Teaching Notes

A quick simultaneous evaluation of $\sum k^3$ and $\sum k^2$

In this short Note I offer a "two for the price of one" evaluation of the standard sums in the title.

Expanding brackets, we have:

$$\left(\sum_{k=1}^{n} k\right)^2 = \sum_{k=1}^{n} k^2 + 2\sum_{k=2}^{n} \big(1 + 2 + \ldots + (k-1)\big)k$$

so, using $\sum_{k=1}^{n} k = \frac{1}{2}n(n+1)$ on both sides,

$$\frac{1}{4}n^2(n+1)^2 = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k^2(k-1) \qquad (1)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} k^3 - \sum_{k=1}^{n} k^2,$$

whence $\sum_{k=1}^{n} k^3 = \frac{1}{4}n^2(n+1)^2$.

But this means that we can rewrite (1) as:

$$\sum_{k=1}^{n} k^3 = \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} (k-1+1)^2(k-1)$$

$$= \sum_{k=1}^{n} k^2 + \sum_{k=1}^{n} (k-1)^3 + 2\sum_{k=1}^{n} (k-1)^2 + \sum_{k=1}^{n} (k-1)$$

so, on combining terms with the same powers,

$$n^3 = 3\sum_{k=1}^{n} k^2 - 2n^2 + \tfrac{1}{2}n(n-1)$$

and $$\sum_{k=1}^{n} k^2 = \tfrac{1}{6}\left(2n^3 + 3n^2 + n\right) = \tfrac{1}{6}n(n+1)(2n+1).$$

10.1017/mag.2022.32

NICK LORD
Tonbridge School,
Tonbridge, Kent TN9 1JP
e-mail: *njl@tonbridge-school.org*

How to impress a chemist (again!)

This note is a sequel to [1] and – spoiler alert – it ends in the same place!

We start with a circular filter paper with a radial slit and, with a paper clip, form it into a right circular cone of adjustable height. If we maximise the volume of this cone, its vertical angle is, as in [1], $\cos^{-1}\left(-\frac{1}{3}\right) = 109.5°$, the bond angle of methane.

https://doi.org/10.1017/mag.2022.32 Published online by Cambridge University Press