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Partial metrisability of continuous posets

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Dedicated to Klaus Keimel on the occasion of his 65th birthday

In this article, we characterise all continuous posets that are partially metrisable in their Scott topology. We present conditions for pmetrisability, which are both necessary and sufficient, in terms of measurements, domain-theoretic bases and, in a more general setting, in terms of radially convex metrics. These conditions, together with their refinements and generalisations, set a natural hierarchy on the class of partially metrised posets. We locate the class of countably-based continuous dcpos within this hierarchy.

1. Introduction

This article is addressed primarily to researchers working in the area of quantitative domain theory and topologists who are interested in weakly separated spaces and generalised distances. Our work answers the question posed in Heckmann (1999): Which continuous posets are partially metrisable in their Scott topology? For topologists, our paper, together with Waszkiewicz (2003a), can serve as a source of many non-trivial examples of spaces equipped with distances generating T_0 topologies.

Our research started with an observation that the self-distance of the partial metric on the poset of formal balls **B**X (Edalat and Heckmann 1998; Heckmann 1999) is a measurement in the sense of Keye Martin (Martin 2000b). Indeed, the self-distance of every pmetric compatible with the Scott topology on a domain is a measurement (Waszkiewicz 2003a). Some time later, Martin and the author noted that the distance induced by a measurement on a domain (see the definitions in Section 2) captures the underlying order, as in the partial metric case. In fact, the distance satisfies all the partial metric axioms except the sharp triangle inequality, which fails in general. However, it can be shown that so-called (weakly) modular measurements are precisely those for which the induced distance is a partial metric. On the other hand, not every partial metric for the Scott topology arises from a modular measurement, the formal ball model with its natural pmetric being an example. Hence, if one wants to characterise partial metrics by their self-distances, a more general construction of a distance from a measurement is needed. Such a construction has been known in topology for a long time, and was presented in an elegant form by Frink (Frink 1937), see Theorem 3.1. A modification of Frink's construction that is best suited to our purposes is due to Künzi and Vajner (Künzi and Vajner 1994), see Theorem 3.2. Our main result, Theorem 3.3, which gives a complete characterisation of pmetrisability of the Scott topology on continuous posets in terms of measurement, is a combination of the ideas outlined above. It is worth pointing out that characterisations of pmetrisability in terms of measurement are very useful, since a measurement on a domain is often easy to construct, understand and use.

Our second characterisation, Theorem 4.2, is a relatively straightforward observation with surprising consequences. The result states that a pmetric for the Scott topology on a continuous poset can be reconstructed from a distance on a basis of the poset. It tells us much about the structure of algebraic domains, since they have a canonical basis of compact elements. Ideal domains, which were described by Martin in Martin (2003), are extreme examples of algebraic domains in which every element is either maximal or compact. They are readily constructed from posets equipped with measurements. For example, the interval order $\mathbf{IR}$ can be restricted in such a way that it becomes ideal but still retains its natural measurement, the length function: see Martin (2003). Our Theorem then makes it easy to see that there is no partial metric on this domain that would induce the Scott topology and, at the same time, would assign the self-distance zero precisely to the maximal elements of the domain: see Example 6.1. This result is indeed surprising, since it shows that there is no general construction of a partial metric p on an arbitrary ω-continuous dcpo P that induces the Scott topology and satisfies the condition $\ker p = \mathrm{Max}(P)$ (which means that the pmetric reduces to a metric exactly at the maximal elements of the domain).

Therefore, it seems more reasonable to look for a pmetric on P that is a metric when restricted to the *constructive* maximal elements $\mathrm{CMax}(P)$. This point of view is advocated by Michael Smyth in Smyth (2002). He proves that every ω-continuous dcpo admits a pmetric of this sort. Generally, as Smyth observes, such a pmetric cannot symmetrically induce the Lawson topology. In Theorem 6.5 we show that this inconvenience disappears if the pmetric topology is strictly weaker than the Scott topology. In order to obtain this result, we use our third characterisation of the pmetrisability of continuous posets, Theorem 5.5.

2. Domains, measurements and partial metrics

Our research arose at the crossroads of domain theory, Steve Matthews' theory of partial metrics and Keye Martin's recent theory of measurements. In what follows we give a brief review of basic definitions and results that are needed for our paper.

Our primary references in domain theory are Abramsky and Jung (1994) and Gierz *et al.* (2003). Let P be a poset. A subset $A \subseteq P$ of P is *directed* if it is non-empty and any pair of elements of A has an upper bound in A. If a directed set A has a supremum, it is denoted $\bigsqcup^{\uparrow} A$. A poset P in which every directed set has a supremum is called a *dcpo*. The subset of maximal elements of a poset P is denoted $\mathrm{Max}(P)$. We say that $x \in P$ *approximates* (*is way-below*) $y \in P$, and write $x \ll y$, if for all directed subsets A of P with $\bigsqcup^{\uparrow} A \in P$, $y \sqsubseteq \bigsqcup^{\uparrow} A$ implies $x \sqsubseteq a$ for some $a \in A$. If $x \ll x$, then x is called a *compact* element. The subset of compact elements of a poset P is denoted $K(P)$. Now, $\twoheaddownarrow x$ is the set of all approximants of x and $\twoheaduparrow x$ is the set of all elements that x approximates. We say that a subset B of a poset P is a (*domain-theoretic*) *basis* for P if for every element x of P, the set $\twoheaddownarrow x \cap B$ is directed with supremum x. A poset is called *continuous* if it has a basis. A poset P is continuous if and only if $\twoheaddownarrow x$ is directed with supremum x, for all

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$x \in P$. A poset is called a *domain* (*continuous domain*) if it is a continuous dcpo. Note that $K(P) \subseteq B$ for any basis B of P. If a poset admits a countable basis, we say that it is ω-continuous, or countably-based. The poset $[0,\infty)^{op}$ is a domain without a least element. We use $\sqsubseteq$ to refer to its order, which is dual to the natural one, $\leqslant$, but we usually prefer to work with the latter.

A subset $U \subseteq P$ of a poset P is *upper* if $x \sqsupseteq y \in U$ implies $x \in U$. The collection of all upper sets of P is a topology, the Alexandroff topology $\alpha(P)$. A subset A of P is *inaccessible by directed suprema* if for all directed sets $D \subseteq P$ such that $\bigsqcup^{\uparrow} D \in P$, whenever $D \cap A = \emptyset$, we have $\bigsqcup^{\uparrow} D \notin A$. Upper sets inaccessible by directed suprema form a topology called the *Scott topology*; it is usually denoted $\sigma(P)$ here (or σ for short). A function $f : P \to Q$ between posets is Scott-continuous if and only if it preserves the order and suprema of directed subsets that exist in P. The collection $\{\twoheaduparrow x \mid x \in P\}$ forms a basis for the Scott topology on a continuous poset P. The topology satisfies only weak separation axioms: it is always T_0 on a poset but T_1 only if the order is trivial. For an introduction to T_0 spaces, see Heckmann (1990). An excellent general reference on topology is Engelking (1989). The *weak topology* on a poset P, $\omega(P)$, is generated by a subbasis $\{P \setminus \uparrow x \mid x \in P\}$, and the *Lawson topology* $\lambda(P)$ is the join $\sigma(P) \vee \omega(P)$ in the lattice of all topologies on P.

We will now give a brief summary of the main elements of Keye Martin's theory of measurements. Our main reference is Martin (2000a). Let P be a poset. For a monotone mapping $\mu : P \to [0,\infty)^{op}$ and $A \subseteq P$, we define

$$\mu(A,\varepsilon) = \{y \in P \mid (\exists x \in A)(y \sqsubseteq x \wedge \mu y < \mu x + \varepsilon)\}.$$

We say that $\mu(A,\varepsilon)$ is the set of elements of P that are *ε-close to A*. If $A = \{x\}$ for $x \in P$, then the set above is denoted $\mu(x,\varepsilon)$. We say that μ *measures* P (or μ *is a measurement on* P) if μ is Scott-continuous and

$$(\forall U \in \sigma)(\forall x \in U)(\exists \varepsilon > 0)\ \mu(x,\varepsilon) \subseteq U.$$

We define $\ker\mu \stackrel{df}{=} \{x \in P \mid \mu x = 0\}$. The kernel is always a G_δ subset of maximal elements of P.

A measurement μ satisfies:

— *the kernel condition* if $\ker\mu = \mathrm{Max}(P)$;

— *the local triangle condition* if for all Scott-compact subsets $K \subseteq P$ and for all Scott-open subsets $U \subseteq P$, if $K \subseteq U$, then there exists $\varepsilon > 0$ such that $\mu(K,\varepsilon) \subseteq U$;

— *weak modularity condition* if for all $a, b, r \in P$ and for all $\varepsilon > 0$, if $a, b \sqsubseteq r$, then there exists $s \in P$ with $s \sqsubseteq a, b$ such that $\mu r + \mu s \leqslant \mu a + \mu b + \varepsilon$.

One can show that every weakly modular measurement satisfies the local triangle condition. Clearly, the kernel condition is independent of the other two. Examples of weakly modular measurements can be found in Table 1 in Section 6. All of them except the last satisfy the kernel condition. The last example in Table 1 describes a measurement for an arbitrary ω-continuous dcpo such that the kernel condition is not satisfied unless the domain has a top element. Finally, for any metric space (X,d), the formal ball model (Edalat and Heckmann 1998) defined as $\mathbf{B}X = \{(x,r) \mid x \in X, r > 0\}$ with $(x,r) \sqsubseteq (y,s)$

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if and only if $d(x, y) \leqslant r - s$ is a continuous poset with a measurement $\mu(x, r) \stackrel{df}{=} r$, which satisfies the local triangle condition but is not weakly modular in general.

Any measurement $\mu : P \to [0, \infty)^{op}$ on a continuous poset P with a least element induces a distance function $p_\mu : P \times P \to [0, \infty)$ by $p_\mu(x, y) \stackrel{df}{=} \inf\{\mu z \mid z \ll x, y\}$. The map p_μ, when considered with codomain $[0, \infty)^{op}$, is Scott-continuous, encodes the order on the poset P: $x \sqsubseteq y$ if and only if $p_\mu(x, y) = p(x, x)$, and the Scott topology: the collection of p_μ-balls $\{B_{p_\mu}(x, \varepsilon) \mid x \in P, \varepsilon > 0\}$, where $B_{p_\mu}(x, \varepsilon) \stackrel{df}{=} \{y \in P \mid p_\mu(x, y) < \mu x + \varepsilon\}$, is a basis for the Scott topology on P.

Waszkiewicz (2001; 2003a) demonstrated a close connection between measurements and generalised distances called partial metrics. A *partial metric* (*pmetric*) on a set X is a map $p : X \times X \to [0, \infty)$ governed by the following axioms due to Steve Matthews (Matthews 1994). For all $x, y, z \in X$:

— *small self-distances*

$$p(x, x) \leqslant p(x, y)$$

— *symmetry*

$$p(x, y) = p(y, x)$$

— *T_0 separation*

$$p(x, y) = p(x, x) = p(y, y) \Rightarrow x = y$$

— *the sharp triangle inequality* $\Delta^\sharp$

$$p(x, y) \leqslant p(x, z) + p(z, y) - p(z, z).$$

The *kernel* of p is the kernel of the induced self-distance $\mu_p x \stackrel{df}{=} p(x, x)$, $x \in P$, that is, $\ker p = \{x \in X \mid \mu_p x = 0\}$. The topology induced by p on X, denoted $\tau_p(X)$, is given by the basis consisting of balls: $B_p(x, \varepsilon) \stackrel{df}{=} \{y \in X \mid p(x, y) \leqslant p(x, x) + \varepsilon\}$ for $x \in X$ and $\varepsilon > 0$. The balls are themselves open in $\tau_p(X)$. Moreover, every pmetric p on X is a continuous mapping from the product topology of $\tau_p(P)$ on $X \times X$ and the Scott topology on $[0, \infty)^{op}$ (note the opposite order) (Heckmann 1999). In particular, a pmetric on a continuous poset P having $\tau_p(P) = \sigma(P)$ is Scott-continuous. In addition, its self-distance $\mu_p : P \to [0, \infty)^{op}$ is a measurement satisfying the local triangle condition (Waszkiewicz 2003b). Every partial metric p on a set X induces a *quasimetric*, $q_p : X \times X \to [0, \infty)$, by $q_p(x, y) \stackrel{df}{=} p(x, y) - \mu_p x$. The topology given by q_p, $\tau_{q_p}(X)$, is the same as the topology of the pmetric $\tau_p(X)$. (Recall that quasimetrics are distances satisfying the triangle inequality with the self-distance zero. They are not symmetric in general.) Finally, every pmetric induces a partial order by $x \sqsubseteq_p y$ if and only if $p(x, y) = p(x, x)$ if and only if $q_p(x, y) = 0$. The induced order is simply the specialisation order of τ_p. If a poset $(P, \sqsubseteq)$ admits a pmetric $p : P \times P \to [0, \infty)$ with $\sqsubseteq = \sqsubseteq_p$, we say that p is compatible with the order on P.

3. Partial metrisability via measurements

There are two constructions of quasimetrics out of more general distances that are crucial for our purposes. The first one is known as Frink's Lemma.

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Theorem 3.1 (Frink). Suppose $d\colon X \times X \to [0, \infty)$ satisfies the following condition:

$$\forall x, y, z\ \forall \varepsilon > 0\ ((d(x, y) < \varepsilon \wedge d(y, z) < \varepsilon) \Rightarrow d(x, z) < 2\varepsilon).$$

Then there is a function $\rho\colon X \times X \to [0, \infty)$ such that for all $x, y, z \in X$:

1. $\rho(x, z) \leqslant \rho(x, y) + \rho(y, z)$
2. $d(x, y)/4 \leqslant \rho(x, y) \leqslant d(x, y)$.

Furthermore, ρ is symmetric if d is.

Proof. Define

$$\rho(a, b) = \inf \left\{ \sum_{i=0}^{n-1} d(x_i, x_{i+1}) \mid n \in \omega, x_i \in X, x_0 = a, x_n = b \right\}. \qquad \square$$

The other construction is a suitable refinement of the method of Frink's Lemma for the case of partial orders. It will be referred to as the Künzi–Vajner construction and can be found in Künzi and Vajner (1994).

Theorem 3.2 (Künzi–Vajner). A poset P admits a partial metric for its Alexandroff topology if and only if there is a monotone function $\mu\colon P \to [0, 1)^{op}$ that satisfies the following condition:

$$(\forall x \in P)(\exists \varepsilon > 0)(\forall z \in \downarrow y \setminus \uparrow x)(\mu z - \mu y \geqslant \varepsilon). \qquad \text{(KV)}$$

Proof (sketch). The self-distance of a pmetric compatible with the Alexandroff topology satisfies (KV). Conversely, let $\mu\colon P \to [0, \infty)^{op}$ be a map that satisfies the condition. A path $\mathscr{W}$ from $x \in P$ to $y \in P$ is a finite sequence of elements of P, $\mathscr{W} = (x_0, x_1, \ldots, x_n)$, such that $x_0 = x$ and $x_n = y$, and $x_{i+1} \in \downarrow x_i \cup \uparrow x_i$ for all $i = 0, 1, \ldots, n-1$. Define the length of the path $\mathscr{W}$ with respect to μ to be the number $l_\mu(\mathscr{W}) = \sum_{i=0}^{n-1} \max\{0, \mu x_{i+1} - \mu x_i\}$. For arbitrary $x \neq y$ in P, define $\rho_\mu(x, y) = \inf\{l_\mu(\mathscr{W}) \mid \mathscr{W}$ is a path from x to $y\}$ and $q(x, y) = \min\{\rho_\mu(x, y), 2 - \mu x\}$. If there is no path between x and y, set $q(x, y) = 2 - \mu x$. Finally, for all x in P define $q(x, x) = 0$. Then q is a quasimetric such that $p(x, y) = q(x, y) + \mu x$ for every $x, y \in P$ is a pmetric for the Alexandroff topology on P. $\square$

Theorem 3.3 (Main characterisation). Let P be a continuous poset with a least element. The following are equivalent:

1. P admits a pmetric compatible with its Scott topology σ.
2. P admits a measurement $\mu\colon P \to [0, \infty)^{op}$ that satisfies the following condition, which we will call *the path condition* from now on:

   (path) For every $U \in \sigma$ and every $x \in U$ there exists $\varepsilon > 0$ such that for every path $\mathscr{W}$ from x to $y \notin U$ we have $l_\mu(\mathscr{W}) \geqslant \varepsilon$.

Proof. (In the proof we will deal with several distance functions; by convention, the topology induced by a distance, say $f\colon X \times X \to [0, \infty)$, will be denoted $\tau(f)$. Note that $U \subseteq X$ is $\tau(f)$-open if for any $x \in U$, there exists $\varepsilon > 0$ such that $x \in B_f(x, \varepsilon) \subseteq U$, where $B_f(x, \varepsilon) = \{y \in X \mid f(x, y) < f(x, x) + \varepsilon\}$.)

Let $p\colon P \times P \to [0,\infty)$ be a pmetric with $\tau(p) = \sigma$. The self-distance of p, denoted $\mu\colon P \to [0,\infty)^{op}$, is a measurement by Waszkiewicz (2003a, Theorem 8). We will show that it satisfies the path condition. Consider the following mappings:

$$\begin{aligned}
q_p(x,y) &\stackrel{df}{=} p(x,y) - \mu x \\
q_\mu(x,y) &\stackrel{df}{=} p_\mu(x,y) - \mu x \\
\rho_p(x,y) &\stackrel{df}{=} \inf\left\{\sum_{i=0}^{n-1} q_\mu(x_i,x_{i+1}) \mid n \in \omega, x_0 = x, x_n = y\right\} \\
\rho_\mu(x,y) &\stackrel{df}{=} \inf\{l_\mu(\mathscr{W}) \mid \mathscr{W} \text{ is a path from } x \text{ to } y\}.
\end{aligned}$$

Observe that by (the proof of) Frink's Lemma, ρ_p is a quasimetric such that $1/4q_p(x,y) \leqslant 1/4q_\mu(x,y) \leqslant \rho_p(x,y)$ for all $x,y \in P$. Moreover, $\rho_p(x,y) \leqslant \rho_\mu(x,y)$, $x,y \in P$, by definition. Therefore $\sigma = \tau(p) = \tau(q_p) \subseteq \tau(q_\mu) \subseteq \tau(\rho_p) \subseteq \tau(\rho_\mu)$; the outermost inclusion, $\sigma \subseteq \tau(\rho_\mu)$, is equivalent to the path condition.

Conversely, suppose that a map $\mu\colon P \to [0,\infty)^{op}$ measures P and satisfies the path condition. We define maps q_μ, ρ_μ as above, and additionally define

$$p'(x,y) \stackrel{df}{=} \rho_\mu(x,y) + \mu x.$$

The path condition is equivalent to the inclusion of topologies: $\sigma \subseteq \tau(\rho_\mu)$. On the other hand, note that

$$\rho_\mu(x,y) = \inf\left\{\sum_{i=0}^{n-1} q_\mu(x_i,x_{i+1}) \mid n \in \omega, (x_0 = x, \ldots, x_n = y) \text{ is a path}\right\}.$$

It is easy to see that ρ_μ is a quasimetric with $\rho_\mu \leqslant q_\mu$, which implies $\tau(\rho_\mu) \subseteq \tau(q_\mu)$. We conclude that $\sigma \subseteq \tau(\rho_\mu) \subseteq \tau(q_\mu) = \sigma$. Hence p' is a partial metric whose self-distance is μ and such that $\tau(p') = \tau(\rho_\mu) = \sigma$. □

The characterisation above generalises previously known partial metrisability results from Waszkiewicz (2003a; 2003b).

Corollary 3.4. Let P be a continuous poset with a least element. Let $\mu\colon P \to [0,\infty)^{op}$ be a monotone mapping. The following are equivalent:

1. $p_\mu(x,y) \stackrel{df}{=} \inf\{\mu z \mid z \ll x,y\}$ is a partial metric for the Scott topology on P.
2. μ is a weakly modular measurement.

Proof. For the interesting direction, we will show that weakly modular measurements satisfy the path condition. Suppose $x \in U$ for some Scott-open subset U of P. Since μ is a measurement, choose $\varepsilon > 0$ such that $\mu(x,\varepsilon) \subseteq U$. Let $\mathscr{W}$ be a path from x to y of length smaller than ε. Using weak modularity, one observes that if (x,y,z) is a subsequence of consecutive elements of $\mathscr{W}$ with $x,z \sqsubseteq y$, then there is $y' \in P$ with $y' \sqsubseteq x,z$ such that a modified path $\mathscr{W}'$ (with y replaced by y') has $l_\mu(\mathscr{W}') \leqslant l_\mu(\mathscr{W}) < \varepsilon$. Inductively, we can find $s \sqsubseteq x,y$ and a path $\mathscr{W}''$ from x to y with $l_\mu(\mathscr{W}'') = \mu s - \mu x < \varepsilon$. Therefore, $s \in \mu(x,\varepsilon) \subseteq U$ and, consequently, $y \in U$, since $s \sqsubseteq y$. The condition (path) is now proved. Also note that from any path $\mathscr{V}$ containing a subsequence (x,y,z) with $x \sqsubseteq y \sqsubseteq z$ or $x \sqsupseteq y \sqsupseteq z$,

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the element y can be erased, resulting in a modified path $\mathscr{V}'$ of the same length. This means, however, that using the two methods of modification described above, any path can be replaced by a path of smaller length containing at most three elements. In fact, the partial metric constructed from a measurement μ (as in the proof of Theorem 3.3) is of the form p_μ. □

Corollary 3.5. Let P be an algebraic poset with a least element. Then the following are equivalent:

1. P admits a partial metric compatible with the Scott topology.
2. P admits a measurement $\mu : P \to [0,\infty)^{op}$ satisfying the local triangle condition.

Proof. For (1) implies (2), consult Waszkiewicz (2003b). For the converse, let $z \in K(P)$ be arbitrary. By the local triangle condition, there exists $\varepsilon > 0$ such that $\mu(\uparrow z, \varepsilon) = \uparrow z$. If $\mathscr{W} = (x_0, \dots, x_n)$ is any path from $x \in \uparrow z$ to $y \notin \uparrow z$, choose $i \in \{0, \dots, n-1\}$ to be the least number such that $x_i \in \uparrow z$ and $x_{i+1} \notin \uparrow z$. Hence, $x_{i+1} \sqsubseteq x_i$. If $\mu x_{i+1} - \mu x_i < \varepsilon$, then $\mu x_{i+1} \in \mu(\uparrow z, \varepsilon) = \uparrow z$, which is a contradiction. Therefore, $l_\mu(\mathscr{W}) \geqslant \mu x_{i+1} - \mu x_i \geqslant \varepsilon$, as required. □

4. Partial metrisability via bases

Our second characterisation gives a necessary and sufficient condition for the Scott topology to be partially metrisable in terms of bases of continuous posets.

Recall that a T_0 topology τ on a poset P is order-consistent if its specialisation order agrees with the order on P and, moreover, τ-open sets are inaccessible by directed suprema.

Lemma 4.1. For a partial metric $p : P \times P \to [0,\infty)$ on a continuous poset P, which is compatible with the order on P, the following are equivalent:

1. $\tau(p)$ is order-consistent.
2. The self-distance $\mu : P \to [0,\infty)^{op}$ of p is Scott-continuous.
3. $\tau(p) \subseteq \sigma(P)$.

Proof. For the equivalence of (1) and (2), use O'Neill (1995, Lemma 3.2). Since $\sigma(P)$ is the finest order-consistent topology, (1) implies (3). Finally, assume (3). For a Scott-open subset V of $[0,\infty)^{op}$, the preimage $\mu^{-1}[V]$ is $\tau(p)$-open and hence Scott-open. This proves (2), as required. □

Theorem 4.2. Let P be a continuous poset with a basis B. Then B is pmetrisable in its relative Scott topology $\sigma_{|B}$ if and only if P is pmetrisable in its Scott topology σ.

Proof. Let p be a pmetric compatible with $\sigma_{|B}$ and let $\mathscr{I}(B)$ denote the rounded ideal completion of B with respect to approximation $\ll$ on B. Note that the map p is continuous with respect to the product topology on $B \times B$ and the Scott topology on $[0,\infty)^{op}$. Hence, it extends to a unique Scott-continuous function $\hat{p} : \mathscr{I}(B) \times \mathscr{I}(B) \to [0,\infty)^{op}$ satisfying $\hat{p}(a,b) = p(a,b)$ for all $(a,b) \in B \times B$. The latter fact together with the continuity of the functions involved, and the continuity of P, imply that all equalities and inequalities

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satisfied by p carry over to $\hat{p}$. Hence $\hat{p}$ is a pmetric on P, which is explicitly given by $\hat{p}(x,y) = \inf\{p(a,b) \mid a \in x \wedge b \in y\}$.

We will show that $x \subseteq y$ in $\mathscr{I}(B)$ if and only if $\hat{p}(x,y) = \hat{p}(x,x)$. If $x \subseteq y$, then $\{p(a,b) \mid a \in x \wedge b \in x\} \subseteq \{p(a,b) \mid a \in x \wedge b \in y\}$, hence $\hat{p}(x,y) \leqslant \hat{p}(x,x)$, and the equality of the distances follows from the small self-distances axiom. Conversely, suppose that $\hat{p}(x,y) = \hat{p}(x,x)$. Let $a \in x$ and $\varepsilon > 0$. There are $c \in x$ and $d \in y$ such that $p(c,d) < \hat{p}(x,y)+\varepsilon = \hat{p}(x,x)+\varepsilon$. Choose $u \in x$ with $a, c \sqsubseteq u$. Then $p(a,d) \leqslant p(a,u)+p(u,c)+p(c,d)-p(u,u)-p(c,c) = p(a,a)+p(c,d)-p(u,u)$. However, $p(c,d) < \hat{p}(x,x)+\varepsilon \leqslant p(u,u)+\varepsilon$, and thus $p(a,d) < p(a,a)+\varepsilon$ for any $\varepsilon > 0$. Therefore $p(a,d) = p(a,a)$, which gives $a \sqsubseteq d$. Since $d \in y$ and y is lower, we have $a \in y$, and consequently, $x \subseteq y$.

Let $\mu\colon B \to [0,\infty)^{op}$ be the self-distance of p. It is continuous, hence it extends uniquely to the Scott-continuous map $\hat{\mu}\colon \mathscr{I}(B) \to [0,\infty)^{op}$, which is the self-distance of $\hat{p}$. Therefore, by the previous paragraph and Lemma 4.1, we have $\tau(\hat{p}) \subseteq \sigma(\mathscr{I}(B))$. Conversely, suppose $a \in x$ for $a \in B$ and $x \in \mathscr{I}(B)$. Choose $b \in x$ with $a \ll b$ and $\varepsilon > 0$ such that $B_p(b,\varepsilon) \cap B \subseteq \mathord{\Uparrow} a \cap B$. Let $z \in B_{\hat{p}}(x,\varepsilon)$. Since we have already proved that $\hat{p}$-balls are Scott-open, there is $c \in B$ with $c \ll z$ and $c \in B_{\hat{p}}(x,\varepsilon)$. We have $p(b,c) = \hat{p}(b,c) \leqslant \hat{p}(b,x)+\hat{p}(x,c)-\hat{\mu}x < \hat{p}(b,x)+\varepsilon = \hat{\mu}b+\varepsilon = \mu b+\varepsilon$. Therefore, $c \in B_p(b,\varepsilon)$ and consequently $c \in \mathord{\Uparrow} a \cap B$. Hence $z \in \mathord{\Uparrow} a$, which proves that $\sigma(\mathscr{I}(B)) \subseteq \tau(\hat{p})$.

The proof the first part is now complete since $\mathscr{I}(B)$ is isomorphic to P.

For the converse, if $\hat{p}$ is a pmetric for the Scott topology on P, then its restriction to B is the desired partial metric for the subspace Scott topology on B. □

5. Partial metrisability via radially convex metrics

Our last characterisation of partial metrisability of continuous posets concerns a more general situation: we only require that the partial metric order agrees with the underlying order. Interestingly, such a requirement alone is equivalent to the existence of a metric, which is also tied to the order, in a slightly more elaborate way.

Definition 5.1. A metric $d\colon P \times P \to [0,\infty)$ on a poset P is *exactly radially convex (erc)* provided $x \sqsubseteq y \sqsubseteq z$ if and only if $d(x,y) + d(y,z) = d(x,z)$.

Clearly, if P has a least element $\perp$, then $x \sqsubseteq y$ if and only if $d(x,y) = d(\perp,y) - d(\perp,x)$.

Example 5.2. Let the product of ω copies of the unit interval, $\mathbb{I}^\omega$, be equipped with a metric

$$d(x,y) = \sum_{i\in\omega} 2^{-i}|x_i - y_i|,$$

where $x = (x_0, x_1, \dots)$, $y = (y_0, y_1, \dots)$. The metric d is erc with respect to the coordinate-wise order.

Proof. Note that $0 = (0,0,\dots)$ is the bottom element of $\mathbb{I}^\omega$. Now, $d(0,x) + d(x,y) = d(0,y)$ if and only if $\sum_{i\in\omega} 2^{-i}(x_i - y_i + |x_i - y_i|) = 0$ if and only if $x_i \leqslant y_i$ for all $i \in \omega$ if and only if $x \sqsubseteq y$. □

Lemma 5.3. Let P be a poset with a least element $\bot$. If $d : P \times P \to [0,\infty)$ is an erc, the map $p : P \times P \to [0,\infty)$ given by

$$p(x,y) \stackrel{df}{=} d(x,y) + \mu x + \mu y$$

for all $x, y \in P$, where $\mu : P \to [0,\infty)^{op}$ is defined as

$$\mu x \stackrel{df}{=} \sup\{d(\bot,z) \mid z \in P\} - d(\bot,x),$$

is a pmetric compatible with the order with the self-distance 2μ. Moreover:

1 If either $\tau(d) = \lambda(P)$, or $\sigma(P) \subseteq \tau(d)$, then μ has the measurement property: for all $x \in P$, $U \in \sigma(P)$, if $x \in U$, then there exists $\varepsilon > 0$ with $\mu(x,\varepsilon) \subseteq U$.

2 The inclusion $\tau(p) \subseteq \sigma(P)$ is equivalent to either:

(a) μ is Scott-continuous.

(b) p is order-consistent.

Proof. The function μ is monotone: if $x \sqsubseteq y$ in P, then $d(\bot,x) + d(x,y) = d(\bot,y)$. Hence $\mu x = \mu\bot - d(\bot,x) = \mu\bot - d(\bot,y) + d(x,y) \geqslant \mu\bot - d(\bot,y) = \mu y$.

Symmetry, the sharp triangle inequality of p and the claim about the self-distance are trivial. For the small self-distances, since $d(\bot,x) - d(\bot,y) \leqslant d(x,y)$ by the triangle inequality, we have $d(x,y) \geqslant \mu y - \mu x$. Adding $\mu x + \mu y$ to both sides of the inequality gives $p(x,y) \geqslant 2\mu y = p(y,y)$. The inequality $p(x,y) \geqslant p(x,x)$ follows by symmetry. Now, $x \sqsubseteq y$ if and only if $d(x,y) = d(\bot,y) - d(\bot,x)$ if and only if $d(x,y) = \mu x - \mu y$ if and only if $p(x,y) = 2\mu x$ if and only if $x \sqsubseteq_p y$. Hence p is compatible with the order. This also proves the T_0 axiom.

The proof of (1) is elementary; part (2) follows from Lemma 4.1. □

Lemma 5.4. Let P be a poset with a least element. Every pmetric $p : P \times P \to [0,\infty)$ that is compatible with the order induces an erc $d : P \times P \to [0,\infty)$ that satisfies $d(\bot,x) = \mu\bot - \mu x$ for every $x \in P$, where μ is the self-distance of p.

Proof. Define $d(x,y) \stackrel{df}{=} 2p(x,y) - p(x,x) - p(y,y)$. It is clear that d is a metric. Note that $\bot \sqsubseteq x$ if and only if $p(\bot,x) = \mu\bot$, which is equivalent to saying that $d(\bot,x) = \mu\bot - \mu x$, for any $x \in P$.

Now we have $x \sqsubseteq y$ if and only if $x \sqsubseteq_p y$ if and only if $p(x,y) = \mu x$ if and only if $d(x,y) = \mu x - \mu y$. However, $\mu x - \mu y = \mu\bot - d(\bot,x) + d(\bot,y) - \mu\bot = d(\bot,y) - d(\bot,x)$. This yields $d(\bot,x) + d(x,y) = d(\bot,y)$, as required. □

We have proved, among other things, the following theorem.

Theorem 5.5. Let P be a continuous poset with a least element. Then the following are equivalent:

1. P admits a partial metric compatible with the order.
2. P admits an exactly radially convex metric.

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Table 1. *Domains with their measurements and partial metrics*

Domain	Measurement μ	Pmetric p_μ
Closed, non-empty intervals of $\mathbb{R}$ with reverse inclusion	length	$p_\mu([a,b],[c,d]) = \max\{b,d\} - \min\{a,c\}$
The powerset of natural numbers with inclusion	$\mu x = \sum_{n \notin x} 2^{-(n+1)}$	$p_\mu(x,y) = \sum_{n \notin x \cap y} 2^{-(n+1)}$
Words over $\{0,1\}$, prefix order	length	length of common prefix
Plotkin's $\mathbb{T}^\omega$ $\{(P,N) \subseteq \omega \times \omega \mid P \cap N = \varnothing\}$ with coordinatewise inclusion	$\mu(P,N) = \sum_{n \notin P \cup N} 2^{-(n+1)}$	$p_\mu((A,B),(C,D)) = \sum_{n \notin (A \cap C) \cup (B \cap D)} 2^{-(n+1)}$
Any ω-continuous poset	$\mu x = 1 - \sum_{\{n \mid x \in U_n\}} 2^{-(n+1)}$ $\{U_n \mid n \in \omega\}$ - basis of open filters	$p_\mu(x,y) = 1 - \sum_{\{n \mid x,y \in U_n\}} 2^{-(n+1)}$

6. Quantitative domains

The characterisations of partially metrised continuous posets given above set up a natural hierarchy of domains: the largest class is formed by partial orders with exactly radially convex metrics. These orders admit compatible partial metrics. We can form a subclass of these by adding the requirement that the self-distance of the compatible pmetric is a measurement. If we require that the measurement satisfies the local triangle condition, we arrive at a third subclass. It is presently unknown if this condition is equivalent to the path condition outside the algebraic case. The fourth sublass is then the class of all continuous orders with measurements satisfying the path condition, which is precisely the class of all continuous posets with pmetrisable Scott topology. Examples of these are orders equipped with weakly modular measurements; in this case the partial metric for the Scott topology has a simple description in terms of measurement, see Table 1.

A preferable way of looking at continuous posets, as advocated in Lawson (1997), is to treat them as computational models of their spaces of maximal elements. This point of view justifies a particular requirement that can be considered when searching for a partial metric for a domain, namely, that the kernel of the pmetric is equal to the space of maximal elements of the domain. It is known that every Scott domain admits such a pmetric for its Scott topology (Waszkiewicz 2003a). We show below that apart from the bounded-complete case, there are domains that cannot be partially metrised by a distance satisfying the kernel condition. The following example is due to Achim Jung.

Example 6.1. Let P be the set of closed, non-empty intervals of the real line $\mathbb{R}$ ordered by a refined version of reverse inclusion (consult Martin (2003) for a general theory of such refinements):

$$x \sqsubseteq y \iff \left[(x = y) \vee \left((y \subseteq x) \wedge l(y) < \frac{1}{2} l(x)\right)\right],$$

for all $x, y \in P$, where $l(\cdot)$ is the length function. Under this order, P is an algebraic dcpo, where each non-maximal element is compact. We claim that there is no partial metric p compatible with the Scott topology such that $\ker p = \mathrm{Max}(P)$.

Proof. To show a contradiction, suppose that such p exists. Theorem 4.2 implies that the basis $K(P)$ is partially metrisable in the relative Scott topology, which is the Alexandroff topology, $\alpha(P)$. By Theorem 3.2, the self-distance μ_p satisfies condition (KV), which is easily seen to be equivalent to

$$[(x_n \to_{\alpha(P)} x) \wedge (\lim p_\mu(x_n, y_n) = \lim \mu x_n)] \Rightarrow (y_n \to_{\alpha(P)} x) \qquad (\mathrm{KV}')$$

for all $x \in K(P)$ and all sequences $(x_n), (y_n) \subseteq K(P)$. However, for $x = [0, 2]$ and $x_n = [2^{-(n+1)}, 2^{1-n}]$, $y_n = [-2^{-(n+1)}, 2^{-(n+1)}]$, where $n \in \omega$, condition (KV′) is violated. □

Therefore, even in the case of countably-based domains, there is no general construction of a partial metric that would satisfy the kernel condition. Following Smyth (2002), we have the following definition.

Definition 6.2. A point x in a continuous poset P is constructively maximal, $x \in \mathrm{CMax}(P)$, if every Lawson neighbourhood of x contains a Scott neighbourhood of x.

An interesting research question is how one should define a class of *quantitative domains of computation*; domains that would serve as semantic models of programming languages that are capable of interpreting quantitative aspects of computing such as the speed of algorithms or their complexity. There are many proposals in the literature, ranging from totally bounded quasi-uniform spaces (Smyth 1991), continuity and $\mathscr{V}$-continuity spaces (Flagg and Kopperman 1997; Flagg and Sünderhauf 2002), and the generalised metric spaces of Bonsangue *et al.* (1998), to domains with partial metrics and measurements. We would like to conclude our paper with a tentative definition of a quantitative domain, much in the spirit of the set of axioms proposed in the concluding section of Smyth (2002). We then prove that every countably-based continuous dcpo is quantitative (see Theorem 6.5 below). This result provides a nice application of all the characterisations of partial metrisability that we have presented in this paper.

Definition 6.3. A poset P is a quantitative domain if:

1. P is a continuous dcpo.
2. P admits a pmetric such that:
   (a) $\tau(p) \subseteq \sigma$.
   (b) Its self-distance is a measurement $\mu : P \to [0, \infty)^{op}$.
   (c) $\ker\mu = \mathrm{CMax}(P)$.
   (d) Its induced metric gives the Lawson topology.

In Lawson (1997), Lawson observed that the maximal elements of a countably-based poset P embed onto maximal elements of the Fell order compactification $F(P)$ of P provided the Scott and Lawson topologies agree at $\mathrm{Max}(P)$. It is known that the latter condition implies that $\mathrm{Max}(P) = \mathrm{CMax}(P)$. In the following lemma, which is in some sense a 'local' version of Lawson (1997, Lemma 3.2), we sharpen Lawson's observation by proving that the constructive maximals of P are in one-to-one correspondence with the maximals of the compactification.

Lemma 6.4. Let $(P, \sqsubseteq)$ be an ω-continuous dcpo. Then $x \in \mathrm{CMax}(P)$ if and only if $i(x) \in \mathrm{Max}(F(P))$, where $i\colon P \hookrightarrow F(P)$ is the Fell order compactification of P.

Proof. Poset P embeds as the coprimes into the lattice Γ of Scott-closed sets via $x \mapsto i(x)$. The Lawson-closure of the image $i[P]$ in Γ is a compact pospace $F(P)$. Assume that $x \in \mathrm{CMax}(P)$ and suppose that there is $y \in F(P)$ such that $i(x) \sqsubset y$. Since Γ is continuous, there is $z \ll y$ such that $z \not\sqsubseteq i(x)$. Since Γ is completely distributive, there is $q \in P$ such that $i(q)$ is coprime in Γ with $i(q) \sqsubseteq z$ and $i(q) \not\sqsubseteq i(x)$, that is, $q \not\sqsubseteq x$. The subset $P \setminus \uparrow q$ is Lawson-open and contains x, hence there is $w \ll x$ such that $\twoheaduparrow w \cap \mathrm{Max}(P)$ misses $\uparrow q \cap \mathrm{Max}(P)$. However, this means that $i(w) \ll i(x)$ in Γ. Since $\twoheaduparrow i(w) \cap \twoheaduparrow i(q)$ is a Lawson neighbourhood of y, we have $s \in P$ with $i(s)$ in this neighbourhood. Since P is a dcpo, there is $m \in \mathrm{Max}(P)$ with $s \sqsubseteq m$. But then $m \in \twoheaduparrow w \cap \uparrow q \cap \mathrm{Max}(P)$, which is a contradiction. Thus we have shown that $i(x) \in \mathrm{Max}(F(P))$.

Conversely, assume $i(x) \in \mathrm{Max}(F(P))$ for $x \in P$. Take $y \not\sqsubseteq x$. Then $\uparrow i(y)$ is a compact subset of $F(P)$ missing $i(x)$. Thus $\downarrow\uparrow i(y)$ is a compact lower set and still misses $i(x)$, because $i(x)$ is maximal. Therefore, the complement of $\downarrow\uparrow i(y)$ in $F(P)$ is an open upper set containing $i(x)$. Its inverse image V by i in P must be an open, upper subset, hence Scott-open, around x. However, if $z \in V$, we have $i(z) \in F(P) \setminus \downarrow\uparrow i(y)$, and thus $i(y) \not\sqsubseteq i(z)$, yielding $y \not\sqsubseteq z$. Therefore, $V \subseteq P \setminus \uparrow y$. We have shown that every subbasic Lawson neighbourhood of x contains a Scott neighbourhood of x, which means that $x \in \mathrm{CMax}(P)$. □

Theorem 6.5. Every ω-continuous dcpo with a least element is a quantitative domain.

Proof. The Fell order compactification $F(P)$ is a compact metrisable pospace, hence admits an erc metric $\overline{d}$ by the Urysohn–Carruth metrisation theorem, which figures as Gierz *et al.* (2003, Exercise VI-1.18), and by Example 5.2. Define $\mu\colon P \to [0,\infty)^{op}$ as $\mu x = \sup\{\overline{d}(i(x), z) \mid x \in z \in F(P)\}$, where $i\colon P \hookrightarrow F(P)$ is in fact $x \mapsto \downarrow x$. By Martin (2000a, Theorem 2.5.1), the map μ is a measurement on P. The restriction of $\overline{d}$ to P is an erc metric on P, call it d, which is compatible with the Lawson topology of P. Note that in the presence of the least element $\perp$ of P, we have $\mu x = \sup\{d(\perp, z) \mid z \in P\} - d(\perp, x)$ for every $x \in P$, which follows almost exclusively from Lawson (1997, Lemma 4.1). Hence, by Lemma 5.3, the distance $p(x, y) = d(x, y) + \mu x + \mu y$ is a pmetric with $\tau(P) \subseteq \sigma$. The self-distance μ_p of p is a measurement, since μ is. Finally, using Lemma 6.4,

$$\begin{aligned}
\mu_p x = 0 &\iff \mu x = 0 \\
&\iff \forall z \in F(P)\ (i(x) \subseteq z \Rightarrow \overline{d}(i(x), z) = 0) \\
&\iff i(x) \in \mathrm{Max}(F(P)) \\
&\iff x \in \mathrm{CMax}(P),
\end{aligned}$$

and thus $\ker \mu_p = \mathrm{CMax}(P)$, as required. □

It is worth poining out that the last result remains valid for countably based domains without least element.

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