \hat{X}\}$ of the polynomial hull $\hat{X}$ of a certain $X \subset \partial\Delta \times \mathbf{C}$; where the fibers of X are Jordan domains in $\mathbf{C}$. He could then apply the description of the boundary of polynomial hulls due to Forstnerič [12] to prove the statement of the extended λ-lemma.

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Holomorphic motions in one complex dimension are still the subject of current research [27], [28] and a first step has been made in [4] to understand *higher dimensional holomorphic motions*. In this paper we continue this investigation as we study the holomorphic motions in higher dimensions that are induced by the analytic foliation of polynomial hulls fibered over the unit disc.

We are going to work in the following setting. Let $X \subseteq \partial\Delta \times \mathbf{C}^n$ be a smoothly varying family of strictly convex smooth domains in $\mathbf{C}^n$, $n \geq 2$ fibered over the unit circle $\partial\Delta$: $X = \bigcup_{\zeta\in\partial\Delta}\{\zeta\} \times X(\zeta)$, where $X(\zeta) \subseteq \mathbf{C}^n$ is a strictly convex smooth domain containing 0 in the interior. X is given by a defining function $r\colon \partial\Delta \times \mathbf{C}^n \to \mathbf{R}$ such that

$$X(\zeta) = \{y \in \mathbf{C}^n : r(\zeta, y) < 0\},\ \zeta \in \partial\Delta.$$

We assume that r is of class $C^{1,\alpha}\big(\partial\Delta, C^{k+1,\alpha}(\mathbf{C}^n)\big)$ where $0 < \alpha < 1$, $k \geq 2$. This means that for a fixed $\zeta \in \partial\Delta$ the function $r(\zeta,\cdot)\colon \mathbf{C}^n \to \mathbf{R}$ is of class $C^{k+1,\alpha}$ and the mapping: $\zeta \to r(\zeta,\cdot)$ from $\partial\Delta$ to $C^{k+1,\alpha}(\mathbf{C}^n)$ is in Hölder class $C^{1,\alpha}$.

The polynomial hull $\hat{X}$ of X in $\mathbf{C}^{n+1}$ is defined by

$$\hat{X} = \{x \in \mathbf{C}^{n+1} : |p(x)| \leq \max_X |p| \text{ for every polynomial } p \text{ on } \mathbf{C}^{n+1}\}.$$

The structure of $\hat{X}$ was studied by several authors (see [25] and the references therein). We start by recalling some relevant results. Let $X(z)$ denote the fiber of $\hat{X}$ over $z \in \overline{\Delta}$:

$$X(z) = \{y \in \mathbf{C}^n : (z, y) \in \hat{X}\}.$$

Let $\pi\colon \mathbf{C} \times \mathbf{C}^n \to \mathbf{C}$ be the projection to the first component and let

$$S = \pi^{-1}(\Delta) \cap \partial\hat{X}$$

denote the boundary of $\hat{X}$ over the unit disc Δ. Consider the following nonlinear Riemann-Hilbert problem:

$$\begin{cases} f\colon \Delta \to \mathbf{C}^n \text{ is holomorphic and continuous on } \overline{\Delta}, \\ f(\zeta) \in \partial X(\zeta), \zeta \in \partial\Delta. \end{cases} \tag{2}$$

Let us denote by $\mathrm{gr}(f) = \{\big(z, f(z)\big) : z \in \overline{\Delta}\}$ the graph of f. It is clear by maximum principle that if f is a solution of (2) then $\mathrm{gr}(f) \subseteq \hat{X}$. Moreover, the boundary S is foliated by such graphs. A quite complete description of this fact is given by Slodkowski in [25] which we next recall:

***Theorem A* (Slodkowski)**

(a) *For every $x_0 \in \partial X(0)$ there exists a unique solution $f = f_{x_0}$ of (2) such that $f(0) = x_0$ which in addition satisfies the following:*

  (i) *f is of class $C^{1,\alpha-0}$ on $\overline{\Delta}$;*

  (ii) *$f(z) \in \partial X(z)$, $z \in \overline{\Delta}$;*

(iii) *there exists a nonvanishing, positive function* $p\colon \partial\Delta \to \mathbf{R}$ *of class* $C^{\frac{1}{2}}$ *such that the dual map of* f

$$\tilde{f}(\zeta) = p(\zeta) r_z\big(\zeta, f(\zeta)\big), \quad \zeta \in \partial\Delta$$

admits a holomorphic extension that does not vanish in Δ, *where* $r_z = (r_{z_1}, \dots, r_{z_n})$.

(b) *Concerning the fibers* $X(z), z \in \Delta$ *we have the following:*

(i) *for any* $x \in \partial X(z)$ *there is a unique* $x_0 \in \partial X(0)$ *such that* $\operatorname{gr}(f_{x_0}) \subseteq S$ *and* $f_{x_0}(z) = x$;

(ii) *the fibers* $X(z)$ *are strictly convex;*

(iii) *the value of the dual map* $\tilde{f}(z)$ *at* z *supports* $X(z)$ *at the point* $f(z)$, *i.e.,*

$$\operatorname{Re}\langle \tilde{f}(z), f(z) - w\rangle \geq 0, \quad w \in X(z),$$

where $\langle a, b\rangle = \sum_i a_i b_i$ *is the complex product in* $\mathbf{C}^n$.

(c) *If* $f\colon \Delta \to \mathbf{C}^n$ *is a solution of (2) such that there exists a dual map* $\tilde{f}$ *of* f *that has a nonvanishing holomorphic extension to* Δ *then* $\operatorname{gr}(f) \subseteq S$.

Recall that $f \in C^{1,\alpha-0}$ means that $f \in C^{1,\beta}$ for each $0 < \beta < \alpha$.

Remark 1.1 In fact in [25] it is only shown that $f \in C^{\beta}(\overline{\Delta}, \mathbf{C}^n)$ for all $\beta < 1$. With our assumptions on r the regularity of f as stated in Theorem A (a)(i) is easily seen by the following standard consideration: Define the set $Y \subseteq \partial\Delta \times \mathbf{C}^n$ by

$$Y = \bigcup_{\zeta \in \partial\Delta} \{\zeta\} \times \partial X(\zeta)$$

and consider the map

$$\begin{aligned} \psi\colon Y &\to \mathbf{C}^{n+1} \times \mathbf{C}P^{n-1} \\ (\zeta, y) &\mapsto \big(\zeta, y, H_y \partial X(\zeta)\big), \end{aligned}$$

where $H_y\partial X(\zeta)$ is the maximal complex hyperplane in $\mathbf{C}^n$ tangent to $\partial X(\zeta)$ at y viewed as a point in $\mathbf{C}P^{n-1}$. Using a result of Webster [31] it is easy to check that $M = \psi(Y)$ is a totally real submanifold in $\mathbf{C}^{n+1} \times \mathbf{C}P^{n-1}$ of class $C^{1,\alpha}$. Consider the map $F\colon \overline{\Delta} \to \mathbf{C}^{n+1} \times \mathbf{C}P^{n-1}$ given by $z \mapsto \big(z, f(z), H_{f(z)}\big)$ where $H_{f(z)}$ is the complex hyperplane $H_{f(z)} = \{w \in \mathbf{C}^n : \langle \tilde{f}(z), w\rangle = 0\}$ viewed as a point in $\mathbf{C}P^{n-1}$. By Theorem A (a)(iii) it follows that F is an analytic disc with $F(\partial\Delta) \subseteq M$. The desired regularity of f follows by the results of Čirca [11].

In the sequel we shall also use the fact that there is a *uniform* $C^{\frac{1}{2}}$-bound on the discs f (see [25, Lemma 1.6]).

Clearly the above result concerns only the regularity of a fixed holomorphic discs that is contained in S. In contrast to the case of one dimensional fibers (see *e.g.* [12]) in the above higher dimensional setting there were no explicit results concerning the global regularity of

S; even though specialists believed that such result must be true (*cf.* [29]). Our first result concerns the regularity of S, and of the mapping:

$$\Psi\colon \overline{\Delta} \times \partial X(0) \to \mathbf{C}^n \quad \text{given by } \Psi(z, x_0) = f_{x_0}(z),$$

where f_{x_0} is the unique disc given in Theorem A with $f_{x_0}(0) = x_0$ and $\operatorname{gr}(f_{x_0}) \subseteq S$.

Theorem 1.1

(a) *The boundary $S = \pi^{-1}(\Delta) \cap \partial\hat{X}$ is a $C^{k,\alpha-0}$ smooth hypersurface in $\mathbf{C}^{n+1}$. Moreover, for each $z \in \Delta$ the boundaries $\partial X(z)$ are $C^{k,\alpha-0}$-smooth hypersurfaces in $\mathbf{C}^n$.*

(b) *The mapping $\Psi(z,\cdot)\colon \partial X(0) \to \partial X(z)$ is a $C^{k,\alpha-0}$ smooth diffeomorphism.*

To comment on this statement let us mention that in [25] it is shown that $\Psi\colon \overline{\Delta} \times \partial X(0) \to \mathbf{C}^n$ is a homeomorphism onto S. A weaker result of regularity than in statement (a) of Theorem 1.1 could be obtained using the theory of partial indices by combining the results of [10] and [14]. To do that one has to assume that the defining function r is of class $C^{1,\alpha}\big(\partial\Delta, C^{k+2}(\mathbf{C}^n)\big)$, $k \geq 2$.

The idea of our proof is based on condition (a)(iii) that is called the *stationarity condition* and is a natural extension to polynomial hulls of Lempert's stationary discs [17], [18] used in the study of the Kobayashi metric in convex domains.

Our point is that the ideas of Lempert from [18] and [20] to study variations of the Kobayashi extremals work (with the appropriate modifications) in the case of polynomial hulls as well.

An important feature for holomorphic motions in one complex variable was the relation with quasiconformal mappings. In the setting several complex variables a natural notion of quasiconformality has been introduced by Korányi and Reimann [15], [16], [23]. According to [16], [23] a $\mathbf{C}^1$ smooth diffeomorphism $F\colon \partial D_1 \to \partial D_2$ between the boundaries of two strictly pseudoconvex domains $D_1, D_2 \subset \mathbf{C}^n$ is a quasiconformal map if it is a contactomorphism. This means that the tangent map F_* preserves the horizontal bundle, *i.e.*, $F_* H\partial D_1 = H\partial D_2$, where the horizontal bundle $H\partial D$ is the maximal complex subbundle of the tangent bundle $T\partial D$. The local distortion of the complex structure is measured by the number $K(x)$ at every $x \in \partial D_1$ and the maximal distortion of F is defined by $K = \sup_{x\in\partial D_0} K(x)$. In [16] and [23] there is a formula to calculate the distortion using a higher dimensional version of the Beltrami differential. It follows that CR-maps—in particular restrictions to the boundary of biholomorphisms of strictly pseudoconvex domains—are 1-quasiconformal which shows that this notion of quasiconformality is most suitable in several complex variables. Let us mention that Bland and Duchamp introduced a similar notion of Beltrami differential in order to study deformations of CR structures, see [9], [8].

In contrast to the powerful Ahlfors-Bers-Bojarski measurable Riemann mapping theorem, in several complex variables there is no complete theory for the existence of quasiconformal maps between boundaries of strictly pseudoconvex domains. A way to create contactomorphisms as time-t maps of certain flows has been studied by Reimann in [23]. In many other places these maps appear in quite different setting (see [24], [19], [4], [5], [6]). In our situation quasiconformal contactomorphisms are induced by the analytic foliation in a natural way, as stated in our next result.

Theorem 1.2 *The mapping* $\Psi\colon \overline{\Delta} \times \partial X(0) \to \mathbf{C}^n$ *has the following properties:*

(a) $\Psi(0,\cdot) = \mathrm{id}\,|_{\partial X(0)}$,
(b) $\Psi(\cdot, x_0)\colon \Delta \to \mathbf{C}^n$ *is holomorphic for every* $x_0 \in \partial X(0)$,
(c) $\Psi(z,\cdot)\colon \partial X(0) \to \partial X(z)$ *is a* $C^{k,\alpha-0}$*-smooth contactomorphism for every* $z \in \Delta$,
(d) there exists a non-negative constant $C = C(X)$, $C < 1$, *such that the maximal quasiconformal distortion* $K(z)$ *of* $\Psi(z,\cdot)$ *is estimated by*

$$K(z) \le \frac{1 + C|z|}{1 - C|z|}.$$

Conditions (a) and (b) are obvious by the definition of Ψ but we include them here to point out the analogy with the one-dimensional holomorphic motions. Observe also that the bound in (d) is similar to (1).

The paper is organized as follows. In the next section we prove Theorem 1.1 and Theorem 1.2, while in Section 3 we study in more details the special case where $X(\zeta) = \frac{1}{\zeta} D$, and D is a strictly convex domain in $\mathbf{C}^2$. In this special case the holomorphic discs are related to the Kobayashi extremals and the bound on the quasiconformality is related to the Lempert invariants (*cf.* [19]). Section 4 is for final remarks.

2 Proofs of Theorem 1.1 and Theorem 1.2

Taking Theorem 1.1 for granted we give first the

Proof of Theorem 1.2 By the second part of statement (a) of Theorem 1.1 we have that $\partial X(z)$ is a $C^{k,\alpha-0}$-smooth hypersurface in $\mathbf{C}^n$ for any $z \in \overline{\Delta}$. Property (c) is proven in view of statement (b) of Theorem 1.1 if we check the contact property of $\Psi(z,\cdot)\colon \partial X(0) \to \partial X(z)$. Let us choose a vector $Y \in H_{x_0}\partial X(0)$. We have to show that $\Psi_*(z,x_0)Y \in H_{\Psi(z,x_0)}\partial X(z)$ where $\Psi_*(z,x_0)\colon T_{x_0}\partial X(0) \to T_{\Psi(z,x_0)}\partial X(z)$ is the tangent map of Ψ with respect to the second variable. In view of statement (b) from Theorem A the vector $\overline{\tilde f} \in \mathbf{C}^n$ is a normal vector to $\partial X(z)$ at the point $f(z) = \Psi(z,x_0)$. Consequently the equation of the tangent space $T_{\Psi(z,x_0)}\partial X(z)$ is given by

$$T_{\Psi(z,x_0)}\partial X(z) = \{Z \in \mathbf{C}^n : \mathrm{Re}\langle \tilde f(z), Z\rangle = 0\} \tag{3}$$

and the equation of the horizontal space $H_{\Psi(z,x_0)}\partial X(z)$ is

$$H_{\Psi(z,x_0)}\partial X(z) = \{Z \in \mathbf{C}^n : \langle \tilde f(z), Z\rangle = 0\}. \tag{4}$$

Therefore we have to show

$$\langle \tilde f(z), \Psi_*(z,x_0)Y\rangle = 0. \tag{5}$$

Choose a smooth curve $\gamma\colon (-\epsilon,\epsilon) \to \partial X(0)$ with the properties $\gamma(0) = x_0$ and $\dot\gamma(0) = \frac{d}{dt}|_{t=0}\gamma(t) = Y$ and consider the function $\tilde\gamma\colon \Delta \times (-\epsilon,\epsilon) \to \mathbf{C}^n$ given by $\tilde\gamma(z,t) = \Psi\bigl(z,\gamma(t)\bigr)$. We have

$$\Psi_*(z,x_0)Y = \frac{d}{dt}|_{t=0}\tilde\gamma(z,t) = v(z),$$

where $v\colon \Delta \to \mathbf{C}^n$ is holomorphic and $v(z) \in T_{\Psi(z,x_0)}\partial X(z)$. Consider the holomorphic function $h\colon \Delta \to \mathbf{C}$ defined by $h(z) = \langle \tilde{f}(z), v(z)\rangle$. Then $h(0) = 0$ since $v(0) = Y \in H_{x_0}\partial X(0)$. Furthermore for all $z \in \Delta$

$$\operatorname{Re} h(z) = \operatorname{Re}\langle \tilde{f}(z), v(z)\rangle = 0 \tag{6}$$

by (3) because $v(z) \in T_{\Psi(z,x_0)}\partial X(z)$. By holomorphicity of h it follows $h \equiv 0$. This means $\langle \tilde{f}(z), \Psi_*(z,x_0)Y\rangle = 0$ and we are done.

Let us show the estimate in (d). Since $\Psi(z,\cdot)\colon \partial X(0) \to \partial X(z)$ is a $C^{k,\alpha-0}$-smooth contactomorphism between strictly pseudoconvex boundaries it is quasiconformal in the sense of Korányi and Reimann.

The quasiconformal distortion $K(z)$ of $\Psi(z,\cdot)$ (*cf.* [15], [16]) is calculated by

$$K(z) = \sup_{x_0 \in \partial X(0)} \frac{1 + \|\mu(z,x_0)\|_L}{1 - \|\mu(z,x_0)\|_L}, \tag{7}$$

where $\mu(z,x_0)$ is a higher dimensional version of the Beltrami differential defined as follows.

Let us consider the complexified horizontal spaces $\mathbf{C}\otimes H_{x_0}\partial X(0)$, $\mathbf{C}\otimes H_{\Psi(z,x_0)}\partial X(z)$ and their splitting into $(1,0)$ and $(0,1)-$subspaces, *i.e.*, the eigenspaces of the standard complex structure J in $\mathbf{C}^n$. For example, the $(1,0)$ and $(0,1)$-spaces of $\mathbf{C}\otimes H_{x_0}\partial X(0)$ are given by

$$\begin{aligned} H^{1,0}_{x_0}\partial X(0) &= \{X - iJX : X \in H_{x_0}\partial X(0)\},\\ H^{0,1}_{x_0}\partial X(0) &= \overline{H^{1,0}_{x_0}\partial X(0)}. \end{aligned} \tag{8}$$

Because $\Psi(z,\cdot)\colon \partial X(0) \to \partial X(z)$ is generally not a CR map, its complexified tangent map

$$\Psi_*(z,x_0)\colon \mathbf{C}\otimes H_{x_0}\partial X(0) \to \mathbf{C}\otimes H_{\Psi(z,x_0)}\partial X(z) \tag{9}$$

does not preserve the $(1,0)$ and $(0,1)-$spaces. The Beltrami differential $\mu(z,x_0)$ will be in our case a matrix that measures the distortion of these subspaces. To be more precise, let us fix a basis $(Z_\alpha)_{\alpha=1,\dots,n-1}$ of $H^{1,0}_{x_0}\partial X(0)$. The image of this basis $(Z_\alpha)_{\alpha=1,\dots,n-1}$ under the tangent map $\Psi_*(z,x_0)$ decomposes into holomorphic and antiholomorphic parts

$$\Psi_*(z,x_0)Z_\alpha = V_\alpha + \overline{W}_\alpha,$$

where $V_\alpha = V_\alpha(z)$, $W_\alpha = W_\alpha(z) \in H^{1,0}_{\Psi(z,x_0)}\partial X(z)$.

Now, the vectors $V_1,\dots,V_{n-1}$ span the whole of $H^{1,0}_{\Psi(z,x_0)}\partial X(z)$ (*cf.* [15]) and consequently there exist coefficients $\mu_{\alpha\beta}$, $\alpha,\beta = 1,\dots,n-1$, such that

$$W_\alpha = \sum_{\beta=1}^{n-1} \mu_{\alpha\beta} V_\beta. \tag{10}$$

The matrix valued function $\mu\colon \Delta \times \partial X(0) \to \mathcal{M}_{n-1}(\mathbf{C})$ is our higher dimensional version of the Beltrami coefficient. The norm of $\mu(z, x_0)$ in (7) is meant to be calculated in terms of the Leviform L of $\partial X(0)$ at the point $x_0 \in \partial X_0$ by

$$\|\mu(z, x_0)\|_L = \sup_{L(Z,Z)=1} L(\mu Z, \mu Z).$$

Notice that the above expression does not depend on the choice of the defining function. Furthermore, observe that for $z = 0$ we have $\Psi(0, \cdot) = \mathrm{id}\,|_{\partial X(0)}$ and thus $\mu(0, x_0) = 0$ for any $x_0 \in \partial X(0)$. By our assumption of strict convexity (in fact strict pseudoconvexity is enough) it follows (see [4] or [15])

$$\|\mu(z, x_0)\|_L = \|\mu(z)\|_L < 1.$$

The estimate in (d) follows now from a version of the Schwarz lemma if we can show that for fixed $x_0 \in \partial X(0)$ the function $\mu := \mu(\cdot, x_0)\colon \Delta \to \mathcal{M}_{n-1}(\mathbf{C})$ is holomorphic. To do that write $\Psi(z, \cdot)\colon \partial X(0) \to \mathbf{C}^n$ as $\Psi = (\Psi^1, \dots, \Psi^n)$ and view $H^{1,0}_{\Psi(z,x_0)}\partial X(z)$ as a subspace of $T^{1,0}\mathbf{C}^n$. Therefore we have

$$V_\alpha = \sum_{k=1}^{n} (Z_\alpha \Psi^k) \frac{\partial}{\partial z_k},$$
$$W_\alpha = \sum_{k=1}^{n} (\overline{Z}_\alpha \Psi^k) \frac{\partial}{\partial z_k}. \tag{11}$$

Notice first that the coefficients $Z_\alpha \Psi^k$ and $\overline{Z}_\alpha \Psi_k$ are holomorphic functions in z.

Combining (11) and (10) we obtain

$$\sum_{k=1}^{n} (\overline{Z}_\alpha \Psi^k) \frac{\partial}{\partial z_k} = \sum_{\beta=1}^{n-1} \mu_{\alpha\beta} \sum_{k=1}^{n} (Z_\beta \Psi^k) \frac{\partial}{\partial z_k},$$

from where it follows that

$$\overline{Z}_\alpha \Psi^k = \sum_{\beta=1}^{n-1} \mu_{\alpha\beta} (Z_\beta \Psi^k), \quad \alpha = 1, \dots, n-1, \quad k = 1, \dots, n. \tag{12}$$

Let us introduce the notations $a_{k\alpha} = \overline{Z}_\alpha \Psi^k$ and $b_{k\alpha} = Z_\alpha \Psi^k$. Since μ is a symmetric matrix (*cf.* [15]) relation (12) can be written as

$$a_{k\alpha} = \sum_{\beta=1}^{n-1} b_{k\beta} \mu_{\beta\alpha}. \tag{13}$$

Introducing the matrices $\mathbf{A} = (a_{k\alpha})$ and $\mathbf{B} = (b_{k\alpha})$, $k = 1, \dots, n$, $\alpha = 1, \dots, n-1$, one can write (13) in matrix form

$$\mathbf{A} = \mathbf{B} \cdot \mu, \tag{14}$$

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where $\mathbf{A}, \mathbf{B}\colon \Delta \to \mathcal{M}_{n,n-1}(\mathbf{C})$ are holomorphic matrix-valued functions.

Since the vectors $V_1(0), \dots, V_{n-1}(0)$ that are columns of $\mathbf{B}(0)$ are independent, there exists an $(n-1) \times (n-1)$-minor $\mathbf{b}(0)$ of $\mathbf{B}(0)$ such that $\det \mathbf{b}(0) \neq 0$. Consider the holomorphic fuction $b\colon \Delta \to \mathbf{C}$, $b(z) = \det \mathbf{b}(z)$. Denote by $\mathcal{P}$ the discrete set of zeros of b in Δ. Consider the equation $\mathbf{a} = \mathbf{b} \cdot \mu$ where $\mathbf{a}$ is the $(n-1) \times (n-1)-$minor of $\mathbf{A}$ satisfying the above relation. As $\mathbf{b}$ is invertible on $\Delta \setminus \mathcal{P}$ we have that

$$\mu(z) = \mathbf{b}^{-1}(z)\mathbf{a}(z), \quad z \in \Delta \setminus \mathcal{P},$$

and so μ is holomorphic on $\Delta \setminus \mathcal{P}$. On the other hand $\|\mu\|_L < 1$, thus each entry $\mu_{\alpha\beta}$ is bounded. Since isolated singularities are removable for bounded holomorphic functions it follows that μ is holomorphic on Δ.

Consequently the function $\mu\colon \Delta \to \mathcal{M}_{n-1}(\mathbf{C})$ is holomorphic with $\|\mu(z)\|_L < 1$ and $\mu(0) = 0$. Moreover, using (11) and the regularity of Ψ given in Theorem 1.1 it is clear from the definition of μ that the function

$$\overline{\Delta} \times \partial X(0) \ni (z, x_0) \mapsto \|\mu(z, x_0)\|_L \in \mathbf{R}$$

is continuous. Consequently there exists a non-negative constant $C < 1$ such that $\|\mu(z)\|_L = \|\mu(z, x_0)\|_L \leq C$ for all $z \in \overline{\Delta}$, $x_0 \in \partial X(0)$. To finish the proof of (d) we need to show that $\|\mu(z)\|_L \leq C|z|$.

To show this estimate it is better to use the first definition of μ. Let us recall that the Leviform L is fixed (independent of z) as we have considered it at the fixed point $x_0 \in \partial X(0)$. It is an easy exercise in linear algebra to see that there exists a diagonal matrix D and a unitary matrix U such that $\|\mu\|_L = \|\tilde{\mu}\|$, where $\tilde{\mu} = V\mu V^{-1}$, $V = DU$ and $\|\cdot\|$ stands for the usual operator norm of matrices. Then $\tilde{\mu}\colon \Delta \to \mathcal{M}_{n-1}(\mathbf{C})$ is holomorphic, $\tilde{\mu}(0) = 0$ and $\|\tilde{\mu}\| < C$ for all $z \in \Delta$. All we need to show is $\|\tilde{\mu}(z)\| \leq C|z|$.

To see this choose a vector $h \in \mathbf{C}^{n-1}$, $\|h\| = 1$. Consider the vector-valued holomorphic function $\phi_h\colon \Delta \to \mathbf{C}^{n-1}$ given by

$$\phi_h(z) = \begin{cases} \frac{1}{z}\tilde{\mu}(z)h, & z \in \overline{\Delta} \setminus \{0\}, \\ \tilde{\mu}'(0)h, & z = 0. \end{cases}$$

By the maximum priciple $\|\phi_h(z)\| \leq C$ which gives $\|\tilde{\mu}(z)h\| \leq C|z|$. Taking supremum over $h \in \mathbf{C}^{n-1}$, $\|h\| = 1$, the estimate $\|\tilde{\mu}(z)\| \leq C|z|$ follows. This together with (7) yields

$$K(z) \leq \frac{1 + C|z|}{1 - C|z|}$$

and the proof is finished. ■

Remark 2.1 A Beltrami differential similar to the one in the above proof was used by Lempert [21] to study the embeddability problem of 3-dimensional CR manifolds. The relation between these two Beltrami differentials was discussed in [4].

Proof of Theorem 1.1 We will follow an idea of L. Lempert to create holomorphic discs near the fixed disc $f = f_{x_0}$ which satisfy (2). This is done by the implicit function theorem using a setup similar to the one in [18]. Our discs will foliate smoothly a neighborhood of f in S proving the smoothness of S. For the convenience of the reader we include a detailed proof even if this results in repeating the reasoning of [18], [20]. Let us start with some preparations.

The first step is to prove that $\langle \tilde{f}, f\rangle \in C^{1,\alpha-0}(\overline{\Delta})$. By Theorem A, $\tilde{f}$ is in $C^{\frac{1}{2}}(\overline{\Delta})$ and f is in $C^{1,\alpha-0}(\overline{\Delta})$. Recall that

$$\tilde{f}(\zeta) = p(\zeta) r_z\big(\zeta, f(\zeta)\big), \quad \zeta \in \partial\Delta,$$

for some $p \in C^{\frac{1}{2}}(\partial\Delta)$, $p\colon \partial\Delta \to \mathbf{R}_+$, and observe that $\operatorname{Re}\langle \tilde{f}, f\rangle > 0$ by Theorem A. This implies that

$$\arg\langle r_z\big(\zeta, f(\zeta)\big), f(\zeta)\rangle = \arg\langle \tilde{f}(\zeta), f(\zeta)\rangle, \quad \zeta \in \partial\Delta,$$

and thus $\arg\langle \tilde{f}, f\rangle \in C^{1,\alpha-0}(\partial\Delta)$ by our smoothness assumption on r. Here we use the fact that the composition of f and r_z is a $C^{1,\alpha-0}$-regular map. (This follows *e.g.* from Lemma 11.2 of [14]; or the reader can easily check it by an argument similar to the one in the proof of Claim 4 below.)

By taking the harmonic extension to Δ of $\operatorname{Im}\log\langle \tilde{f}, f\rangle = \arg\langle \tilde{f}, f\rangle$ we obtain that $\operatorname{Im}\log\langle \tilde{f}, f\rangle \in C^{1,\alpha-0}(\overline{\Delta})$. By Privalov's theorem we obtain $\langle \tilde{f}, f\rangle \in C^{1,\alpha-0}(\overline{\Delta})$.

According to Lemma 3.2 in [18] there is a function $\sigma\colon \overline{\Delta} \to \mathbf{C}\setminus\{0\}$, holomorphic in Δ and of class $C^{\frac{1}{2}}(\overline{\Delta})$ such that $\sigma\tilde{f} \in C^{1,\alpha-0}(\overline{\Delta})$. Write

$$\sigma = \langle \sigma\tilde{f}, f\rangle \frac{1}{\langle \tilde{f}, f\rangle}.$$

Using that f, $\sigma\tilde{f}$ and $\langle \tilde{f}, f\rangle$ are $C^{1,\alpha-0}$-regular it follows that $\sigma \in C^{1,\alpha-0}(\overline{\Delta})$. Similarly, we can write $\tilde{f} = \frac{1}{\sigma}(\sigma\tilde{f})$ to conclude $\tilde{f} \in C^{1,\alpha-0}(\overline{\Delta})$.

Since $\tilde{f}$ is nonvanishing there is no loss of generality to assume that its first two coordinates $\tilde{f}_1$ and $\tilde{f}_2$ do not vanish simultanously. As $\sigma\tilde{f} \in C^{1,\alpha-0}(\overline{\Delta})$ there exist holomorphic functions $g_1, g_2 \in C^{1,\alpha-0}(\overline{\Delta})$ such that $\tilde{f}_1 g_1 + \tilde{f}_2 g_2 = \frac{1}{\sigma}$. We define now the holomorphic matrix

$$H_f = \begin{pmatrix} f_1 & -\tilde{f}_2 & -g_1\tilde{f}_3 & \cdots & -g_1\tilde{f}_n \\ f_2 & -\tilde{f}_1 & -g_2\tilde{f}_3 & \cdots & -g_2\tilde{f}_n \\ f_3 & 0 & \frac{1}{\sigma} & \cdots & 0 \\ \vdots & \vdots & \vdots & & \vdots \\ f_n & 0 & 0 & \cdots & \frac{1}{\sigma} \end{pmatrix}. \tag{15}$$

Clearly H_f is of class $C^{1,\alpha-0}(\overline{\Delta})$ and we claim that $\det H_f \neq 0$. Indeed, the last $n-1$ columns are independent among themselves and they are orthogonal to $\tilde{f}$ whereas the first column is not in view of $\operatorname{Re}\langle \tilde{f}, f\rangle > 0$ on $\overline{\Delta}$.

Let us introduce some notations. Choose a number s with $0 < 2s < \alpha$ and denote by X_n the space of $\mathbf{C}^n$-valued functions in $C^s(\partial\Delta)$. Let Y_n ($\overline{Y}_n$) be the subspace of X_n

consisting of functions having holomorphic (antiholomorphic) extension to Δ, and let $\overline{Y}_n^0 = \{f \in \overline{Y}_n : f(0) = 0\}$. Furthermore we denote by $\pi \colon X_n \to \overline{Y}_n^0$ the projection

$$\pi\Big(\sum_{-\infty}^{\infty} a_k \zeta^k\Big) = \sum_{-\infty}^{-1} a_k \zeta^k.$$

Finally, by Q we denote the set of functions $g \colon \partial\Delta \to \mathbf{R}$ of class C^s.

Fix $x_0 \in \partial X(0)$ and let $(f_0, p_0) \in Y_n \times Q$ be as in statement (a) of Theorem A. Without loss of generality assume that $x_0 = (1, 0, \dots, 0) \in \mathbf{C}^n$ and consider the real $(2n-1)$-dimensional parameter space

$$\mathcal{T} = \{\tau \in \mathbf{C}^n : \tau = x_0 + (i\tau_0, \tau_1 + i\tau_2, \dots, \tau_{2n-3} + i\tau_{2n-2}), \tau_i \in \mathbf{R}, |\tau_i| \le \epsilon\}.$$

We are going to look for a triple $(f_\tau, p_\tau, \lambda_\tau) \in Y_n \times Q \times \mathbf{R}$ such that $(f_{x_0}, p_{x_0}, \lambda_{x_0}) = (f_0, p_0, 1)$ and $(f_\tau, p_\tau, \lambda_\tau)$ solves for each $\tau \in \mathcal{T}$ the system

$$\begin{aligned} r\big(\zeta, f_\tau(\zeta)\big) &= 0, \\ \pi\Big(p_\tau(\zeta) r_z\big(\zeta, f_\tau(\zeta)\big)\Big) &= 0, \\ f_\tau(0) - \lambda_\tau \tau &= 0. \end{aligned} \tag{16}$$

Statement (c) of Theorem A guarantees that the solutions of the first two equations of (16) are necessarily in S. The last equation realizes the local parametrization of $\partial X(0)$ by τ. Notice that the middle equation of (16) is equivalent to

$$\pi\Big(p_\tau(\zeta) H^t(\zeta) r_z\big(\zeta, f_\tau(\zeta)\big)\Big) = 0, \tag{17}$$

where H is the matrix from (15) corresponding to f_0. The advantage of multiplying by $H^t(\zeta)$ is that the first component $\langle \tilde{f}, f\rangle$ of the new column vector is nonvanishing when $f_\tau = f_0$. This implies that the same holds for f near f_0 and condition (17) is equivalent to requiring that the quotient of an arbitrary component of $H^t(r_z \circ f_\tau)$ and its first component have a holomorphic extension to Δ where $(r_z \circ f_\tau)(\zeta) = r_z\big(\zeta, f_\tau(\zeta)\big)$. Let us denote by $(z)_1$ the first component of a vector $z \in \mathbf{C}^n$ and by $[z]$ the vector in $\mathbf{C}^{n-1}$ formed by the last $n-1$ components.

We are now to set up the implicit function theorem in question. Consider the mapping

$$\begin{aligned} &\Phi \colon \mathcal{T} \times Y_n \times \mathbf{R} \to Q \times \overline{Y}_{n-1}^0 \times \mathbf{C}^n \\ &\Phi(\tau, f, \lambda) = \left(r \circ f, \pi \frac{[H^t(r_z \circ f)]}{\big(H^t(r_z \circ f)\big)_1}, f(0) - \lambda\tau \right). \end{aligned} \tag{18}$$

More exactly Φ is defined in a neighborhood of $(\tau_0 = x_0, f_0 = f_{x_0}, 1) \subseteq \mathcal{T} \times Y_n \times \mathbf{R}$.

It is easy to see that there exists an implicit function

$$\begin{aligned} \psi \colon \mathcal{T} &\to Y_n \times \mathbf{R} \\ \tau &\mapsto (f_\tau, \lambda_\tau), \end{aligned} \tag{19}$$

such that

$$\Phi\big(\tau, \psi(\tau)\big) = \Phi(\tau, f_\tau, \lambda_\tau) = (0, 0, 0). \tag{20}$$

Namely, as $X(0)$ is strictly convex (Theorem A (b)(ii)) and $0 \in \operatorname{int} X(0)$ we clearly find for $\tau \in \mathcal{T}$ a value λ_τ such that $x(\tau) = \lambda_\tau \tau \in \partial X(0)$, and f_τ is the mapping $f_\tau = f_{x(\tau)}$ given by Theorem A (a). By what we said above it is clear that f_τ and λ_τ satisfy (20).

Recall now from the *a priori* estimate in Lemma 1.6 of [25] that all disks f_{x_0}, $x_0 \in \partial X(0)$, are $C^{\frac{1}{2}}$-bounded by some uniform constant C. Using this fact we can prove the following

Claim 1 The mapping ψ from (19) is continuous at each point $\tau \in \mathcal{T}$.

Proof The continuity of $\tau \to \lambda_\tau$ is clear by the strict convexity of $\partial X(0)$. To prove the continuity of $\tau \to f_\tau$, assume by contradiction there were a sequence $\tau_i \to \tau$ such that $f_i = f_{\tau_i}$ did not converge to $f = f_\tau$. All f_i lie in the bounded set $\mathcal{B}(C) = \{g \in C^{\frac{1}{2}} : \|g\|_{\frac{1}{2}} \le C\}$. As the embedding $\iota\colon C^{\frac{1}{2}} \hookrightarrow C^s$ is compact (see *e.g.* [1, Theorem 1.31]) the set $\iota\big(\mathcal{B}(C)\big)$ is precompact in C^s and hence there exists a subsequence of f_i (again denoted by f_i) such that $f_i \to g$ in C^s for some $g \in C^s$ with $g \neq f$. On the other hand, we clearly have that $g(0) = f(0) \in \partial X(0)$, and g satisfies the Riemann-Hilbert problem (2) since all f_i do. Hence $g = f$ by Theorem A (a), a contradiction. ■

We are going to show now that the regularity of ψ can considerably be increased. We shall use an argument similar to the one in [20]. For this purpose let us introduce the set $Z_n = \{f \in Y_n \cap C^{1,\alpha}(\partial\Delta, \mathbf{C}^n) : \|f\|_{\frac{1}{2}} \le C\}$ endowed with the C^s-norm of Y_n. The idea is now to restrict the mapping Φ from (18) to $\mathcal{T} \times Z_n \times \mathbf{R}$ and apply the following proposition proved in [20].

Proposition B *Let X, Y, A be Banach spaces, $Z \subseteq X$ a convex subset, and $\Phi\colon Z \times Y \to A$ be a $C^{k,\beta}$ mapping ($k \ge 1$, $0 < \beta < 1$). Assume that for a point $(z_0, y_0) \in Z \times Y$ we have $\Phi(z_0, y_0) = 0$ and for a suitable choice of $\partial\Phi/\partial z$*

$$\frac{\partial\Phi}{\partial z}(x_0, y_0)\colon X \to A$$

is an isomorphism. If ψ is a continuous mapping of some neighborhood U of y_0 in Y into Z with $\psi(y_0) = z_0$ and $\Phi\big(\psi(y), y\big) = 0$, $y \in U$, then Ψ is of class $C^{k,\beta}$ near y_0.

In this proposition we used the following notion of $C^{k,\beta}$-regularity. A mapping $R\colon Z \to A$ is of class C^β ($0 < \beta < 1$) if $\|R(z_1) - R(z_2)\|_A \le \text{const.}\, \|z_1 - z_2\|_X$ for $z_1, z_2 \in Z$. R is differentiable at $z \in Z$ if there is a continuous linear mapping $dR(z)\colon X \to A$ such that $\|R(z_1) - R(z) - dR(z)(z_1 - z)\|_A = o(\|z_1 - z\|_X)$ as $z_1 \to z$, $z_1 \in Z$. (Clearly, dR need not be unique.) The classes $C^{k,\beta}$ are then defined in the usual recursive way.

We apply Proposition B to our mappings Φ and ψ from (18) and (19). The continuity of ψ was already proven in Claim 1 above. Next, we are going to check the other conditions of Proposition B.

We start with

Claim 2 The partial derivative

$$L = \frac{\partial \Phi}{\partial(f,\lambda)}\Big|_{(x_0,f_0,1)} : Y_n \times \mathbf{R} \to Q \times \overline{Y}^0_{n-1} \times \mathbf{C}^n$$

is invertible.

Proof The linear mapping L applied to $(f,\lambda) \in Y_n \times \mathbf{R}$ is

$$\begin{aligned} L(f,\lambda) &= \frac{d}{dt}\Big|_{t=0} \Phi(x_0, f_0 + tf, 1 + t\lambda) \\ &= \left(2\operatorname{Re}\langle r_z \circ f_0, f\rangle, \pi\frac{[H^t(r_{zz}\circ f_0)f + H^t(r_{z\bar{z}}\circ f_0)\overline{f}]}{\big(H^t(r_z \circ f_0)\big)_1}, f(0) - \lambda x_0\right). \end{aligned} \tag{21}$$

The proof of the invertibility of L is quite similar to the considerations in Section 4 in [18]. We omit the details. ■

The proof of the regularity of Φ is a bit more technical. First, we need

Claim 3 Let $\rho \colon \mathbf{C}^n \to \mathbf{C}$ be function of class $C^{k,\alpha}$. The mapping $R \colon Z_n \to X_1$ defined by $R(g)(\zeta) = \rho\big(g(\zeta)\big)$ is of class $C^{k,\alpha-2s}$.

The proof is a slight modification of the proof of Proposition 1 in [20] that is easily established. We need Claim 3 to prove

Claim 4 Let $k \geq 1$ and $\rho \in C^{1,\alpha}\big(\partial\Delta, C^{k,\alpha}(\mathbf{C}^n,\mathbf{C})\big)$. The mapping $R \colon Z_n \to C^s(\partial\Delta,\mathbf{C})$ given by

$$R(f)(\zeta) = \rho\big(\zeta, f(\zeta)\big), \quad \zeta \in \partial\Delta,$$

is of class $C^{k,\alpha-2s}$.

Proof We first show that

$$\rho \in C^{1,\alpha}\big(\partial\Delta, C^{k,\alpha}(\mathbf{C}^n,\mathbf{C})\big) \quad \text{implies } \rho \in C^{1,\alpha}(\partial\Delta \times \mathbf{C}^n, \mathbf{C}). \tag{22}$$

To see this, we prove first that from $\sigma \in C^\alpha\big(\partial\Delta, C^\alpha(\mathbf{C}^n,\mathbf{C})\big)$ it follows $\sigma \in C^\alpha(\partial\Delta \times \mathbf{C}^n, \mathbf{C})$. Let $\sigma \in C^\alpha\big(\partial\Delta, C^\alpha(\mathbf{C}^n,\mathbf{C})\big)$. As $\zeta \to \sigma(\zeta,\cdot)$ is of class C^α we have for $\zeta_1,\zeta_2 \in \partial\Delta$, $z \in \mathbf{C}^n$

$$|\sigma(\zeta_1,z) - \sigma(\zeta_2,z)| \leq \|\sigma(\zeta_1,\cdot) - \sigma(\zeta_2,\cdot)\|_\alpha \leq C_1|\zeta_1 - \zeta_2|^\alpha.$$

By the continuity of the mapping $\zeta \to \|\sigma(\zeta,\cdot)\|_\alpha$ we get that for $\zeta \in \partial\Delta$, $z_1,z_2 \in \mathbf{C}^n$

$$|\sigma(\zeta,z_1) - \sigma(\zeta,z_2)| \leq \|\sigma(\zeta,\cdot)\|_\alpha\, |z_1 - z_2|^\alpha \leq C_2|z_1 - z_2|^\alpha$$

where $C_2 = \sup_{\zeta\in\partial\Delta} \|\sigma(\zeta,\cdot)\|_\alpha$. Combining these estimates we get

$$\begin{aligned} |\sigma(\zeta_1,z_1) - \sigma(\zeta_2,z_2)| &\leq |\sigma(\zeta_1,z_1) - \sigma(\zeta_1,z_2)| + |\sigma(\zeta_1,z_2) - \sigma(\zeta_2,z_2)| \\ &\leq \max(C_1,C_2)(|\zeta_1 - \zeta_2|^\alpha + |z_1 - z_2|^\alpha), \end{aligned}$$

proving $\sigma \in C^\alpha(\partial\Delta \times \mathbf{C}^n, \mathbf{C})$.

Taking partial derivatives of ρ with respect to ζ and $z = (z_1, \dots, z_n)$ the implication (22) is now easily checked.

To prove Claim 4 let us first assume $k = 1$. Consider the mapping

$$\iota \colon Z_n \to \big(C^{1,\alpha}(\partial\Delta, \partial\Delta \times \mathbf{C}^n), \|\cdot\|_s\big),$$
$$\iota(f)(\zeta) = \big(\zeta, f(\zeta)\big)$$

Clearly ι is C^∞. As ρ is in $C^{1,\alpha}(\partial\Delta \times \mathbf{C}^n, \mathbf{C})$ and $R(f) = \rho \circ \iota(f)$ we conclude from Claim 3 that R is of class $C^{1,\alpha-2s}$.

Let now $k = 2$. Observe that the first derivative of R is given by

$$dR \colon Z_n \times C^s(\partial\Delta, \mathbf{R}^{2n}) \to C^s(\partial\Delta, \mathbf{C}),$$
$$dR(f,h)(\zeta) = \sum_{j=1}^{n} h_j(\zeta)\frac{\partial\rho}{\partial z_j}\big(\zeta, f(\zeta)\big) + \bar{h}_j(\zeta)\frac{\partial\rho}{\partial \bar{z}_j}\big(\zeta, f(\zeta)\big)$$

But as $\frac{\partial\rho}{\partial z_j}, \frac{\partial\rho}{\partial \bar{z}_j} \in C^{1,\alpha}\big(\partial\Delta, C^{k-1,\alpha}(\mathbf{C}^n, \mathbf{C})\big)$ and $k = 2$ we still have $\frac{\partial\rho}{\partial z_j}, \frac{\partial\rho}{\partial \bar{z}_j} \in C^{1,\alpha}(\partial\Delta \times \mathbf{C}^n, \mathbf{C})$ and the above argument can be repeated to get that dR is of class $C^{1,\alpha-2s}$. The proof for $k > 2$ is an obvious induction. ■

Using Claim 4 and observing that each component of r_z is in $C^{1,\alpha}\big(\partial\Delta, C^{k,\alpha}(\mathbf{C}^n, \mathbf{C})\big)$ it follows that

$$\Phi|_{\mathcal{T}\times Z_n \times \mathbf{R}} \colon \mathcal{T} \times Z_n \times \mathbf{R} \to Q \times \overline{Y}^0_{n-1} \times \mathbf{C}^n$$

is of class $C^{k,\alpha-2s}$.

We are now in position to apply Proposition B to the mappings $\Phi|_{\mathcal{T}\times Z_n\times\mathbf{R}}$ and $\psi \colon \mathcal{T} \to Z_n \times \mathbf{R}$. We get that ψ is of class $C^{k,\alpha-2s}$ and hence the solutions $f_\tau, \tau \in \mathcal{T}$, of the equation

$$\Phi(\tau, f_\tau, \lambda_\tau) = 0$$

have the property that $\mathrm{gr}(f_\tau)$ foliate S $C^{k,\alpha-2s}$-smoothly in a neighborhood of f_0. The first part of statement (a) of Theorem 1.1 follows now since s can be chosen arbitrarily small.

The smoothness of S does not automatically imply that the fibers $\partial X(z)$ are smooth hypersurfaces. The smoothness of $\partial X(0)$ follows from the $C^{k,\alpha-0}$-smoothness of the implicit function ψ. Namely, the mapping $\tau \to \lambda_\tau \tau$ yields a $C^{k,\alpha-0}$-smooth (local) parametrization of $\partial X(0)$, and it is easily checked that its (real) Jacobian has maximal rank $2n-1$. The proof of the smoothness of $\partial X(z)$ for $z \in \Delta$, $z \neq 0$, is postponed to the very end.

For the proof of the second statement of Theorem 1.1 notice that $\Psi(z,\cdot) \colon \partial X(0) \to \partial X(z)$ is a bijection by Theorem A. It is $C^{k,\alpha-0}$-smooth by the first statement of Theorem 1.1 and the C^∞-smoothness of the evaluation map. We need to check that the map

$$\Psi_*(z, x_0) \colon T_{x_0}\partial X(0) \to \mathbf{C}^n \tag{23}$$

has maximal (real) rank $2n-1$ at any point $(z,x_0) \in \overline{\Delta} \times \partial X(0)$. This is equivalent to the fact that the holomorphic function $V\colon \overline{\Delta} \to \mathbf{C}^n$ given by

$$V(z) = \Psi_*(z,x_0)V_0, \quad z \in \Delta, \tag{24}$$

does not vanish on $\overline{\Delta}$ for any nonzero $V_0 \in T_{x_0}\partial X(0)$. Consider the function $\phi\colon \Delta \times \mathfrak{T} \to \mathbf{C}^n$ given by $\phi(z,\tau) = f_\tau(\zeta)$, where f_τ are coming from the implicit function theorem. It is clear that the nonvanishing of V from (24) is equivalent to

$$W(z) = \phi_*(z,x_0)W_0 \tag{25}$$

does not vanish for any $W_0 \in T_{x_0}\mathfrak{T} = i\mathbf{R} \times \mathbf{C}^{n-1}$, $W_0 \neq 0$. On the other hand recall that

$$\Phi\big(\tau, \phi(z,\tau), \lambda(\tau)\big) = 0.$$

Taking the directional derivative with respect to τ in the above equation at $\tau = x_0$ and in direction W_0 we obtain

$$\frac{\partial \Phi}{\partial(f,\lambda)}\Big|_{(x_0,f_0,1)}(W,\Lambda) + \frac{\partial \Phi}{\partial \tau}\Big|_{(x_0,f_0,1)}(W_0) = 0$$

where $\Lambda = \lambda_*(x_0)W$. From (18) it follows that

$$\frac{\partial \Phi}{\partial \tau}\Big|_{(x_0,f_0,1)}(W_0) = (0,0,-W_0).$$

Using this and (21) we obtain that the function $z \mapsto W(z)$ satisfies the equations

$$\begin{gathered} \operatorname{Re}\langle r_z \circ f_0, W\rangle = 0, \\ \pi\left(\frac{[H^t(r_{zz}\circ f_0)W + H^t(r_{z\bar{z}}\circ f_0)\overline{W}]}{\big(H^t(r_z\circ f_0)\big)_1}\right) = 0, \\ W(0) - \Lambda x_0 = W_0. \end{gathered} \tag{26}$$

On the other hand $W(0) = \phi_*(0,x_0)W_0$ and by $\phi(0,\tau) = f_\tau(0) = \lambda(\tau)\tau$ we have $W(0) = \Lambda x_0 + W_0$. So the last equation is an identity. Introducing $U = H^{-1}W$, (26) becomes

$$\operatorname{Re}(U_1\langle \tilde{f}, f_0\rangle) = 0, \quad A[\overline{U}] + B[U] + \varphi \in Y_{n-1}, \tag{27}$$

where φ is a multiple of some vector-valued function by U_1. By the convexity of the fibres the matrix-valued functions $A, B\colon \partial\Delta \to \mathcal{M}_{n-1}(\mathbf{C})$ satisfy

$$z^t A(\zeta)\bar{z} > |z^t B(\zeta) z| \tag{28}$$

for any $z \in \mathbf{C}^{n-1}$, $\zeta \in \partial\Delta$. This follows the same way as in [18].

Let us assume that $U(\zeta_0) = 0$ for some $\zeta_0 \in \overline{\Delta}$. By the first equation of (27) we obtain $U_1 \equiv 0$ and thus $\varphi \equiv 0$. Consequently $[U]$ is solution of

$$A[\overline{U}] + B[U] \in Y_{n-1}. \tag{29}$$

By Lemma 4.2 of [18] (29) has a unique solution. Since $[U](\zeta_0) = 0$ we conclude that $[U] \equiv 0$. Consequently $U \equiv 0$ which is clearly a contradiction to $W(0) = \Lambda x_0 + W_0 \neq 0$.

Because $\Psi(z,\cdot)\colon \partial X(0) \to \partial X(z)$ is a $C^{k,\alpha-0}$-smooth diffeomorphism and $\partial X(0)$ is $\mathbf{C}^{k,\alpha-0}$-smooth the smoothness of $\partial X(z)$, $z \neq 0$, follows. This finishes the proof of Theorem 1.1. ∎

3 Beltrami Differential and Lempert Invariants

We are going to study next the case when the fibers $X(\zeta), \zeta \in \partial\Delta$, are rotations of a fixed strictly convex, smooth domain $D \in \mathbf{C}^2$, *i.e.*, $X(\zeta) = \frac{1}{\zeta}D$, $\zeta \in \partial\Delta$. Slodkowski observed (*cf.* [25]) that the discs $g\colon \Delta \to \mathbf{C}^n$ whose graphs foliate the boundary of the hull are of the form

$$g(z) = \begin{cases} \frac{f(z)}{z}, & z \neq 0 \\ f'(0), & z = 0, \end{cases}$$

where $f\colon \Delta \to \mathbf{C}^n$ are the Kobayashi extremal discs of the domain D. The 0-fiber $X(0)$ becomes the Kobayashi indicatrix of D:

$$X(0) = I_D = \{f'(0) \mid f\colon \Delta \to D, f \text{ is holomorphic}, f(0) = 0\}.$$

For $v \in \partial I_D$ we denote by f_v the unique extremal disc with $f_v(0) = 0$, $f_v'(0) = v$. I_D is a completely circled smooth domain, and the mapping

$$\Psi := \Psi(1,\cdot)\colon \partial I_D \to \partial D$$
$$v \mapsto f_v(1)$$

is a contact diffeomorphism. This is the restriction to the boundary of the so-called circular representation of D.

Choosing $v \in \partial I_D$ and a nonvanishing section $Y(\zeta)$ of $H^{1,0}\partial I_D|_{\zeta v}$, $\zeta \in \partial\Delta$, the Beltrami differential μ of Ψ along the circle $\zeta v, \zeta \in \partial\Delta$, becomes a function $\mu\colon \partial\Delta \to \mathbf{C}$ defined by

$$\Psi_* Y(\zeta) = Y_1(\zeta) + \bar{\mu}(\zeta)\overline{Y}_1(\zeta), \tag{30}$$

where $Y_1(\zeta)$ is the projection of $\Psi_* Y$ onto $H^{1,0}_{\Psi(\zeta v)}\partial D$. μ depends on the choice of the section Y but $|\mu|$ does not and $\|\mu\|_\infty < 1$.

The aim of this section is to relate μ to the biholomorphic invariants introduced by Lempert [19] that we briefly recall.

For v as above and the corresponding extremal disk f_v there exists a normalizing biholomorphic mapping $\Phi\colon D \to D_0$ such that

$$f_0(z) = (\Phi \circ f_v)(z) = (z, 0), \quad z \in \Delta,$$

and the defining function of D_0 can be written in a neighborhood of $f_0(\overline{\Delta})$ as

$$r_0(z_1, z_2) = -1 + |z_1|^2 + |z_2|^2 + \operatorname{Re} B(z_1) z_2^2 + O(3). \tag{31}$$

Here $O(3)$ denotes a term bounded by a constant times $\operatorname{dist}^3\big((z_1, z_2), f(\partial\Delta)\big)$. The function $\beta\colon \partial\Delta \to \mathbf{C}$, $\beta(\zeta) = \zeta^2 B(\zeta)$, (or rather its antiholomorphic part) is a biholomorphic invariant that we call *Lempert invariant* associated to the extremal disk f_v. The indicatrix together with the Lempert invariants of all extremal disks form a complete set of holomorphic invariants (see [19]). The meaning of β is to measure the deviation of D from I_D. The convexity of D implies that the distance of β to H^∞ is less than 1:

$$\operatorname{dist}(\beta, H^\infty) = \inf_{h \in H^\infty} \|h - \beta\|_\infty < 1.$$

We are going to estimate the size of $|\mu|$ by $\operatorname{dist}(\beta, H^\infty)$. This is done via the solutions of the following Riemann-Hilbert problem which we call *Lempert problem*.

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Definition 3.1 We say that $U_a\colon \Delta \to \mathbf{C}$, $U_a \in H^\infty(\Delta) \cap C(\overline{\Delta})$, solves the Lempert problem with $U_a(0) = a \in \mathbf{C}$ and the function $\phi\colon \partial\Delta \to \mathbf{C}$ given by

$$\phi(\zeta) = \overline{U}_a(\zeta) + \beta(\zeta)U_a(\zeta) \tag{32}$$

has a holomorphic extension to Δ.

Since $\operatorname{dist}(\beta, H^\infty) < 1$, for given $a \in \mathbf{C}$ the Lempert problem (32) admits a unique solution U_a that is smooth up to $\partial\Delta$, and if $a \neq 0$ then U_a has no zeros in $\bar{\Delta}$ (see [18]).

We are now in position to formulate the following

Proposition 3.2 *The size $|\mu|$ of the Beltrami differential is related to the Lempert problem by the formula*

$$|\mu(\zeta)| = \frac{|V_1(\zeta) + iV_2(\zeta)|}{|V_2(\zeta) - iV_2(\zeta)|}, \tag{33}$$

where V_1, V_2 are the solutions $V_1 = U_1$ and $V_2 = U_i$, i.e., V_1, V_2 solve the Lempert problem with $V_1(0) = 1$, $V_2(0) = i$.

From this follows the estimate

Corollary 3.3

a) We have

$$\|\mu\|_\infty \geq \operatorname{dist}(\beta, H^\infty) \tag{34}$$

with equality if and only if $|\mu|$ is constant on $\partial\Delta$.

b) β extends holomorphically to Δ if and only if $\mu \equiv 0$.

Proof of Proposition 3.2 The first step is to calculate the action of Ψ_* on a non-vanishing section of the bundle $H_{\zeta v}\partial I$ over the circle $\zeta v, \zeta \in \partial\Delta$. We do this by considering a variation of extremal disks.

As the size $|\mu|$ of the Beltrami differential is a holomorphic invariant it is no loss of generality to assume that $v = (0, 1)$, $f = f_v$ is in normal form

$$f(z) = (z, 0)$$

and the defining function of D is locally given by (31). Consider a curve $v(t)$ in ∂I_D through v tangent to $W_0 \in H_v\partial I_D$:

$$v(t) = v + tW_0 + O(t^2)$$

Let $f^t = f_{v(t)}$ be the corresponding family of extremal disks. Write

$$f^t = f + tF + O(t^2), \quad F = (F_1, F_2),$$

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where $F\colon \Delta \to \mathbf{C}^2$ is holomorphic and $F(0) = 0$. Extremality of f^t and contact property of Ψ imply that $W_0 = (0, W_2)$ and $F_1 \equiv 0$ (see [19]). Geometrically this means that the complex lines $H_{\zeta\nu}\partial I_D$, $H_{f(\zeta)}\partial D$ are simply the vertical lines $\{0\} \times \mathbf{C} \subseteq \mathbf{C}^2$.

It also follows that the holomorphic function $V(z) = F_2(z)/z$ has the property that

$$\phi(\zeta) = \overline{V}(\zeta) + \beta(\zeta)V(\zeta), \quad \zeta \in \partial\Delta,$$

has a holomorphic extension to Δ. Observe that $V(0) = F'(0) = W_2$. In other words, V solves the Lempert problem (32) with $V(0) = W_2$.

With these preparations we can now calculate

$$\begin{aligned}(\Psi_*)_\nu W_0 = (\Psi_*)_\nu\big((0, W_2)\big) &= \frac{d}{dt}\Big|_{t=0} \Psi\big(\nu + t(0, W) + O(t^2)\big)\\ &= \frac{d}{dt}\Big|_{t=0} f^t(1) = F(1) = \big(0, V(1)\big).\end{aligned}$$

Consider the section $W(\zeta) = \zeta W_0$ along the circle $\zeta\nu, \zeta \in \partial\Delta$. Using the fact that $\Psi(\zeta\nu) = f_{\zeta\nu}(1) = f_\nu(\zeta)$ one obtains as above:

$$(\Psi_*)_{\zeta\nu} W(\zeta) = \big(0, \zeta V(\zeta)\big). \tag{35}$$

The next step is to calculate the Beltrami differential of Ψ at the points $\zeta\nu$ choosing $Y = Y(\zeta)$ in (30) to be

$$Y = W - \mathbf{i} \otimes iW \in H^{1,0}_{\zeta\nu}\partial I_D.$$

Here we write $\mathbf{i}$ for the imaginary unit coming from the complexification of $H\partial I_D$ and $H\partial D$ to distinguish it from the complex structure tensor of the horizontal bundles $H\partial I$ and $H\partial D$. Now we have

$$\Psi_*(W - \mathbf{i} \otimes iW) = U_1 - \mathbf{i} \otimes U_2 \in \mathbf{C} \otimes H\partial D$$

where $U_1(\zeta), U_2(\zeta) \in H_{\Psi(\zeta\nu)}\partial D$. If we make the special choice $W(\zeta) = (0, \zeta)$, *i.e.*, $W_0 = (0, 1)$, we get by (35):

$$\begin{aligned} U_1(\zeta) &= (\Psi_*)_{\zeta\nu} W(\zeta) = \big(0, \zeta V_1(\zeta)\big),\\ U_2(\zeta) &= (\Psi_*)_{\zeta\nu} iW(\zeta) = \big(0, \zeta V_2(\zeta)\big),\end{aligned} \tag{36}$$

where V_1, V_2 are the solutions of the Lempert problem (32) with initial values $V_1(0) = 1$ and $V_2(0) = i$.

On the other hand, by the definition of the Beltrami differential $\mu(\zeta) = \mu\big(\zeta\nu, Y(\zeta)\big)$ we have

$$\Psi_*(W - \mathbf{i} \otimes iW) = Y_1 + \bar{\mu}\overline{Y}_1, \tag{37}$$

where $Y_1(\zeta) \in H^{1,0}_{\Psi(\zeta\nu)}\partial D$, *i.e.*, it is of the form $Y_1 = X - \mathbf{i} \otimes iX$ for $X(\zeta) \in H_{\Psi(\zeta\nu)}\partial D$. If we write μ as $\mu = \mu_1 + \mathbf{i}\mu_2$ then (37) reads

$$\Psi_*(W - \mathbf{i} \otimes iW) = (X + \mu_1 X + \mu_2 iX) - \mathbf{i} \otimes (iX - \mu_1 iX + \mu_2 X),$$

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and hence

$$U_1 = \Psi_* W = X + \mu_1 X + \mu_2 iX, \qquad U_2 = \Psi_* iW = i(X - \mu_1 X - \mu_2 iX). \tag{38}$$

Recall now that $H_{\Psi(\zeta v)}\partial D = \{0\} \times \mathbf{C}$ and so $X = (0, X_2)$. Using (36) and (38) we obtain

$$\zeta V_1(\zeta) = X_2(\zeta) + \big(\mu_1(\zeta) + i\mu_2(\zeta)\big)X_2(\zeta) = X_2(\zeta) + \mu(\zeta)X_2(\zeta),$$
$$\zeta V_2(\zeta) = i\Big(X_2(\zeta) - \big(\mu_1(\zeta) + i\mu_2(\zeta)\big)X_2(\zeta)\Big) = i\big(X_2(\zeta) - \mu(\zeta)X_2(\zeta)\big),$$

which imply that

$$\mu(\zeta)\big(V_1(\zeta) - iV_2(\zeta)\big) = V_1(\zeta) + iV_2(\zeta). \tag{39}$$

From this and the fact that V_1, V_2 solve (32) it follows that $U = V_1 - iV_2$ has the property that the function

$$\overline{U}(\zeta) + \beta(\zeta)\mu(\zeta)U(\zeta), \quad \zeta \in \partial\Delta,$$

admits a holomorphic extension to Δ. In other words, U solves the above Lempert problem, where $\beta\mu$ appears instead of β.

Recall that $\operatorname{dist}(\beta, H^\infty) < 1$. By the holomorphicity of μ and $\|\mu\|_\infty < 1$ we see that $\operatorname{dist}(\beta\mu, H^\infty) < 1$. Since $U(0) = V_1(0) - iV_2(0) = 2$ we conclude (see the remark after Definition 3.1) that U has no zeros on $\overline{\Delta}$. Hence from (39)

$$\mu(\zeta) = \frac{V_1(\zeta) + iV_2(\zeta)}{V_1(\zeta) - iV_2(\zeta)}. \tag{40}$$

As the absolute value of μ does not depend on the choice of the section W this proves Propostition 3.2. ■

Proof of Corollary 3.3 To prove the estimate in part a) of Corollary 3.3 consider the functions

$$\phi_1(\zeta) = \overline{V}_1(\zeta) + \beta(\zeta)V_1(\zeta),$$
$$\phi_2(\zeta) = \overline{V}_2(\zeta) + \beta(\zeta)V_2(\zeta),$$

for $\zeta \in \partial\Delta$. They admit a holomorphic extension to Δ, again denoted by ϕ_1, ϕ_2. Dividing the equation

$$\overline{V_1(\zeta) + iV_2(\zeta)} + \beta(\zeta)\big(V_1(\zeta) - iV_2(\zeta)\big) = \phi_1(\zeta) - i\phi_2(\zeta) \tag{41}$$

by $V_1(\zeta) - iV_2(\zeta)$ we get

$$|\mu(\zeta)| = \left|\frac{\overline{V_1(\zeta) + iV_2(\zeta)}}{V_1(\zeta) - iV_2(\zeta)}\right| = \left|\frac{\phi_1(\zeta) - i\phi_2(\zeta)}{V_1(\zeta) - iV_2(\zeta)} - \beta(\zeta)\right|. \tag{42}$$

Because the function $h\colon \Delta \to \mathbf{C}$, $h(\zeta) = \big(\phi_1(\zeta) - i\phi_2(\zeta)\big)/\big(V_1(\zeta) - iV_2(\zeta)\big)$ is in H^∞ and smooth on $\overline{\Delta}$ there exists $\zeta_0 \in \partial\Delta$ such that $|\mu(\zeta_0)| = |h(\zeta_0) - \beta(\zeta_0)| \geq \operatorname{dist}(\beta, H^\infty)$ and (34) follows.

It remains to prove the necessary and sufficient condition for equality in (34). Assume first that $\|\mu\|_\infty = \operatorname{dist}(\beta, H^\infty)$. By (42) it follows that h is the nearest H^∞-function to β. Then it follows (see *e.g.* [13, p. 135]) that $|h - \beta|$ and hence also $|\mu|$ is constant on $\partial\Delta$.

To prove the other direction assume that $|\mu| = c > 0$ on $\partial\Delta$. We have to show that $\operatorname{dist}(\beta, H^\infty) = c$. Assume by contradiction that we had $\operatorname{dist}(\beta, H^\infty) < c$. Then there exists a function $h_1 \in H^\infty$ such that $\|\beta - h_1\|_\infty < c$. Observe now that V_1 and V_2 are also solutions to the Lempert problem with $\beta - h_1$ instead of β. Consequently the function $\phi\colon \partial\Delta \to \mathbf{C}$,

$$\phi = \overline{V_1 + iV_2} + (\beta - h_1)(V_1 - iV_2), \tag{43}$$

has a holomorphic extension to Δ. Using (40) and since $|\mu(\zeta)| = c \neq 0$ for $\zeta \in \partial\Delta$ we can write (43) as

$$\phi = \overline{V_1 + iV_2} + \frac{\beta - h_1}{\mu}(V_1 + iV_2).$$

On the other hand

$$\operatorname{dist}\left(\frac{\beta - h_1}{\mu}, H^\infty\right) \leq \frac{\|\beta - h_1\|_\infty}{\inf_{\zeta\in\partial\Delta} |\mu(\zeta)|} = \frac{1}{c}\|\beta - h_1\|_\infty < 1.$$

Now observe that $U = V_1 + iV_2$ is a solution of the above Lempert problem with $\frac{\beta - h_1}{\mu}$ instead of β. But $U(0) = 0$ and thus $U \equiv 0$ which is a contradiction to $|\mu(\zeta)| = c \neq 0$ by (40). This finishes the proof of part a).

For part b) of Corollary 3.3 notice that if the Lempert invariant $\beta(\zeta)$ has a holomorphic extension to Δ the solutions V_1, V_2 of (32) are constant, $V_1 \equiv 1$ and $V_2 \equiv i$, and hence $\mu = 0$ on $\partial\Delta$ by (40). The converse is clear from part a). ∎

Example 1 Let $t < 1$ be a positive number. We construct a strictly convex, smooth domain $D \in \mathbf{C}^2$ for which

$$\sup_{\nu \in I_D} \|\mu_\nu\|_\infty \geq t. \tag{44}$$

Let $r\colon \mathbf{C}^2 \to \mathbf{R}$ be given by

$$r(z_1, z_2) = -1 + |z_1|^2 + |z_2|^2 + t \operatorname{Re} \bar{z}_1^3 z_2^2 + C|z_2|^4.$$

Consider the holomorphic disk $f\colon \Delta \to \mathbf{C}^2$, $f(z) = (z, 0)$. It is not hard to check that r is strictly convex in a neighborhood of $f(\overline{\Delta})$. Choose a positive number η so small and the constant C so large that r is strictly convex in the interior of the bidisk

$$B_\eta = \{z \in \mathbf{C}^2 : |z_1|^2 \leq 1 + \eta, |z_2|^2 \leq \eta\}$$

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and $r|_{\partial B_\eta} > 0$. It follows that $D = \{z \in B_\eta : r(z) < 1\}$ is a smooth strictly convex domain. It is straightforward to check that f is a stationary map for D and hence extremal. (See [17] for the definition of stationarity.) The Lempert invariant of f is $\beta(\zeta) = t\bar{\zeta}$. One can explicitely solve the corresponding Lempert problem and calculate that $|\mu_\nu(\zeta)| = t$ for $\nu = (1, 0)$ and (44) follows.

Example 2 For the Lempert invariant $\beta(\zeta) = t(\bar{\zeta} + \bar{\zeta}^2)$, $0 < t < \frac{1}{2}$, the corresponding Beltrami differential can also be calculated explicitely. In this case $|\mu(\zeta)|$, $\zeta \in \partial\Delta$, is not a constant and so one has the strict inequality

$$\|\mu\|_\infty > \operatorname{dist}(\beta, H^\infty).$$

4 Final Remarks

Remark 4.1 In [23] it was shown that flows of contactomorphisms $\Psi_s \colon \partial D_0 \to \partial D_s$ of boundaries of strictly pseudoconvex smooth domains can always be extended to the inside of the domains as symplectic mappings with respect to the Bergman forms. In particular our maps $\Psi(z, \cdot) \colon \partial X(0) \to \partial X(z)$ can be extended to maps $\tilde{\Psi}(z, \cdot) \colon \overline{X(0)} \to \overline{X(z)}$ that are symplectic inside and such that $\tilde{\Psi}(z, \cdot)|_{\partial X(0)} = \Psi(z, \cdot)$.

Remark 4.2 Example 1 in Section 3 shows that there are configurations X where the constant C from Theorem 1.2 (d) is arbitrarily close to 1. Can C be equal to 1 if we allow $X(\zeta)$ to be not strictly convex for certain ζ (say of zero measure in $\partial\Delta$)? This is *e.g.* the case when X is defined via the smooth function

$$r(\zeta, z_1, z_2) = |z_1|^2 + |\zeta - 1|^2 |z_2|^2 + |z_2|^4 - 1.$$

Most of the results of Theorem A still hold in such degenerate cases. However a difficulty appears from the fact that for $\zeta = 1$ the fiber $X(1)$ is no longer strictly convex, hence the inequality (28) does not hold. Moreover, the fiber $X(1)$ is not even strictly pseudoconvex and so the submanifold M from Remark 1.1 is not totally real. Consequently, our considerations as well as the ones in [14] and [10] do not apply.

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