The infinitary lambda calculus of the infinite eta Böhm trees

P A U L A   S E V E R I and F E R - J A N   D E   V R I E S

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In this paper, we introduce a strong form of eta reduction called *etabang* that we use to construct a confluent and normalising infinitary lambda calculus, of which the normal forms correspond to Barendregt's infinite eta Böhm trees. This new infinitary perspective on the set of infinite eta Böhm trees allows us to prove that the set of infinite eta Böhm trees is a model of the lambda calculus. The model is of interest because it has the same local structure as Scott's D_∞-models, i.e. two finite lambda terms are equal in the infinite eta Böhm model if and only if they have the same interpretation in Scott's D_∞-models.

1. Introduction

In the classical finitary lambda calculus (Barendregt 1984), one can express that the fixed point combinator Y $(= \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)))$ can reduce to terms of the form $\lambda f.f^n((\lambda x.f(xx))(\lambda x.f(xx)))$, for any $n > 0$, but one cannot express that Y has an infinite reduction to the infinite term $\lambda f.f^\omega$, where f^ω is convenient shorthand for the infinite term $f(f(f(\ldots)))$.

In the infinitary lambda calculus, the set Λ of finite λ-terms is extended to explicitly include infinite terms such as $\lambda f.f^\omega$ and the notation allows for finite and infinite reductions. This makes it possible to define the concept of Böhm tree directly in the notational framework of the infinitary lambda calculus, in contrast to Barendregt (1984) where Böhm trees are defined with their own notational machinery.

Infinitary lambda calculus allows an alternative definition of the notion of tree as normal form. Figure 1 summarises the correspondences between the infinitary lambda calculi and the trees which have been studied so far. All these calculi include a notion of ⊥-reduction and they are all proved to be confluent and normalising before except for the one on the last row (Berarducci 1996; Kennaway and de Vries 2003; Kennaway et al. 1995a, 1997; Severi and de Vries 2002; Severi and de Vries 2011). From any infinitary lambda calculus which is confluent and normalising, we can construct a model of the finite lambda calculus by defining the interpretation of a term to be exactly the (infinite) normal form of that term (or equivalently the tree of that term).

The infinitary lambda calculi sketched in the first four rows of Figure 1 are variations of $\lambda^\infty_{\beta\perp} = (\Lambda^\infty_\perp, \longrightarrow_{\beta\perp})$. By changing the ⊥-rule, we obtain different notions of trees. If we take the terms without head normal form (HNF) as meaningless terms, then we obtain an infinitary lambda calculus which is confluent and normalising. The normal form of a term in this calculus correspond to the Böhm tree of this term. The collection of normal forms

	Reduction Rules	Normal forms
β-rule	$\perp$-rule for terms without head normal form	Böhm trees
β-rule	$\perp$-rule for terms without weak head normal form	Lévy-Longo trees
β-rule	$\perp$-rule for terms without top normal form	Berarducci trees
β-rule	$\perp$-rule parametrised by a set of weakly meaningless terms	Parametric trees
β-rule $\quad \eta$-rule	$\perp$-rule for terms without head normal form	η-Böhm trees
β-rule $\quad \eta!$-rule	$\perp$-rule for terms without head normal form	$\infty\eta$-Böhm trees

Fig. 1. Trees as infinite normal forms.

of this calculus forms a model of the lambda beta calculus, better known as Barendregt's Böhm model (Barendregt 1984). Similarly, by reducing terms without weak HNF to $\perp$, we capture the notion of Lévy-Longo tree (Kennaway and de Vries 2003; Kennaway et al. 1995a, 1997) and this gives rise to the model of Lévy-Longo trees. Also by reducing terms without top HNF to $\perp$, we capture the notion of Berarducci tree (Berarducci 1996; Kennaway and de Vries 2003; Kennaway et al. 1995a, 1997) which gives rise to the model of Berarducci trees.

The infinitary lambda calculi $\lambda_{\beta\perp}^\infty$ with a $\perp$-rule parametric on a set of (weakly) meaningless terms encompasses the previous three cases (Kennaway and de Vries 2003; Severi and de Vries 2011). This method to construct models of the lambda beta calculus is quite flexible as there is ample choice for the set of meaningless terms (Severi and de Vries 2005a,b; Severi and de Vries 2011). Because the collection of sets of weakly meaningless terms is uncountable, we get an uncountable class of models which are not continuous (Severi and de Vries 2005a).

The infinitary lambda calculus $\lambda_{\beta\perp\eta}^\infty = (\Lambda_\perp^\infty, \longrightarrow_{\beta\perp\eta})$ sketched in the last but one row incorporates the η-rule (Severi and de Vries 2002). This calculus captures the notion of η-Böhm tree, which can be described as the eta-normal form of a Böhm tree, and gives rise to an extensional model of the lambda calculus that has the same local structure as Coppo, Dezani and Zacchi's filter model D_∞^* (Coppo et al. 1987).

The last row in Figure 1 represents the contribution of this paper. The infinitary lambda calculus $\lambda_{\beta\perp\eta!}^\infty = (\Lambda_\perp^\infty, \longrightarrow_{\beta\perp\eta!})$ is constructed with the $\eta!$-rule, a strengthening of the η-rule. The notion of $\eta!$-reduction is based on the observation that the explicit syntactic characterisation of infinite eta expansions in the definition of infinite eta Böhm trees in Barendregt (1984) can be succinctly redefined as *strongly converging* eta-expansions in the terminology of infinitary rewriting. The power of $\eta!$-reduction is such that it reduces the Böhm tree of J to I, see Figure 2. The main complication of this paper will be to prove that $\lambda_{\beta\perp\eta!}^\infty$ is confluent and normalising. As direct consequences of this result, we will first obtain an alternative definition of the notion of $\infty\eta$-Böhm tree of a lambda term as its normal form in $\lambda_{\beta\perp\eta!}^\infty$ which is more compact than the one in Bakel et al. (2002); Barbanera et al. (1998); Barendregt (1984). Second, we can show that the set of $\infty\eta$-Böhm trees is an extensional model of the finite lambda calculus. The model of

FINITE λ-TERM	BÖHM TREE	η-BÖHM TREE	$\infty\eta$-BÖHM TREE
$\mathsf{I} = \lambda x.x$	$\mathsf{BT}(\mathsf{I}) = \mathsf{I}$	$\eta\mathsf{BT}(\mathsf{I}) = \mathsf{I}$	$\infty\eta\mathsf{BT}(\mathsf{I}) = \mathsf{I}$
$1 = \lambda xy.xy$	$\mathsf{BT}(1) = 1$	$\eta\mathsf{BT}(1) = \mathsf{I}$	$\infty\eta\mathsf{BT}(1) = \mathsf{I}$
$\mathsf{J} = \mathsf{Y}(\lambda fxy.x(fy))$	$\mathsf{BT}(\mathsf{J}) = \lambda xy_0.x$	$\eta\mathsf{BT}(\mathsf{J}) = \lambda xy_0.x$	$\infty\eta\mathsf{BT}(\mathsf{J}) = \mathsf{I}$

$$
\begin{array}{ccc}
& \lambda xy_0.x & \lambda xy_0.x \\
& | & | \\
& \lambda y_1.y_0 & \lambda y_1.y_0 \\
& | & | \\
& \lambda y_2.y_1 & \lambda y_2.y_1 \\
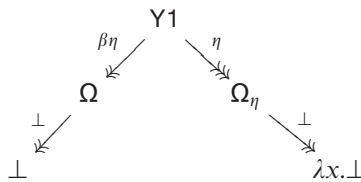& \vdots & \vdots
\end{array}
$$

Fig. 2. Difference between the notions of Böhm, η-Böhm and $\infty\eta$-Böhm trees.

$\infty\eta$-Böhm trees is of interest because it has the same local structure as Scott's D_∞-models, i.e. two finite lambda terms have the same normal form in $\lambda^\infty_{\beta\perp\eta!}$ if and only if they are equal in D_∞ Hyland (1975) and Wadsworth (1976).

It may appear at first sight that extending an infinitary lambda calculus with η or $\eta!$ should not be complicated. However, the two lambda calculi of Lévy-Longo and Berarducci trees do not seem to accept any variations on the $\perp$-rule without losing confluence. There is a critical pair between the η-rule ($\eta!$-rule) and the $\perp$-rule for terms without weak HNF:

$$
\begin{array}{ccc}
& \lambda x.\Omega x & \\
{}^\perp\swarrow & & \searrow^\eta \\
\lambda x.\perp & & \perp
\end{array}
$$
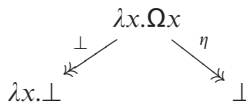
The $\perp$-step follows from the fact that the term Ωx has no weak HNF. This pair can be completed only if $\lambda x.\perp \longrightarrow_\perp \perp$ which is true only for the $\perp$-rule that equates terms without HNF. For a counterexample of confluence for $\beta\perp\eta$ and $\beta\perp\eta!$ where the $\perp$-rule equates terms without top normal form, we use the term $\Omega_\eta = \lambda x_0.(\lambda x_1.(\ldots)x_1)x_0$. Similar to Ω which β-reduces to itself in only one step, this term η-reduces to itself in only one step. The term Ω_η can be obtained as the fixed point of $1 = \lambda xy.xy$. The body of the outermost abstraction in Ω_η is root active (it always reduces to a β-redex) and hence $\Omega_\eta \longrightarrow_\perp \lambda x.\perp$. The span

$$
\begin{array}{ccccc}
& & \mathsf{Y}1 & & \\
& {}^{\beta\eta}\swarrow & & \searrow^\eta & \\
& \Omega & & \Omega_\eta & \\
{}^\perp\swarrow & & & & \searrow^\perp \\
\perp & & & & \lambda x.\perp
\end{array}
$$

can only be joined if $\lambda x.\perp \longrightarrow_\perp \perp$.

1.1. *Outline of this paper*

Section 2 recalls some notions of infinitary lambda calculus and introduces the definition of $\eta!$-reduction. Section 3 studies properties of mainly $\longrightarrow_{\eta!}$ and $\longrightarrow_{\eta^{-1}}$ on their own. Section 4 proves two strip lemmas for $\eta!$ and β. Section 5 proves that outermost $\perp$-reduction

commutes with η!. Section 6 proves confluence and normalisation of the infinitary lambda calculus $\lambda_{\beta\perp\eta!}^{\infty}$. Section 7 explains in detail the connection between the infinite eta Böhm trees and the normal forms in $\lambda_{\beta\perp\eta!}^{\infty}$. Section 8 shows that the set of the normal forms of $\lambda_{\beta\perp\eta!}^{\infty}$ is an extensional model of the finite lambda calculus.

2. Infinitary lambda calculus

2.1. *The set $\Lambda_{\perp}^{\infty}$ of finite and infinite terms*

Infinitary lambda calculus provides a single framework for finite lambda terms and infinite terms. Infinite extensions of finite lambda calculus were introduced around 1994 following similar developments in first order term rewriting initiated by Dershowitz and Kaplan (Berarducci 1996; Kennaway et al. 1997). As starting point for this paper, we are interested in one particular extension $\lambda_{\beta\perp}^{\infty}$ of the finite lambda calculus defined in Kennaway et al. (1997), namely the extension in which the normal forms correspond to the Böhm trees of Barendregt (1984). The set $\Lambda_{\perp}^{\infty}$ of finite and infinite terms of $\lambda_{\beta\perp}^{\infty}$ can conveniently be defined as metric completion of the finite terms for a suitable chosen metric. In spirit, this construction goes back at least to Arnold and Nivat (1980). The metric context will also be used to define transfinite converging reductions.

We will now briefly recall this construction from Kennaway et al. (1997). Throughout, we assume familiarity with basic notions and notations from Barendregt (1984).

Definition 2.1 (Set $\Lambda_{\perp}$ of finite lambda terms with $\perp$). Let $\Lambda_{\perp}$ be the set of finite λ-terms given by the inductive grammar:

$$M ::= \perp \mid x \mid (\lambda x M) \mid (MM),$$

where x is a variable from some fixed countable set of variables $\mathcal{V}$.

We follow the usual conventions on syntax. Terms and variables will respectively be written with (super- and subscripted) letters M, N and x, y, z. Terms of the form $(M_1 M_2)$ and $(\lambda x M)$ will respectively be called applications and abstractions. A context $C[\ ]$ is a term with a hole in it, and $C[M]$ denotes the result of filling the hole by the term M, possibly capturing some free variables of M.

Notation 2.1. We will also use the following abbreviations for terms in $\Lambda_{\perp}$:

$$\mathsf{I} = \lambda x.x \qquad \Omega = (\lambda x.xx)\lambda x.xx \quad \mathsf{K} = \lambda xy.x$$
$$1 = \lambda xy.xy \quad \mathsf{J} = \mathsf{Y}(\lambda fxy.x(fy)) \quad \mathsf{Y} = \lambda f.(\lambda x.f(xx))(\lambda x.f(xx)).$$

Definition 2.2 (Subterm at a certain position). Let $M \in \Lambda_{\perp}$ and p be any finite sequence of 0, 1 and 2's. We will use $\langle\rangle$ for the empty sequence. The subterm $M|_p$ of a term $M \in \Lambda_{\perp}$ at position p (if there is one) is defined by induction as usual:

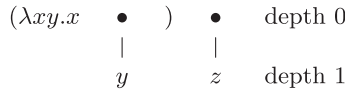$$M|_{\langle\rangle} = M \quad (\lambda x M)|_{0p} = M|_p \quad (MN)|_{1p} = M|_p \quad (MN)|_{2p} = N|_p.$$

$$(\lambda xy.x \quad \bullet \quad) \quad \bullet \qquad \text{depth } 0$$
$$| \qquad \quad | \qquad$$
$$y \qquad \quad z \qquad \text{depth } 1$$

Fig. 3. Graph representation of $(\lambda x.xy)z$ that respects our notion of depth.

Definition 2.3 (Depth of a subterm at a certain position). Let $M \in \Lambda_\perp$. The length $L(p)$ of a position p is the number of 2's in p. The depth at which a subterm N occurs in M is the length of the position p of N in M.

Figure 3 shows a graph representation of $(\lambda xy.xy)z$ that respects our notion of depth.

We define now the truncation of a term M at depth n as the term obtained by replacing all subterms at depth n by $\perp$.

Definition 2.4 (Truncation). Let $M \in \Lambda_\perp$. The truncation of M at depth n is defined by induction on M as follows.

$$M^0 = \perp \qquad x^{n+1} = x \qquad \perp^{n+1} = \perp$$
$$(MN)^{n+1} = M^{n+1}N^n \quad (\lambda x.M)^{n+1} = \lambda x.M^{n+1}$$

Note that for truncating an abstraction $\lambda x.M$ at depth $n+1$, we truncate the body M at the same depth $n+1$. For the application MN, we truncate the argument N at depth n but the operator M is truncated at depth $n+1$. For example, $(\lambda x.xy)^1 = \lambda x.x\perp$.

Definition 2.5 (Metric). Let $M, N \in \Lambda_\perp$. We define a metric on $\Lambda_\perp$ as follows: $d(M, N) = 0$, if $M = N$ and $d(M, N) = 2^{-m}$, where $m = max\{n \in \mathbb{N} \mid M^n = N^n\}$.

For example, if $M = (x(y(zu))$ and $N = (x(y(zv))$ then $d(M, N) = 2^{-3}$.

Definition 2.6 (Set $\Lambda_\perp^\infty$ of finite and infinite terms). The set $\Lambda_\perp^\infty$ is defined as the *metric completion* of the set of finite lambda terms $\Lambda_\perp$ with respect to the metric d.

From now on, $M, N, \ldots$ will be assumed to belong to $\Lambda_\perp^\infty$ unless we state otherwise. By construction $\Lambda_\perp^\infty$ contains all Böhm trees. For example the term $x(x(x \ldots))$ belongs to $\Lambda_\perp^\infty$. But $\Lambda_\perp^\infty$ does not contain the Lévy-Longo tree $\lambda x.\lambda x. \ldots$, nor the Berarducci tree or $((\ldots)x)xs$ (Abramsky and Ong 1993; Berarducci 1996; Kennaway et al. 1997; Lévy 1976; Longo 1983).

Notation 2.2. We will also use the following abbreviations for terms in $\Lambda_\perp^\infty$:

$$\mathsf{J}_\infty = \lambda xy_0.x(\lambda y_1.y_0(\lambda y_2.y_1(\ldots))) \quad \mathsf{E}_y = \lambda y_1.y(\lambda y_2.y_1(\ldots)) \text{ for any } y \in \mathcal{V}.$$

Note that $\mathsf{J}_\infty = \lambda xy_0.x\mathsf{E}_{y_0}$ and $\mathsf{E}_{y_0} = \lambda y_1.y_0\mathsf{E}_{y_1}$.

Definitions 2.2–2.4 can all be extended to infinite terms in $\Lambda_\perp^\infty$ in the obvious way. The notion of depth of the hole in $C[\ ]$ can be defined in the same way as the depth of a subterm at a certain position (see Definition 2.3).

Definition 2.7 (Prefix). Let $M, N \in \Lambda_\perp^\infty$. We say that M is a prefix of N (we write $M \preceq N$), if M is obtained from N by replacing some subterms of N by $\perp$

2.2. *Converging reductions*

In this section, we define the notion of strongly converging reduction.

Definition 2.8 (Reduction). We call a binary relation $\longrightarrow_\rho$ on $\Lambda_\perp^\infty$ a reduction relation, if $\longrightarrow_\rho$ is closed under contexts, that is, if $M \longrightarrow_\rho N$ implies $C[M] \longrightarrow_\rho C[N]$.

Definition 2.9 (Infinitary lambda calculus). If $\longrightarrow_\rho$ is a reduction relation on $\Lambda_\perp^\infty$, then we $(\Lambda_\perp^\infty, \longrightarrow_\rho)$ an infinitary lambda calculus. Instead of $(\Lambda_\perp^\infty, \longrightarrow_\rho)$, we may write λ_ρ^∞.

For an infinitary lambda calculus λ_ρ^∞ and ordinals α, we define reduction sequences of any transfinite ordinal length α.

Definition 2.10 (Strongly converging reductions (Kennaway and de Vries 2003)). A strongly convergent ρ-reduction sequence of length α (an ordinal) is a sequence $\{M_\beta \mid \beta \leqslant \alpha\}$ of terms in $\Lambda_\perp^\infty$ such that $M_\beta \longrightarrow_\rho M_{\beta+1}$ for all $\beta < \alpha$, besides $M_\lambda = \lim_{\beta<\lambda} M_\beta$ for every limit ordinal $\lambda \leqslant \alpha$ and $\lim_{i\to\lambda} d_i = \infty$ where d_i is the depth of the redex contracted at $M_i \longrightarrow_\rho M_{i+1}$ for every limit ordinal $\lambda \leqslant \alpha$.

Strongly converging reduction is a key concept of infinite rewriting (Kennaway et al. 1995b, 1997) that generalises and includes finite reduction. Intuitively, an infinite reduction is strongly converging when the depth of the position of the application of the reduction rules goes to infinity along the reduction sequence. Cauchy converging reduction sequence do not behave so nicely as strongly converging reductions (Kennaway et al. 1997; Simonsen 2004). Hence, strongly converging reduction is the natural notion of reduction to study. This preference is reflected in the next notation.

Notation 2.3.
$M \longrightarrow_\rho N$ denotes a one-step reduction from M to N;
$M \longrightarrow\!\!\!\!\rightarrow_\rho N$ denotes a finite reduction from M to N;
$M \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\rightarrow_\rho N$ denotes a strongly converging reduction from M to N.
$M \longrightarrow_{=,\rho} N$ denotes equality or one-step reduction from M to N.

We will sometimes write the depth of the contracted redex on top of the arrows. For example, $M \xrightarrow{m}_\rho N$ denotes a reduction step where the contracted redex is at depth m.

Many notions of finite lambda calculus extend now more or less straightforwardly to an infinitary lambda calculus λ_ρ^∞. A term M in λ_ρ^∞ is a *ρ-normal form* if there is no N in λ_ρ^∞ such that $M \longrightarrow_\rho N$, and M *has a ρ-normal form* if $M \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\rightarrow_\rho N$ for some ρ-normal form N.

Definition 2.11. Let $\lambda_\rho^\infty = (\Lambda_\perp^\infty, \longrightarrow_\rho)$.
λ_ρ^∞ is *confluent* if $\,_\rho\!\!\leftarrow\!\!\!\!\!\leftarrow\!\!\!\!\leftarrow \circ \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\rightarrow_\rho \subseteq \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\rightarrow_\rho \circ \,_\rho\!\!\leftarrow\!\!\!\!\!\leftarrow\!\!\!\!\leftarrow$.
λ_ρ^∞ is *normalising* if for all $M \in \Lambda_\perp^\infty$ there exists an N in ρ-normal form such that $M \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\rightarrow_\rho N$.
Let α be an ordinal. λ_ρ^∞ is *α-compressible*, if for all M, N such that $M \longrightarrow\!\!\!\!\!\rightarrow\!\!\!\!\rightarrow_\rho N$ there exists a reduction from M to N of length at most α.

If λ_ρ^∞ is confluent and normalising, it induces a total function, denoted by nf_ρ, from $\Lambda_\perp^\infty$ to $\Lambda_\perp^\infty$ such that $\mathsf{nf}_\rho(M)$ gives *the ρ-normal form* of M. The set of ρ-normal forms over $\Lambda_\perp^\infty$ is denoted by $\mathsf{nf}_\rho(\Lambda_\perp^\infty)$ and the set of ρ-normal forms over Λ is denoted by $\mathsf{nf}_\rho(\Lambda)$.

2.3. *The basic reductions: β, η, η^{-1} and $\perp$-reductions*

We will now extend several notions of reductions on finite lambda calculus to infinite terms. The β-reduction, denoted by $\longrightarrow_\beta$, is the smallest reduction on $\Lambda_\perp^\infty$ closed under the rule:

$$(\lambda x.M)N \longrightarrow M[x := N] \quad (\beta).$$

The β_h-reduction, denoted by $\longrightarrow_{\beta_h}$, is the β-reduction restricted to head redexes, i.e. $\lambda x_1 \ldots x_n.(\lambda x.P)QN_1 \ldots N_k \longrightarrow_{\beta_h} \lambda x_1 \ldots x_n.P[x := Q]N_1 \ldots N_k$.

The η-reduction, denoted by $\longrightarrow_\eta$, is the smallest reduction on $\Lambda_\perp^\infty$ closed under the η-rule:

$$\frac{x \notin FV(M)}{\lambda x.Mx \to M} \,(\eta).$$

The η^{-1}-reduction (or the η-expansion), denoted by $\longrightarrow_{\eta^{-1}}$, is the smallest reduction on $\Lambda_\perp^\infty$ closed under the η^{-1}-rule:

$$\frac{x \notin FV(M)}{M \to \lambda x.Mx} \,(\eta^{-1}).$$

We now define the $\perp$-rule. The variant that we will use in this paper is the one that equates terms that have no HNF. The $\perp$-rule is necessary because the infinitary lambda calculus with only β-reduction is not confluent. For example (Berarducci 1996),

$$\Omega_\mathsf{I} \xrightarrow{\;\;}_\beta \mathsf{I}(\Omega_\mathsf{I}) \xrightarrow{\;\;}_\beta \mathsf{I}(\mathsf{I}(\Omega_\mathsf{I}))_\beta \xrightarrow{\;\;} \quad \ldots \qquad \mathsf{I}^\omega$$
$$\beta\Big\downarrow$$
$$\Omega$$

where $\Omega = (\lambda x.xx)\lambda x.xx$, $\mathsf{I} = \lambda x.x$ and $\Omega_\mathsf{I} = (\lambda x.\mathsf{I}(xx))(\lambda x.\mathsf{I}(xx))$.

Let $M \in \Lambda_\perp^\infty$. We say that M is in *HNF*, if M is of the form $\lambda x_1 \ldots x_n.yN_1 \ldots N_k$. We say that *$M$ has a HNF* (or is *head normalising*), if there exists N in HNF such that $M \longrightarrow_\beta N$. The terms Ω and $\lambda x.\perp x$ are examples of terms without HNF.

The $\perp$-reduction, denoted by $\longrightarrow_\perp$, is the smallest reduction on $\Lambda_\perp^\infty$ closed under the $\perp$-rule:

$$\frac{M \text{ has no head normal form}}{M \longrightarrow \perp} \,(\perp).$$

Next, we define the notion of outermost $\perp$-redex as a maximal subterm without HNF. For example, the term $M = x((\lambda y.\Omega y)z)$ has four $\perp$-redexes, i.e. Ω, Ωy, $\lambda y.\Omega y$ and $(\lambda y.\Omega y)z$ but only the latter is an outermost $\perp$-redex.

We will also need a variation of the $\perp$-reduction, called $\perp_{out}$-reduction, that contracts only outermost $\perp$-redexes and which is not closed under contexts. The $\perp_{out}$-reduction, denoted by $\longrightarrow_{\perp_{out}}$, is defined as the smallest binary relation on $\Lambda_\perp^\infty$ such that $C[M] \longrightarrow_{\perp_{out}} C[\perp]$ whenever M is an outermost $\perp$-redex of $C[M]$.

2.4. *The new reduction: η!-reduction*

We will now introduce the notion of η!-rule. It is inspired by Barendregt's $\infty\eta$ construction on Böhm trees (Barendregt 1984). With the current knowledge of infinite rewriting, we see that this relation $\leqslant_\eta$ on Böhm trees is nothing else but an alternative definition for strongly converging η^{-1}-reduction. For η-expansions, strong convergence ensures that the expanded terms remain within $\Lambda_\perp^\infty$ and are finitely branching. Thus, we define the η!-rule on $\Lambda_\perp^\infty$ as:

$$\frac{x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N \qquad x \notin FV(M)}{\lambda x.MN \to M} \; (\eta!)$$

where $\longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}}$ denotes *strongly converging η-expansion*. The η!-reduction, denoted by $\longrightarrow_{\eta!}$, is the smallest reduction on $\Lambda_\perp^\infty$ closed under the η!-rule.

This η!-rule does not occur in the finite lambda calculus. Note that the original notion $\leqslant_\eta$ in Barendregt (1984) is defined on $\beta\perp$-normal forms (Böhm trees) only, while η-expansion $\longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}}$ applies to any term in $\Lambda_\perp^\infty$. It is easy to see that $\leqslant_\eta$ and $\longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}}$ coincide on the set of $\beta\perp$-normal forms. Hence $\longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}}$ is an extension of $\leqslant_\eta$ to the set of all lambda terms $\Lambda_\perp^\infty$.

The strength of the new η!-reduction can be demonstrated on the Böhm tree of Wadsworth's term J mentioned above. The Böhm tree of J is represented by the term $\mathsf{J}_\infty = \lambda xy_0.x(\lambda y_1.y_0(\lambda y_2.y_1(\ldots)))$. We see that J_∞ is of the form $\lambda xy_0.x\mathsf{E}_{y_0}$ where $\mathsf{E}_{y_0} = \lambda y_1.y_0(\lambda y_2.y_1(\ldots))$. The term E_{y_0} is the limit of a strongly converging η-expansion of y_0:

$$y_0 \longrightarrow_{\eta^{-1}} (\lambda y_1.y_0 y_1) \longrightarrow_{\eta^{-1}} (\lambda y_1.y_0(\lambda y_2.y_1 y_2)) \longrightarrow_{\eta^{-1}} \ldots \mathsf{E}_{y_0}$$

Therefore J_∞ reduces to I in a single η!-step, while J_∞ is not even a η-redex.

2.5. *The infinitary calculus $\lambda_\perp^\infty$*

The infinitary calculus $\lambda_\perp^\infty$ has some straightforward properties worthwhile to state on their own which have not been stated explicitly before.

Theorem 2.1 (Confluence, normalisation and compression of $\perp$). The infinitary
lambda calculus $\lambda_\perp^\infty$ is confluent, normalising and ω-compressible. Moreover, $M \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}}$ $\mathsf{nf}_\perp(M)$ for all $M \in \Lambda_\perp^\infty$.

Proof. Confluence follows from Lemma 26 in Kennaway et al. (1999). Depth-first left-most $\perp$-reduction is clearly a normalising strategy. Since the depth-first left-most strategy contracts only outermost redexes, we have that $M \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}} \mathsf{nf}_\perp(M)$. It is not difficult to show ω-compression by adapting the proof of the compression lemma for λ_β^∞ in Kennaway et al. (1997) (quite similar to our later proof of Lemma 3.4). $\qquad\square$

2.6. *The infinitary lambda calculus $\lambda_{\beta\perp}^\infty$*

In this section, we collect some properties of the infinitary lambda calculus $\lambda_{\beta\perp}^\infty$ that will be used later. Crucial is the following theorem from Kennaway et al. (1997) and Kennaway and de Vries (2003).

Theorem 2.2 (Confluence, normalisation and compression of $\beta\perp$). The infinitary lambda calculus $\lambda^\infty_{\beta\perp}$ is confluent, normalising, ω-compressible and satisfies $\beta\perp$-postponement: if $M \longrightarrow\!\!\!\twoheadrightarrow_{\beta\perp} N$, then $M \longrightarrow\!\!\!\twoheadrightarrow_\beta Q \longrightarrow\!\!\!\twoheadrightarrow_\perp N$ for some $Q \in \Lambda^\infty_\perp$.

Corollary 2.1 (Existence of $\perp_{out}$-reduction). For all $M \in \Lambda^\infty_\perp$, we have that $M \longrightarrow\!\!\!\twoheadrightarrow_\beta N \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}} \mathsf{nf}_{\beta\perp}(M)$.

Proof. The previous theorem implies that for any $M \in \Lambda^\infty_\perp$ there has a reduction to normal form $\mathsf{nf}_{\beta\perp}(M)$ in $\lambda^\infty_{\beta\perp}$. By postponement, this reduction factors into $M \longrightarrow\!\!\!\twoheadrightarrow_\beta N \longrightarrow\!\!\!\twoheadrightarrow_\perp \mathsf{nf}_{\beta\perp}(M)$. This implies that $\mathsf{nf}_{\beta\perp}(M)$ is a normal form of N in $\lambda^\infty_\perp$. Hence, $N \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}} \mathsf{nf}_\perp(N)$ and $\mathsf{nf}_\perp(N) = \mathsf{nf}_{\beta\perp}(M)$ by Theorem 2.1. $\qquad\square$

Lemma 2.1 (β-reducing a prefix). Let $M, N \in \Lambda^\infty_\perp$. If $M \leq N$ and $M \longrightarrow\!\!\!\twoheadrightarrow_\beta M'$, then there exists N' such that $N \longrightarrow\!\!\!\twoheadrightarrow_\beta N'$ and $M' \leq N'$.

Proof. By induction on the length of the reduction sequence. $\qquad\square$

Theorem 2.3 (Monotonicity). Let $M, N \in \Lambda^\infty_\perp$. If $M \leq N$, then $\mathsf{nf}_{\beta\perp}(M) \leq \mathsf{nf}_{\beta\perp}(N)$.

Proof. Let $M, N \in \Lambda^\infty_\perp$ such that $M \leq N$. We prove that $\mathsf{nf}_{\beta\perp}(M) \leq \mathsf{nf}_{\beta\perp}(N)$. By normalisation of $\beta\perp$ and postponement of $\perp$ over β (Theorem 2.2), we have that there exists M' such that $M \longrightarrow\!\!\!\twoheadrightarrow_\beta M' \longrightarrow\!\!\!\twoheadrightarrow_\perp \mathsf{nf}_{\beta\perp}(M)$. By Lemma 2.1, we have that $N \longrightarrow\!\!\!\twoheadrightarrow_\beta N'$ and $M' \leq N'$ for some N'. Next, we prove that for all n, $(\mathsf{nf}_{\beta\perp}(M'))^n \leq (\mathsf{nf}_{\beta\perp}(N'))^n$ by induction on n. The base case $n = 0$ is trivial. Suppose $n = h + 1$. We have three cases:

Case $M' = \perp$. Then $(\mathsf{nf}_{\beta\perp}(M'))^n = \perp \leq (\mathsf{nf}_{\beta\perp}(N'))^n$.

Case $M' = \lambda x_1 \ldots x_n.y\ P_1 \ldots P_k$. Then $N' = \lambda x_1 \ldots x_n.y\ Q_1 \ldots Q_k$.

$$\begin{aligned}
(\mathsf{nf}_{\beta\perp}(M'))^n &= \lambda x_1 \ldots x_n.y\ (\mathsf{nf}_{\beta\perp}(P_1))^{h-k} \ldots (\mathsf{nf}_{\beta\perp}(P_k))^h \\
&\leq \lambda x_1 \ldots x_n.y\ (\mathsf{nf}_{\beta\perp}(Q_1))^{h-k} \ldots (\mathsf{nf}_{\beta\perp}(Q_k))^h \quad \text{by induction hypothesis} \\
&= (\mathsf{nf}_{\beta\perp}(N'))^n.
\end{aligned}$$

Case $M' = \lambda x_1 \ldots x_n.(\lambda y.R)SQ_1 \ldots Q_k$. Since $M' \longrightarrow\!\!\!\twoheadrightarrow_\perp \mathsf{nf}_{\beta\perp}(M)$, M' cannot have a HNF. Hence, $(\mathsf{nf}_{\beta\perp}(M'))^n = \perp \leq (\mathsf{nf}_{\beta\perp}(N'))^n$. $\qquad\square$

Lemma 2.2 (Increasing truncations). Let $M, N \in \Lambda^\infty_\perp$. If $M \longrightarrow_{\beta\perp} N$ then for all n there exists $m \geq n$ such that $\mathsf{nf}_{\beta\perp}(M^m) \geq \mathsf{nf}_{\beta\perp}(N^n)$.

The proof can be found in the appendix.

Theorem 2.4 (Approximation). Let $M \in \Lambda^\infty_\perp$. For all n, there exists $m \geq n$ such that $\mathsf{nf}_{\beta\perp}(M^m) \geq (\mathsf{nf}_{\beta\perp}(M))^n$.

Proof. By Theorem 2.2, there exists a strongly convergent reduction sequence of length ω from M to $\mathsf{nf}_{\beta\perp}(M)$:

$$M = M_0 \longrightarrow_{\beta\perp} M_1 \longrightarrow_{\beta\perp} M_2 \ldots \mathsf{nf}_{\beta\perp}(M).$$

Since this reduction sequence is strongly convergent, for all n there exists M_i such that $(\mathsf{nf}_{\beta\perp}(M))^n = (M_i)^n$. By Lemma 2.2, there exist $m = m_i \geqslant m_{i-1} \geqslant \ldots m_0 = n$ such that

$$
\begin{aligned}
\mathsf{nf}_{\beta\perp}(M^m) = \mathsf{nf}_{\beta\perp}(M_0^{m_i}) &\geqslant \mathsf{nf}_{\beta\perp}(M_1^{m_{i-1}}) \\
&\cdots \\
&\geqslant \mathsf{nf}_{\beta\perp}(M_i^{m_0}) = \mathsf{nf}_{\beta\perp}((\mathsf{nf}_{\beta\perp}(M))^n) = \mathsf{nf}_{\beta\perp}(M^n).
\end{aligned}
$$

$\square$

3. Properties of η!-reduction

Before we will deal with the interaction of η!-reduction with β- and $\perp$-reduction in the further sections, we will study a number of useful properties of η!-reduction and η-expansion. First, we show that any η!-reduction is strongly converging. Next, we will demonstrate that $\lambda_{\eta!}^{\infty}$ and $\lambda_{\eta^{-1}}^{\infty}$ are dual calculi in the sense that strongly converging η-expansion and strongly converging η!-reduction are each others inverse (cf. Lemma 3.2). This allows us to prove that strongly converging η!-reductions and strongly converging η^{-1}-reductions can be compressed to reductions of length at most ω. It also permits us to prove that the steps of a strongly converging η^{-1}-reduction can be ordered according to their depth. Finally, we will show in this section that $\lambda_{\eta!}^{\infty}$ is confluent and normalising.

3.1. *Strong convergence of η!*

We will prove that any η!-reduction (and hence η-reduction) starting from a term in $\Lambda_{\perp}^{\infty}$ is strongly converging. This is a direct result of our choice of depth used in the metric completion $\Lambda_{\perp}^{\infty}$ of $\Lambda_{\perp}$. The infinite term Ω_{η} is an example of a term that is not in $\Lambda_{\perp}^{\infty}$. Clearly, Ω_{η} η-reduces to itself by contraction of the η-redex at its root. Therefore, Ω_{η} can perform infinitely many η-reductions at depth zero, and hence, it is not strongly converging.

Lemma 3.1 (Strong convergence of η!). Any η!-reduction in $\Lambda_{\perp}^{\infty}$ is strongly convergent.

Proof. Strong convergence of η! reduction follows by a counting argument. For $M \in \Lambda_{\perp}^{\infty}$, let $|M^n|$ denote the number of abstractions in M^n. The number $|M^n|$ decreases by one, if we contract an η!-redex in M at depth n and it remains equal if we contract an η!-redex at depth $m > n$. Suppose by contradiction that we have a transfinite η!-reduction sequence that is not strongly convergent, that is, suppose we have a reduction $M_0 \longrightarrow_{\eta!} M_1 \longrightarrow_{\eta!} \ldots$ in which infinitely many reductions occur at depth n. Then, infinitely many inequalities in the sequence

$$|M_0^n| \geqslant |M_1^n| \geqslant |M_2^n| \geqslant \ldots$$

are strict, which is impossible. Hence, the limit of the depth of the contracted redexes in any sequence $M_0 \longrightarrow_{\eta!} M_1 \longrightarrow_{\eta!} \ldots$ goes to infinity at each limit ordinal $\leqslant \alpha$. This implies that all η!-reduction sequences are strongly converging. $\square$

In contrast to η!-reduction, η-expansion need not be strongly converging. For instance, the following infinite sequence of η-expansions is not Cauchy, as the distance between

any two terms in this sequence in this sequence is always 1.

$$M \longrightarrow_{\eta^{-1}} \lambda y_0.My_0 \longrightarrow_{\eta^{-1}} \lambda y_0 y_1.My_0 y_1 \longrightarrow_{\eta^{-1}} \ldots$$

3.2. Relation between η^{-1} and $\eta!$

Next, we will show that strongly converging η-expansion is the inverse of strongly converging $\eta!$-reduction: $(\longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}})^{-1} = \longrightarrow\!\!\!\twoheadrightarrow_{\eta!}$. In general, $\eta!$-reduction may need less steps than its inverse. For example, while an infinite number of eta expansions is necessary to reach E_x starting from x, the reverse $\eta!$-reduction can be done in only one step.
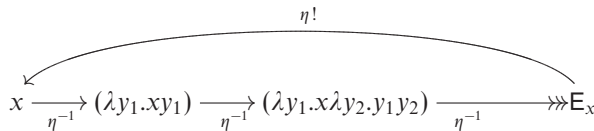

We will make frequent use of this inverse relationship. The proof of the inverse relationship (Theorem 3.1) will follow from some smaller results and ω-compression lemmas for $\eta!$ and η^{-1}. These compression lemmas will simplify many later proofs.

Lemma 3.2 (Inverse of one-step reduction). Let M, N in $\Lambda^{\infty}_{\perp}$.

1. If $M \xrightarrow{n}_{\eta^{-1}} N$, then $N \xrightarrow{n}_{\eta!} M$.   2 If $M \xrightarrow{n}_{\eta!} N$, then $N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M$.

Proof. The first statement is trivial. The second statement follows directly from the definitions of $\eta!$ and η^{-1} as illustrated in the next diagram.
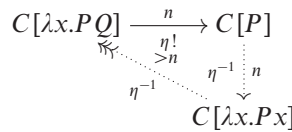

If the depth of the $\eta!$-redex in $M \longrightarrow_{\eta!} N$ is n then the η^{-1}-redexes in $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ occur at least at depth n. $\square$

Lemma 3.3 (Inverse reductions restricted to ω-length). Let M, N in $\Lambda^{\infty}_{\perp}$.

1. If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ is of length at most ω, then $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$.
2. If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$ is of length at most ω, then $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M$.

Proof. We only prove the first item using induction on the length α of the reduction sequence from M to N. The proof of the second item is similar.

The base case $\alpha = 0$ is trivial. The successor case $\alpha = n + 1$ follows easily from Lemma 3.2 and the induction hypothesis, as shown in the next diagram:
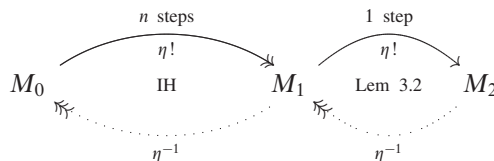
Limit case $\alpha = \omega$. By strong convergence, the number of steps at certain depth n is finite. We can, then, always split the sequence by depth as follows.

$$M = M_0 \xrightarrow[\eta!]{\geq 0} M_1 \xrightarrow[\eta!]{\geq 1} M_2 \xrightarrow{\quad\quad\quad} M_\omega = N.$$

Now consider the last step occurring at depth 0 in this sequence. The position of its redex is still present in all terms that follow M_1, including M_ω. By reversing this last $\eta!$-step at depth 0 in the limit M_ω, we construct the following diagram:

$$M = M_0 \xrightarrow[\eta!]{\geq 0} C[\lambda x.PQ] \xrightarrow[\eta!]{0} C[P] = M_1 \xrightarrow[\eta!]{>0} M_\omega = C'[P']$$

$$\begin{array}{c} {\geq 0} \Big\downarrow {\eta^{-1}} \quad\quad {\geq 0} \Big\downarrow {\eta^{-1}} \\ C[\lambda x.PQ] \xrightarrow[\eta!]{>0} C'[\lambda x.P'Q] \end{array}$$

We repeat this process for each step at depth 0 and obtain a term N_1 such that

$$M \xrightarrow[\eta!]{\geq 0} M_1 \xrightarrow[\eta!]{>0} N$$

$$\xrightarrow[\eta!]{>0} \quad {\geq 0} \Big\downarrow {\eta^{-1}} $$

$$N_1$$

Since all steps in the $\eta!$-reductions sequence from M to N_1 occur at depth greater than 0, the terms M and N_1 coincide at depth 0.

Repeating the above argument on the reduction sequence $M \xrightarrow[\eta!]{\geq 1} N_1$, we find a term N_2 such that $N_1 \xrightarrow[\eta^{-1}]{\geq 1} N_2$ and $M \xrightarrow[\eta!]{>1} N_2$. Moreover, M and N_2 coincide up to depth 1. Hence, we obtain an infinite η^{-1}-reduction sequence from N as indicated in the next diagram:

$$\begin{array}{c} M \cdots\cdots\cdots \overset{=}{} \cdots\cdots \\ {\eta!}\Big\downarrow {>0} \quad {>1} \quad {\eta!} \quad {>2} \\ N \xrightarrow[\eta^{-1}]{} N_1 \xrightarrow[\eta^{-1}]{\geq 1} N_2 \xrightarrow[\eta^{-1}]{\geq 2} N_3 \cdots\cdots\cdots\cdots N_\omega \end{array}$$

Because the reduction sequence from N is strongly converging, it has a limit, say N_ω. Since each term N_i coincides with M up to depth i, the limit N_ω of this sequence is exactly M. □

Lemma 3.4 (Compression for η^{-1} and $\eta!$). Strongly converging reduction is ω-compressible in $\lambda^\infty_{\eta^{-1}}$ and $\lambda^\infty_{\eta!}$.

Proof. First, we consider $\lambda^\infty_{\eta!}$. The proof proceeds by transfinite induction on the length of the reduction sequence. By a general argument (Kennaway and de Vries 2003; Kennaway et al. 1995b), it is sufficient to prove that a sequence of length $\omega + 1$ can be compressed into one of length ω. Without loss of generality, we may suppose that we have a strongly

convergent $\eta!$-reduction sequence of length $\omega + 1$ as follows:

$$
\begin{array}{ccccccc}
\lambda x.M_0 N_0 & \xrightarrow{\eta!} & \lambda x.M_1 N_1 & \xrightarrow{\eta!} & \lambda x.M_2 N_2 & \xrightarrow{\eta!}\!\!\!\!\twoheadrightarrow & \lambda x.M_\omega N_\omega \\
{\scriptstyle\eta!}\downarrow & & {\scriptstyle\eta!}\downarrow & & {\scriptstyle\eta!}\downarrow & & {\scriptstyle\eta!}\downarrow \\
M_0 & \dashrightarrow[{=,\eta!}] & M_1 & \dashrightarrow[{=,\eta!}] & M_2 & \dashrightarrow[{=,\eta!}]\!\!\!\!\twoheadrightarrow & M_\omega
\end{array}
$$

Note that $M_i \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M_\omega$ and $N_i \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N_\omega$ for all i. By Lemma 3.3, $N_\omega \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N_i$. Since $\lambda x.M_\omega N_\omega$ is an $\eta!$-redex, we have that $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N_\omega$. Hence, $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N_\omega \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N_i$ and all terms $\lambda x.M_i N_i$ in the top row are $\eta!$-redexes. Contracting them, we obtain the terms of the bottom row. The reduction in the bottom row is the projection of the reduction in the top row. This way we obtain a sequence of length ω from $\lambda x.M_0 N_0$ to M_ω.

The proof of compression of $\lambda^\infty_{\eta^{-1}}$ is similar, but without appeal to Lemma 3.3. $\qquad\square$

Lemma 3.4 allows us to remove the conditions on length in Lemma 3.3.

Theorem 3.1 (Inverse reductions in $\Lambda^\infty_\perp$). $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ if and only if $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$.

Thus, we have shown at the main result of this section that strongly converging $\eta!$-reduction is the inverse of strongly converging η^{-1}-reduction.

3.3. *Confluence of $\eta!$*

In this section, we will show that $\eta!$-reduction is confluent. The main ingredients of the proof are Local Confluence and the Strip Lemma for $\eta!$.

Lemma 3.5 (Preservation of $\eta!$-redexes by $\eta!$). If $\lambda x.MN$ is an $\eta!$-redex and $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N'$, then $\lambda x.MN'$ is also an $\eta!$-redex.

To prove the previous lemma, we use the next lemma, the proof of which can be found in the appendix.

Lemma 3.6 (Preservation of η-expansions of x after $\eta!$). If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M'$, then $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

Lemma 3.7 (Local $\eta!$-confluence). Given $M \longrightarrow_{\eta!} M_1$ and $M \longrightarrow_{\eta!} M_2$, there exists M_3 such that the next diagram holds:

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\;m\;}_{\eta!} & M_1 \\
{\scriptstyle n}\downarrow{\scriptstyle\eta!} & & {\scriptstyle n}\downarrow{\scriptstyle =,\eta!} \\
M_2 & \dashrightarrow[{=,\eta!}]{m} & M_3
\end{array}
$$

Proof. By case analysis on the relative positioning of the $\eta!$-redexes. For example, if the $\eta!$-redex $\lambda y.PQ$ occurs inside the argument N of the other $\eta!$-redex $\lambda x.MN$, i.e. $M_0 = C_1[\lambda x.MC_2[\lambda y.PQ]]$. By Lemma 3.5, $C_2[P]$ is an η-expansion of x and we can

construct the diagram:

$$C_1[\lambda x.M[C_2[\lambda y.PQ]]] \xrightarrow{\quad m \quad}_{\eta!} C_1[\lambda x.M[C_2[P]]]$$

with vertical arrows labeled $n \mid \eta!$ on the left and $n \mid \eta!$ on the right, going down to

$$C_1[M] \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots C_1[M]$$

$\square$

Note that the annotation of the reduction depths in the previous diagram implies that the depth of an $\eta!$-redex in a term does not change when another $\eta!$-redex in the term is contracted.

Lemma 3.8 (Strip lemma for $\eta!$). Given a strongly converging reduction $M \twoheadrightarrow_{\eta!} P$ and a one-step reduction $M \longrightarrow_{\eta!} N$, then we can construct the next diagram with elementary local confluence tiles:

$$
\begin{array}{ccc}
M & \xrightarrow{\quad \eta! \quad} & P \\
m \downarrow \eta! & & m \downarrow =,\eta! \\
N & \cdots\cdots_{=,\eta!}\cdots\cdots & Q
\end{array}
$$

Proof. By Lemma 3.4 (Compression lemma) We can assume that the sequence has length ω.

$$
\begin{array}{ccccccccc}
M_0 & \xrightarrow{n_0}_{\eta!} & M_1 & \xrightarrow{n_1}_{\eta!} & M_2 & \xrightarrow{n_2}_{\eta!} & M_3 & \ldots & M_\omega \\
m \downarrow \eta! & & m \downarrow =,\eta! & & m \downarrow =,\eta! & & m \downarrow =,\eta! & & m \downarrow =,\eta! \\
N_0 & \dashrightarrow{n_0}_{\eta!} & N_1 & \dashrightarrow{n_1}_{\eta!} & N_2 & \dashrightarrow{n_2}_{\eta!} & N_3 & \ldots & ?
\end{array}
$$

Using Lemma 3.7, we can complete all the subdiagrams except for the limit case. The constructed reduction $N_0 \longrightarrow_{\eta!} N_1 \longrightarrow_{\eta!} N_2 \longrightarrow_{\eta!} \ldots$ is strongly converging, say with limit N_ω. Either the vertical $\eta!$-reduction $M_0 \longrightarrow_{\eta!} N_0$ got cancelled out in one of the applications of Local Confluence or not. If it gets cancelled out, then, from that moment on, all vertical reductions are reductions of length 0, implying that M_ω is equal to the limit N_ω. Or the vertical $\eta!$-reduction $M_0 \longrightarrow_{\eta!} N_0$ did not get cancelled out, implying that its residual is present in M_k, for all $k \geqslant 0$. That is, all M_k with $k \geqslant 0$ are of the form $C_k[\lambda x.S_k T_k]$, where all the $C_k[\ ]$ have the hole at the same position at depth m, and all N_k with $k \geqslant 0$ are of the form $C_k[S_k]$. The limit term M_ω is of the form $C_\omega[\lambda x.S_\omega T_\omega]$ and the hole of C_ω is also at depth m. By Lemma 3.5, $\lambda x.S_\omega T_\omega$ is an $\eta!$-redex. Contracting this redex in the limit M_ω, we obtain $C_\omega[S_\omega]$ which is equal to the limit N_ω of the bottom sequence. $\square$

Theorem 3.2 ($\eta!$-Confluence). The infinitary calculus $\lambda^\infty_{\eta!}$ is confluent.
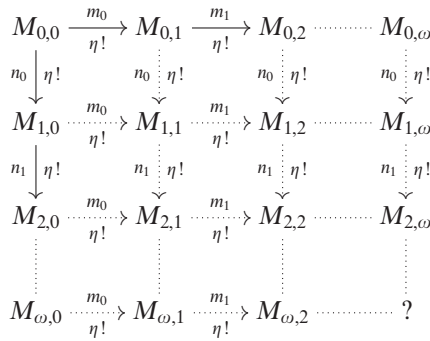
Proof. Confluence of $\lambda^\infty_{\eta!}$ can be shown by a simultaneous induction on the length of the two given coinitial $\eta!$-reductions. By compression (Lemma 3.4), we may assume that these reductions are at most of length ω, so here we don't need transfinite induction.

The induction proof makes use of so called tiling diagrams (Kennaway and de Vries 2003), which can be constructed using the induction hypothesis, Lemma 3.7 (Local Confluence) and Lemma 3.8 (Strip Lemma). The important thing to note is that the depth of an η!-redex in a term does not change when we contract an η!-redex elsewhere in the term.

The double limit case is more involved. In that case, we can construct the tiling diagram shown below. The induction hypothesis allows us to construct all proper subtiling diagrams. It remains to show that the bottom row reduction and the right-most column reduction strongly converge to the same limit.

Clearly, by the fact that all subtiles are depth preserving, both the bottom row reduction and the right-most column reduction inherit the strong convergence property from respectively the top row and the left-most column reductions. Using strong convergence, we can show that for any k there exists k_1, k_2, such that for all $i \geqslant k_1$ and $j \geqslant k_2$ the terms $M_{i,j}$ have the same prefix up to depth k. Hence, the limits of the bottom row reduction and the right-most column reduction are the same.

Alternatively, one can check that the conditions of the general tiling diagram theorem in Kennaway and de Vries (2003) are satisfied to conclude that both limits are the same.

$$
\begin{array}{ccccccc}
M_{0,0} & \xrightarrow[\eta!]{m_0} & M_{0,1} & \xrightarrow[\eta!]{m_1} & M_{0,2} & \cdots\cdots & M_{0,\omega} \\
{\scriptstyle n_0}\Big\downarrow{\scriptstyle \eta!} & & {\scriptstyle n_0}\Big\downarrow{\scriptstyle \eta!} & & {\scriptstyle n_0}\Big\downarrow{\scriptstyle \eta!} & & {\scriptstyle n_0}\Big\downarrow{\scriptstyle \eta!} \\
M_{1,0} & \dashrightarrow[\eta!]{m_0} & M_{1,1} & \dashrightarrow[\eta!]{m_1} & M_{1,2} & \cdots\cdots & M_{1,\omega} \\
{\scriptstyle n_1}\Big\downarrow{\scriptstyle \eta!} & & {\scriptstyle n_1}\Big\downarrow{\scriptstyle \eta!} & & {\scriptstyle n_1}\Big\downarrow{\scriptstyle \eta!} & & {\scriptstyle n_1}\Big\downarrow{\scriptstyle \eta!} \\
M_{2,0} & \dashrightarrow[\eta!]{m_0} & M_{2,1} & \dashrightarrow[\eta!]{m_1} & M_{2,2} & \cdots\cdots & M_{2,\omega} \\
\vdots & & \vdots & & \vdots & & \\
M_{\omega,0} & \dashrightarrow[\eta!]{m_0} & M_{\omega,1} & \dashrightarrow[\eta!]{m_1} & M_{\omega,2} & \cdots\cdots & ?
\end{array}
$$

$\square$

In a similar way, one can prove that strongly converging η-expansion is confluent. We skip the proof, as we don't need this result in this paper.

3.4. *Normalisation of η!-reduction*

We finish this section showing that $\lambda^\infty_{\eta!}$-calculus is normalising in contrast to $\lambda^\infty_{\eta^{-1}}$-calculus which is not normalising.

Theorem 3.3 (Normalisation of $\lambda^\infty_{\eta!}$). The infinitary lambda calculus $\lambda^\infty_{\eta!}$ is depth-first left-most normalising.

Proof. Let M_0 be some lambda term in $\lambda^\infty_{\eta!}$. Consider the reduction $M_0 \longrightarrow_{\eta!} M_1 \longrightarrow_{\eta!} M_2\ldots$ in which each M_{i+1} is obtained from its predecessor M_i by contracting the depth-first left-most η!-redex in M_i. By Lemma 3.1, this reduction is strongly converging. If it is finite, then the last term is an η!-normal form. If it is infinite, then by strong convergence

it has a limit M_ω. By a reductio ad absurdum M_ω must be an η!-normal form as well: For suppose M_ω contains an η!-redex $\lambda x.PX$ at some position p. Then, by strong convergence, there is an M_n in the reduction that contains a subterm of the form $\lambda x.P'X'$ at position p, while all reduction steps after M_n take place at depth greater than the depth of $\lambda x.P'X'$. Hence $X' \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} X$, and so $X \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X'$ by Lemma 3.1. We also have that $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X$, because $\lambda x.PX$ is an η!-redex. Therefore, $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X'$. Thus, $\lambda x.P'X'$ must also be an η!-redex in M_n. Since the later reductions steps in $M_n \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M_\omega$ take place at greater depth than $\lambda x.P'X'$. This contradicts the fact that the reduction $M_0 \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M_\omega$ is depth-first left-most. $\qquad\square$

The combination of the previous result with the confluence of η!-reduction give us uniqueness of normal forms as corollary:

Corollary 3.1 (Uniqueness of η!-normal forms). Each lambda term in $\lambda_{\eta!}^\infty$ has a unique η!-normal form.

4. Commutation properties for β and η!

In this section, we will prove various instances of commutation of β and η! to be used in the proof of confluence of $\lambda_{\beta\perp\eta!}^\infty$.

To prove local commutation of one step β and one step η! we need a preservation result which in turn is a consequence of the next lemma, the proof of which can be found in the appendix.

Lemma 4.1. If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$ and $N \longrightarrow\!\!\!\twoheadrightarrow_\beta N'$, then $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N'$.

Lemma 4.2 (Preservation of η!-redexes by β). If $\lambda x.MN$ is an η!-redex and $N \longrightarrow\!\!\!\twoheadrightarrow_\beta N'$, then $\lambda x.MN'$ is also an η!-redex.

Lemma 4.3 (Local commutation of β and η!). If $M_0 \longrightarrow_{\eta!} M_1$ and $M_0 \longrightarrow_\beta M_2$, then there exists an M_3 such that the next diagram holds:

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\ \ m\ \ }_{\eta!} & M_1 \\
{\scriptstyle n}\Big\downarrow{\scriptstyle \beta} & & {\scriptstyle n}\Big\downarrow{\scriptstyle =,\beta} \\
M_2 & \xtwoheadrightarrow[\eta!]{\ \geq m-1\ } & M_3
\end{array}
$$

Proof. Suppose M_0 can do both a β-reduction and an η!-reduction at respectively depths n and m. We prove only one case. The β-redex is inside the expanded variable term of the η!-redex, that is M_0 is of the form $C_1[\lambda x.MN]$ and $N = C_2[(\lambda y.P)Q]$.

$$
\begin{array}{ccc}
C_1[\lambda x.MN] & \xrightarrow{\ \ m\ \ }_{\eta!} & C_1[M] \\
{\scriptstyle n}\Big\downarrow{\scriptstyle \beta} & & \Big\vdots \\
C_1[\lambda x.MN'] & \xrightarrow[\eta!]{\ \ m\ \ } & C_1[M]
\end{array}
$$

By Lemma 4.2, we have that if $N \longrightarrow_\beta N'$ then $\lambda x.MN'$ is also an η!-redex. $\qquad\square$

Next, we prove the strip lemma for one step β over $\eta!$.

Lemma 4.4 (Strip Lemma for $\longrightarrow_\beta$ over $\longrightarrow\!\!\!\!\twoheadrightarrow_{\eta!}$). Given $M \longrightarrow_\beta P$ and $M \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta!} N$, then there exists Q such that:

$$
\begin{array}{ccc}
M & \xrightarrow{\quad\eta!\quad} & \!\!\!\!\twoheadrightarrow N \\
{\scriptstyle m}\downarrow{\scriptstyle\beta} & & {\scriptstyle m}\vdots{\scriptstyle =,\beta} \\
P & \cdots\!\!\!\!\twoheadrightarrow & Q \\
& {\scriptstyle\eta!} &
\end{array}
$$

Proof. The proof is similar to the proof of the strip lemma for $\eta!$. By Lemma 3.4 (Compression Lemma), we can assume that the sequence has length at most ω.

$$
\begin{array}{ccccccccc}
M = M_0 & \xrightarrow[\eta!]{n_0} & M_1 & \xrightarrow[\eta!]{n_1} & M_2 & \xrightarrow[\eta!]{n_2} & M_3 & \cdots & M_\omega = N \\
{\scriptstyle m}\downarrow{\scriptstyle\beta} & & {\scriptstyle m}\vdots{\scriptstyle =,\beta} & & {\scriptstyle m}\vdots{\scriptstyle =,\beta} & & {\scriptstyle m}\vdots{\scriptstyle =,\beta} & & {\scriptstyle m}\vdots{\scriptstyle =,\beta} \\
N_0 & \cdots\!\!\!\!\twoheadrightarrow_{\eta!}^{\geqslant n_0-1} & N_1 & \cdots\!\!\!\!\twoheadrightarrow_{\eta!}^{\geqslant n_1-1} & N_2 & \cdots\!\!\!\!\twoheadrightarrow_{\eta!}^{\geqslant n_2-1} & N_3 & \cdots & ?
\end{array}
$$

Using Lemma 4.3, we can complete all the subdiagrams except for the limit case. Either the vertical β-reduction got cancelled out in one of the applications of Local Confluence or not. If it gets cancelled out, then from that moment on all vertical reductions are reductions of length 0, implying that M_ω is equal to the limit N_ω. Or, if the vertical β-reduction did not get cancelled out, then a residual of the β-redex in M is present in all terms M_k. Hence $M_k = C_k[(\lambda x.P_k)Q_k]$ for all $k \geqslant 0$, and in all $C_k[\ ]$ the hole occurs at the same position at depth m, so that all N_k are of the form $C_k[P_k]$. This holds for the limit terms as well. Contracting this residual in the limit M_ω gives us the limit N_ω. $\quad\square$

Lemma 4.5 (Strip lemma for $\longrightarrow_{\eta!}$ over $\longrightarrow\!\!\!\!\twoheadrightarrow_\beta$). Let X be in $\beta\bot$-normal form. If $M = C[\lambda x.M_0 X] \longrightarrow_{\eta!} C[M_0] = P$ and $M \longrightarrow\!\!\!\!\twoheadrightarrow_\beta N$, then there exists Q such that

$$
\begin{array}{ccc}
M & \xrightarrow{\quad\beta\quad} & \!\!\!\!\twoheadrightarrow N \\
\downarrow{\scriptstyle\eta!} & & \vdots{\scriptstyle\eta!} \\
P & \cdots\!\!\!\!\twoheadrightarrow & Q \\
& {\scriptstyle\beta} &
\end{array}
$$

The proof can be found in the appendix.

5. Commutation of $\eta!$ and $\bot_{out}$

Full commutation of $\eta!$ and $\bot$ does not hold. Already local commutation of η and $\bot$ goes wrong (cf. Severi and de Vries (2002)) when the contracted $\bot$-redex is not outermost. Take for instance $\Omega \xleftarrow{}_{\eta!} \lambda x.\Omega x \rightarrow_\bot \lambda x.\bot$. However, for proving confluence of $\lambda^\infty_{\beta\bot\eta!}$ it is sufficient that $\eta!$-reduction commutes with $\bot_{out}$-reduction. Recall that $\longrightarrow_{\bot_{out}}$ is the reduction that replaces an outermost subterm without HNF by $\bot$. The proof of this commutation property then follows the familiar pattern.

Lemma 5.1 (Local $\eta!\perp_{out}$-commutation). If $M_0 \longrightarrow_{\eta!} M_1$ and $M_0 \longrightarrow_{\perp_{out}} M_2$, there exists M_3 such that

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\ m\ }_{\eta!} & M_1 \\
{\scriptstyle \perp_{out}} \downarrow {\scriptstyle n} & & {\scriptstyle \perp_{out}} \vdots {\scriptstyle n} \\
M_2 & \underset{=,\eta!}{\dashrightarrow}^{m} & M_3
\end{array}
$$

The proof can be found in the appendix.

Lemma 5.2 (Strip lemma for $\longrightarrow_{\perp_{out}}$ over $\longrightarrow\!\!\!\twoheadrightarrow_{\eta!}$). Given a one-step reduction $M \longrightarrow_{\perp_{out}} N$ and a strongly converging reduction $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} P$, then there exists Q such that:

$$
\begin{array}{ccc}
M & \xrightarrow{\hspace{4cm}}\!\!\!\twoheadrightarrow & P \\
& {\scriptstyle \eta!} & \\
{\scriptstyle \perp_{out}} \downarrow {\scriptstyle n} & & {\scriptstyle \perp_{out}} \vdots {\scriptstyle n} \\
N & \dashrightarrow\!\!\!\twoheadrightarrow & Q \\
& {\scriptstyle \eta!} &
\end{array}
$$

The proof can be found in the appendix.

Lemma 5.3 (Strip lemma for $\longrightarrow_{\eta!}$ over $\longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}}$). Given a one-step reduction $M \longrightarrow_{\eta!} N$ and a strongly converging reduction $M \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}} P$, then there exists Q such that:

$$
\begin{array}{ccc}
M & \xrightarrow{\hspace{4cm}}\!\!\!\twoheadrightarrow & P \\
& {\scriptstyle \perp_{out}} & \\
{\scriptstyle \eta!} \downarrow {\scriptstyle n} & & {\scriptstyle =,\eta!} \vdots {\scriptstyle n} \\
N & \dashrightarrow\!\!\!\twoheadrightarrow & Q \\
& {\scriptstyle \perp_{out}} &
\end{array}
$$

The proof can be found in the appendix.

Theorem 5.1 ($\eta!\perp_{out}$-commutation). Strongly converging $\eta!$-reduction commutes with strongly converging $\perp_{out}$-reduction: $\twoheadleftarrow\!\!\!\longleftarrow_{\eta!} \circ \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}} \subseteq \longrightarrow\!\!\!\twoheadrightarrow_{\perp_{out}} \circ \twoheadleftarrow\!\!\!\longleftarrow_{\eta!}$

Proof. As for the confluence of $\eta!$ (Theorem 3.2), but using Lemmas 5.1–5.3 instead. $\square$

6. Confluence and normalisation of $\beta\perp\eta!$

We are now ready to prove the main results of this paper concerning confluence and normalisation of the infinite extensional lambda calculus $\lambda^{\infty}_{\beta\perp\eta!}$.

Theorem 6.1 (Preservation of $\beta\perp$-normal forms by $\eta!$). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ and M is a $\beta\perp$-normal form, then N is a $\beta\perp$-normal form.

The proof can be found in the appendix.

Theorem 6.2 (β-normalisation of η-expansion of x). If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X$, then $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_{\beta}(X)$.

The proof can be found in the appendix.

Theorem 6.3 (Projecting $\beta\bot\eta!$-reductions onto $\eta!$-reductions via $\mathsf{nf}_{\beta\bot}$). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\beta\bot\eta!} N$, then $\mathsf{nf}_{\beta\bot}(M) \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(N)$ and the next diagram commutes:

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\ \beta\bot\eta!\ }\!\!\!\twoheadrightarrow & M_\alpha \\
{\scriptstyle \beta\bot}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta\bot} \\
\mathsf{nf}_{\beta\bot}(M_0) & \xrightarrow[\ \eta!\ ]{}\!\!\!\twoheadrightarrow & \mathsf{nf}_{\beta\bot}(M_\alpha)
\end{array}
$$

Proof. We prove by induction on α that if $M_0 \longrightarrow\!\!\!\twoheadrightarrow_{\beta\bot\eta!} M_\alpha$, then $\mathsf{nf}_{\beta\bot}(M_0) \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(M_\alpha)$.

Case $\alpha = 0$. This is trivial. Case $\alpha = \gamma + 1$. By the induction hypothesis, we have that $\mathsf{nf}_{\beta\bot}(M_0) \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(M_\gamma)$ We distinguish two cases depending on the last step:

1. If the last step is a $\beta\bot$-reduction step, we have $\mathsf{nf}_{\beta\bot}(M_\gamma) = \mathsf{nf}_{\beta\bot}(M_{\gamma+1})$ by Theorem 2.2.

$$
\begin{array}{ccccc}
M_0 & \xrightarrow{\ \beta\bot\eta!\ }\!\!\!\twoheadrightarrow & M_\gamma & \xrightarrow{\ \beta\bot\ } & M_{\gamma+1} \\
{\scriptstyle \beta\bot}\Big\downarrow & {\scriptstyle (\text{IH})} & {\scriptstyle \beta\bot}\Big\downarrow & {\scriptstyle (\text{thma 2.2})} & \Big\downarrow{\scriptstyle \beta\bot} \\
\mathsf{nf}_{\beta\bot}(M_0) & \xrightarrow[\ \eta!\ ]{}\!\!\!\twoheadrightarrow & \mathsf{nf}_{\beta\bot}(M_\delta) & =\!\!=\!\!= & \mathsf{nf}_{\beta\bot}(M_{\delta+1})
\end{array}
$$

2. If the last step is an $\eta!$-reduction step, then we first normalise the term X of the $\eta!$-redex in $M_\gamma = C[\lambda x.PX]$. By Theorem 6.2, $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_\beta(X)$. Hence, $C[\lambda x.P\,\mathsf{nf}_\beta(X)]$ $\eta!$-reduces to $C[P]$. Then, we split the $\beta\bot$-reduction sequence from $C[\lambda x.P\mathsf{nf}_\beta(X)]$ to the normal form $\mathsf{nf}_{\beta\bot}(M_\gamma)$ into a β-reduction sequence followed by a $\bot_{out}$-reduction sequence using Theorems 2.2 and 2.1. This is depicted in the diagram:

$$
\begin{array}{ccccc}
M_0 & \xrightarrow{\ \beta\bot\eta!\ }\!\!\!\twoheadrightarrow & M_\gamma = C[\lambda x.PX] & \xrightarrow{\ \eta!\ } & C[P] = M_{\gamma+1} \\
& & {\scriptstyle \beta}\Big\downarrow \quad {\scriptstyle (\text{thma 6.2})} & & \\
& & C[\lambda x.P\ \mathsf{nf}_\beta(X)] & \xrightarrow{\ \eta!\ } & C[P] = M_{\gamma+1} \\
{\scriptstyle \beta\bot}\Big\downarrow & {\scriptstyle (\text{IH})} & {\scriptstyle \beta}\Big\downarrow \quad {\scriptstyle (\text{lema 4.5})} & & \Big\downarrow{\scriptstyle \beta} \\
& & N_1 & \cdots\!\!\!\xrightarrow{\ \eta!\ }\!\!\!\twoheadrightarrow & N_2 \\
& & {\scriptstyle \bot_{out}}\Big\downarrow \quad {\scriptstyle (\text{thma 5.1})} & & \Big\downarrow{\scriptstyle \bot_{out}} \\
\mathsf{nf}_{\beta\bot}(M_0) & \cdots\!\!\!\xrightarrow[\ \eta!\ ]{}\!\!\!\twoheadrightarrow & \mathsf{nf}_{\beta\bot}(M_\gamma) & \cdots\!\!\!\xrightarrow[\ \eta!\ ]{}\!\!\!\twoheadrightarrow & \mathsf{nf}_{\beta\bot}(M_{\gamma+1})
\end{array}
$$

Next, applying first the Strip Lemma 4.5 for $\longrightarrow_{\eta!}$ with $\longrightarrow\!\!\!\twoheadrightarrow_\beta$ and secondly the full $\eta!\bot_{out}$-Commutation Theorem 5.1, we find a term Q such that $\mathsf{nf}_{\beta\bot}(M_\gamma) \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} Q$ and $C[P] \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} \circ \longrightarrow\!\!\!\twoheadrightarrow_{\bot_{out}} Q$. Since $\mathsf{nf}_{\beta\bot}(M_\gamma)$ is a $\beta\bot$-normal form, so is Q by Theorem 6.1. Hence, by unicity of $\beta\bot$-normal forms in $\lambda_{\beta\bot}^\infty$ (Theorem 2.2), we find $Q = \mathsf{nf}_{\beta\bot}(M_{\gamma+1})$.

Case $\alpha = \lambda$. By induction hypothesis, we have $\mathsf{nf}_{\beta\bot}(M_0) \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} \mathsf{nf}_{\beta\bot}(M_\gamma)$ for all $\gamma < \lambda$.

$$
\begin{array}{ccccccc}
M_0 & \xrightarrow{\ \beta\bot\eta!\ } & M_1 & \xrightarrow{\ \beta\bot\eta!\ } & M_2 & \twoheadrightarrow & M_\lambda \\
{\scriptstyle \beta\bot}\Big\downarrow & & {\scriptstyle \beta\bot}\Big\downarrow & & {\scriptstyle \beta\bot}\Big\downarrow & & \Big\downarrow{\scriptstyle \beta\bot} \\
\mathsf{nf}_{\beta\bot}(M_0) & \xrightarrow[\ \eta!\ ]{} & \mathsf{nf}_{\beta\bot}(M_1) & \xrightarrow[\ \eta!\ ]{} & \mathsf{nf}_{\beta\bot}(M_2) & \!\!\!\twoheadrightarrow N \stackrel{?}{=} & \mathsf{nf}_{\beta\bot}(M_\lambda)
\end{array}
$$

Since all η!-reduction sequences are strongly convergent (Theorem 3.1), the bottom reduction sequence is strongly convergent, and hence has a limit, say N. To conclude that N is in fact $\mathsf{nf}_{\beta\perp}(M_\lambda)$, it suffices to prove that for all n, $N^n = (\mathsf{nf}_{\beta\perp}(M_\lambda))^n$.

By Approximation Theorem 2.4, there exist $m \geqslant n$ such that $(M_\lambda)^m \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\perp}$ $\mathsf{nf}_{\beta\perp}((M_\lambda)^m) = P \geq (\mathsf{nf}_{\beta\perp}(M_\lambda))^n$. Since $M_0 \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\perp\eta!} M_\lambda$ is strongly convergent, we have for some large enough γ_0 that for all $\gamma \geqslant \gamma_0$, $(M_\gamma)^m = (M_\lambda)^m$. Hence, for all $\gamma \geqslant \gamma_0$, we have that $P = \mathsf{nf}_{\beta\perp}((M_\lambda)^m) = \mathsf{nf}_{\beta\perp}((M_\gamma)^m) \leq \mathsf{nf}_{\beta\perp}(M_\gamma)$ by Monotonicity of $\mathsf{nf}_{\beta\perp}$ (Theorem 2.3). And so, from $\mathsf{nf}_{\beta\perp}(M_{\gamma_0})$ onwards, P is a prefix of all terms of the bottom reduction sequence. Hence, P is a prefix of their limit N. Therefore, $N^n = P^n = (\mathsf{nf}_{\beta\perp}(M_\lambda))^n$. $\qquad\square$

Theorem 6.4 (Confluence and normalisation of $\beta\perp\eta!$). The infinite extensional lambda calculus $\lambda^\infty_{\beta\perp\eta!}$ is confluent and normalising.

Proof. We first prove confluence. Suppose $M_0 \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\perp\eta!} M_1$ and $M_0 \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\perp\eta!} M_2$. Then by Theorem 6.3, we project these $\beta\perp\eta!$-reductions onto $\mathsf{nf}_{\beta\perp}(M_0) \longrightarrow\!\!\!\!\twoheadrightarrow_\eta \mathsf{nf}_{\beta\perp}(M_1)$ and $\mathsf{nf}_{\beta\perp}(M_0) \longrightarrow\!\!\!\!\twoheadrightarrow_\eta \mathsf{nf}_{\beta\perp}(M_2)$. The $\eta!$-Confluence Theorem 3.2 then gives us the term M_3 such that $\mathsf{nf}_{\beta\perp}(M_1) \longrightarrow\!\!\!\!\twoheadrightarrow_\eta M_3$ and $\mathsf{nf}_{\beta\perp}(M_1) \longrightarrow\!\!\!\!\twoheadrightarrow_\eta M_3$. The next diagram illustrates this proof.
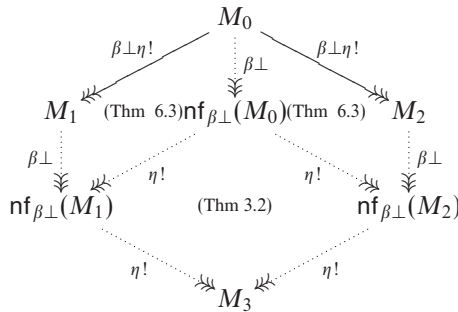
Second, normalisation of $\lambda^\infty_{\beta\perp\eta!}$ follows from normalisation of $\beta\perp$-reduction (Theorem 2.2) and normalisation of $\eta!$-reduction (Theorem 3.3): given a term M we $\beta\perp$-reduce first to $\mathsf{nf}_{\beta\perp}(M)$ and then we $\eta!$-reduce further to $\mathsf{nf}_{\eta!}(\mathsf{nf}_{\beta\perp}(M))$. $\qquad\square$

This implies that $\beta\perp\eta!$- reduction is $\omega + \omega$-compressible. More importantly, it implies follows that:

Corollary 6.1 (Uniqueness of $\beta\perp\eta!$-normal forms). The extensional infinite lambda calculus $\lambda^\infty_{\beta\perp\eta!}$ has unique normal forms.

7. Infinite eta Böhm trees as normal forms

In this section, we will see that the infinite eta Böhm tree of a lambda term M denoted by $\infty\eta\mathsf{BT}(M)$ is nothing else than the $\eta!$-normal form of $\mathsf{BT}(M)$, the Böhm tree of M, which in turn is nothing else than $\mathsf{nf}_{\beta\perp}(M)$.

We begin with the definition of Böhm tree formulated as a term in $\Lambda^\infty_\perp$. The original notion of Böhm tree defined in Barendregt (1984) for finite terms applies to infinite terms as well.

Definition 7.1 (Böhm trees). Let $M \in \Lambda_\perp^\infty$. The Böhm tree of a term M (denoted by $\mathsf{BT}(M)$) is defined by co-recursion as follows.

$$\mathsf{BT}(M) = \perp \qquad\qquad\qquad\qquad \text{if } M \text{ has no HNF}$$
$$\mathsf{BT}(M) = \lambda x_1 \ldots \lambda x_n.y \; \mathsf{BT}(M_1) \ldots \mathsf{BT}(M_m) \quad \text{if } M \longrightarrow_\beta \lambda x_1 \ldots \lambda x_n.y M_1 \ldots M_m$$

We can read this from the definition that an algorithm starting from the root of a term M calculates the Böhm tree of M layer by layer. Since the subterms of the Böhm tree are either a HNF or $\perp$, it is clear that possibly infinite output $\mathsf{BT}(M)$ of the algorithm is the (unique) $\beta\perp$- normal form of M.

Remark 7.1. We define Böhm trees as terms in the infinitary lambda calculus and this definition is given co-recursively. In Barendregt (1984) Definition 10.1.4, a Böhm tree is defined as a function from a set of sequences or positions to a set Σ of labels. Up to a change of representation, Definition 7.1 is very similar to the informal definition of Böhm trees given in Definition 10.1.3 in Barendregt (1984).

As stated before, one purpose of the infinitary lambda calculus $\lambda_{\beta\perp}^\infty = (\Lambda_\perp^\infty, \longrightarrow_{\beta\perp})$ is to capture the notion of Böhm trees as normal forms inside the calculus (see Figure 1):

Theorem 7.1 (Böhm trees as $\beta\perp$-normal forms). $\mathsf{BT}(M) = \mathsf{nf}_{\beta\perp}(M)$, for all $M \in \Lambda_\perp^\infty$.

Proof. It is easy to see that $M \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\beta\perp} \mathsf{BT}(M)$ and that $\mathsf{BT}(M)$ is in $\beta\perp$-normal form. It follows from Theorem 2.2 that $\mathsf{BT}(M) = \mathsf{nf}_{\beta\perp}(M)$. $\qquad\qquad\square$

Next, we redefine the $\infty\eta$-construction of Barendregt (1984) using the notation of strongly converging reduction.

Definition 7.2 (Infinite eta Böhm trees). We define $\infty\eta$ on Böhm trees in $\mathsf{BT}(\Lambda_\perp^\infty)$ co-inductively as follows:

$$\infty\eta(\perp) = \perp$$
$$\infty\eta(\lambda x_1 \ldots \lambda x_n.y \; M_1 \ldots M_m) = \infty\eta(\lambda x_1 \ldots \lambda x_{n-1}.y \; M_1 \ldots M_{m-1})$$
$$\text{if } x_n \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} M_m \text{ and } \quad x_n \notin FV(y \; M_1 \ldots M_{m-1})$$
$$\infty\eta(\lambda x_1 \ldots \lambda x_{n-1}.y \; M_1 \ldots M_{m-1}) = \lambda x_1 \ldots \lambda x_n.y \; \infty\eta(M_1) \ldots \infty\eta(M_m) \quad \text{otherwise}$$

This $\infty\eta$-construction contracts layer by layer all the $\eta!$-redexes in the Böhm tree $\mathsf{BT}(M)$ of a term M, so that the result is its $\beta\perp\eta$-normal form.

Barendregt's original definition in Barendregt (1984) Proposition 10.2.15 of the infinite eta expansion differs slightly from the above definition.

It uses the order $\leqslant_\eta$ on Böhm trees instead of $\longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}}$. To prove that our definition of infinite eta Böhm trees coincides with the definition in Barendregt (1984), it suffices to prove that the relations $\leqslant_\eta$ and $\longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}}$ coincide on $\beta\perp$-normal forms. We leave the proof for the reader.

Lemma 7.1. Let M, N be in $\beta\perp$-normal form. Then, $M \leqslant_\eta N$ if and only if $M \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} N$.

Theorem 7.2 (Infinite eta Böhm trees are $\beta\perp\eta!$-normal forms). Let $M \in \Lambda_\perp^\infty$.

1. If M is in $\beta\perp$-normal form then $\mathsf{nf}_{\eta!}(M) = \infty\eta(M)$.   2 $\mathsf{nf}_{\beta\perp\eta!}(M) = \infty\eta(\mathsf{BT}(M))$.

Proof. For the first part, it is not difficult to prove that M η!-reduces to $\infty\eta(M)$ and that $\infty\eta(M)$ is in η!-normal form. By Corollary 3.1, the η!-normal form is unique, hence $\infty\eta(M) = \mathsf{nf}_{\eta!}(M)$. The second part follows from Theorems 6.4 and 7.1 and the previous part. $\qquad\square$

8. Conclusion: the $\infty\eta$-Böhm trees are a model of the lambda calculus

In Barendregt (1984), the Böhm trees got a role on their own, when with help of the continuity theorem it was shown that the set of Böhm trees can be enriched to become a model of the finite lambda calculus λ_β. The original definition of Böhm tree used the language of labelled trees. In that approach, lambda terms and Böhm trees live in different worlds, because lambda terms got defined with the usual syntax definition. Using infinitary lambda calculus (Kennaway et al. 1997), this separation no longer exists. Hence, it can be shown directly from the confluence and normalisation properties of $\lambda_{\beta\perp}^\infty$ that the Böhm trees are a model of the finite lambda calculus λ_β.

Now, we have shown in this paper that $\lambda_{\beta\perp\eta!}^\infty$ is confluent and normalising, we can prove that the infinite eta Böhm trees are a model of $\lambda_{\beta\eta}^\infty$. This is a new result that could not have been proved with a variation of continuity argument used in Barendregt (1984) to prove that the Böhm trees for a model of the finite lambda calculus. In Barendregt (1984), the proof that the set $\mathfrak{B} = \mathsf{BT}(\Lambda)$ of Böhm-like trees is a λ-model uses continuity of the context operator. However, this appeal to continuity is not possible for $\infty\eta$-Böhm trees, because neither the abstraction nor the application are continuous (Severi and de Vries 2005a). For instance, take $\lambda x.y\perp$ and $\lambda x.yx$. Then $\lambda x.y\perp \leq \lambda x.yx$, but $\infty\eta\mathsf{BT}(\lambda x.y\perp) = \lambda x.y\perp \not\leq y = \infty\eta\mathsf{BT}(\lambda x.yx)$.

From the $\infty\eta$-Böhm trees of the finite lambda terms in Λ, we will now construct an extensional model $\mathfrak{B}_{\infty\eta}$ of the finitary lambda calculus, using the properties of $\lambda_{\beta\perp\eta!}^\infty$.

Definition 8.1. The $\infty\eta$-Böhm model is a tuple $(\mathfrak{B}_{\infty\eta}, \bullet, [\![\ ]\!])$ where $\mathfrak{B}_{\infty\eta}$ denotes the set $\infty\eta\mathsf{BT}(\Lambda)$ of $\infty\eta$-Böhm trees over Λ or equivalently the set $\mathsf{nf}_{\beta\perp\eta!}(\Lambda)$ of $\beta\perp\eta$!-normal forms of finite lambda terms, the application $\bullet : \mathfrak{B}_{\infty\eta} \times \mathfrak{B}_{\infty\eta} \to \mathfrak{B}_{\infty\eta}$ is defined by $M\bullet N = \infty\eta\mathsf{BT}(MN)$ for all M, N in $\mathfrak{B}_{\infty\eta}$ and for each map ρ from variables to $\mathfrak{B}_{\infty\eta}$, the interpretation $[\![M]\!]_\rho : \mathfrak{B}_{\infty\eta} \to \mathfrak{B}_{\infty\eta}$ is defined by $[\![M]\!]_\rho = \infty\eta\mathsf{BT}(M^\rho)$, where M^ρ is the result of simultaneously replacing each free variable x in M by $\rho(x)$.

Definition 8.2. Let ρ be a map from variables to $\mathfrak{B}_{\infty\eta}$. We define $\rho(x := N)$ as a map from variables to $\mathfrak{B}_{\infty\eta}$ such that $\rho(x := N)(x) = N$ and $\rho(x := N)(y) = \rho(y)$ for every $y \neq x$.

We will now show that $\mathfrak{B}_{\infty\eta}$ is a syntactic λ-model in the sense of Barendregt (1984), Definitions 5.3.1. and 5.3.2. The proof can be found in the appendix.

Theorem 8.1. $(\mathfrak{B}_{\infty\eta}, \bullet, [\![\ ]\!])$ is a syntactical model of the lambda calculus, that is, if for all ρ:

1. $[\![x]\!]_\rho = \rho(x)$, $[\![MN]\!]_\rho = [\![M]\!]_\rho\bullet[\![N]\!]_\rho$ and $[\![\lambda x.M]\!]_\rho\bullet P = [\![M]\!]_{\rho(x:=P)}$.
2. $\rho \mid FV(M) = \rho' \mid FV(M)$ implies $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$.
3. if $[\![M]\!]_{\rho(x:=P)} = [\![N]\!]_{\rho(x:=P)}$ for all $P \in \mathfrak{B}_{\infty\eta}$, then $[\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$.
4. $[\![\lambda xy.xy]\!]_\rho = [\![\lambda x.x]\!]_\rho$.

Acknowledgements

Appendix A. Infinitary Lambda Calculus

These sections in the appendix contain all omitted proofs.

The following lemma is proved by induction on the depth of the hole in the context.

Lemma A.1. Let $C[M] \in \Lambda_\perp^\infty$ and d the depth of the hole in C. If $n > d$ then $(C[M])^n = C^n[M^{n-d}]$. Otherwise $(C[M])^n = C^n$ is a term without a hole.

Lemma A.2. Let $P, Q \in \Lambda_\perp^\infty$. Then, $P^n[x := Q^n] \geq (P[x := Q])^n$.

Proof. This is proved by induction on the lexicographically ordered pair $(n, \|P^n\|)$ where $\|P^n\|$ is the number of symbols of P^n. Suppose $n > 0$ and $P = P_1 P_2$. Then

$$\begin{aligned}
P^n[x := Q^n] &= P_1^n[x := Q^n]P_2^{n-1}[x := Q^n] &&\text{by Definition 2.4} \\
&\geq P_1^n[x := Q^n]P_2^{n-1}[x := Q^{n-1}] \\
&\geq (P_1[x := Q])^n(P_2[x := Q])^{n-1} &&\text{by induction hypothesis} \\
&= (P[x := Q])^n &&\text{by Definition 2.4}
\end{aligned}$$

$\square$

Lemma A.3 (Lemma 2.2). Let $M, N \in \Lambda_\perp^\infty$. If $M \longrightarrow_{\beta\perp} N$ then for all n there exists $m \geq n$ such that $\mathsf{nf}_{\beta\perp}(M^m) \geq \mathsf{nf}_{\beta\perp}(N^n)$.

Proof. Suppose $M = C[P] \to_\perp C[\perp] = N$. Let d be the position of the hole in $C[[\;]]$. By Lemma A.1, if $n \leq d$ then $M^n = C^n = N^n$. Otherwise, $M^n = C^n[P^{n-d}] \geq C^n[\perp] = N^n$. By Monotonicity (Theorem 2.3), we have that $\mathsf{nf}_{\beta\perp}(M^n) \geq \mathsf{nf}_{\beta\perp}(\mathsf{nf}_{\beta\perp}(N^n))$.

Suppose $M = C[(\lambda x.P)Q] \longrightarrow_\beta C[P[x := Q]] = N$. Let d be the position of the hole in $C[[\;]]$ and assume $k = n - d > 0$.

$$\begin{aligned}
(C[(\lambda x.P)Q])^{n+2} &= C^{n+2}[(\lambda x.P^k)Q^{k+1}] &&\text{by Lemma A.1} \\
&\longrightarrow_\beta C^{n+2}[P^k[x := Q^{k+1}]] \\
&\geq C^n[P^k[x := Q^k]] \\
&\geq C^n[(P[x := Q])^k] &&\text{by Lemma A.2} \\
&= (C[P[x := Q]])^n &&\text{by Lemma A.1}
\end{aligned}$$

By Confluence (Theorem 2.2) and Monotonicity (Theorem 2.3), we have that $\mathsf{nf}_{\beta\perp}(M^{n+2}) \geq \mathsf{nf}_{\beta\perp}(\mathsf{nf}_{\beta\perp}(N^n))$. $\square$

We also need a variation of Lemma 2.1 with β_h-reduction instead of $\perp_{out}$-reduction.

Lemma A.4 (β_h-reducing a prefix). Let $M, N \in \Lambda_\perp^\infty$. If $M \preceq N$ and $M \longrightarrow_{\beta_h} M'$, then there exists N' such that $N \longrightarrow_{\beta_h} N'$ and $M' \preceq N'$.

Proof. By induction on $M \longrightarrow_{\beta_h} M'$. $\square$

Theorem A.1. Let $M \in \Lambda_\perp^\infty$. The following statements are equivalent:

1. There exists a HNF N, such that $M \longrightarrow_\beta N$.

2. There exists a HNF N', such that $M \longrightarrow\!\!\!\!\rightarrow_\beta N'$.
3. There exists a HNF N'', such that $M \longrightarrow\!\!\!\!\rightarrow_{\beta_h} N''$.
4. There exists a HNF N''', such that $M \xrightarrow{0}\!\!\!\!\rightarrow_\beta N'''$.

Proof. ($i \Rightarrow ii$). Kennaway et al. (1997) Suppose there exists N in HNF such that $M \longrightarrow\!\!\!\!\rightarrow_\beta N$. We can assume that the length of this reduction is ω by Theorem 2.2. Since $\longrightarrow\!\!\!\!\rightarrow_\beta$ is strongly convergent, we have that there exists N'' such that

$$M \xrightarrow{\geq 0}\!\!\!\!\rightarrow_\beta N'' \xrightarrow{>0}\!\!\!\!\rightarrow_\beta N.$$

It is easy to show that N'' is in HNF.

($ii \Rightarrow iii$). By Theorem 2.2, $M \longrightarrow\!\!\!\!\rightarrow_\beta N'' \longrightarrow\!\!\!\!\rightarrow_{\beta\perp} \mathsf{nf}_{\beta\perp}(M)$. Since N'' is in HNF, so is $\mathsf{nf}_{\beta\perp}(M)$. We truncate the normal form of M at depth 1 and apply the well-known results on head normalisation in finite lambda calculus. By Theorem 2.4, there exists $m > 1$ such that $\mathsf{nf}_{\beta\perp}(M^m) \geq (\mathsf{nf}_{\beta\perp}(M))^1$. Hence, $\mathsf{nf}_{\beta\perp}(M^m)$ is in HNF because $(\mathsf{nf}_{\beta\perp}(M))^1$ is in HNF. By Theorem 2.2 and the fact that $\longrightarrow\!\!\!\!\rightarrow_{\beta\perp}$ is strongly convergent, we have that

$$M^m \xrightarrow{\geq 0}\!\!\!\!\rightarrow_\beta P \xrightarrow{>0}\!\!\!\!\rightarrow_\beta Q \longrightarrow\!\!\!\!\rightarrow_\perp \mathsf{nf}_{\beta\perp}(M^m) \geq (\mathsf{nf}_{\beta\perp}(M))^1.$$

Since the term $M^m \in \Lambda_\perp$ is a finite λ-term and the reduction $M^m \xrightarrow{\geq 0}\!\!\!\!\rightarrow_\beta P$ is finite, we can now apply Theorem 8.3.11 of Barendregt (1984). We have that $M^m \longrightarrow\!\!\!\!\rightarrow_{\beta_h} N$ for some N in HNF. By Lemma A.4, there exists N'' such that $M \longrightarrow\!\!\!\!\rightarrow_{\beta_h} N''$ and $N \leq N''$. Since N is in HNF, so is N''.

($iii \Rightarrow iv$) and ($iv \Rightarrow i$) are trivial. $\square$

Appendix B. Preservation of η-Expansions of x after η!

By postponing the η^{-1} (or the η!) steps at greater depth, we can re-order the steps in an η^{-1} (or an η!) reduction sequence by increasing order of depth.

Lemma B.1 (Postponing η^{-1}-steps at greater depth). Let $j < i$. Then, an η^{-1}-reduction step at depth i can be postponed over an η^{-1}-reduction step at depth j, that is:

$$
\begin{array}{ccc}
M_1 & \xrightarrow{\ i\ }_{\eta^{-1}} & M_2 \\
{\scriptstyle\eta^{-1}}\downarrow{\scriptstyle j} & & {\scriptstyle\eta^{-1}}\downarrow{\scriptstyle j} \\
M_3 & \dashrightarrow[\eta^{-1}]{\ i\ } & M_4
\end{array}
$$

Proof. Suppose $M = C[P] \xrightarrow{\ i\ }_{\eta^{-1}} C[\lambda x.Px]$. Since $j < i$, the η^{-1}-redex at depth j occurs either in C or in P but it cannot occur in x. We have that

$$
\begin{array}{ccc}
C[P] & \xrightarrow{\ i\ }_{\eta^{-1}} & C[\lambda x.Px] \\
{\scriptstyle\eta^{-1}}\downarrow{\scriptstyle j} & & {\scriptstyle\eta^{-1}}\downarrow{\scriptstyle j} \\
C'[P'] & \dashrightarrow[\eta^{-1}]{\ i\ } & C'[\lambda x.P'x]
\end{array}
$$

$\square$

As a consequence of the previous lemma, we obtain:

Lemma B.2 (Sorting η^{-1}-reduction sequences by order of depth). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$, then there is either a finite reduction

$$M = M_0 \xrightarrow{\ 0\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} M_1 \xrightarrow{\ 1\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} M_2 \xrightarrow{\ 2\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} \ldots M_{n-1} \xrightarrow{\ n-1\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} M_n = N$$

or an infinite reduction

$$M = M_0 \xrightarrow{\ 0\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} M_1 \xrightarrow{\ 1\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} M_2 \xrightarrow{\ 2\ }\!\!\!\twoheadrightarrow_{\eta^{-1}} \ldots M_\omega = N$$

Lemma B.3 (Postponing η!-steps at greater depth). Let $j < i$. There are only two possible situations that can occur when we postpone an η!-reduction step at depth i over an η!-reduction step at depth j

$$
\begin{array}{ccc}
M_1 \xrightarrow[\eta!]{\ i\ } M_2 & \qquad & M_1 \xrightarrow[\eta!]{\ i\ } M_2 \\
\eta!\downarrow j \qquad \eta!\downarrow j & & \searrow_{\eta!}^{j} \quad \eta!\downarrow j \\
M_3 \xdashrightarrow[\eta!]{\ i\ } M_4 & & M_4
\end{array}
$$
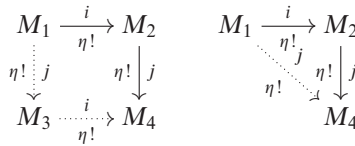
Proof. The situation of the second tile occurs when the term M_1 contains a term $\lambda x.PQ$ at depth j and the η!-redex at depth i is inside Q:

$$
\begin{array}{ccc}
C[\lambda x.PQ] & \xrightarrow[\eta!]{\ i\ } & C[\lambda x.PQ'] \\
& \searrow_{\eta!}^{j} & \eta!\downarrow j \\
& & C[P]
\end{array}
$$
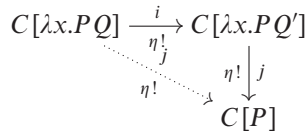
We know that Q' is an infinite η-expansion of x and we have to show that so is Q. Since Q' is obtained from Q by applying only one step of η!-reduction, by Lemma 3.2, we can reverse the reduction from Q to Q'. Hence, $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q' \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q$. □

As a consequence of the previous lemma, we obtain:

Lemma B.4 (Sorting η!-reduction sequences by order of depth). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$, then there is either a finite reduction

$$M = M_0 \xrightarrow{\ 0\ }\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{\ 1\ }\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{\ 2\ }\!\!\!\twoheadrightarrow_{\eta!} \ldots M_{n-1} \xrightarrow{\ n-1\ }\!\!\!\twoheadrightarrow_{\eta!} M_n = N.$$

or an infinite reduction

$$M = M_0 \xrightarrow{\ 0\ }\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{\ 1\ }\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{\ 2\ }\!\!\!\twoheadrightarrow_{\eta!} \ldots M_\omega = N.$$

If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$, then not all abstractions in M have to be η!-redexes. For example

$$x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} \lambda yz.(\lambda u.x\mathsf{E}_y\mathsf{E}_u)\mathsf{E}_z = M.$$

The first abstraction λy is not an η!-redex. In spite of this, it is possible to undo all the η^{-1}-steps by doing only a finite number of steps of η! at depth 0. This is proved in the following lemma which will be used to prove Commutation of β and η!.

Lemma B.5 (Inverting the expansion of a variable). If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$, then $M \xrightarrow{0}\!\!\!\twoheadrightarrow_{\eta!} x$ and there is a single free occurrence of x in M at depth 0.

Proof. By Theorem 3.1, $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ implies that $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} x$. By Lemma 3.4 (Compression Lemma) and Lemma B.4 (Sorting by order of depth), we have that there is either a finite reduction

$$M = M_0 \xrightarrow{0}\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{1}\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{2}\!\!\!\twoheadrightarrow_{\eta!} \ldots M_{n-1} \xrightarrow{n-1}\!\!\!\twoheadrightarrow_{\eta!} M_n = N = x.$$

or an infinite reduction

$$M_0 \xrightarrow{0}\!\!\!\twoheadrightarrow_{\eta!} M_1 \xrightarrow{1}\!\!\!\twoheadrightarrow_{\eta!} M_2 \xrightarrow{2}\!\!\!\twoheadrightarrow_{\eta!} \ldots M_\omega = N = x.$$

In both cases, the following case analysis allows us to conclude that $M_1 = x$.

1. If M_1 is a variable, then the rest of the reduction sequence from M_1 onwards is empty, so that $M_1 = x$.
2. Suppose $M_1 = (PQ)$. Then, all $\eta!$-reducts from M_1 are applications (including N). This contradicts the fact that N is a variable.
3. Suppose $M_1 = \lambda x.P$. Then either all reducts from M_1 are abstractions (including N) or this abstraction disappears because it is contracted by an $\eta!$-redex. Neither case is possible. The first option contradicts the fact that N is a variable. The second option contradicts the fact that in the reduction from M_1 to N we contract only redexes at depth strictly greater than 0.

$\square$

Local Confluence and the Strip Lemma for $\eta!$ depend on the following lemma that says that $\eta!$-redexes are preserved by certain $\eta!$-reduction sequences.

Lemma B.6. If $x \xrightarrow{<n}\!\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \xrightarrow{n}\!\!\!\twoheadrightarrow_{\eta!} M'$, then $N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

Proof. Assume $x \xrightarrow{<n}\!\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \xrightarrow{n}\!\!\!\rightarrow_{\eta!} M'$. We will show that $N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

From depth considerations, it follows that the abstraction of the $\eta!$-redex contracted in $M \longrightarrow_{\eta!} M'$ got created after N in the reduction $x \xrightarrow{<n}\!\!\!\twoheadrightarrow_{\eta^{-1}} N \xrightarrow{\geq n}\!\!\!\twoheadrightarrow_{\eta^{-1}} M \xrightarrow{n}\!\!\!\rightarrow_{\eta!} M'$. Since η^{-1} does not change the depth of any subterm once the $\eta!$-redex is created, its depth remains fixed. By omitting the η-expansion step that created the abstraction of the $\eta!$-redex plus all the subsequent η-expansions from y to Q, we construct the reduction sequence at the bottom:

$$x \xrightarrow[\eta^{-1}]{<n}\!\!\!\twoheadrightarrow N \xrightarrow[\eta^{-1}]{n}\!\!\!\twoheadrightarrow C'[P'] \xrightarrow[\eta^{-1}]{n} C'[\lambda y.P'y] \xrightarrow[\eta^{-1}]{\geq n}\!\!\!\twoheadrightarrow C[\lambda y.PQ] = M$$

with the diagonal $\xrightarrow[\eta^{-1}]{\geq n}\!\!\!\twoheadrightarrow$ and the vertical $\eta! \downarrow n$ to $\twoheadrightarrow C[P] = M'$

The more general statement of the lemma follows by repeated application of the above.

$\square$

Theorem B.1 (Lemma 3.6). If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} M'$, then $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M'$.

Proof. By compression of $\eta!$-reduction we may assume that the reduction $M \longrightarrow\!\!\!\rightarrow_{\eta!} M'$ is at most ω steps long. By Theorem B.2, this reduction sequence can be sorted. Two possible situations can arise:

Case $M \longrightarrow\!\!\!\rightarrow_{\eta!} M'$ is finite. We illustrate the proof for a sequence of length 3.
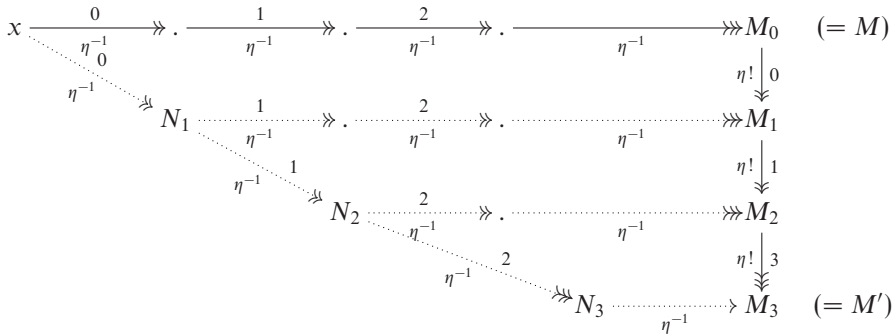
By Lemma B.6, we have $x \longrightarrow\!\!\!\rightarrow_{\eta^{-1}} M_1$. By Lemma B.2, there exists N_1 such that $x \xrightarrow{0}_{\eta^{-1}}$ $N_1 \xrightarrow{>0}_{\eta^{-1}} M_1$. By Lemma B.6, we have that $N_1 \xrightarrow{>0}_{\eta^{-1}} M_2$. Hence, we get $x \xrightarrow{0}_{\eta^{-1}}$ $N_1 \xrightarrow{>0}_{\eta^{-1}} M_2$. By Lemma B.2, there exists N_2 such that $x \xrightarrow{0}_{\eta^{-1}} N_1 \xrightarrow{1}_{\eta^{-1}} N_2 \xrightarrow{>1}_{\eta^{-1}}$ M_2. Again by Lemma B.6, we have that $N_2 \xrightarrow{>1}_{\eta^{-1}} M_3$. Once more by Lemma B.2, there exists N_3 such that $x \xrightarrow{0}_{\eta^{-1}} N_1 \xrightarrow{1}_{\eta^{-1}} N_2 \xrightarrow{2}_{\eta^{-1}} N_3 \xrightarrow{>2}_{\eta^{-1}} M_3$.

Case $M \longrightarrow\!\!\!\rightarrow_{\eta!} M'$ has length ω. By repeated application of the previous argument, we can construct the diagonal sequence as shown in the following diagram:
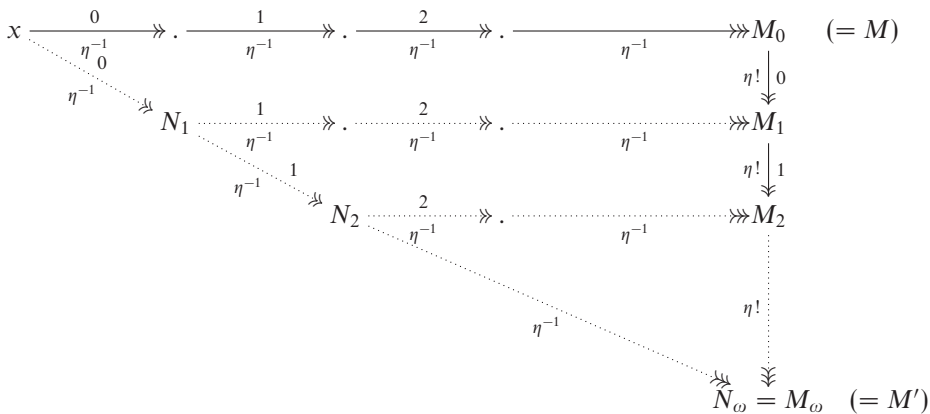
By construction, the diagonal sequence is strongly convergent and has a limit, say N_ω. It is easy to see that the limits M_ω and N_ω are the same, because for all k we have N_{k+1} and M_{k+1} have the same truncation to depth k. $\square$

Appendix C. Preservation of η-Expansions of x after β

For the proof of local commutation, we need to prove preservation of $\eta!$-redexes under β-reduction. The proof follows the same pattern as the proof of preservation of $\eta!$-redexes under $\eta!$-reduction (Lemma 3.5).

Lemma C.1. If $x \xrightarrow{<n}_{\eta^{-1}} N \xrightarrow{\geq n}_{\eta^{-1}} M$ and $M \xrightarrow{\geq n}_{\beta} M'$ then $N \xrightarrow{\geq n}_{\eta^{-1}} M'$.

Proof. Assume $x \xrightarrow{<n}_{\eta^{-1}} N \xrightarrow{\geq n}_{\eta^{-1}} M$ and $M \xrightarrow{\geq n}_{\beta} M'$. We will show $N \xrightarrow{\geq n}_{\eta^{-1}} M'$. Since η^{-1} does not change the depth of any term, once the β-redex is created, its depth remains fixed. There are only two ways in which η-expansions can create a β-redex:

1. The application of the β-redex is created before its abstraction in the η-expansion. This happens as follows:

$$x \xrightarrow[\eta^{-1}]{<n} N \xrightarrow[\eta^{-1}]{\geq n} C'[P'Q'] \xrightarrow[\eta^{-1}]{\geq n} C'[(\lambda y.P'y)Q'] \xrightarrow[\eta^{-1}]{\geq n} C[(\lambda y.P)Q] = M$$

$$\xrightarrow[\eta^{-1}]{\geq n} C[P[y := Q]] = M' \quad \beta \downarrow \geq n$$

Since $P'y \longrightarrow_{\eta^{-1}} P$ and $Q' \longrightarrow_{\eta^{-1}} Q$ and $y \notin FV(P')$, we have that

$$P'Q' = (P'y)[y := Q'] \longrightarrow_{\eta^{-1}} P[y := Q]$$

2. The abstraction in the β-redex gets created before its application in the η-expansion. This happens as follows:

$$x \xrightarrow[\eta^{-1}]{<n} N \xrightarrow[\eta^{-1}]{\geq n} C'[\lambda y.P'] \xrightarrow[\eta^{-1}]{\geq n} C'[\lambda z.(\lambda y.P')z] \xrightarrow[\eta^{-1}]{\geq n} C[\lambda z.(\lambda y.P)Q] = M$$

$$\xrightarrow[\eta^{-1}]{\geq n} C[\lambda z.P[y := Q]] = M' \quad \beta \downarrow \geq n$$

Since $P' \longrightarrow_{\eta^{-1}} P$ and $z \longrightarrow_{\eta^{-1}} Q$, we have that

$$\lambda y.P' =_{\alpha} \lambda z.P'[y := z] \longrightarrow_{\eta^{-1}} \lambda z.P[y := Q]$$

$\square$

Theorem C.1 (Lemma 4.1). If $x \longrightarrow_{\eta^{-1}} M$ and $M \longrightarrow_{\beta} M'$, then $x \longrightarrow_{\eta^{-1}} M'$.

Proof. By strong convergence, we can assume that the β-reduction sequence is of the form $M_0 \xrightarrow{\geq 0}_{\beta} M_1 \xrightarrow{\geq 1}_{\beta} M_2 \ldots$. Now, we can proceed similarly as in the proof of Theorem B.1 while exploiting Lemmas B.2 and C.1 instead. $\square$

Appendix D. Strip lemma for one step $\eta!$ over infinitely many β's

The full strip lemma for β over $\eta!$ is harder than the strip lemma for $\eta!$ over β (see Lemma 4.4). The difficulty lies in the fact that, due to overlap, the residuals of an $\eta!$ redex are not always immediately $\eta!$ redexes themselves. We illustrate this with an example. Consider $M = (\lambda x.zX)Q$, where $x \longrightarrow_{\eta^{-1}} \lambda y_1 y_2 y_3.xy_1 y_2 y_3 = X$ and Q is some arbitrary

term. What are the residuals of the $\eta!$-redex $\lambda x.zX$ in M after contracting the β-redex $(\lambda x.zX)Q$? We have that

$$M = (\lambda x.zX)Q \longrightarrow_\beta z(\lambda y_1 y_2 y_3.Qy_1 y_2 y_3)$$
$$\longrightarrow_{\eta!} z(\lambda y_1 y_2.Qy_1 y_2)$$
$$\longrightarrow_{\eta!} z(\lambda y_1.Qy_1)$$
$$\longrightarrow_{\eta!} zQ$$

Only the first of these consecutive $\eta!$-redexes is readily present in $\lambda y_1 y_2 y_3.Qy_1 y_2 y_3$. From the next two redexes, only their λ's are present in $\lambda y_1 y_2 y_3.Qy_1 y_2 y_3$. These lambda's are $\eta!$-redexes 'in waiting.' The residuals in $\lambda y_1 y_2 y_3.Qy_1 y_2 y_3$ of the original $\eta!$-redex $\lambda x.zX$ will be the three abstractions $\lambda y_3.Qy_1 y_2 y_3$, $\lambda y_2 y_3.Qy_1 y_2 y_3$ and $\lambda y_1 y_2 y_3.Qy_1 y_2 y_3$ in spite of the fact that not all of them are $\eta!$-redexes. We make this precise with underlining. To track the residuals of $\lambda x.zX$, we will not only underline the λ of $\lambda x.zX$ but also all the λ's in X, i.e. $(\underline{\lambda} x.z\underline{X})Q$ where $\underline{X} = \underline{\lambda} y_1 \underline{\lambda} y_2 \underline{\lambda} y_3.xy_1 y_2 y_3$.

To simplify matters a bit, we will not do this in full generality. Instead, we will do this only with $\eta!$-redexes of the form $\lambda x.MN$ where N is an η-expansion of x which is in $\beta\bot$-normal form, because such expansions have a straightforward format:

Lemma D.1. If $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$ and X is in $\beta\bot$-normal form, then X is of the form $\lambda y_1 \ldots y_n.xY_1 \ldots Y_n$, where $x \neq y_i$, $y_i \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ and Y_i is in $\beta\bot$-normal form for each $1 \leqslant i \leqslant n$.

Proof. The reduction steps in $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta!} X$ can be sorted by depth with Lemma B.2, so that we may assume without loss of generality that $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta!} X$ is of the form $x \xrightarrow{0}_{\eta^{-1}} X_1 \xrightarrow{>0}_{\eta^{-1}} X$. Because X is a $\beta\bot$-normal form, X_1 must be of the form $\lambda y_1 \ldots y_n.xy_1 \ldots y_n$ as can be shown with a proof by induction on n: if the expansions steps at depth 0 would be executed at other positions than the sequence of positions that leads to $\lambda y_1 \ldots y_n.xy_1 \ldots y_n$ a β-redex would be introduced and all further terms in the sequence would contain a β-redex, contradicting the normal form of the final term X.

Now, because the deeper reductions in $X_1 \xrightarrow{>0}_{\eta^{-1}} X$ do not alter the left spine of X_1, it follows that also X must be of the shape $\lambda y_1 \ldots y_n.xY_1 \ldots Y_n$ where $y_i \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ and Y_i is in $\beta\bot$-normal form for each $1 \leqslant i \leqslant n$. □

In the proof of the restricted strip lemma for one step $\eta!$ over β, we will employ the underlining technique of Barendregt (1992) to track the residuals of $\eta!$-redexes of the particular form $\lambda x.PX$ where X is an η-expansion of x in $\beta\bot$-normal form. To introduce this technique in the infinitary setting, we extend the set $\Lambda^\infty_\bot$ to $\underline{\Lambda^\infty_\bot}$ which will contain underlined terms of the following form only:

$$\underline{\lambda} y_1 \ldots \underline{\lambda} y_n.M\underline{Y_1} \ldots \underline{Y_n}.$$

where $Y_i \in \Lambda^\infty_\bot$ is in $\beta\bot$-normal form, Y_i is an η-expansion of y_i and $\underline{Y_i}$ is obtained by underlining all λs in Y_i. for all $1 \leqslant i \leqslant n$.

Definition D.1 (Family of sets $\underline{\mathcal{E}}_x$ for $x \in \mathcal{V}$). We define a family of sets $\underline{\mathcal{E}}_x$ on $x \in \mathcal{V}$ by simultaneous induction

$$X ::= x \mid \underline{\lambda}x_1 \dots \underline{\lambda}x_n.xX_1 \dots X_n.$$

where, $X_i \in \underline{\mathcal{E}}_{x_i}$ for all $1 \leqslant i \leqslant n$.

Definition D.2 (Set $\underline{\Lambda}_\perp$ of underlined finite lambda terms with $\perp$). We define the set $\underline{\Lambda}_\perp$ of underlined finite λ-terms by induction

$$M ::= \perp \mid x \mid (\lambda x M) \mid (MM) \mid \underline{\lambda}x_1 \dots \underline{\lambda}x_n.MX_1 \dots X_n.$$

where, $x \in \mathcal{V}$, $x_i \notin FV(M)$ and $X_i \in \underline{\mathcal{E}}_{x_i}$ for all $1 \leqslant i \leqslant n$.

The metric d on Λ can be easily extended to terms in $\underline{\Lambda}_\perp$ and in each of the $\underline{\mathcal{E}}_x$.

Definition D.3 (Sets of underlined finite and infinite terms).

1. Let $x \in \mathcal{V}$. The set $\underline{\mathcal{E}}_x^\infty$ is the *metric completion* of the set $\underline{\mathcal{E}}_x$ with respect to the metric d.
2. The set $\underline{\Lambda}_\perp^\infty$ is the *metric completion* of the set of underlined finite lambda terms $\underline{\Lambda}_\perp$ with respect to the metric d.

Now, we are ready to define underlined η!- and β-reduction.

Definition D.4.

1. Let $\longrightarrow_{\underline{\eta}!}$ be the smallest binary relation on $\underline{\Lambda}_\perp^\infty$ containing the rule

$$\frac{X \in \underline{\mathcal{E}}_x^\infty \qquad x \notin FV(M)}{\underline{\lambda}x.MX \to M} \ (\underline{\eta}!),$$

and closed under contexts.

2. Let $\longrightarrow_{\underline{\beta}}$ be the smallest binary relation on $\underline{\Lambda}_\perp^\infty$ containing the following rule:

$$(\underline{\lambda}x.P)Q \to P[x := Q] \quad (\underline{\beta}),$$

and closed under contexts.

The definition of $\longrightarrow_{\underline{\beta}}$ is correct in the sense that $\underline{\Lambda}_\perp^\infty$ is closed under underlined β-reduction: one sees easily that $X[x := Q] \in \underline{\Lambda}_\perp^\infty$ holds for any $X \in \underline{\mathcal{E}}_x^\infty$ and any $Q \in \underline{\Lambda}_\perp^\infty$.

We will frequently use situations, where $\overline{X_i \in \underline{\mathcal{E}}_x^\infty}$ and $x_i \notin FV(\overline{M})$ for all $1 \leqslant i \leqslant n$, in which case we have the reductions:

$$\underline{\lambda}x_1 \dots \underline{\lambda}x_n.MX_1 \dots X_n \longrightarrow\!\!\!\!\rightarrow_{\underline{\eta}!} M$$

and

$$(\underline{\lambda}x_1\underline{\lambda}x_2 \dots \underline{\lambda}x_n.MX_1X_2 \dots X_n)Q \to_{\underline{\beta}} \underline{\lambda}x_2 \dots \underline{\lambda}x_n.MX_1[x_1 := Q]X_2 \dots X_n.$$

We will denote the union of $\longrightarrow_\beta$ and $\longrightarrow_{\underline{\beta}}$ by $\longrightarrow_{\beta\underline{\beta}}$.

As in the finitary setting, we need mechanisms to remove the underlining:

Definition D.5 (Removing underlinings). Let $M \in \underline{\Lambda}_\perp^\infty$.

1. We define $|M| \in \Lambda_\perp^\infty$ as the result of removing all the underlinings in M.

2. We define $\varphi(M) \in \Lambda_{\perp}^{\infty}$ as the result of contracting all $\underline{\eta}!$-redexes from M by co-recursion as follows.

$$\varphi(x) = x$$
$$\varphi(PQ) = \varphi(P)\varphi(Q)$$
$$\varphi(\lambda x.P) = \lambda x.\varphi(P)$$
$$\varphi(\underline{\lambda} x_1 \ldots \underline{\lambda} x_n.MX_1 \ldots X_n) = \varphi(M).$$

Note that $\varphi(M)$ is in $\underline{\eta}!$-normal form for all $M \in \Lambda_{\perp}^{\infty}$.

For example, $\varphi((\underline{\lambda} x.z\overline{X})\mathsf{I}) = z\mathsf{I}$ where $X = \underline{\lambda} y_1 \underline{\lambda} y_2 \underline{\lambda} y_3.xy_1 y_2 y_3$.

Lemma D.2.

1. If $X \in \mathcal{E}_x^{\infty}$, then $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} |X|$.
2. Let X be a $\beta\perp$-normal form. If $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$, then $\underline{X} \in \mathcal{E}_x^{\infty}$ where $\underline{X}$ is the result of underlining all abstractions in X.

Proof.

1. Suppose $X \in \mathcal{E}_x^{\infty}$. It is not difficult to show that $X = \underline{\lambda} y_1 \ldots \underline{\lambda} y_n.xY_1 \ldots Y_n$ and $Y_i \in \mathcal{E}_{y_i}^{\infty}$ for all $1 \leqslant i \leqslant n$ using Definitions D.1 and D.3. We consider the η^{-1}-reduction sequence: $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} \lambda y_1 \ldots \lambda y_n.xy_1 \ldots y_n$. We repeat a similar argument for each Y_i with $1 \leqslant i \leqslant n$ as we did for X. This process can be repeated ad infinitum to obtain an η^{-1}-strongly converging reduction sequence from x to $|X|$.
2. Suppose $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. We construct a Cauchy sequence $M_1, M_2, \ldots$ of terms in $\mathcal{E}_x$ whose limit is $\underline{X}$ using Lemma D.1. By construction, the limit $\underline{X}$ is an element of $\mathcal{E}_x^{\infty}$. The first term M_1 in this sequence is x which belongs to $\mathcal{E}_x$. By Lemma D.1, we have that $X = \lambda y_1 \ldots \lambda y_n.xY_1 \ldots Y_n$ and $y_i \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ for each $1 \leqslant i \leqslant n$. The second term M_2 of the sequence is $\underline{\lambda} y_1 \ldots \underline{\lambda} y_n.xy_1 \ldots y_n$ which belongs to $\mathcal{E}_x$. We repeat this process to construct all the terms in the sequence. The limit of this sequence is $\underline{X}$ and it belongs to $\mathcal{E}_x^{\infty}$ by Definition D.3.

□

Lemma D.3. If $M \longrightarrow\!\!\!\!\twoheadrightarrow_{\underline{\eta}!} N$, then $|M| \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta!} |N|$.

Proof. This is proved by induction on the length of the reduction sequence. We only prove it for a reduction sequence of length 1. Suppose $\underline{\lambda} x.MX \longrightarrow_{\underline{\eta}!} M$. Then, $X \in \mathcal{E}_x^{\infty}$ and $x \notin FV(M)$. By Lemma D.2(i), we have that $x \longrightarrow\!\!\!\!\twoheadrightarrow_{\eta^{-1}} |X|$ and hence, $\lambda x.MX \longrightarrow_{\eta!} M$. □

The next lemma is a straightforward consequence of the definition of φ.

Lemma D.4. Let $X \in \mathcal{E}_x^{\infty}$. Then, $\varphi(X) = x$.

Lemma D.5. Let $M \in \Lambda_{\perp}^{\infty}$. Then, there exists a reduction of length at most ω such that $M \longrightarrow\!\!\!\!\twoheadrightarrow_{\underline{\eta}!} \varphi(M)$.

Proof. Contraction of the $\underline{\eta}!$-redexes using a depth-first left-most strategy gives a reduction $M \longrightarrow\!\!\!\!\twoheadrightarrow_{\underline{\eta}!} \varphi(M)$ of length at most ω. □

Lemma D.6 (φ on substitutions). Let $M, N \in \underline{\Lambda}^\infty_\perp$. Then, $\varphi(M[x := N]) = \varphi(M)[x := \varphi(N)]$.

Proof. We prove that $(\varphi(M[x := N]))^n = (\varphi(M)[x := \varphi(N)])^n$ for all n by induction on (n, m) where n is the depth of the truncation and m is the number of abstractions and applications in M at depth n. $\qquad\square$

Lemma D.7 (φ on one step β). Let $M \in \underline{\Lambda}^\infty_\perp$.

1. If $M \xrightarrow{n}_\beta N$, then $\varphi(M) \xrightarrow{n}_\beta \varphi(N)$.
2. If $M \longrightarrow_{\underline{\beta}} N$, then $\varphi(M) = \varphi(N)$.

Proof. In both cases, we proceed by induction on the pair (n, m) where n is the depth of the redex in M and m is the number of abstractions and applications in M at depth n.

1. Suppose $n = 0$. Then, $M = (\lambda x.P)Q \xrightarrow{0}_\beta P[x := Q] = N$ and we apply Lemma D.6. The case $n > 0$ follows by induction hypothesis.
2. The case $n > 0$ follows by induction hypothesis. Suppose $n = 0$. Since the only occurrence of x_1 in $(\underline{\lambda}x_2 \ldots \underline{\lambda}x_n.M_0 X_1 X_2 \ldots X_n)$ is in X_1, we have that

$$
\begin{aligned}
M &= (\underline{\lambda}x_1 \ldots \underline{\lambda}x_n.M_0 X_1 \ldots X_n)Q \\
&\longrightarrow_{\underline{\beta}} (\underline{\lambda}x_2 \ldots \underline{\lambda}x_n.M_0 X_1[x_1 := Q]X_2 \ldots X_n) \\
&= N
\end{aligned}
$$

Since $X_1 = \underline{\lambda}y_1 \ldots y_k.x_1 Y_1 \ldots Y_k \in \underline{\mathcal{E}}^\infty_{x_i}$, we have that

$$
\begin{aligned}
\varphi(M) &= (\underline{\lambda}x_1 \ldots \underline{\lambda}x_n.M_0 X_1 \ldots X_n)Q & \text{by definition of } \varphi \\
&= \varphi(M_0)\varphi(Q) & \text{by definition of } \varphi \\
&= \varphi(M_0)x_1[x_1 := \varphi(Q)] & \\
&= \varphi(N)\varphi(X_1)[x_1 := \varphi(Q)] & \text{by Lemma D.4} \\
&= \varphi(N)\varphi(X_1[x_1 := Q]) & \text{by Lemma D.6}
\end{aligned}
$$

$\qquad\square$

The function φ does not preserve truncations, i.e. $\varphi(M^n) \neq \varphi(M)^n$. For example, take $M = \underline{\lambda}x.y(\underline{\lambda}z.xz)$. We will define a notion of quasi-truncation which is preserved by φ. The quasi-truncation of a term at depth n truncates the term at depth n except for the η-expansions X in an $\underline{\eta}!$-redex.

Definition D.6 (Quasi-truncation). We define quasi-truncation of M at depth n by induction on the lexicographically ordered pair (n, m) where m is the number of abstractions and applications at depth n:

$$
\begin{aligned}
[\perp]^n &= \perp \\
[M]^0 &= \perp \\
[x]^{n+1} &= x \\
[\lambda x.M]^{n+1} &= \lambda x.[M]^{n+1} \\
[MN]^{n+1} &= [M]^{n+1}[N]^n \\
[\underline{\lambda}x_1 \ldots \underline{\lambda}x_k.MX_1 \ldots X_k]^n &= \underline{\lambda}x_1 \ldots \underline{\lambda}x_k.[M]^n X_1 X_2 \ldots X_k
\end{aligned}
$$

For example, take $M = \underline{\lambda}x.y(y(y\dots))(\underline{\lambda}z.xz)$. Then, $M^1 = \underline{\lambda}x.y\bot\bot$ and $[M]^1 = \underline{\lambda}x.y\bot(\underline{\lambda}z.xz)$. Note that $([M]^n)^n = M^n$, for all $M \in \underline{\Lambda}^\infty_\bot$. The function φ preserves quasi-truncations:

Lemma D.8 (Preservation of quasi-truncations). $\varphi([M]^n) = \varphi(M)^n$.

Proof. This is proved by induction on (n, m) where n is the depth of the truncation and m is the number of abstractions and applications in M at depth n. $\square$

Lemma D.9 (φ on many β-steps). Let $M \in \underline{\Lambda}^\infty_\bot$. If $M \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\underline{\beta}} N$ has length at most ω, then $\varphi(M) \longrightarrow\!\!\!\!\twoheadrightarrow_\beta \varphi(N)$.

Proof. We prove it by induction on the length of $M \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\underline{\beta}} N$. The finite case follows from Lemma D.7. We prove the case when the length is ω. The following diagram can be constructed by repeated applications of Lemma D.7. Since φ preserves the depth of the contracted redex, we have that the bottom sequence is strongly convergent and the limit exists which is P.

$$M = M_0 \xrightarrow[\beta\underline{\beta}]{n_0} M_1 \xrightarrow[\beta\underline{\beta}]{n_1} M_2 \xrightarrow[\beta\underline{\beta}]{n_2} \qquad \dots \qquad M_\omega = N$$

$$\varphi(M_0) \xdashrightarrow[=,\beta]{n_0} \varphi(M_1) \xdashrightarrow[=,\beta]{n_1} \varphi(M_2) \xdashrightarrow[=,\beta]{n_2} \qquad \dots \qquad P \stackrel{?}{=} \varphi(M_\omega)$$

It remains to prove that $P = \varphi(M_\omega)$. By strong convergence, there exists n_0 such that for all $n \geqslant n_0$, $(M_n)^k = (M_\omega)^k$ and $(\varphi(M_n))^k = P^k$. Now, $[M_n]^k = [M_\omega]^k$ because $[M_n]^k$ and $[M_\omega]^k$ are obtained from $(M_n)^k$ and $(M_\omega)^k$ by adding terms of the form $X \in \underline{\mathcal{E}}^\infty_x$ which are in β-normal form. We have

$$\begin{aligned} P^k &= (\varphi(M_n))^k \text{ by strong convergence} \\ &= \varphi([M_n]^k) \text{ by Lemma D.8} \\ &= \varphi([M_\omega]^k) \text{ because } [M_n]^k = [M_\omega]^k \\ &= (\varphi(M_\omega))^k \text{ by Lemma D.8} \end{aligned}$$

$\square$

Lemma D.10 (Lemma 4.5).
Let X be in $\beta\bot$-normal form. If $M = C[\lambda x.M_0 X] \longrightarrow_{\eta!} C[M_0] = P$ and $M \longrightarrow\!\!\!\!\twoheadrightarrow_\beta N$, then there exists Q such that

$$\begin{array}{ccc} M & \xrightarrow{\quad\beta\quad}\!\!\!\!\!\twoheadrightarrow & N \\ \downarrow{\scriptstyle\eta!} & & \Vdash{\scriptstyle\eta!} \\ P & \dashrightarrow\!\!\!\!\!\twoheadrightarrow & Q \\ & \beta & \end{array}$$

Proof. Let X be an η-expansion of x such that X is a $\beta\bot$-normal form. Suppose $M = C[\lambda x.M_0 X]$. In order to track the residuals of this $\eta!$-redex, we consider the term $M' = C[\underline{\lambda}x.M_0\underline{X}]$ where $\underline{X}$ is the result of underlining all abstractions in X. Then, $\underline{X} \in \underline{\mathcal{E}}^\infty_x$

by Lemma D.2(ii). The reduction $M \longrightarrow\!\!\!\!\twoheadrightarrow_\beta N$ is lifted to $M' \longrightarrow\!\!\!\!\twoheadrightarrow_{\beta\underline{\beta}} N'$. Using Lemmas D.3, D.5 and D.9, we obtain the following diagram:

$$
\begin{array}{ccc}
M = C[\lambda x.M_0 X] & \xrightarrow{\quad\beta\quad} & N \\
\end{array}
$$
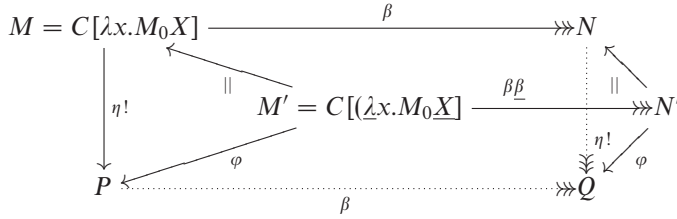
Appendix E. Commutation properties of β and η^{-1}

In this section, we will study some precise commutation properties of β and η^{-1}. We need these properties to prove that η-expansions of HNFs again have a HNF. As a consequence $\eta!$-reduction preserves $\perp_{out}$-redexes, which plays a crucial role in the proof of the commutation property for $\eta!$ and $\perp_{out}$ in Section 5.

E.1. *Strip Lemmas for One Step β_0 Over η^{-1}*

In this subsection, we concentrate on the strip lemmas for one-step β-reductions that takes place at depth 0 over η-expansion. There is a slight complication, because η-expansions can create β-redexes as shown by the next example.
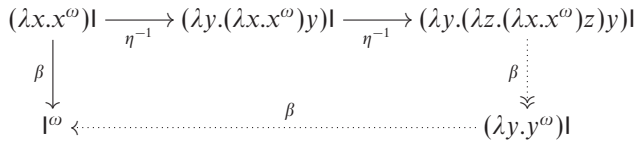
In the above η^{-1}-reduction sequence, we have created two extra β-redexes which should be contracted to get a common reduct. These extra β-redexes are of a special nature, for which we will introduce the notion of β_0-reduction.

Definition E.1 (β_0-reduction). We will introduce the notation $M \longrightarrow_{\beta_0} N$ for the situation where M is of the form $C[(\lambda x.P)Q]$ and N is of the form $C[P[x := Q]]$, while the hole $[\,]$ in $C[\,]$ occurs at depth 0 and the variable x occurs at depth 1 and exactly once in P.

Note that $\longrightarrow_{\beta_0}$ is a restricted form $\xrightarrow{\;0\;}_\beta$.

Examples of β_0-redexes are: $(\lambda x.yx)\mathsf{I}, (\lambda x.y(\lambda z.xz))\mathsf{I}$ and $(\lambda xy.y(x\mathsf{K}))\mathsf{I}$. Nonexamples are $(\lambda x.xy)\mathsf{I}$ and $(\lambda x.yxx)\mathsf{I}$, as in the former the variable x does not occur at depth 1 and in the latter the variable x occurs twice.

The next diagram shows the usefulness of this restricted form of β-reduction in the context of a strip lemma of one-step β over η^{-1}-reduction. Consider the η^{-1}-reduction

sequence $(\lambda x.x^\omega)\mathsf{I} \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} (\lambda y_1 y_2.(\lambda x.x^\omega)(\lambda z.y_1 z)y_2)\mathsf{I} = N$. Then

$$
\begin{array}{ccccc}
(\lambda x.x^\omega)\mathsf{I} & \xrightarrow[\eta^{-1}]{\quad 0 \quad}\!\!\!\!\!\twoheadrightarrow & (\lambda y_1 y_2.(\lambda x.x^\omega)y_1 y_2)\mathsf{I} & \xrightarrow[\eta^{-1}]{\quad 1 \quad} & (\lambda y_1 y_2.(\lambda x.x^\omega)(\lambda z.y_1 z)y_2)\mathsf{I} \\
\Big\downarrow {\scriptstyle\beta} & & \Big\downarrow {\scriptstyle\beta_0} & & \Big\downarrow {\scriptstyle\beta_0} \\
& & \lambda y_2.(\lambda x.x^\omega)\mathsf{I}y_2 & \xrightarrow[\eta^{-1}]{\quad 1 \quad} & \lambda y_2.(\lambda x.x^\omega)\mathsf{1}y_2 \\
& & \Big\downarrow {\scriptstyle\beta} & & \Big\downarrow {\scriptstyle\beta} \\
\mathsf{I}^\omega & \xrightarrow[\eta^{-1}]{\quad 0 \quad} & (\lambda y_2.\mathsf{I}^\omega y_2) & \xrightarrow[\eta^{-1}]{\quad 1 \quad}\!\!\!\!\!\twoheadrightarrow & (\lambda y_2.\mathsf{1}^\omega y_2)
\end{array}
$$

Lemma E.1 (Local commutation for one step β_0 and one step η^{-1}). Given $M_0 \longrightarrow_{\eta^{-1}} M_1$ and $M_0 \longrightarrow_{\beta_0} M_2$, there exists M_3 such that either one of the following diagrams hold:

$$
\begin{array}{ccc}
M_0 \xrightarrow[\eta^{-1}]{\;m\;} M_1 & \qquad\qquad & M_0 \xrightarrow[\eta^{-1}]{\;0\;} M_1 \\
\Big\downarrow {\scriptstyle\beta_0} \quad (1) \quad \Big\downarrow {\scriptstyle\beta_0} & & \Big\downarrow {\scriptstyle\beta_0} \quad (2) \quad \begin{array}{c}\Big\downarrow {\scriptstyle\beta_0} \\ N \\ \Big\downarrow {\scriptstyle\beta_0}\end{array} \\
M_2 \xdashrightarrow[\eta^{-1}]{\;m\;} M_3 & & M_2 \dashrightarrow M_3
\end{array}
$$

Proof. A term M_0 in $\Lambda_\perp^\infty$ can contain a β_0-redex $(\lambda y.P)Q$ at depth 0 and an η^{-1}-redex N at depth m in exactly one of the following situations:

1. The β_0-redex $(\lambda y.P)Q$ and the η^{-1}-redex N are not nested, i.e. $M_0 = C[(\lambda y.P)Q, N]$. This results in an instance of Diagram (1).
2. The β_0-redex is inside the η^{-1}-redex, that is M_0 is of the form $C_1[N]$, where $N \equiv C_2[(\lambda y.P)Q]$. This case results in an instance of Diagram (1) too.
3. The η^{-1}-redex is part of the body of the abstraction $\lambda y.P$ of the β_0-redex, i.e. $P \longrightarrow_{\eta^{-1}} P'$. Since η^{-1} does not affect the depth of the variable y, $(\lambda y.P')Q$ remains a β_0-redex and we have

$$
\begin{array}{ccc}
C[(\lambda y.P)Q] & \xrightarrow[\eta^{-1}]{\;m\;} & C[(\lambda y.P')Q] \\
\Big\downarrow {\scriptstyle\beta_0} & & \Big\downarrow {\scriptstyle\beta_0} \\
C[P[y := Q]] & \xdashrightarrow[\eta^{-1}]{\;m\;} & C[P'[y := Q]]
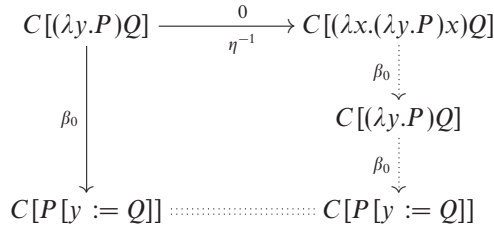\end{array}
$$

which is an instance of Diagram (1).
4. The η^{-1}-redex is part of the argument Q of the β_0-redex. This results in the following:

$$
\begin{array}{ccc}
C[(\lambda y.P)Q] & \xrightarrow[\eta^{-1}]{\;m\;} & C[(\lambda y.P)Q'] \\
\Big\downarrow {\scriptstyle\beta_0} & & \Big\downarrow {\scriptstyle\beta_0} \\
P[y := Q] & \xdashrightarrow[\eta^{-1}]{\;m\;} & P[y := Q']
\end{array}
$$

which corresponds to Diagram (1). Since the variable y occurs only once in P, we need only one η^{-1}-step from $P[y := Q]$ to $P[y := Q']$. And because the depth of this variable is 1, the depth of that η^{-1}-redex in $P[y := Q]$ is m.
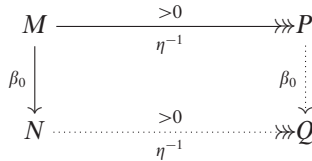
5. Only if $m = 0$, the η^{-1}-redex coincides with the abstraction $\lambda y.P$ of the β_0-redex, that is M_0 is of the form $C[NQ]$ where $N \equiv \lambda y.P$.
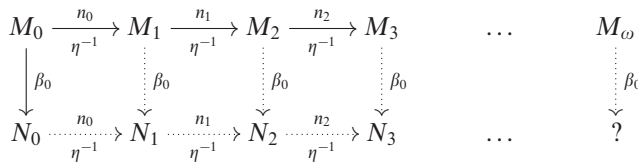
$$C[(\lambda y.P)Q] \xrightarrow[\eta^{-1}]{0} C[(\lambda x.(\lambda y.P)x)Q]$$

$$\begin{array}{ccc}
& & \beta_0 \downarrow \\
& & C[(\lambda y.P)Q] \\
& & \beta_0 \downarrow \\
\beta_0 \downarrow & & \\
C[P[y := Q]] \cdots\cdots\cdots\cdots C[P[y := Q]]
\end{array}$$

The above is an instance of Diagram (2).

$\square$

Lemma E.2 (Strip lemma for $\longrightarrow_{\beta_0}$ over $\longrightarrow\!\!\!\rightarrow_{\eta^{-1}}$ at $m > 0$).

$$\begin{array}{ccc}
M & \xrightarrow[\eta^{-1}]{>0} & \!\!\!\rightarrow P \\
\beta_0 \downarrow & & \downarrow \beta_0 \\
N & \cdots\!\!\!\rightarrow_{\eta^{-1}}^{>0} & \!\!\!\rightarrow Q
\end{array}$$

Proof. By Lemma 3.4 (Compression Lemma), we can assume that the η^{-1}-reduction sequence has length at most ω. If the length is finite, then the result follows by repeated application of Diagram (1) of Lemma E.1. Diagram (2) does not apply as the η-expansions are performed at depth greater than 0. When the length is ω we construct the diagram:

$$\begin{array}{ccccccccc}
M_0 & \xrightarrow[\eta^{-1}]{n_0} & M_1 & \xrightarrow[\eta^{-1}]{n_1} & M_2 & \xrightarrow[\eta^{-1}]{n_2} & M_3 & \cdots & M_\omega \\
\downarrow \beta_0 & & \downarrow \beta_0 & & \downarrow \beta_0 & & \downarrow \beta_0 & & \downarrow \beta_0 \\
N_0 & \cdots\!\!\rightarrow_{\eta^{-1}}^{n_0} & N_1 & \cdots\!\!\rightarrow_{\eta^{-1}}^{n_1} & N_2 & \cdots\!\!\rightarrow_{\eta^{-1}}^{n_2} & N_3 & \cdots & ?
\end{array}$$

Using Diagram (1) of Lemma E.1, we can complete all the subdiagrams except for the limit case. Since the η-expansions are performed at depth greater than 0, all M_k with $k \geqslant 0$ are of the form $C_k[(\lambda x.P_k)Q_k]$, where all the $C_k[\ ]$ have the hole at the same position at depth 0, and all P_k have exactly one occurrence of x at depth 1. The limit term is of the form $C_\omega[(\lambda x.P_\omega)Q_\omega]$. The hole of C_ω occurs also at depth 0 and x occurs only once in P_ω and at depth 1 because η-expansions do not introduce variables and the existing variables remain at the same depth. Hence, the residual remains a β_0-redex in the limit. Contracting this redex in the limit M_ω reduces to $C_\omega[P_\omega[x := Q_\omega]]$ which is equal to the limit N_ω of the bottom sequence. $\square$

E.2. *Strip Lemma for One Step β at Depth 0 Over η^{-1} Reduction*

We will now prove the strip lemma of $\xrightarrow{0}_\beta$ with $\longrightarrow_{\eta^{-1}}$ using the results of the previous subsection.

Lemma E.3 (Local commutation of $\xrightarrow{0}_\beta$ and $\longrightarrow_{\eta^{-1}}$). Given $M_0 \xrightarrow{m}_{\eta^{-1}} M_1$ and $M_0 \xrightarrow{0}_\beta M_2$, there exists M_3 such that one of the following diagrams hold:
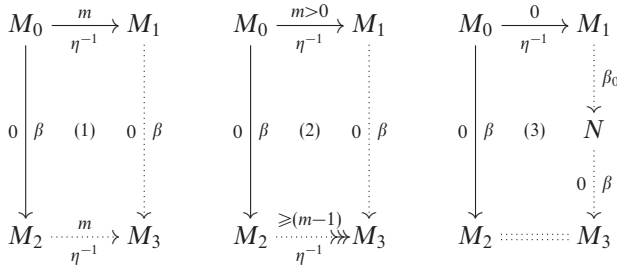
Note the extra information in Diagram (3): the first step of constructed down reduction in Diagram (3) is a β_0-reduction.

Proof. A term M_0 can contain a β-redex $(\lambda y.P)Q$ at depth 0 and an η^{-1}-redex N at depth m. A case analysis leads to the following exhaustive list of possible positions of β-redex and the η^{-1}-redex relative to each other:

1. The β-redex $(\lambda y.P)Q$ and the η^{-1}-redex N are not nested, i.e. $M_0 = C[(\lambda y.P)Q, N]$. This case leads to Diagram (1).
2. The β-redex is inside the η^{-1}-redex, that is M_0 is of the form $C_1[N]$, where $N \equiv C_2[(\lambda y.P)Q]$. This case leads to Diagram (1).
3. The η^{-1}-redex is part of the body of the abstraction $\lambda y.P$ of the β-redex. This case leads to Diagram (1).
4. The η^{-1}-redex is part of the argument Q of the β-redex. This case can only happen if $m > 0$ and it results in an instance of Diagram (2).
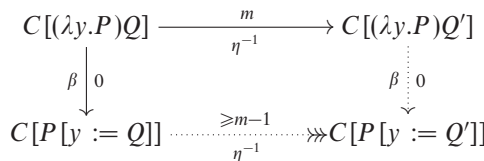
   If the variable y occurs infinite times in P, then we need ω-steps to get $P[y := Q']$ from $P[y := Q]$. If there is some occurrence of y at depth 0, then there will be some η^{-1}-redex in $P[y := Q]$ at depth $m - 1$.
5. The η^{-1}-redex coincides with the abstraction $\lambda y.P$ of the β-redex, that is M_0 is of the form $C[NQ]$ where $N \equiv \lambda y.P$. If this happens, M must be 0. It leads to the following

instance of Diagram (3):

$$\begin{array}{ccc}
C[(\lambda y.P)Q] & \xrightarrow{\;\;\;0\;\;\;}_{\eta^{-1}} & C[(\lambda x.(\lambda y.P)x)Q] \\
{\scriptstyle 0}\;\Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta_0} \\
& & C[(\lambda y.P)Q] \\
& & {\scriptstyle 0}\;\Big\downarrow{\scriptstyle \beta} \\
C[P[y := Q]] & \cdots\cdots\cdots\cdots & C[P[y := Q]]
\end{array}$$

Note that in the first step on the right vertical line, the contracted outermost β-redex that got created by the η-expansion is a β_0-redex. Again, note the informative role of β_0 in the formulation of the lemma.

$\square$

The previous local commutation lemma generalises to a finite strip lemma of $\xrightarrow{0}_{\beta}$ over $\twoheadrightarrow_{\eta^{-1}}$.

Lemma E.4 (Finite strip lemma of $\xrightarrow{0}_{\beta}$ at depth 0 over $\twoheadrightarrow_{\eta^{-1}}$ at depth 0). Given a one-step reduction $M \xrightarrow{0}_{\beta} P$ and a finite reduction $M \xrightarrow{0}_{\eta^{-1}} N$, then there exists Q and Q_0 such that

$$\begin{array}{ccc}
M & \xrightarrow{\;\;\;0\;\;\;}_{\eta^{-1}} & N \\
{\scriptstyle 0}\;\Big\downarrow{\scriptstyle \beta} & & \Big\downarrow{\scriptstyle \beta_0} \\
& & Q_0 \\
& & {\scriptstyle 0}\;\Big\downarrow{\scriptstyle \beta} \\
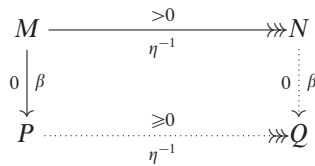P & \xrightarrow[\;\;\;0\;\;\;]{\eta^{-1}} & Q
\end{array}$$

Proof. By induction on the finite length of the η^{-1}-reduction sequence. We show the induction step.

$$\begin{array}{ccccc}
M & \xrightarrow{\;\;0\;\;}_{\eta^{-1}} & N & \xrightarrow{\;\;0\;\;}_{\eta^{-1}} & N' \\
& & {\scriptstyle \beta_0}\Big\downarrow & {\scriptstyle (\text{Lem. E.1})} & \Big\downarrow{\scriptstyle \beta_0} \\
& & Q_0 & \xrightarrow{\;\;0\;\;}_{\eta^{-1}} & R_0 \\
{\scriptstyle 0}\Big\downarrow{\scriptstyle \beta} & {\scriptstyle (\text{IH})} & & & \Big\downarrow{\scriptstyle =,\beta_0} \\
& & {\scriptstyle 0}\Big\downarrow{\scriptstyle \beta} & {\scriptstyle (\text{Lem. E.3})} & Q'_0 \\
& & & {\scriptstyle (1),(3)} & {\scriptstyle 0}\Big\downarrow{\scriptstyle \beta} \\
P & \xrightarrow[\;\;0\;\;]{\eta^{-1}} & Q & \xrightarrow[=,\eta^{-1}]{\;\;0\;\;} & Q'
\end{array}$$

The top right square follows from a repeated application of Lemma E.1. In the bottom right square, we apply Diagrams (1) and (3) of Lemma E.3. Note that Diagram (2) does not apply because β and η^{-1} are performed at the same depth. $\qquad\square$
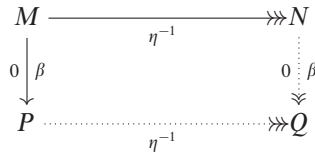
Next, we prove the strip lemma for one-step β reduction over many step η^{-1}.

Lemma E.5 (Strip lemma for $\xrightarrow{0}_\beta$ over $\twoheadrightarrow_{\eta^{-1}}$ at depth greater than 0**).** If $M \xrightarrow{0}_\beta P$ and $M \overset{>0}{\twoheadrightarrow}_{\eta^{-1}} N$, then there exists Q such that

$$
\begin{array}{ccc}
M & \xrightarrow[\eta^{-1}]{>0} & N \\
{\scriptstyle 0}\big\downarrow{\scriptstyle \beta} & & {\scriptstyle 0}\big\downarrow{\scriptstyle \beta} \\
P & \underset{\eta^{-1}}{\overset{\geqslant 0}{\dashrightarrow}} & Q
\end{array}
$$

Proof. Similar to Lemma E.2 using Lemma E.3 Diagrams (1) and (2). $\qquad\square$

Lemma E.6 (Strip lemma for β at depth 0 **over η^{-1}).** Given a one-step reduction $M \xrightarrow{0}_\beta P$ and a strongly converging reduction $M \twoheadrightarrow_{\eta^{-1}} N$, then there exists Q such that

$$
\begin{array}{ccc}
M & \xrightarrow[\eta^{-1}]{} & N \\
{\scriptstyle 0}\big\downarrow{\scriptstyle \beta} & & {\scriptstyle 0}\big\downarrow{\scriptstyle \beta} \\
P & \underset{\eta^{-1}}{\dashrightarrow} & Q
\end{array}
$$

Proof. By Lemma B.2, we can assume that the η^{-1}-reduction sequence is of the form $M \xrightarrow{0}_{\eta^{-1}} M_1 \overset{>0}{\twoheadrightarrow}_{\eta^{-1}} N$. The proof is sketched in the following diagram.

$$
\begin{array}{ccccc}
M & \xrightarrow[\eta^{-1}]{0} & M_1 & \xrightarrow[\eta^{-1}]{>0} & N \\[4pt]
\Big| & & \vdots\,{\scriptstyle\beta_0} & {\scriptstyle(\text{Lem. E.2})} & \vdots\,{\scriptstyle\beta_0} \\[6pt]
{\scriptstyle 0}\,\beta\;{\scriptstyle(\text{Lem. E.4})} & \cdots\cdots & & \xrightarrow[\eta^{-1}]{>0} & \\[6pt]
\Big| & & {\scriptstyle 0}\,\vdots\,{\scriptstyle\beta} & {\scriptstyle(\text{Lem. E.5})} & {\scriptstyle 0}\,\vdots\,{\scriptstyle\beta} \\[6pt]
P & \underset{\eta^{-1}}{\overset{0}{\dashrightarrow}} & P_1 & \underset{\eta^{-1}}{\overset{\geqslant 0}{\dashrightarrow}} & Q
\end{array}
$$

$\qquad\square$
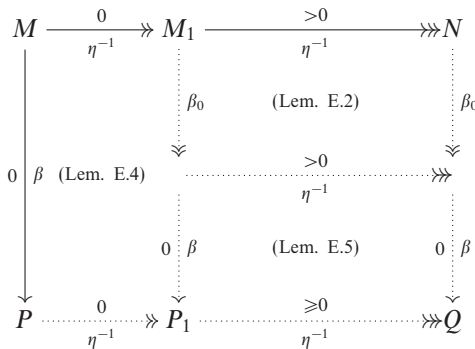
E.3. *Commutation Properties for Restricted β Reduction and η-expansion*

In this subsection, we will consider a particular instance of β-reduction in order to deal with the β-redexes created by η-expansions from truncated HNFs. For example

$$\lambda x.x \bot \bot \longrightarrow_{\eta^{-1}} \lambda y.(\lambda x.x \bot \bot)y$$
$$\lambda x.x \bot \bot \longrightarrow_{\eta^{-1}} \lambda x.(\lambda y.x \bot y)\bot.$$

In the first example, we see that the argument of the β-redex that we have created is a variable, while in the second example the argument is $\bot$.

In fact, we will define two instances of β-reduction, called respectively β_v-reduction, and β_V-reduction in order to deal with β-redexes created by η-expansions starting from a HNF.

Definition E.2 (β_v and β_V-reductions). Let $C[\ ]$ be a context with the hole at depth 0.

1. We define β_v-reduction by $C[(\lambda x.P)Q] \longrightarrow_{\beta_v} C[P[x := Q]]$ if either $Q \equiv y$ for some variable y or $Q \equiv \bot$.
2. We define β_V-reduction by $C[(\lambda x.P)Q] \longrightarrow_{\beta_V} C[P[x := Q]]$ if Q is an η-expansion of either some variable y or $\bot$.

Note that $M \longrightarrow_{\beta_v} N$ implies $M \longrightarrow_{\beta_V} N$ implies $M \overset{0}{\longrightarrow}_\beta N$ for all $M, N \in \Lambda_\bot^\infty$.

For example, $(\lambda x.x^\omega)\bot$ and $(\lambda x.x^\omega)y$ are β_v-redexes and also β_V-redexes while the terms $(\lambda x.x^\omega)(\lambda z.\bot z)$ and $(\lambda x.x^\omega)(\lambda z.yz)$ are β_V-redexes but they are not β_v-redexes.

We defined β_V because β_v and η^{-1} do not commute if the η^{-1}-step is performed at depth greater than 0. The following diagram can be completed because the right vertical line is a β_V-step.
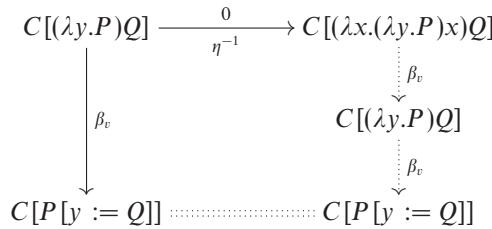
$$
\begin{array}{ccc}
(\lambda x.x \bot)y & \xrightarrow{\ \ \ 1\ \ \ }_{\eta^{-1}} & (\lambda x.x \bot)(\lambda z.yz) \\
\Big\downarrow{\scriptstyle \beta_v} & & \Big\downarrow{\scriptstyle \beta_V} \\
y\bot & \cdots\cdots\xrightarrow{\ 0\ }_{\eta^{-1}}\cdots\cdots & (\lambda z.yz)\bot
\end{array}
$$

Lemma E.7 (Local commutation of β_v and η^{-1} at depth 0). If $M_0 \longrightarrow_{\eta^{-1}} M_1$ and $M_0 \longrightarrow_{\beta_v} M_2$, then there exists an M_3 such that one of the following diagrams holds:

$$
\begin{array}{ccc}
M_0 & \xrightarrow[\eta^{-1}]{0} & M_1 \\
\Big\downarrow{\scriptstyle \beta_v}\ \ (1) & & \vdots\ {\scriptstyle \beta_v} \\
M_2 & \cdots\xrightarrow[\eta^{-1}]{0}\cdots\!\!> & M_3
\end{array}
\qquad
\begin{array}{ccc}
M_0 & \xrightarrow[\eta^{-1}]{0} & M_1 \\
& & \vdots\ {\scriptstyle \beta_v} \\
\Big\downarrow{\scriptstyle \beta_v}\ \ (2) & & N \\
& & \vdots\ {\scriptstyle \beta_v} \\
M_2 & \cdots\cdots\cdots & M_3
\end{array}
$$

Proof. Suppose M_0 can do both a β_v-redex $(\lambda y.P)Q$ at depth 0 and η^{-1}-redex N at depth 0. The only possible situations in which this can happen are as follows:
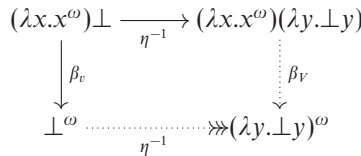
1. The β_v-redex $(\lambda y.P)Q$ and the η^{-1}-redex N are not nested, i.e. $M_0 = C[(\lambda y.P)Q, N]$. This case leads to Diagram (1).
2. The β_v-redex is inside the η^{-1}-redex, that is M_0 is of the form $C_1[N]$, where $N \equiv C_2[(\lambda y.P)Q]$. This case leads to Diagram (1).
3. The η^{-1}-redex is part of the body of the abstraction $\lambda y.P$ of the β_v-redex. This case leads to Diagram (1).
4. The η^{-1}-redex cannot be part of the argument Q of the β_v-redex, because the η^{-1}-redex occurs at depth 0.
5. The η^{-1}-redex coincides with the abstraction $\lambda y.P$ of the β_v-redex, that is M_0 is of the form $C[NQ]$, where $N \equiv \lambda y.P$.

$$
\begin{array}{ccc}
C[(\lambda y.P)Q] & \xrightarrow[\eta^{-1}]{\quad 0 \quad} & C[(\lambda x.(\lambda y.P)x)Q] \\
\Big\downarrow{\scriptstyle \beta_v} & & \Big\downarrow{\scriptstyle \beta_v} \\
& & C[(\lambda y.P)Q] \\
& & \Big\downarrow{\scriptstyle \beta_v} \\
C[P[y := Q]] & \cdots\cdots\cdots & C[P[y := Q]]
\end{array}
$$

This case leads to Diagram (2). Note that $C[(\lambda x.(\lambda y.P)x)Q]$ contains a β_v-step and a β_0-step. The β_0-reduction contracts the outermost redex created by the η-expansion and the β_v-reduction contracts the innermost one.

$\square$

Later, we will need a sort of strip lemma of β_v over η^{-1} where the right vertical line is a β_V-step. For example

$$
\begin{array}{ccc}
(\lambda x.x^\omega)\bot & \xrightarrow[\eta^{-1}]{\quad} & (\lambda x.x^\omega)(\lambda y.\bot y) \\
\Big\downarrow{\scriptstyle \beta_v} & & \Big\downarrow{\scriptstyle \beta_V} \\
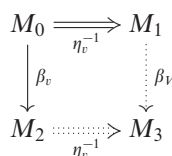\bot^\omega & \cdots\cdots\twoheadrightarrow_{\eta^{-1}} & (\lambda y.\bot y)^\omega
\end{array}
$$

For the bottom horizontal line, we will define a parallel reduction called η_v^{-1}-reduction which replaces some of the variables and $\bot$'s in a term by their η-expansions.

Definition E.3 (Parallel η_v^{-1}-reduction). We define η_v^{-1}-reduction on $\Lambda_\bot^\infty$ as follows: $M \Longrightarrow_{\eta_v^{-1}} N$ if N is obtained from M by replacing each variable x by a term X such that $x \twoheadrightarrow_{\eta^{-1}} X$ and each $\bot$ by a term B such that $\bot \twoheadrightarrow_{\eta^{-1}} B$.

For example, $\lambda x.y\bot\bot x \Longrightarrow_{\eta_v^{-1}} \lambda x.yB_1B_2X$ where $X = \lambda z.xz$, $B_1 = \lambda x.\bot x$ and $B_2 = \lambda x.\bot X$. Note that the variable y has been replaced by itself, i.e. it has not changed.

Lemma E.8. If $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$ and $M_1 \longrightarrow_{\beta_v} M_2$, then there exists a term M_3 such that:

$$
\begin{array}{ccc}
M_0 & \Longrightarrow_{\eta_v^{-1}} & M_1 \\
\Big\downarrow{\scriptstyle \beta_v} & & \Big\downarrow{\scriptstyle \beta_V} \\
M_2 & \cdots\cdots\!\!\Longrightarrow_{\eta_v^{-1}} & M_3
\end{array}
$$

Proof. Suppose $M_0 = C[(\lambda x.P)Q]$ and $(\lambda x.P)Q$ is a β_v-redex. Then, Q is either a variable y or $\perp$. The following diagram can be completed

$$C[(\lambda x.P)Q] \underset{\eta_v^{-1}}{\Longrightarrow} C'[(\lambda x.P')Q']$$

$$\downarrow \beta_v \qquad\qquad\qquad \downarrow \beta_V$$

$$C[P[x := Q]] \underset{\eta_v^{-1}}{\dashrightarrow} C'[P'[x := Q']]$$

Clearly $C[(\lambda x.P')Q'] \longrightarrow_{\beta_V} C'[P'[x := Q']]$. Since $Q \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$, either $y \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$ or $\perp \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$ and $(\lambda x.P')Q'$ is a β_V-redex.

Next, we show that $C[P[x := Q]] \Longrightarrow_{\eta_v^{-1}} C'[P'[x := Q']]$. Note that P' is obtained from P by replacing some of the variables or $\perp$'s by their η-expansions. Suppose that x has been replaced in P by its η-expansion. There are now two options for Q:

1. If $Q = y$, then $P[x := Q] \equiv P[x := y]$ and $y = Q \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$. We replace y by $X[x := Q']$. We have that $y \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X[x := Q']$ because $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X$ and $y \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$.
2. If $Q = \perp$, then $P[x := Q] \equiv P[x := \perp]$ and $\perp = Q \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$. We replace all the $\perp$'s of $P[x := \perp]$ obtained from substituting x by $\perp$ by $X[x := Q']$. We have that $\perp \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X[x := Q']$ because $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X$ and $\perp \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q'$. $\square$

Appendix F. Preservation of head normalisation by $\eta!$ and η^{-1}

In this section, we prove that the property 'having a reduction to HNF' is preserved both by $\eta!$ and η^{-1}. Both properties will be used in the proof of local commutation of $\eta!$ and $\perp_{out}$.

F.1. *Preservation of head normalisation by $\eta!$*

Lemma F.1. If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ and M is a HNF, then N is a HNF as well.

Proof. The reduction $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ can be sorted by Lemma B.4, so that it starts with a finite number of reductions at depth 0 followed by deeper reductions. Clearly, each of the depth 0 reductions preserves the HNF. And the remaining deeper reductions cannot alter the left spine of the resulting normal form. $\square$

Theorem F.1 (Preservation of head normalisation by $\eta!$). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ and M has a HNF, then so does N.

Proof. Suppose M has a β-reduces to a HNF H. Then, it has a finite β-reduction to HNF as well by strong convergence. By repeated application of Lemma 4.4, we construct the diagram:

$$M \xrightarrow{\eta!} \!\!\!\twoheadrightarrow N$$
$$\beta \downarrow \quad \text{(Lem 4.4)} \quad \downarrow \beta$$
$$H \underset{\eta!}{\dashrightarrow}\!\!\!\twoheadrightarrow H'$$

By Lemma F.1, H' is in HNF. $\qquad\square$

Using the above theorem, we can prove the following result:

Theorem F.2. (Theorem 6.1). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} N$ and M is a $\beta\bot$-normal form, then N is a $\beta\bot$-normal form.

Proof. Using Theorem 3.1, it is equivalent to prove that if $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and N is not a $\beta\bot$-normal form then M is not a $\beta\bot$-normal form. Suppose that $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and N is not a $\beta\bot$-normal form. We have two cases:

1. Suppose N is not a β-normal form. Then, N has a subterm that is a β-redex. It is easy to show by induction on the length of the η^{-1}-reduction sequence that, if $N \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M$ and N contains a β-redex then so does M.
2. Suppose N is not a $\bot$-normal form. Then, $N = C[P]$ for some P that has no HNF. Then, there exists P_0 and C_0 such that $M = C_0[P_0]$ and $P \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} P_0$. By Theorem 3.1, $P_0 \longrightarrow\!\!\!\twoheadrightarrow_{\eta!} P$. By Theorem F.1, P_0 does not have HNF. But that implies that M contains a $\bot$-redex.
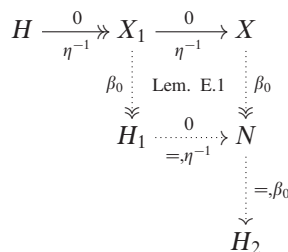
$\qquad\square$

F.2. *Preservation of head normalisation by η^{-1}*

This is more involved than for $\eta!$ because of the obvious complication that when $M \longrightarrow_{\eta^{-1}} N$ and M is in HNF, then N may not be in HNF. Consider as an example of this the HNF xPQ which η^{-1}-reduces to the term $(\lambda y.xPy)Q$. The latter term is not a HNF itself. However, in this case as well in the general case, there is a further β-reduction to HNF.

Lemma F.2 (Head normalisation of η-expansions of a variable). Let x, y_1, ..., y_n be all different variables. If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} X$ then $X \longrightarrow\!\!\!\twoheadrightarrow_{\beta_0} \lambda y_1 \ldots y_n.xY_1 \ldots Y_n$, where $y_i \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ and x does not occur free in Y_i for $1 \leqslant i \leqslant n$.

Proof. First, we consider a special instance of the lemma. Suppose that $x \overset{0}{\longrightarrow\!\!\!\twoheadrightarrow}_{\eta^{-1}} X$. By induction on the length of this reduction, it follows that $X \longrightarrow\!\!\!\twoheadrightarrow_{\beta_0} \lambda y_1 \ldots y_n.xy_1 \ldots y_n$. Because, if the length is zero, we are done, and if the length is non-zero, then $x \overset{0}{\longrightarrow}_{\eta^{-1}} X_1 \overset{0}{\longrightarrow\!\!\!\twoheadrightarrow}_{\eta^{-1}} X$. Now, by induction hypothesis we get $X_1 \longrightarrow\!\!\!\twoheadrightarrow_{\beta_0} H_1 \equiv \lambda y_1 \ldots y_n.xy_1 \ldots y_n$, so that, with by a repeated use of Lemma E.1, we obtain the diagram:

$$
\begin{array}{ccccc}
H & \overset{0}{\underset{\eta^{-1}}{\longrightarrow\!\!\!\twoheadrightarrow}} & X_1 & \overset{0}{\underset{\eta^{-1}}{\longrightarrow}} & X \\
 & {}_{\beta_0}\Big\downarrow & {\scriptstyle\text{Lem. E.1}} & {}_{\beta_0}\Big\downarrow & \\
 & & H_1 & \overset{0}{\underset{=,\eta^{-1}}{\dashrightarrow}} & N \\
 & & & & \Big\downarrow{\scriptstyle =,\beta_0} \\
 & & & & H_2
\end{array}
$$

We distinguish four type of positions where an η-expansion at depth 0 can be performed in H_1:

1. At the position of the subterm $xy_1 \ldots y_n$. Then, N is the HNF $\lambda y_1 \ldots y_n z.xy_1 \ldots y_n z$ and $H_2 = N$.
2. Between two applications. Then $N \equiv \lambda y_1 \ldots y_n.(\lambda z.xy_1 \ldots y_i z)y_{i+1} \ldots y_n$, so that $N \longrightarrow_{\beta_0} H_1$ and $H_2 = H_1$.
3. Before an abstraction. Then $N \equiv \lambda y_1 \ldots y_i z.(\lambda y_{i+1} \ldots y_n.xy_1 \ldots y_n)z$, so that $N \longrightarrow_{\beta_0} H_1$ and $H_2 = H_1$.

We are now ready for the general case. Assume $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. By Lemma B.2, we can assume that this η^{-1}-reduction is of the form $x \xrightarrow{\ 0\ }_{\eta^{-1}} X_1 \xrightarrow{\ >0\ }\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. By the above, there exists some H_1 such that $X_1 \longrightarrow_{\beta_0} H_1 = \lambda y_1 \ldots y_n.xy_1 \ldots y_n$. Then with a repeated use of Lemma E.2, we obtain the diagram:

$$
\begin{array}{ccc}
x \xrightarrow{\quad 0 \quad}_{\eta^{-1}} X_1 & \xrightarrow{\quad >0 \quad}_{\eta^{-1}}\!\!\!\!\!\twoheadrightarrow X \\
\beta_0 \Big\downarrow \quad \text{Lem. E.2} \quad \beta_0 \Big\downarrow \\
H_1 \xrightarrow[\eta^{-1}]{\quad >0 \quad}\!\!\!\!\!\twoheadrightarrow H_2
\end{array}
$$

Since all the redexes in the bottom η^{-1}-reduction occur at depth greater than 0, H_2 is a HNF of the form $\lambda y_1 \ldots y_n.xY_1 \ldots Y_n$ where $y_i \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ for all $1 \leqslant i \leqslant n$. $\qquad\square$

The previous lemma has an important consequence: we do not need the $\bot$-rule to obtain the β-normal form of an η-expansion of a variable.

Theorem F.3 (Theorem 6.2).
  If $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$, then $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_\beta(X)$.

Proof. Suppose $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. Then by Lemma F.2, X has a finite β-reduction to a HNF $\lambda y_1 \ldots y_n.xY_1 \ldots Y_n$, where $y_i \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ for each $1 \leqslant i \leqslant n$. We can now repeat Lemma F.2 and β-reduce all the Y_i's to HNFs $\lambda z_1 \ldots z_{n_i}.y_i Z_1 \ldots Z_{n_i}$. And we can continue the same process on the Z_j. In every step of this process, we reveal a new layer of the β-normal form of X. In this way, we construct a strongly converging β-reduction from X to its β-normal form $\mathsf{nf}_\beta(X)$. Hence, $X \longrightarrow\!\!\!\!\!\twoheadrightarrow_\beta \mathsf{nf}_\beta(X)$. By Theorem C.1, we have that $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} \mathsf{nf}_\beta(X)$. $\qquad\square$

Lemma F.3 (Head normalisation of applications of η-expansions of x). If $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$, then $XN_1 \ldots N_k$ is head normalising for any $N_1, \ldots, N_k \in \Lambda_\bot^\infty$, and x occurs free as head variable in $XN_1 \ldots N_k$.

Proof. Suppose $x \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} X$. By Lemma F.2, $X \longrightarrow\!\!\!\!\!\twoheadrightarrow_\beta \lambda y_1 \ldots y_m.xY_1 \ldots Y_m$ where $y_i \neq x$ and $y_i \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Y_i$ for all $1 \leqslant i \leqslant m$. We have two cases:

1. Case $m \leqslant k$. Then

$$
\begin{aligned}
XN_1 \ldots N_k &\longrightarrow_\beta (\lambda y_1 \ldots y_m.xY_1 \ldots Y_m)N_1 \ldots N_k \\
&\longrightarrow_\beta xY_1^* \ldots Y_m^* N_{m+1} \ldots N_k
\end{aligned}
$$

where $Y_i^* = Y_i[y_i := N_i]$ for all $1 \leqslant i \leqslant m$.

2. Case $m > k$. Then

$$XN_1 \ldots N_k \longrightarrow_\beta (\lambda y_1 \ldots y_m.xY_1 \ldots Y_m)N_1 \ldots N_k$$
$$\longrightarrow_\beta \lambda y_{k+1} \ldots y_m.xY_1^* \ldots Y_k^* Y_{k+1} \ldots Y_m$$

where $Y_i^* = Y_i[y_i := N_i]$ for all $1 \leqslant i \leqslant k$.

$\square$

The η-expansions of a variable only create β_0-redexes but the η-expansions of an arbitrary HNF may also create β_v-redexes. For example, if $H = \lambda x.xP$ then, we have several cases depending on where the η-expansion in H is performed:

1. In the subterm xP, i.e. $N = \lambda xz.xPz$. Then, N is in HNF.
2. Between applications, i.e. $N = \lambda x.(\lambda z.xz)P$. So, we have $N \longrightarrow_{\beta_0} H$.
3. Before the abstraction, i.e. $N = \lambda z.(\lambda x.xP)z$, in which case $N \longrightarrow_{\beta_v} H$.

The combination of β_0 and β_v (or β_V) does not have nice commuting properties with respect to η-expansions of HNFs. In order to prove head normalisation of η-expansions of an arbitrary HNF H, we will consider the truncation of H at depth 1. Because then η-expansions can create only β_V-redexes in such truncations.

Lemma F.4 (Head normalisation of η-expansions of truncated hnf). Let $H \equiv \lambda x_1 \ldots x_m.x\bot \ldots \bot$. If $H \xrightarrow{0}_{\eta^{-1}} M$, then there exists H' such that $M \longrightarrow_{\beta_v} H'$ and $H' \equiv \lambda x_1 \ldots x_m y_1 \ldots y_n.x\bot \ldots \bot y_1 \ldots y_n$ where $y_i \neq x$ for all $1 \leqslant i \leqslant m$.

Proof. This is proved by induction on the length of the reduction. We prove the successor case. Suppose $H \xrightarrow{0}_{\eta^{-1}} M_1 \xrightarrow{0}_{\eta^{-1}} M_2$. By induction hypothesis, there exists H_1 in HNF such that $M_1 \longrightarrow_{\beta_v} H_1$. By a repeated use of Lemma E.7, we obtain the following diagram:

$$
\begin{array}{ccccc}
H & \xrightarrow[\eta^{-1}]{0} & M_1 & \xrightarrow[\eta^{-1}]{0} & M_2 \\
& & \Big\downarrow_{\beta_v} \quad \text{Lem. E.7} & & \Big\downarrow_{\beta_v} \\
& & H_1 & \xdashrightarrow[\eta^{-1}]{0} & N \\
& & & & \Big\downarrow_{=,\beta_v} \\
& & & & H_2
\end{array}
$$

By induction hypothesis, $H_1 \equiv \lambda x_1 \ldots x_m y_1 \ldots y_n.x\bot \ldots \bot y_1 \ldots y_n$ where $x \neq y_i$ for $1 \leqslant i \leqslant n$. We distinguish four type of places where the η-expansion in H_1 can take place:

1. At the subterm $x\bot \ldots \bot y_1 \ldots y_n$, i.e.

$$N \equiv \lambda x_1 \ldots x_n y_1 \ldots y_n z.x\bot \ldots \bot y_1 \ldots y_n z.$$

Then, N is in HNF and $H_2 = N$.

2. Between two applications, i.e.

$$N \equiv \lambda x_1 \ldots x_n y_1 \ldots y_n.(\lambda z.x\bot \ldots \bot z)\bot \ldots \bot y_1 \ldots y_n.$$

Then, $N \longrightarrow_{\beta_v} H_1$ and $H_2 = H_1$.

3. Before one of the λy_i, e.g.

$$N \equiv \lambda x_1 \ldots x_n y_1 \ldots y_i z.(\lambda y_{i+1} \ldots y_n.x\bot \ldots \bot y_1 \ldots y_n)z.$$

Then, $N \longrightarrow_{\beta_v} H_1$ and $H_2 = H_1$.

4. Before one of the λx_i, e.g. $N \equiv \lambda x_1 \ldots x_i z.(\lambda x_{i+1} \ldots x_m y_1 \ldots y_n.x\bot \ldots \bot y_1 \ldots y_n)z$. Then, $N \longrightarrow_{\beta_v} H_1$ and $H_2 = H_1$.

$\square$

Lemma F.5 (Parallel η-expansions). Let $N \in \Lambda_\bot^\infty$ be the truncation of some term at depth 1. If $N \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} M_0 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_1$, then $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$.

Proof. Assume that $N = M^1$ for some M in $\Lambda_\bot^\infty$ and suppose $N \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} M_0$. Then, both N and M_0 have only subterms at depth 1 that are either a variable or $\bot$. Suppose further that $M_0 \xrightarrow{>0}\!\!\twoheadrightarrow_{\eta^{-1}} M_1$. Since these η-expansion steps are performed at depth 1 or deeper, we find that each subterm at depth 1 in M_1 is an η-expansion of a subterm at the same position in M_0. Hence, $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$. $\square$

Lemma F.6 (Approximation for η^{-1}). If $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$, then there is a P such that $M^1 \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} P$ where $N^1 \leq P \leq N$.

Proof. By Lemma 3.4, we can assume that the η^{-1}-reduction sequence has at most length ω and by Lemma B.2 we can assume it is sorted by increasing order of depth. Suppose the reduction sequence is finite, i.e.

$$M = M_0 \xrightarrow{n_0}_{\eta^{-1}} M_1 \xrightarrow{n_1}_{\eta^{-1}} M_2 \xrightarrow{n_2}_{\eta^{-1}} \ldots \xrightarrow{n_m}_{\eta^{-1}} M_m.$$

We construct a reduction sequence from M^1 of the form:

$$M^1 = P_0 \xrightarrow{n_0}_{=,\eta^{-1}} P_1 \xrightarrow{n_1}_{=,\eta^{-1}} P_2 \xrightarrow{n_2}_{=,\eta^{-1}} \ldots \xrightarrow{n_m}_{=,\eta^{-1}} P_m.$$

such that $(M_i)^1 \leq P_i \leq M_i$, for all $0 \leq i \leq m$ by induction on m.

The case $m = 0$ is trivial. Next, consider the successor case $m = k + 1$. So, suppose $M = M_0 \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} M_k = C[N] \xrightarrow{n_k}_{\eta^{-1}} M_{k+1} = C[\lambda x.Nx]$.

We have two possibilities depending on the depth of the η^{-1}-step:

1. Suppose the depth n_k of the η^{-1}-step is 0, i.e. $M = M_0 \xrightarrow{0}\!\!\twoheadrightarrow_{\eta^{-1}} M_k = C[N] \xrightarrow{0}_{\eta^{-1}} C[\lambda x.Nx] = M_{k+1}$. By induction hypothesis, $(M_k)^1 \leq P_k \leq M_k$. Since $(M_k)^1 \leq P_k$ and the hole in C occurs at depth 0, we have that there exist N_1 and C_1 such that $P_k = C_1[N_1]$ where $C^1 \leq C_1$ and $N^1 \leq N_1$. Since $P_k \leq M_k$, we also have that $C_1 \leq C$ and $N_1 \leq N$. By setting $P_{k+1} = C_1[\lambda x.N_1 x]$, we have that

$$P_k = C_1[N_1] \xrightarrow{0}_{\eta^{-1}} C_1[\lambda x.N_1 x] = P_{k+1}.$$

where $(M_{k+1})^1 = C^1[\lambda x.N^1\bot] \leq C_1[\lambda x.N_1 x] = P_{k+1} \leq C[\lambda x.Nx] = M_{k+1}$.

2. Suppose the depth n_k of the η^{-1}-step is greater than 0. Then, $M = M_0 \xrightarrow{\leq n_k}\!\!\twoheadrightarrow_{\eta^{-1}} M_k = C[N] \xrightarrow{n_k}_{\eta^{-1}} M_{k+1} = C[\lambda x.Nx]$. Since $P_k \leq M_k$ (by induction hypothesis), P_k is obtained from M_k by replacing some of its subterms by $\bot$. We have two possibilities:

a. A subterm of M_k containing N is replaced by $\bot$ in P_k, i.e. $M_k = C[N] = C'[C''[N]]$ and $P_k = C'_1[\bot]$ for some C'_1 such that $C'_1 \leq C'$. We set $P_{k+1} = P_k = C'_1[\bot]$. Note that we also have that $(M^{k+1})^1 = (M_k)^1$ because the position of the hole in C occurs at depth n_k greater than 0. Hence, $(M_{k+1})^1 = (M_k)^1 \leq P_k = P_{k+1} = C'_1[\bot] \leq C'[C''[\lambda x.Nx]] = C[\lambda x.Mx] = M_{k+1}$.

b. Otherwise, $M_k = C[N] = C'[C''[N]]$ and $P_k = C_1[N] = C'_1[C''_1[N_1]]$ where $C'_1 \leq C'$, $C''_1 \leq C''$, $N_1 \leq N$ and the holes of C' and C'_1 occur at depth 1. By setting $P_{k+1} = C_1[\lambda x.N_1x]$, we have that

$$P_k = C_1[N_1] \xrightarrow{n_k}_{\eta^{-1}} C_1[\lambda x.N_1 x] = P_{k+1}.$$

By induction hypothesis, $(M_k)^1 \leq P_k$ and $C'^1[\bot] = (M_k)^1 \leq P_k = C'_1[C''_1[N_1]]$. Hence, $C'^1 \leq C'_1$. Hence, $(M_{k+1})^1 = (M_k)^1 \leq C'^1[\bot] \leq C'_1[C''_1[\lambda x.N_1 x]] = P_{k+1} \leq C[\lambda x.Nx] = M_{k+1}$.

Finally consider the limit case. Suppose we have a strongly convergent reduction of length ω:

$$M = M_0 \xrightarrow{n_0}_{\eta^{-1}} M_1 \xrightarrow{n_1}_{\eta^{-1}} M_2 \xrightarrow{n_2}_{\eta^{-1}} \ldots M_\omega$$

By induction, we can construct the infinite reduction that performs the η-expansions at the same depth:

$$M^1 = P_0 \xrightarrow{n_0}_{=,\eta^{-1}} P_1 \xrightarrow{n_1}_{=,\eta^{-1}} P_2 \xrightarrow{n_2}_{=,\eta^{-1}} \ldots P_\omega$$

The above sequence is strongly convergent and hence, it has a limit P_ω.

We first prove that $(M_\omega)^1 \leq P_\omega$. There exists n_0 such that for all $n \geq n_0$, $(M_n)^1 = (M_\omega)^1$ and $(P_n)^1 = (P_\omega)^1$. By induction, $(M_n)^1 \leq P_n$. Hence, $(M_\omega)^1 = (M_n)^1 \leq (P_n)^1 = (P_\omega)^1 \leq P_\omega$.

We prove that $P_\omega \leq M_\omega$ by showing that $(P_\omega)^k \leq (M_\omega)^k$ for all k. For any k, there exists n_0 such that for all $n \geq n_0$, $(M_n)^k = (M_\omega)^k$ and $(P_n)^k = (P_\omega)^k$. By induction, $P_n \leq M_n$. Hence, $(P_\omega)^k = (P_n)^k \leq (M_n)^k = (M_\omega)^k$. $\qquad \square$

Lemma F.7 (Head normalisation of η-expansions of HNFs). If H is a HNF and $H \twoheadrightarrow_{\eta^{-1}} M$, then M is head normalising.

Proof. Let $H = \lambda x_1 \ldots x_m.y N_1 \ldots N_k$. We consider the truncation $H^1 = \lambda x_1 \ldots x_m.y \bot \ldots \bot$ of H at depth 1. If $H \twoheadrightarrow_{\eta^{-1}} M$, then there exists $M_1 \leq M$ such that $H^1 \twoheadrightarrow_{\eta^{-1}} M_1$ by Lemma F.6. We will show that M_1 is head normalising. By Lemma B.2, we can assume that the η^{-1}-reduction is of the form $H^1 \xrightarrow{0}_{\eta^{-1}} M_0 \xrightarrow{>0}_{\eta^{-1}} M_1$. By Lemma F.5, we have that $M_0 \Longrightarrow_{\eta_v^{-1}} M_1$. By Lemma F.4, we have that $M_0 \twoheadrightarrow_{\beta_v} H_0$ and H_0 is in HNF.
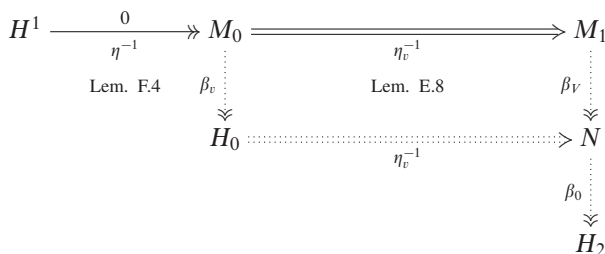
Suppose $H_0 = \lambda y_1 \ldots y_n.zP_1 \ldots P_l$. Then, $N = \lambda y_1 \ldots y_n.ZQ_1 \ldots Q_l$ where $z \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Z$ and $P_i \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} Q_i$ for $1 \leqslant i \leqslant l$. By Lemma F.3, we have that N is head normalising. Hence, M_1 is also head normalising. By monotonicity of $\mathsf{nf}_{\beta\perp}$, we have that $\mathsf{nf}_{\beta\perp}(M_1) \preceq \mathsf{nf}_{\beta\perp}(M)$ and hence, if M_1 is head normalising, so is M. $\qquad\square$

Theorem F.4 (Preservation of head normalisation by η^{-1}). Let $M \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} N$. If M has a HNF, so does N.

Proof. Suppose M β-reduces to a HNF H. By Lemma A.1, $M \xrightarrow{\;0\;}\!\!\!\!\twoheadrightarrow_\beta H$. Then

$$
\begin{array}{ccc}
M & \xrightarrow[\eta^{-1}]{\hspace{2cm}}\!\!\!\!\!\twoheadrightarrow & N \\[2pt]
\beta \Big\downarrow {\scriptstyle 0} & {\scriptstyle (\text{Lem E.6})}\quad \beta \Big\downarrow {\scriptstyle 0} & \\[6pt]
H & \cdots\!\!\!\!\!\twoheadrightarrow & Q \\
& {\scriptstyle \eta^{-1}} &
\end{array}
$$

By Lemma F.7, Q has a HNF. $\qquad\square$

As a consequence of the above theorem and Theorem 3.1, we have the following:

Corollary F.1 (Preservation of $\perp$-redexes by $\eta!$). Let $M \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta!} N$. If M does not have a HNF, neither does N.

In order to prove that the η-expansions of a variable do not contain subterms without HNF, we will need the following theorem:

Theorem F.5 (Preservation of subterm head normalisation by η^{-1}). Let $M \longrightarrow\!\!\!\!\!\twoheadrightarrow_{\eta^{-1}} N$. If all subterms of M are head normalising, then all subterms of N are head normalising too.

Proof. We prove it by induction on the length of the reduction sequence. First, we consider the one-step case. Suppose $M = C[P] \longrightarrow_{\eta^{-1}} C[\lambda x.Px] = N$ and all subterms of M are head normalising. Let Q be a subterm of N at position q. We do a case analysis:

1. Q is a subterm of $\lambda x.Px$.

   a. $Q = x$. This case is trivial.

   b. $Q = Px$. Since P is a subterm of M, we have that P is head normalising, i.e. $P \longrightarrow_\beta \lambda y_1 \ldots y_k.zP_1 \ldots P_n$. Hence, $Q = Px \longrightarrow_\beta \lambda y_2 \ldots y_k.(zP_1 \ldots P_n)[y_1 := x]$ is also head normalising.

   c. $Q = \lambda x.Px$. This case is similar to the previous case.

   d. Q is a subterm of P in N, then so it is in M as well, and therefore head normalising.

2. Q is not an subterm of $\lambda x.Px$. Then at the same position q we find a possible different subterm Q' in M. By assumption Q' is head normalising.

   a. P is a subterm of Q', then $Q' \longrightarrow_{\eta^{-1}} Q$ and hence by Theorem F.4, we see that Q is head normalising.

   b. P is not a subterm of Q', then $Q = Q'$, and so Q is head normalising.

Hence, we have found that all subterms of N are head normalising.

For any finite reduction $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$ we have that, if all subterms of M are head normalising, then so are all of N. By η^{-1}-compression Lemma 3.4, the remaining situation we have to consider is a reduction $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$ of length ω. So consider a subterm Q of N at some depth n. By strong convergence, there is $M \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} C[Q'] \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$ so that Q' occurs at depth n in $C[Q']$ and $Q' \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} Q$. Assuming that all subterms of M are head normalising, it follows from induction hypothesis that all subterms of $C[Q']$ are head normalising, in particular Q'. Again, by Theorem F.4, we find that Q is head normalising. □

Corollary F.2 (Preservation of $\eta!$-redexes by $\bot$). If $x \longrightarrow\!\!\!\twoheadrightarrow_{\eta^{-1}} N$ then N does not contain any subterm without HNF. Hence, if $\lambda x.MN$ is an $\eta!$-redex and $\lambda x.MN \longrightarrow\!\!\!\twoheadrightarrow_{\bot} \lambda x.M'N'$, then $N = N'$ and $\lambda x.M'N'$ is an $\eta!$-redex.

Appendix G. Commutation of $\eta!$ and $\bot_{out}$

Lemma G.1 (Lemma 5.1).

If $M_0 \longrightarrow_{\eta!} M_1$ and $M_0 \longrightarrow_{\bot_{out}} M_2$, there exists M_3 such that

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\ \ m\ \ }_{\eta!} & M_1 \\
{\scriptstyle \bot_{out}}\downarrow{\scriptstyle n} & & {\scriptstyle \bot_{out}}\vdots{\scriptstyle n} \\
M_2 & \dashrightarrow[=,\eta!]{\ \ m\ \ } & M_3
\end{array}
$$

Proof. Suppose M_0 can do both a $\bot_{out}$-reduction and an $\eta!$-reduction. Out of the potentially five relative positions of these two redexes only three are actually possible:

1. The $\bot_{out}$-redex U and the $\eta!$-redex $\lambda x.MN$ are not nested, i.e. $M_0 = C[U, \lambda x.MN]$. We have to show that if U is the outermost redex of $M_0 = C[U, \lambda x.MN]$, then U is also outermost in $M_1 = C[U, M]$. Using Theorem F.1, U cannot be a subterm of a term without HNF. This results in the following diagram.

$$
\begin{array}{ccc}
M_0 & \xrightarrow{\ \ m\ \ }_{\eta!} & M_1 \\
{\scriptstyle \bot_{out}}\downarrow{\scriptstyle n} & & {\scriptstyle \bot_{out}}\vdots{\scriptstyle n} \\
M_2 & \dashrightarrow[\eta!]{\ \ m\ \ } & M_3
\end{array}
$$

2. The $\bot_{out}$-redex is inside the first term of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.C_2[U]N]$. Using Theorem F.1, similarly to the previous case, we can show that U is outermost in $M_1 = C_1[C_2[U]]$. This results in the same diagram as the previous case.

3. The $\eta!$-redex is inside the $\perp_{out}$-redex. Corollary F.1 and Theorem F.1 ensure that the contracted term is still a $\perp_{out}$-redex. This results in the following diagram.

$$
\begin{array}{ccc}
C[U] & \xrightarrow{\ m\ }_{\eta!} & C[U'] \\
\perp_{out} \downarrow n & & \perp_{out} \downarrow n \\
C[\perp] & \cdots\cdots & C[\perp]
\end{array}
$$

4. The $\perp_{out}$-redex is inside the expanded variable term of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.MC_2[U]]$. This option is impossible by Corollary F.2.
5. The $\perp_{out}$-redex is the body of the $\eta!$-redex, that is M_0 is of the form $C_1[\lambda x.U]$, where $U \equiv MN$. This possibility is excluded because U would not be an outermost $\perp$-redex. □

Lemma G.2 (Lemma 5.2).

Given a one-step reduction $M \longrightarrow_{\perp_{out}} N$ and a strongly converging reduction $M \longrightarrow\!\!\!\!\rightarrow_{\eta!} P$, then there exists Q such that

$$
\begin{array}{ccc}
M & \xrightarrow{\qquad\qquad \eta! \qquad\qquad} & \!\!\!\!\!\rightarrow P \\
\perp_{out} \downarrow n & & \perp_{out} \downarrow n \\
N & \cdots\cdots\cdots_{\eta!}\cdots\cdots\cdots & \!\!\!\!\!\rightarrow Q
\end{array}
$$

Proof. By Lemma 3.4 (Compression Lemma), we can assume that the sequence has length ω.

$$
\begin{array}{ccccccccc}
M_0 & \xrightarrow{\ n_0\ }_{\eta!} & M_1 & \xrightarrow{\ n_1\ }_{\eta!} & M_2 & \xrightarrow{\ n_2\ }_{\eta!} & M_3 & \cdots & M_\omega \\
m \downarrow \perp_{out} & & m \downarrow \perp_{out} & & m \downarrow \perp_{out} & & m \downarrow \perp_{out} & & m \downarrow \perp_{out} \\
N_0 & \xrightarrow[=,\eta!]{\ n_0\ } & N_1 & \xrightarrow[=,\eta!]{\ n_1\ } & N_2 & \xrightarrow[=,\eta!]{\ n_2\ } & N_3 & \cdots & ?
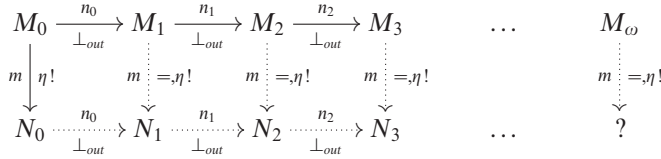\end{array}
$$

Using Lemma 5.1, we can complete all the subdiagrams except for the limit case. All the M_k are of the form $C_k[U_k]$ with the hole at the same position at depth m. The subterm U_0 is an outermost $\perp$-redex in M_0. All other U_k are terms without HNF by Corollaries F.1 and F.2. The limit term is of the form $C_\omega[U_\omega]$ and the hole of C_ω occurs also at depth m. By Corollary F.1, U_ω does not have HNF and by Theorem F.1, it cannot be a subterm of a term without HNF and, hence, it is a $\perp_{out}$-redex. Contracting this redex in the limit M_ω reduces to $C_\omega[\perp]$ which is equal to the limit N_ω of the bottom sequence. □

Lemma G.3 (Lemma 5.3).

Given a one-step reduction $M \longrightarrow_{\eta!} N$ and a strongly converging reduction $M \longrightarrow\!\!\!\!\rightarrow_{\perp_{out}} P$, then there exists Q such that

$$
\begin{array}{ccc}
M & \xrightarrow{\qquad\qquad \perp_{out} \qquad\qquad} & \!\!\!\!\!\rightarrow P \\
\eta! \downarrow n & & =,\eta! \downarrow n \\
N & \cdots\cdots\cdots_{\perp_{out}}\cdots\cdots\cdots & \!\!\!\!\!\rightarrow Q
\end{array}
$$

Proof. By (Compression Lemma 3.4 we can assume that the length of the sequence is ω.

Using Lemma 5.1, we can complete all the subdiagrams except for the limit case. Either the vertical $\eta!$-reduction got cancelled out in one of the applications of Local Commutation or not. If it gets cancelled out, then from that moment on all vertical reductions are reductions of length 0, implying that M_ω is equal to the limit N_ω. Or the vertical $\eta!$-reduction did not get cancelled out, implying that its residual is present in M_k for all $k \geqslant 0$. That is all M_k with $k \geqslant 0$ are of the form $C_k[\lambda x.S_k T_k]$, where all the $C_k[\ ]$ have the hole at the same position at depth m, and all N_k with $k \geqslant 0$ are of the form $C_k[S_k]$. The limit term is of the form $C_\omega[\lambda x.S_\omega T_\omega]$ and the hole of C_ω occurs also at depth m. By Corollary F.2, $\lambda x.S_\omega T_\omega$ is an $\eta!$-redex. Contracting this redex in the limit M_ω reduces to $C_\omega[S_\omega]$ which is equal to the limit N_ω of the bottom sequence. $\qquad\square$

Appendix H. $\infty\eta$-Böhm tree as a lambda model

Theorem H.1 (Theorem 8.1). $(\mathfrak{B}_{\infty\eta}, \bullet, [\![\ ]\!])$ is a syntactical model of the lambda calculus, that is, it satisfies for all ρ:

1. $[\![x]\!]_\rho = \rho(x)$
2. $[\![MN]\!]_\rho = [\![M]\!]_\rho \bullet [\![N]\!]_\rho$
3. $[\![\lambda x.M]\!]_\rho \bullet P = [\![M]\!]_{\rho(x:=P)}$
4. $\rho \mid FV(M) = \rho' \mid FV(M)$ implies $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$
5. if $[\![M]\!]_{\rho(x:=P)} = [\![N]\!]_{\rho(x:=P)}$ for all $P \in \mathfrak{B}_{\infty\eta}$, then $[\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$
6. $[\![\lambda xy.xy]\!]_\rho = [\![\lambda x.x]\!]_\rho$

Proof. First a remark: note that, because $(\lambda x.M)^\rho = \lambda x.M^{\rho(x:=x)}$, we have $[\![\lambda x.M]\!]_\rho = \infty\eta\mathsf{BT}(\lambda x.M^{\rho(x:=x)})$.

1. $[\![x]\!]_\rho = \infty\eta\mathsf{BT}(x^\rho) = \infty\eta\mathsf{BT}(\rho(x)) = \rho(x)$, as $\rho(x) \in \mathfrak{B}_{\infty\eta}$.
2. We have that
$$\begin{aligned}
[\![MN]\!]_\rho &= \infty\eta\mathsf{BT}((MN)^\rho) \\
&= \infty\eta\mathsf{BT}(M^\rho N^\rho) \\
&= \infty\eta\mathsf{BT}(\infty\eta\mathsf{BT}(M^\rho)\infty\eta\mathsf{BT}(N^\rho)) \text{ by Theorem 6.4} \\
&= \infty\eta\mathsf{BT}([\![M]\!]_\rho[\![N]\!]_\rho) \\
&= [\![M]\!]_\rho \bullet [\![N]\!]_\rho
\end{aligned}$$
3. For arbitrary P in $\mathfrak{B}_{\infty\eta}$, we have that
$$\begin{aligned}
[\![\lambda x.M]\!]_\rho \bullet P &= \infty\eta\mathsf{BT}([\![\lambda x.M]\!]_\rho P) \\
&= \infty\eta\mathsf{BT}(\infty\eta\mathsf{BT}(\lambda x.M^{\rho(x:=x)})P) \\
&= \infty\eta\mathsf{BT}((\lambda x.M^{\rho(x:=x)})P) \text{ by Theorem 6.4} \\
&= \infty\eta\mathsf{BT}(M^{\rho(x:=x)(x:=P)}) \text{ by } \beta\text{-reduction and Theorem 6.4} \\
&= [\![M]\!]_{\rho(x:=P)}
\end{aligned}$$

4. $\rho \mid FV(M) = \rho' \mid FV(M)$ implies $M^\rho = M^{\rho'}$, hence $\infty\eta\mathsf{BT}(M^\rho) = \infty\eta\mathsf{BT}(M^{\rho'})$, and so $[\![M]\!]_\rho = [\![M]\!]_{\rho'}$.

5. Similar to the proof of 18.3.10.i in Barendregt (1984):

$\forall P \in \mathfrak{B}_{\infty\eta}\, [\![M]\!]_{\rho(x:=P)} = [\![N]\!]_{\rho(x:=P)}$

$\Rightarrow\ [\![M]\!]_{\rho(x:=x)} = [\![N]\!]_{\rho(x:=x)}$

$\Rightarrow\ \infty\eta\mathsf{BT}(M^{\rho(x:=x)}) = \infty\eta\mathsf{BT}(N^{\rho(x:=x)})$

$\Rightarrow\ \infty\eta\mathsf{BT}(\lambda x.\infty\eta\mathsf{BT}(M^{\rho(x:=x)})) = \infty\eta\mathsf{BT}(\lambda x.\infty\eta\mathsf{BT}(N^{\rho(x:=x)}))$ by Theorem 6.4

$\Rightarrow\ \infty\eta\mathsf{BT}(\lambda x.M^{\rho(x:=x)}) = \infty\eta\mathsf{BT}(\lambda x.N^{\rho(x:=x)})$ by Theorem 6.4

$\Rightarrow\ \infty\eta\mathsf{BT}((\lambda x.M)^\rho) = \infty\eta\mathsf{BT}((\lambda x.N)^\rho)$

$\Rightarrow\ [\![\lambda x.M]\!]_\rho = [\![\lambda x.N]\!]_\rho$

6. Since $\lambda xy.xy \longrightarrow_{\eta!} \lambda x.x$, we have that $\infty\eta\mathsf{BT}((\lambda xy.xy)^\rho) = \infty\eta\mathsf{BT}((\lambda x.x)^\rho)$ and therefore $[\![\lambda xy.xy]\!]_\rho = [\![\lambda x.x]\!]_\rho$. $\qquad\square$

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