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ASYMPTOTIC BEHAVIOUR OF A CLASS OF RESOURCE COMPETITION BIOLOGY SPECIES SYSTEM BY THE FRACTIONAL BROWNIAN MOTION

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Abstract

We analyse the asymptotic behaviour of a biological system described by a stochastic competition model with n species and k resources (chemostat model), in which the species mortality rates are influenced by the fractional Brownian motion of the extrinsic noise environment. By constructing a Lyapunov functional, the persistence and extinction criteria are derived in the mean square sense. Some examples are given to illustrate the effectiveness of the theoretical result.

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1. Introduction

A class of resource competition biology species system (RCBSS) models have been developed [3, 11]. Consider a well-known competition model with n species and k resources. The dynamical behaviour of the phytoplankton species depends on the availability of resources. On the other hand, the availability of the resources depends on the amount of resource provision and the quantity of resources used by the species. A competition model for n species and k resources can be written as [3]

$$\begin{cases} \dfrac{dN_i(t)}{dt} = N_i(t)(\mu_i(R_1, R_2, \ldots, R_k) - m_i), & i = 1, 2, \ldots, n, \\ \dfrac{dR_j(t)}{dt} = D(S_j - R_j(t)) - \displaystyle\sum_{i=1}^{n} C_{ji}\mu_i(R_1, R_2, \ldots, R_k)N_i(t), & j = 1, 2, \ldots, k, \end{cases} \tag{1.1}$$

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where $N_i(t)$ is the population abundance of species i, $R_j(t)$ is the availability of resource j, m_i is the specific mortality rate of species i, D is the system's turnover rate, S_j is the supply concentration of resource j, c_{ji} is the content of resource j in species i, and $\mu_i(R_1, R_2, \ldots, R_k)$ is the specific growth rate of species i as a function of the resource availabilities. Here $\mu_i(R_1, R_2, \ldots, R_k)$ is given by the model [3, 11]

$$\mu_i(R_1, R_2, \ldots, R_k) = \min\Big(\frac{r_i R_1(t)}{K_{1i} + R_1(t)}, \ldots, \frac{r_i R_k(t)}{K_{ki} + R_k(t)}\Big),$$

where r_i is the maximum specific growth rate of species i, and K_{ji} is the half-saturation constant for resource j of species i.

Applications of model (1.1) have been reported in the literature [1, 5, 9]. While the Monod model [9] has been widely used for steady-state growth rates, Li and Smith [7] used the Lotka–Volterra model for a system with two competing species and two limiting resources. A simple model was used to describe the competition between two microbial species [2, 11]. It was shown that in normal oligotrophic soil conditions, oscillatory phenomena are always observed. In the existing literature, most researchers have not considered fractional Brownian motion (FBM) with extrinsic noise for persistence and extinction analysis.

Taking into account the effect of a randomly fluctuating environment, we introduce randomness into model (1.1). We assume that the specific mortality rate m_i is disturbed with $m_i \to m_i + \alpha_i \dot{B}_i^H(t)$, where B_i^H is FBM with Hurst parameter $H \in (0, 1)$, and B_i^H and B_j^H are independent ($i \neq j$). The parameter α_i^2 is nonnegative and denotes the intensity of the stochastic noise which is used to depict the volatility of random perturbations. Then we replace m_i in system (1.1) by $m_i + \alpha_i \dot{B}_i^H(t)$ to obtain the following model with stochastic perturbation:

$$\begin{cases} dN_i(t) = N_i(t)(\mu(R_1, R_2, \ldots, R_k) - m_i)\, dt - \alpha_i N_i(t)\, dB_i^H(t), & i = 1, 2, \ldots, n, \\ dR_j(t) = D(S_j - R_j(t))\, dt - \displaystyle\sum_{i=1}^{n} C_{ji}\mu(R_1, R_2, \ldots, R_k) N_i(t)\, dt, & j = 1, 2, \ldots, k. \end{cases} \tag{1.2}$$

We investigate the new stochastic RCBSS model (1.2). Using the Lyapunov functional, we conduct a persistence and extinction analysis to derive sufficient conditions for this model. The rest of the paper is organized as follows. Section 2 derives the sufficient conditions for persistence and extinction after choosing an appropriate Lyapunov functional. Section 3 provides examples to investigate the persistence and extinction of the RCBSS model. Some concluding remarks are given in Section 4.

2. Persistence and extinction

To investigate the stochastic resource competition dynamics (1.2), we need to show that this model has a unique global positive solution. Let (Ω, F, P) be a complete probability space with filtrations $\{F_t\}_{t\geq 0}$ satisfying the usual conditions.

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Theorem 2.1. *For any initial value $(N_i(0), R_j(0)) \in R^{n+k}$, a solution $(N_i(t), R_j(t)) \in R_+^{n+k}$ exists with probability one, namely, $N_i(t) \in R_+^n$, $i \in (1, 2, \ldots, n)$, and $R_j(t) \in R_+^k$, $j \in (1, 2, \ldots, k)$.*

Proof. Note that the coefficients in equation (1.2) do not satisfy the linear growth condition, thus, there exists a unique local solution $t \in [0, \tau_e)$, where τ_e is the explosion time. Assume that $m_0 > 0$ is sufficiently large, such that $N_i(0)$ and $R_j(0)$, $i = 1, 2, \ldots, n$, $j = 1, 2, \ldots, k$, all lie in the interval $[1/m_0, m_0]$. For each integer $m \geq m_0$, we define the stopping time

$$\tau_m = \inf\Big\{t \in [0, \tau_e) \,\Big|\, \min_{1\leq i\leq n, 1\leq j\leq k}(N_i(t), R_j(t)) \leq \frac{1}{m} \text{ or } \max_{1\leq i\leq n, 1\leq j\leq k}(N_i(t), R_j(t)) \geq m\Big\}.$$

As usual, we set $\inf \emptyset = \infty$, where $\emptyset$ denotes the empty set. Clearly, τ_m is increasing. Set $\tau_\infty = \lim_{m\to\infty} \tau_m$, where $0 \leq \tau_\infty \leq \tau_e$ almost surely (a.s.). If we show that $\tau_\infty = \infty$ a.s., then $\tau_e = \infty$ and the solution remains in R_+^{n+k} for all $t \geq 0$ a.s. If this statement is false, then there is a pair of constants $T > 0$ and $0 < \varepsilon < 1$, such that $P\{\tau_\infty \leq T\} > \varepsilon$. Hence there is an integer $m_1 \geq m_0$ such that

$$P\{\tau_m \leq T\} \geq \varepsilon \quad \text{for all } m \geq m_1. \tag{2.1}$$

By the comparison principle [10], it easy to obtain that for all $t \leq \tau_e$,

$$N_i(t) \vee R_j(t) \leq C_1, \tag{2.2}$$

where C_1 is a positive constant.

Define a Lyapunov functional for model (1.2) as

$$V(N_i(t), R_j(t)) = \sum_{i=1}^{n}\Big[N_i(t) - b_i - b_i \log\Big(\frac{N_i(t)}{b_i}\Big)\Big] + \sum_{j=1}^{k}\Big[R_j(t) - a_j - a_j \log\Big(\frac{R_j(t)}{a_j}\Big)\Big],$$

where a_j $(1 \leq j \leq k)$, b_i $(1 \leq i \leq n)$ are positive constants.

By Itô's formula [6],

$$\begin{aligned} dV \leq{}& \sum_{i=1}^{n} b_i m_i\,dt + \sum_{i=1}^{n} N_i(t)\mu_i(R_1, R_2, \ldots, R_k)\,dt - \sum_{i=1}^{n} N_i(t) m_i\,dt \\ &- \sum_{i=1}^{n} b_i\mu_i(R_1, R_2, \ldots, R_k)\,dt + \sum_{j=1}^{k} DS_j\,dt - \sum_{i=1}^{n}\sum_{j=1}^{k} C_{ji}\mu_i(R_1, R_2, \ldots, R_k)N_i(t)\,dt \\ &+ \sum_{j=1}^{k} Da_j\,dt + \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{a_j}{R_j(t)} C_{ji}\mu_i(R_1, R_2, \ldots, R_k)N_i(t)\,dt \\ &- \sum_{i=1}^{n} Ht^{2H-1}\alpha_i^2\,dt - \sum_{i=1}^{n}\Big(1 - \frac{b_i}{N_i(t)}\Big)\alpha_i N_i(t)\,dB_i^H(t). \end{aligned} \tag{2.3}$$

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Let

$$\begin{aligned}LV &= \sum_{i=1}^{n} b_i m_i + \sum_{i=1}^{n} N_i(t)\mu_i(R_1, R_2, \ldots, R_k) - \sum_{i=1}^{n} N_i(t) m_i \\ &\quad - \sum_{i=1}^{n} b_i \mu_i(R_1, R_2, \ldots, R_k) + \sum_{j=1}^{k} DS_j - \sum_{i=1}^{n}\sum_{j=1}^{k} C_{ji}\mu_i(R_1, R_2, \ldots, R_k)N_i(t) \\ &\quad + \sum_{j=1}^{k} Da_j + \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{a_j}{R_j(t)} C_{ji}\mu_i(R_1, R_2, \ldots, R_k)N_i(t) - \sum_{i=1}^{n} Ht^{2H-1}\alpha_i^2.\end{aligned}$$

Choose a_i and b_j such that, for $1 \le i \le n$, $1 \le j \le k$,

$$\sum_{i=1}^{n} N_i(t)\mu_i(R_1, R_2, \ldots, R_k) - \sum_{i=1}^{n} b_i \mu_i(R_1, R_2, \ldots, R_k) \le 0$$

and

$$-\sum_{i=1}^{n}\sum_{j=1}^{k} C_{ji}\mu_i(R_1, R_2, \ldots, R_k)N_i(t) + \sum_{i=1}^{n}\sum_{j=1}^{k} \frac{a_j}{R_j(t)} C_{ji} r_i \mu_i(R_1, R_2, \ldots, R_k)N_i(t) \le 0.$$

When $t \le \tau_e$, using condition (2.2), we have

$$LV \le C_2, \tag{2.4}$$

where C_2 is a positive constant. For $1 \le i \le n$ and $1 \le j \le k$, (2.3) and (2.4) yield

$$\int_0^{\tau_m \wedge T} dV(N_i(t), R_j(t)) \le \int_0^{\tau_m \wedge T} C_2\, ds - \sum_{i=1}^{n} \int_0^{\tau_m \wedge T} \Big(1 - \frac{b_i}{N_i(s)}\Big)\alpha_i N_i(s)\, dB_i^H(s). \tag{2.5}$$

By the property of the standard FBM $B_i^H(t)$, evaluating the means of the both sides of inequality (2.5) for $1 \le i \le n, 1 \le j \le k$, yields

$$E[V(N_i(\tau_m \wedge T), R_j(\tau_m \wedge T))] \le V(N_i(0), R_j(0)) + C_2 T.$$

We set $\Omega_m = \{\tau_m \le T\}$ for $m \ge m_1$, and by (1.2) we have $P(\Omega_m) \ge \varepsilon$. Note that for every $\omega \in \Omega_m$, at least one of $N_i(\tau_m, \omega)$, $R_i(\tau_m, \omega)$, $1 \le i \le n, 1 \le j \le k$, equals either m or $1/m$. Consequently,

$$\begin{aligned}&V(N_i(\tau_m \wedge T), R_j(\tau_m \wedge T), 1 \le i \le n, 1 \le j \le k) \\ &\quad \ge \min_{1\le j\le k}\Big\{m - a_j - a_j \log\frac{m}{a_j}, \frac{1}{m} - a_j - a_j \log\frac{1}{m a_j}\Big\} \\ &\quad \wedge \min_{1\le i\le n}\Big\{m - b_i - b_i \log\frac{m}{b_i}, \frac{1}{m} - b_i - b_i \log\frac{1}{m b_i}\Big\}.\end{aligned} \tag{2.6}$$

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It follows from (2.1) and (2.6) that

$$\begin{aligned} &V(N_i(0), R_j(0), 1 \le i \le n, 1 \le j \le k) + C_2 T \\ &\ge \min_{1\le j\le k}\Big\{m - a_j - a_j \log\frac{m}{a_j}, \frac{1}{m} - a_j - a_j\log\frac{1}{ma_j}\Big\} \\ &\wedge \min_{1\le i\le n}\Big\{m - b_i - b_i \log\frac{m}{b_i}, \frac{1}{m} - b_i - b_i\log\frac{1}{mb_i}\Big\}, \end{aligned}$$

where $I_{\Omega_m}(\omega)$ is the indicator function of Ω_m. Letting $m \to \infty$ leads to the contradiction that $V(N_i(0), R_j(0), 1 \le i \le n, 1 \le j \le k) + C_2 T = \infty$. So $\tau_\infty = \infty$ a.s. □

We mainly investigate the asymptotic behaviour of the stochastic model (1.2). In an analogous way to the corresponding proof presented by Li and Zhu [8],

$$\lim_{t\to\infty} \frac{\int_0^t R_j(s)\,ds}{t} \le M, \quad j = 1, 2, \dots, k, \tag{2.7}$$

when Hurst parameter $H \le 1/2$, and M is a positive constant.

THEOREM 2.2. *If $S_j > \{\int_0^t R_j(s)\,ds\}/t$ and $0 < H \le 1/2$, for any given initial value $(N_i(0), R_j(0)) \in R_+^{n+k}$, the solution of stochastic model (1.2) satisfies the condition*

$$\liminf_{t\to\infty} \frac{1}{t}\int_0^t EN_i(s)\,ds \ge -\frac{D}{\bar{C}m}(S_j - M) > 0 \quad a.s.\ 1 \le i \le n,$$

where $m = \max_{1\le i\le n} m_i$ and $\bar{C} = \max_{i=1,\dots,n,j=1,\dots,k} C_{ji}$. That is, the population is persistent in the mean with probability one.

PROOF. Define the function $V(N_i(t)) = \sum_{i=1}^n N_i(t)$. By Itô's formula,

$$dV = \sum_{i=1}^n (\mu_i(R_1, R_2, \dots, R_k)N_i(t) - m_i N_i(t))\,dt - \sum_{i=1}^n \alpha_i N_i(t)\,dB_i^H(t). \tag{2.8}$$

Integrating both sides of equation (2.8) from 0 to t, and dividing by t, we get

$$\begin{aligned} \sum_{i=1}^n \frac{N_i(t)}{t} - \sum_{i=1}^n \frac{N_i(0)}{t} &= \sum_{i=1}^n \frac{\int_0^t \mu_i(R_1, R_2, \dots, R_k)N_i(s)\,ds}{t} \\ &\quad - \sum_{i=1}^n m_i \frac{\int_0^t N_i(s)\,ds}{t} - \frac{\sum_{i=1}^n \alpha_i \int_0^t N_i(s)\,dB_i^H(s)}{t}, \end{aligned} \tag{2.9}$$

and integrating both sides of the second equation in equation (1.2) from 0 to t yields

$$R_j(t) - R_j(0) = D\Big(S_j t - \int_0^t R_j(s)\,ds\Big) - \sum_{i=1}^n C_{ji}\int_0^t \mu(R_1, R_2, \dots, R_k)N_i(s)\,ds.$$

Note that

$$\sum_{i=1}^n \int_0^t \mu(R_1, R_2, \dots, R_k)N_i(s)\,ds \ge -\frac{1}{\bar{C}}R_j(t) + \frac{D}{\bar{C}}S_j t - \frac{D}{\bar{C}}\int_0^t R_j(s)\,ds, \tag{2.10}$$

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where $\bar{C} = \max_{i=1,\dots,n,j=1,\dots,k} C_{ji}$. From (2.9) and (2.10) we get

$$\sum_{i=1}^{n} m_i \frac{\int_0^t N_i(s)\,ds}{t} \geq -\frac{D}{\bar{C}} \frac{\int_0^t R_j(s)\,ds}{t} - \frac{1}{\bar{C}} \frac{R_j(t)}{t} + \frac{D}{\bar{C}} S_j - \sum_{i=1}^{n} \frac{N_i(t)}{t} + \sum_{i=1}^{n} \frac{N_i(0)}{t} - \frac{\sum_{i=1}^{n} \alpha_i \int_0^t N_i(s)\,dB_i^H(s)}{t}.$$

Using [4, Lemma 5.1],

$$\liminf_{t\to\infty} \sum_{i=1}^{n} \frac{\int_0^t EN_i(s)\,ds}{t} \geq -\frac{DM}{\bar{C}m} + \frac{DS_j}{\bar{C}m} > 0,$$

where $m = \max_{1\leq i\leq n} m_i$, $\bar{C} = \max_{i=1,\dots,n,j=1,\dots,k} C_{ji}$ and M is as given in (2.7). □

Theorem 2.3. *If $r_i M < \max_{1\leq j\leq k} K_{ji} m_i$ and $0 < H \leq 1/2$, $1 \leq i \leq n$, then for any given initial value $(N_i(0), R_j(0)) \in R^{n+k}$, the solution of stochastic model (1.2) satisfies the condition*

$$\limsup_{t\to\infty} \frac{\log N_i(t)}{t} < 0 \quad a.s.\ 1 \leq i \leq n,$$

namely, the species go extinct with probability one.

Proof. Define the function $V(N_i(t)) = \sum_{i=1}^{n} \log N_i(t)$. By Itô's formula,

$$dV = \sum_{i=1}^{n} (\mu_i(R_1, R_2, \dots, R_k) - m_i) - \sum_{i=1}^{n} H t^{2H-1} \alpha_i^2. \tag{2.11}$$

Integrating both sides of equation (2.11) from 0 to t, and dividing by t, yields

$$\sum_{i=1}^{n} \frac{\log(N_i(t))}{t} - \sum_{i=1}^{n} \frac{\log(N_i(0))}{t} = \sum_{i=1}^{n} \frac{\int_0^t \mu_i(R_1, R_2, \dots, R_k)\,ds}{t} - \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} \frac{\alpha_i^2}{2} t^{2H-1} - \sum_{i=1}^{n} \frac{\alpha_i \int_0^t dB_i^H(s)}{t}.$$

Notice that $\mu_i(R_1, R_2, \dots, R_k) \leq r_i R_i(t)/\{K_{ji} + R_i(t)\}$, for all $i \in 1, 2, \dots, n$, and

$$\sum_{i=1}^{n} \frac{\log(N_i(t))}{t} - \sum_{i=1}^{n} \frac{\log(N_i(0))}{t} \leq \sum_{i=1}^{n} \frac{r_i}{tK_{ji}} \int_0^t R_i(s)\,ds - \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} \frac{\alpha_i^2}{2} t^{2H-1} - \sum_{i=1}^{n} \frac{\alpha_i \int_0^t dB^H(s)}{t}.$$

Using the assumption condition

$$\sum_{i=1}^{n} \frac{\log(N_i(t))}{t} \leq \sum_{i=1}^{n} \frac{\log(N_i(0))}{t} + \sum_{i=1}^{n} \frac{r_i M}{K_{ji}} - \sum_{i=1}^{n} m_i - \sum_{i=1}^{n} \frac{\alpha_i \int_0^t dB_i^H(s)}{t},$$

and letting $t \to \infty$, when $H < 1/2$, by the law of large numbers [6], we have

$$\limsup_{t\to\infty} \sum_{i=1}^{n} \frac{\alpha_i \int_0^t dB_i^H(s)}{t} = 0 \quad \text{a.s.}$$

which yields

$$\limsup_{t\to\infty} \frac{\log N_i(t)}{t} \le \sum_{i=1}^{n} \frac{r_i M}{K_{ji}} - \sum_{i=1}^{n} m_i < 0 \quad \text{a.s.} \qquad \square$$

3. Illustrative examples

EXAMPLE 3.1. This example involves six species and three resources. We use $r_i = 1$, $m_i = D = 0.25$, $S_1 = 6$, $S_2 = 10$, $S_3 = 14$, $H = 1/4$, $\alpha_i = 0.3$, $i = 1, 2, 3$, and matrices K and C used in the competition model are given by

$$K = \begin{pmatrix} 1.00 & 0.90 & 0.30 & 1.04 & 0.34 & 0.77 \\ 0.30 & 1.00 & 0.90 & 0.71 & 1.02 & 0.76 \\ 0.90 & 0.30 & 1.00 & 0.46 & 0.34 & 1.07 \end{pmatrix},$$

$$C = \begin{pmatrix} 0.04 & 0.07 & 0.04 & 0.10 & 0.03 & 0.02 \\ 0.08 & 0.08 & 0.10 & 0.10 & 0.05 & 0.17 \\ 0.14 & 0.10 & 0.10 & 0.16 & 0.06 & 0.14 \end{pmatrix},$$

where k_{ij} and c_{ij} are elements of matrices K and C, respectively. The initial conditions are $R_j = S_j$ at $t = 0$ and $N_i = 0.1 + i/100$ for all species i present at $t = 0$. Figure 1 plots the time series of species abundance and demonstrates the persistence behaviours of the six species while the species coexist; the concentrations of species 4–6 are higher than those of species 1–3.

EXAMPLE 3.2. This example also involves six species and three resources. We use $r = (0.87, 0.88, 0.85, 0.90, 0.86, 0.88)$, r_i is in vector r, $m_i = 0.60$, $D = 0.25$, $S_i = 1$, $N(0) = (0, 5, 0.4, 0.6, 0.3, 0.2, 0.1)$, $H = 1/4$, $\alpha_i = 0.4$, $i = 1, 2, 3$, and

$$K = \begin{pmatrix} 1.52 & 1.49 & 1.46 & 1.52 & 1.59 & 1.44 \\ 1.47 & 1.55 & 1.42 & 1.53 & 1.58 & 1.56 \\ 1.48 & 1.45 & 1.61 & 1.51 & 1.50 & 1.54 \end{pmatrix}.$$

The initial conditions are $R_j = 30$ at $t = 0$. Figure 2 plots the time series of species abundance and demonstrates the extinction behaviours.

4. Conclusions

We derive the persistence and extinction criteria for new resources for a stochastic competition model by the fractional Brownian motion of the extrinsic noise environment. The mathematical derivation of the asymptotic behaviour for random factors becomes significantly more challenging than the determinate resources for a competition model in the existing literature.

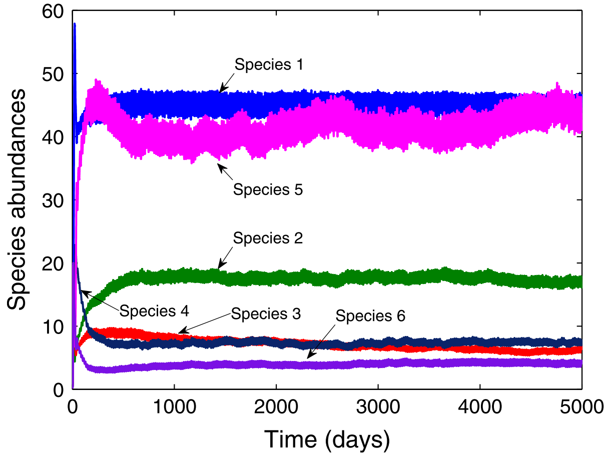

FIGURE 1. Time series of six species with three resources (colour available online).

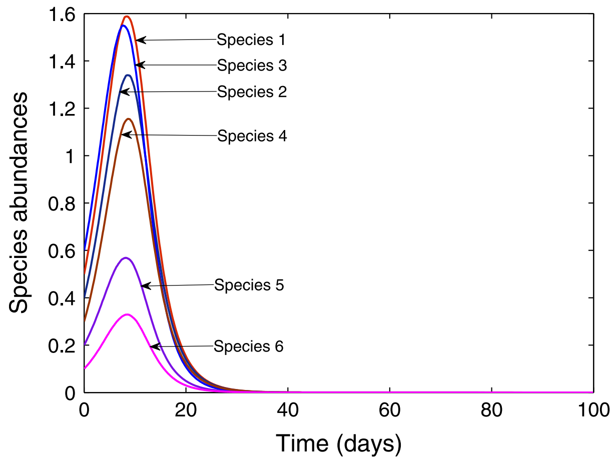

FIGURE 2. Time series of six species with three resources (colour available online).

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