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Point-source scalar turbulence

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The statistics of a passive scalar randomly emitted from a point source is investigated analytically for the Kraichnan ensemble. Attention is focused on the two-point equal-time scalar correlation function, a statistical indicator widely used both in experiments and in numerical simulations. The only source of inhomogeneity/anisotropy is the injection mechanism, the advecting velocity being here statistically homogeneous and isotropic. The main question we address is on the possible existence of an inertial range of scales and a consequent scaling behaviour. The question arises from the observation that for a point source the injection scale is formally zero and the standard cascade mechanism cannot thus be taken for granted. We find from first principles that an intrinsic integral scale, whose value depends on the distance from the source, emerges as a result of sweeping effects. For separations smaller than this integral scale a standard forward cascade occurs. This is characterized by a Kolmogorov–Obukhov power-law behaviour as in the homogeneous case, except that the dissipation rate is also dependent on the distance from the source. Finally, we also find that the combined effect of a finite inertial-range extent and of inhomogeneities causes the emergence of subleading anisotropic corrections to the leading isotropic term, that are here quantified and discussed.

1. Introduction

Turbulent fields whose statistical properties are invariant under translation and rotation in space are particularly interesting to theorists. They represent a formally simple setup allowing neglect of additional 'complications' introduced by the boundaries of the system or external driving mechanisms. However, in most situations of interest, such complications might play a crucial role in determining the statistical properties of the turbulent field. For example, this is the case in channel-flow turbulence, where statistical invariance under translation and rotation (i.e. homogeneity and isotropy) is restricted to a small region around the centre of the channel, while it is totally lost close to the walls (Toschi *et al.* 1999).

In the last few years, important achievements have been made, both in the method of analysis and the level of comprehension, in relation to the effects on the small-scale statistics (i.e. the statistics at scales much smaller than the integral scale) of anisotropic large-scale contributions (see Biferale & Procaccia 2005, for a review). The situation is much less clear regarding the role of inhomogeneities in determining the statistics of small-scale turbulence. In particular, it is not clear if inhomogeneities, activated by boundaries and/or initial conditions, might disappear at small scales owing to cascade processes, which tend to eliminate the detailed memory of the large-scale dynamics. Understanding the above aspects is relevant to applications related, for example, to small-scale subgrid parameterizations. Modern approaches to closure problems commonly use scaling exponents as the basic ingredients to build subgrid-scale models. The best examples are the fractal (Scotti & Meneveau 1997) and multifractal (Basu, Foufoula-Georgiou & Porté-Agel 2004) interpolation schemes (for a general review highlighting the role of scaling exponents in subgrid parameterizations, see Meneveau & Katz 2000). How to define scaling exponents in the presence of anisotropies became clear only very recently (Biferale & Procaccia 2005). The situation is different in the presence of inhomogeneities, where, up to now, basic questions related to the existence of scaling behaviour, and thus of scaling exponents, still have to be seriously addressed.

The main aim of our paper is to give quantitative answers to the above questions. To do that, we will focus on passive scalar turbulence, where the scalar is randomly emitted from a point source. This is the simplest way to mimic the release of a pollutant from a chimney. The injection mechanism is thus intimately inhomogeneous and the question of how this inhomogeneity eventually reflects on the small-scale scalar statistics can be addressed. In order to carry out the study in analytical terms, we will assume a white-in-time, homogeneous, random process to model the velocity-field statistics (Kraichnan 1968, 1994).

The paper is organized as follows. In § 2 we formulate the problem in the context of the Kraichnan advection model with inhomogeneous forcing and we adopt simple mathematical techniques to obtain the equation for the two-point equal-time scalar correlation function, whose general solution is provided in § 3 for a point-source emission. In § 4 we focus on the process of local cascade which represents an interesting example of the persistence of inhomogeneity at small scales. Section 5 deals with the interplay between inhomogeneity, anisotropy and finite-size effects, which provides a correction to the dominant isotropic behaviour. Conclusions and possible future developments follow in § 6. The Appendix is devoted to the reformulation of the problem in a finite box in the presence of periodic boundary conditions.

2. Basic equations

2.1. *Kraichnan advection model*

Let us consider a passive scalar field $\theta(\boldsymbol{x}, t)$ transported by a turbulent flow:

$$\partial_t \theta + \boldsymbol{v} \cdot \boldsymbol{\partial} \theta = \kappa \partial^2 \theta + f, \tag{2.1}$$

where κ is the molecular diffusivity. The incompressible velocity field $\boldsymbol{v}(\boldsymbol{x}, t)$ is assumed statistically homogeneous and isotropic, whereas the source term $f(\boldsymbol{x}, t)$ is allowed not to be invariant under translations: a relevant example is provided by the emission of a tracer from a point source, located e.g. at the origin.

Let us now specialize to the Kraichnan ensemble (Kraichnan 1968, 1994), where the velocity is a Gaussian, zero-average, white-in-time field with two-point correlation function $\langle v_\mu(\boldsymbol{x}_1, t_1) v_\nu(\boldsymbol{x}_2, t_2)\rangle = D_{\mu\nu}(\boldsymbol{x}_1 - \boldsymbol{x}_2)\delta(t_1 - t_2)$, where

$$D_{\mu\nu}(\boldsymbol{r}) = D_0 \delta_{\mu\nu} - d_{\mu\nu}(\boldsymbol{r}). \tag{2.2}$$

The second-order moment of the velocity increments is given by

$$d_{\mu\nu}(\boldsymbol{r}) = D_1 r^\xi \left[(d + \xi - 1)\delta_{\mu\nu} - \xi \frac{r_\mu r_\nu}{r^2} \right] \tag{2.3}$$

for $r = |\boldsymbol{r}|$ smaller than the integral scale of the velocity field (L_v), above which $d_{\mu\nu}(\boldsymbol{r})$ saturates to an almost constant value with order of magnitude $D_1 L_v^{\xi}$. Consequently, since the correlation $D_{\mu\nu}(\boldsymbol{r})$ must vanish for $r \to \infty$, the relation $D_0 \sim D_1 L_v^{\xi}$ holds. Here, d is the space dimension ($\geqslant 2$) and ξ is the scaling exponent, describing the degree of roughness present in the velocity field, lying in the interval $(0, 2)$. As ξ increases the velocity field becomes increasingly smoother and eventually differentiable at $\xi = 2$.

The two-point equal-time correlation function $C(\boldsymbol{x}_1, \boldsymbol{x}_2, t) = \langle \theta(\boldsymbol{x}_1, t)\theta(\boldsymbol{x}_2, t)\rangle$ may be expressed as a function of the centre of mass $\boldsymbol{z} = (\boldsymbol{x}_1 + \boldsymbol{x}_2)/2$ and of the separation $\boldsymbol{r} = \boldsymbol{x}_1 - \boldsymbol{x}_2$. In these coordinates the equation for the correlation function $C(\boldsymbol{r}, \boldsymbol{z}, t)$ follows from the application of a standard functional method of stochastic calculus called Gaussian integration by parts (Furutsu 1963; Novikov 1965; Donsker 1964). The equation is

$$\partial_t C = [2\kappa\delta_{\mu\nu} + d_{\mu\nu}(\boldsymbol{r})]\frac{\partial^2 C}{\partial r_\mu \partial r_\nu} + \frac{(D_0 + 2\kappa)\delta_{\mu\nu} + D_{\mu\nu}(\boldsymbol{r})}{4}\frac{\partial^2 C}{\partial z_\mu \partial z_\nu} + F, \tag{2.4}$$

where $F(\boldsymbol{r}, \boldsymbol{z})$ represents the correlator $\langle \theta(\boldsymbol{x}_1, t) f(\boldsymbol{x}_2, t) + \theta(\boldsymbol{x}_2, t) f(\boldsymbol{x}_1, t)\rangle$.

Two cases appear to be relevant, also in connection with applications: a constant emission from a point source and an emission random in time (but still punctual in space). The former case turns out to be quite cumbersome to attack by analytical methods and is still under investigation. Here, we shall focus on a Gaussian, zero-average, white-in-time forcing representing a random emission from the origin. Namely, $f(\boldsymbol{x}, t) = f_0(t)\delta(\boldsymbol{x})$ with $\langle f(\boldsymbol{x}_1, t_1) f(\boldsymbol{x}_2, t_2)\rangle = F_0\delta(\boldsymbol{x}_1)\delta(\boldsymbol{x}_2)\delta(t_1 - t_2)$, so that $F(\boldsymbol{r}, \boldsymbol{z}) = F_0\delta(\boldsymbol{r})\delta(\boldsymbol{z})$. As we shall see in detail, this case is amenable to analytical treatment.

2.2. *Fourier transform and* SO(d) *decomposition*

Let us now come back to (2.4). Fourier transforming it in $\boldsymbol{z}$ and defining

$$\hat{C}(\boldsymbol{r}, \boldsymbol{q}, t) = \int \mathrm{d}^d z\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z}} C(\boldsymbol{r}, \boldsymbol{z}, t), \qquad \hat{F}(\boldsymbol{r}, \boldsymbol{q}) = \int \mathrm{d}^d z\, \mathrm{e}^{-\mathrm{i}\boldsymbol{q}\cdot\boldsymbol{z}} F(\boldsymbol{r}, \boldsymbol{z}),$$

we obtain

$$\partial_t \hat{C} = [2\kappa\delta_{\mu\nu} + d_{\mu\nu}(\boldsymbol{r})]\frac{\partial^2 \hat{C}}{\partial r_\mu \partial r_\nu} - \frac{(D_0 + 2\kappa)\delta_{\mu\nu} + D_{\mu\nu}(\boldsymbol{r})}{4} q_\mu q_\nu \hat{C} + \hat{F}. \tag{2.5}$$

In the present case, the forcing-correlation transformed $\hat{F} = F_0\delta(\boldsymbol{r})$ is independent of the wavenumber.

Equation (2.5) is differential only in $\boldsymbol{r}$ and is algebraic in the centre-of-mass wavenumber $\boldsymbol{q}$. The second term on the right-hand side represents the inhomogeneous contribution and consistently vanishes for $q = 0$, which is equivalent to an average all over the space. It is convenient to rewrite its $\boldsymbol{r}$-dependent coefficient in the following way:

$$-\frac{(D_0 + 2\kappa)\delta_{\mu\nu} + D_{\mu\nu}(\boldsymbol{r})}{4} = -\left[\frac{D_0 + \kappa}{2} - \frac{(d-1)(d+\xi)}{4d} D_1 r^{\xi}\right]\delta_{\mu\nu} + \frac{\xi}{4d} D_1 r^{\xi}\left(\delta_{\mu\nu} - d\frac{r_\mu r_\nu}{r^2}\right). \tag{2.6}$$

Substituting it back, it is clear that the last term in (2.6) generates the only contribution in (2.5) not invariant under rotations of $\boldsymbol{r}$, because it gives rise to a scalar product between $\boldsymbol{r}$ and $\boldsymbol{q}$ that mixes different angular sectors. However, at separations $r \ll L_v$, a simplification is possible, since, in that case, the order of magnitude of $d_{\mu\nu}(\boldsymbol{r}) \approx D_1 r^{\xi}$ is negligible with respect to $D_0 \sim D_1 L_v^{\xi}$. Therefore, $D_{\mu\nu}(\boldsymbol{r}) \simeq D_0 \delta_{\mu\nu}$ and the right-hand side of (2.6) simplifies to $-(D_0+\kappa)\delta_{\mu\nu}/2$. Note that, when r is of the order of (or larger than) L_v, a coupling between anisotropy and inhomogeneity takes place: we shall come back to this point in § 5, where the consequences of keeping the full form (2.6) into account will be discussed. Here, we concentrate on the case $r \ll L_v$ in the stationary state with vanishing diffusivity† and we can thus consider the simpler equation

$$d_{\mu\nu}(\boldsymbol{r})\frac{\partial^2 \hat{C}}{\partial r_\mu \partial r_\nu} - \frac{1}{2} D_0 q^2 \hat{C} + \hat{F} = 0. \tag{2.7}$$

A dimensional-analysis balance between the first and the second term in (2.7) leads to the introduction of a new scale

$$\ell_q = \left[\frac{q^2 D_0}{2(d-1)D_1}\right]^{-1/(2-\xi)},$$

which is associated with the strength of the scalar inhomogeneities and measures the separation above which they become relevant. A decomposition into the spherical harmonics (Biferale & Procaccia 2005),

$$\hat{C}(\boldsymbol{r},\boldsymbol{q}) = \sqrt{\Omega}\sum_{l,m} \hat{C}_{l,m}(r,\boldsymbol{q}) Y_{l,m}(\Phi), \qquad \hat{F}(\boldsymbol{r},\boldsymbol{q}) = \sqrt{\Omega}\sum_{l,m} \hat{F}_{l,m}(r,\boldsymbol{q}) Y_{l,m}(\Phi),$$

with Φ denoting the solid angle associated with $\boldsymbol{r}$ and Ω its overall value, yields the following equation for $\hat{C}_{l,m}(r,\boldsymbol{q})$ in each sector:

$$r^{-(d-1)}\partial_r r^{d+\xi-1}\partial_r \hat{C}_l - \frac{(d+\xi-1)l(d-2+l)}{d-1} r^{-2}\hat{C}_l - \ell_q^{-(2-\xi)}\hat{C}_l + \varphi_l = 0. \tag{2.8}$$

Note that, because of foliation on l and degeneration, we have dropped the dependence on the subscript m and we have introduced the rescaled forcing $\varphi_l(r) = \hat{F}_l(r,\boldsymbol{q})/(d-1)D_1$ (independent of $\boldsymbol{q}$ because of the punctual nature of the source).

3. General solution

The general solution of (2.8), as a function of r and ℓ_q, reduces to the zero mode (Martins Afonso & Sbragaglia 2005)

$$\hat{C}_l(r;\ell_q) = w^{-\nu_0}[A_l K_{\nu_l}(w) + B_l I_{\nu_l}(w)], \tag{3.1}$$

where

$$w = 2(2-\xi)^{-1}(r/\ell_q)^{(2-\xi)/2}, \quad \nu_l = [(d+\xi-2)^2 + 4(d+\xi-1)l(d-2+l)/(d-1)]^{1/2}/(2-\xi).$$

To determine the coefficients A_l and B_l, one can approximate the Dirac δ by a Heaviside Θ (i.e. $\delta(r) = \lim_{L\to 0}[\Theta(L-r)/L]$), exploiting the vanishing of φ_l in all the anisotropic sectors $l \neq 0$:

$$\delta(\boldsymbol{r}) = \frac{r^{-(d-1)}}{\Omega}\delta(r) = \lim_{L\to 0}\frac{r^{-(d-1)}}{\Omega L}\Theta(L-r) \Rightarrow \varphi_{l=0}(r) = \lim_{L\to 0}\frac{F_0 r^{-(d-1)}}{(d-1)D_1 \Omega L}\Theta(L-r).$$

† Attention should, in principle, be paid to the limit of vanishing diffusive scale, but for the rough flows ($\xi \neq 2$) considered here no commutation problem arises with the limit of vanishing forcing correlation length L.

It must be stressed that, in this way, the point source is obtained as a limit of a forcing having positive Corrsin integral $Q_0 \equiv \int \mathrm{d}^d\boldsymbol{r}\, \hat{F}(\boldsymbol{r},\boldsymbol{q}) = F_0$. The case $Q_0 = 0$ (Falkovich & Fouxon 2005; Celani & Seminara 2005, 2006) will be left for future investigation.

Studying the solution for $r < L$ also, matching the solution $\hat{C}_l$ and its first derivative in $r = L$, imposing regularity for small r and vanishing for large r (Martins Afonso & Sbragaglia 2005), and finally taking the limit $L \to 0$, one finds $B_l = 0\ \forall l$ and

$$A_l = \left(\frac{2-\xi}{2}\right)^{\nu_0} \ell_q^{(d-\xi+2)/2} \lim_{L\to 0}\int_0^W \mathrm{d}\omega\, \varphi_l(\rho)\omega^{\nu_0+1} I_{\nu_l}(\omega) = \delta_{l,0} k_\dagger \frac{F_0}{D_1}\ell_q^{2-d-\xi},$$

where $\omega \equiv w|_{r=\rho} = 2(2-\xi)^{-1}(\rho/\ell_q)^{(2-\xi)/2}$, $W \equiv w|_{r=L} = 2(2-\xi)^{-1}(L/\ell_q)^{(2-\xi)/2}$ and $k_\dagger = 2(2-\xi)^{-d/(2-\xi)}/(d-1)\Omega\Gamma(\nu_0+1)$ ($\Gamma(\cdot)$ being Euler's function).

In the pseudospectral space $(\boldsymbol{r},\boldsymbol{q})$ the scalar-correlation transformed thus coincides with its isotropic projection and depends only on the moduli r and q (i.e. ℓ_q) as

$$\hat{C}(r;\ell_q) = k_\dagger \frac{F_0}{D_1}\ell_q^{-(d+\xi-2)/2} r^{-(d+\xi-2)/2} K_{\nu_0}(w).$$

Back in the physical space, the correlation is thus independent of the angle between $\boldsymbol{r}$ and $\boldsymbol{z}$ and is a function of r and z only:

$$C(r,z) = k_\ddagger \frac{F_0}{D_1}\left(\frac{D_1}{D_0}\right)^{d/2} r^{d\xi/2-2d-\xi+2}\left[1+\frac{\Omega(2-\xi)^2}{4\pi}\frac{D_1}{D_0}z^2 r^{-(2-\xi)}\right]^{-\frac{d(4-\xi)}{2(2-\xi)}+1}. \tag{3.2}$$

For $d = 2$, $k_\ddagger = k_\dagger 2^{-2}\pi^{-1}(2-\xi)^{(4-\xi)/(2-\xi)}\Gamma[2/(2-\xi)]$, whereas for $d = 3$, $k_\ddagger = k_\dagger 2^{-1}\pi^{-3/2}(2-\xi)^{(7-2\xi)/(2-\xi)}\Gamma[3/2+(1+\xi)/(2-\xi)]$.

Note that the behaviour $C \sim r^{-(d+\xi-2)}$, which is typical of the homogeneous situation (Falkovich, Gawędzki & Vergassola 2001) for $r \gg L$, is not observed in this case (even if $L = 0$), unless one integrates the correlation over the whole space, thus averaging out the inhomogeneity and defining the contribution in the homogeneous 'sector'. In the pseudo-spectral space, this last operation is equivalent to considering $q = 0$ ($\Rightarrow \ell_q \to \infty \Rightarrow w = 0$) and thus corresponds to taking into account only the leading term in the development of $K_{\nu_0}(w)$ for small arguments.

4. Local cascade and small-scale persistence of inhomogeneity

Recalling that the ratio D_1/D_0 appearing in (3.2) is of the order of $L_v^{-\xi}$, two opposite developments are worth persuing, corresponding to small or large values of the quantity

$$s \equiv \left(\frac{z}{r}\right)^2\left(\frac{r}{L_v}\right)^\xi. \tag{4.1}$$

This defines a new characteristic length scale,

$$\mathscr{L}_z \equiv z^{2/(2-\xi)} L_v^{-\xi/(2-\xi)}, \tag{4.2}$$

dependent (monotonically) on z and whose meaning as the local integral scale of scalar turbulence will be clear shortly.

For small s (i.e. $r \gg \mathscr{L}_z$) the correlation is approximated by a power law in r, $C \sim L_v^{-d\xi/2} r^{-d(4-\xi)/2+2-\xi}$. On the other hand, for large s (i.e. $r \ll \mathscr{L}_z$), a power law in z is found, and r appears only in subleading terms:

$$C(r,z) \sim \langle\theta^2(z)\rangle - \text{const.}\times\epsilon(z)r^{2-\xi}. \tag{4.3}$$

Here, $\langle\theta^2(z)\rangle \equiv (F_0/D_1)L_v^{\xi(d+\xi-2)/(2-\xi)}z^{2-d(4-\xi)/(2-\xi)}$ and $\epsilon(z) \equiv \langle\theta^2(z)\rangle/\mathscr{L}_z^{2-\xi}$ represent the scalar variance and the local dissipation, respectively.

Expression (4.3) proves that, at scales smaller than $\mathscr{L}_z$ (i.e. for r sufficiently smaller than z, according to the value of L_v), the increment of the two-point equal-time correlation function has the same behaviour as the homogeneous case (power law in r with exponent $2-\xi$: see e.g. Falkovich *et al.* 2001) at scales smaller than the forcing integral scale L. This result suggests that a cascade-like mechanism might be present also in this case, even if here $r > L$ by construction.

This possibility can be supported by the following considerations. The velocity field sweeps the scalar, initially concentrated where it was released, and generates structures which, $\forall \boldsymbol{x}_1$, are correlated on the scale $\boldsymbol{x}_1$. This amounts to saying that correlations between each point $\boldsymbol{x}_1$ and the origin $\boldsymbol{x}_2 = \boldsymbol{0}$ are created. In the centre-of-mass frame of reference, this means that at every point $\boldsymbol{z}$ $(= \boldsymbol{x}_1/2)$ a local cascade can then take place, starting from separations r sufficiently smaller than z.

The quantity $\mathscr{L}_z$ thus plays the role of an effective local forcing correlation length: the word 'local' here refers to the fact that, whereas in the homogeneous situation the prefactors are expressed in terms of constant quantities, in this case a dependence of $\langle\theta^2(z)\rangle$ and $\epsilon(z)$ on the point still persists. It is also worth noticing from (4.2) that, for very rough flows ($\xi \to 0$), $\mathscr{L}_z$ becomes proportional to z, consistently with the impossibility of a consistent definition of L_v. On the other hand, for almost smooth flows ($\xi \to 2$), one finds $\mathscr{L}_z \sim (z/L_v)^{2/(2-\xi)}$, critically dependent on whether the centre-of-mass distance from the source lies within the velocity correlation range.

This physical interpretation of local cascade can easily be supported mathematically by considering the physical-space counterpart of (2.7), i.e. the simplified form of (2.4) with the usual approximations $r \ll L_v$ and $\kappa = 0 = \partial_t$:

$$d_{\mu\nu}(\boldsymbol{r})\frac{\partial^2 C}{\partial r_\mu \partial r_\nu} + \frac{D_0}{2}\frac{\partial^2 C}{\partial z_\mu \partial z_\mu} + F = 0. \tag{4.4}$$

In the homogeneous case, the absence of any dependence on $\boldsymbol{z}$ gives rise to the convective-range balance

$$d_{\mu\nu}(\boldsymbol{r})\frac{\partial^2 C}{\partial r_\mu \partial r_\nu} = -F(r); \tag{4.5}$$

the derivative with respect to r of the left-hand side of (4.5) vanishes in the presence of a constant corresponding right-hand side, as is often the case. On the other hand, with point-source forcing, away from the origin, the balance (4.4) should be written as

$$d_{\mu\nu}(\boldsymbol{r})\frac{\partial^2 C}{\partial r_\mu \partial r_\nu} = -\frac{D_0}{2}\frac{\partial^2 C}{\partial z_\mu \partial z_\mu}, \tag{4.6}$$

but the vanishing of the derivative of the left-hand side of (4.6) still occurs for r sufficiently smaller than z. This is shown in figure 1, which also reflects how this interpretation has its validity limit affected by a change in the ratio r/L_v appearing in the adimensional parameter s (4.1). One should indeed remember that the three scales r, z and L_v appear in a non-trivial way in s, whose magnitude is the key point of approximation (4.3) and its consequences.

The present case thus represents an interesting example of a situation in which inhomogeneity persists at small scales, as the behaviour $C \sim r^{2-\xi}$ observed at small r does not correspond to what was defined at the end of § 3 to be interpreted as the homogeneous counterpart ($C \sim r^{-(d+\xi-2)}$).

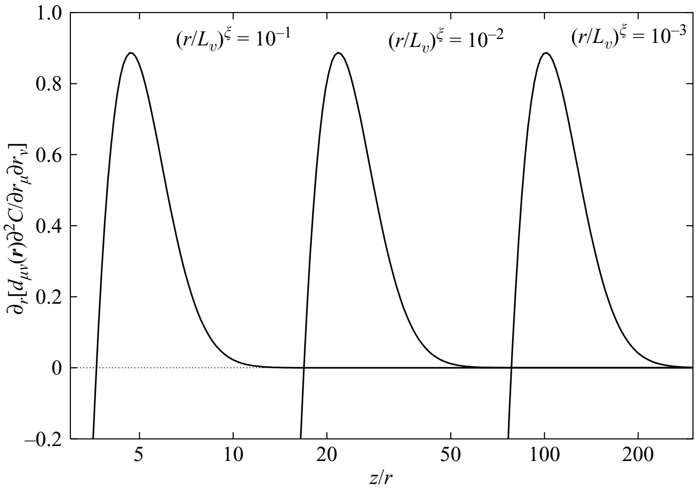

FIGURE 1. Derivative of the left-hand side of equation (4.6) with respect to r, plotted *vs* z/r for $\xi = 4/3$ and $d = 3$. It is evident how the ratio r/L_v, labelling the three curves, affects the limits of the range in which approximation (4.3) is valid and a constant flux holds.

5. Finite-size effects and anisotropic contributions

A comment is required about the relevance of the so-called *finite-size effects*. In other words, one would like to quantify the error deriving from the approximation $r \ll L_v$, which was used to simplify (2.6) and thus to decouple inhomogeneity from anisotropy. This quantification becomes possible, if one proceeds in the following way. First, note that, after the decomposition into spherical harmonics, no more foliation takes place. Namely, the equation for the isotropic sector is still a closed one (with the appearance of a new term, according to the contribution within square brackets in (2.6)),

$$r^{-(d-1)}\partial_r\left(r^{d+\xi-1}\partial_r\hat{C}_0\right) - \ell_q^{-(2-\xi)}\left[1 - \frac{(d+\xi)D_1}{2d(d-1)D_0}r^\xi\right]\hat{C}_0 + \varphi_{l=0}(r) = 0,$$

and gives

$$\hat{C}_0 \propto r^{-(d+\xi-2)/2}K_{\nu_0}(w)\left[1 + O\left(\frac{r}{L_v}\right)^\xi\right], \tag{5.1}$$

but $\hat{C}_0$ now enters the equation for $l = 2$ as a forcing term (the $l = 1$ sector remains unforced because this procedure only couples even sectors, as can be deduced by decomposition (2.6)). A simple power-counting operation is possible in Fourier space for $r \ll \ell_q$, where (specifying the order of the error in the development of $K_{\nu_0}(w)$ for small arguments)

$$\hat{C}_0 \propto r^{-(d+\xi-2)}\left[1 + O\left(\frac{r}{\ell_q}\right)^{2-\xi}\right]. \tag{5.2}$$

As a result, in this regime one easily obtains

$$\hat{C}_2 \sim L_v^{-\xi}\ell_q^{-(2-\xi)}r^{-(d+\xi-4)} \sim \left(\frac{r}{L_v}\right)^\xi\left(\frac{r}{\ell_q}\right)^{2-\xi}\hat{C}_0. \tag{5.3}$$

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Equation (5.3) shows that the first excited anisotropic sector carries a factor, with respect to the isotropic solution, given by the product of the corrections in (5.1) and in (5.2). Its interpretation is thus very simple and useful: at the lowest order, the most relevant anisotropic correction derives from the coupling of finite-size effects ($O(r/L_v)^{\xi}$) and inhomogeneities ($O(r/\ell_q)^{2-\xi}$).

It can also be shown that, in the opposite situation ($r \gg \ell_q$), $\hat{C}_2$ is still given by the right-hand side of (5.3) but without the factor $(r/\ell_q)^{2-\xi}$. This implies that, back in the physical space, the leading contribution is always the isotropic one, provided that $r \ll L_v$. The higher-l anisotropic terms $\hat{C}_4$, $\hat{C}_6$, etc. are indeed smaller and smaller, because they are forced by the (small) quantities $\hat{C}_2$, $\hat{C}_4$, etc. respectively.

Such anisotropic corrections are expected to play a non-negligible role only when the scales r and z are comparable, but not when either is much greater than the other. An example of the former case is provided, for $z = r/2$, by the comparison between the situations $\boldsymbol{z} \parallel \boldsymbol{r}$ (where one of the two points in which the correlation is calculated lies at the source) and $\boldsymbol{z} \perp \boldsymbol{r}$ (where both points are $\sqrt{2}z$ away from the origin): a difference must clearly exist, but cannot be caught by the isotropic function $C_0(r, z)$ and turns out to be subdominant. On the other hand, if $r \gg z$ the two points are almost symmetric with respect to the origin, and if $r \ll z$ their relative separation is much smaller than their distance from the source: in both cases, a rotation of $\boldsymbol{r}$ with fixed $\boldsymbol{z}$ would change little. Of course, the problem is always invariant under rigid rotations of the whole space (and thus of both vectors $\boldsymbol{r}$ and $\boldsymbol{z}$) around the origin.

6. Conclusions and perspectives

The dynamics of a passive scalar released from a point source has been investigated in this paper, as a prototype of inhomogeneity. Focusing on the Kraichnan advection model with Gaussian, white-in-time and zero-mean forcing, it has been possible to study analytically the two-point equal-time scalar correlation function and to prove the persistence of inhomogeneity at small scales, in the spirit of the local cascade process described in § 4. Still to be understood in more detail is the interplay between inhomogeneity, anisotropy and finite-size effects mentioned in § 5.

An interesting open problem is the extension of our calculation to higher-order scalar correlation functions, with the aim of corroborating our results obtained for the two-point correlation function.

Another natural extension of the present work would be the study of a point source not satisfying the aforementioned hypotheses, e.g. a constant-in-time scalar emission. This situation is already under investigation, and a fully analytical study should be completed by numerical results. A related issue is understanding how these results should change in the presence of smooth flows ($\xi = 2$).

Inhomogeneities were here limited to the forcing term, while the velocity field was assumed homogeneous and isotropic. It would be of interest to reformulate the problem with velocity ensembles not invariant under translation, so as to deal with a completely inhomogeneous situation. Moreover, whenever the Kraichnan model is used, the question of a possible extension to more realistic flows arises: numerical simulations would thus be required to test the present results with solutions of the Navier–Stokes equation.

Lastly, a coarse-grained description might be applied to the present case, with the aim of finding large-scale closed equations and an explicit exact parameterization for the small scales, accounting for the effects of inhomogeneities. The infrared limit of the theory for the passive scalar has already been introduced with good results in the homogeneous isotropic case (Martins Afonso *et al.* 2003; Martins Afonso, Celani & Mazzino 2004; Celani, Martins Afonso & Mazzino 2005, 2006). The extension to inhomogeneous cases is in order, in particular using the point-source problem as a paradigm.

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Appendix. Periodicity and discrete spectrum

It is interesting to reformulate the point-source problem in a finite d-dimensional box of side a with periodic boundary conditions, which is equivalent to considering an infinite d-dimensional grid of point sources with mesh size a. It can easily be shown that, upon Fourier transforming, the forcing-correlation spectrum is no longer continuum and flat, but is active only on a discrete set of wavenumbers ($\boldsymbol{q_k} = 2\pi \boldsymbol{k}/a,\ \forall \boldsymbol{k} \in \mathbb{Z}^d$) with uniform intensities. In order to reconstruct the correlation function $C(\boldsymbol{r}, z)$ it is thus sufficient to analyse the discrete values $\hat{C}(\boldsymbol{r}, \boldsymbol{q_k})$ (we will only focus on the isotropic sector $l = 0$).

For $\boldsymbol{k} = \boldsymbol{0} = \boldsymbol{q_k}$, the pure homogeneous scaling behaviour $\hat{C} \sim r^{-(d+\xi-2)}$ is obviously found. For $\boldsymbol{k} \neq \boldsymbol{0}$, this power law is replaced by Bessel functions, which can in turn be expanded in Taylor series, with a result analogous to (5.2). That is, for each wavenumber the leading behaviour for small r is always given by the same homogeneous contribution, but for a fixed r the corrections to such a term become more and more relevant with growing modulus of $\boldsymbol{q_k}$. When antitransforming, one would be tempted to extrapolate such leading behaviour from each mode and conclude that $C \sim r^{-(d+\xi-2)}$ for small r also upon superposition. This is clearly not the case, as the behaviour $C \sim r^{2-\xi}$ is found for small r. In other words, it is not possible to exchange the Taylor and Fourier series, corresponding to the power expansion of the Bessel function and to the discrete antitransform respectively, because of the absence of uniform convergence.

Note that, here, we did not assume any power-law behaviour for each mode (differently from what is usually done in each anisotropic sector upon SO(d) decomposition), coherently with the result that for each wavenumber we obtain a Bessel function, i.e. an infinite superposition of power laws.

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