The boundary of hyperbolicity for Hénon-like families

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Abstract. We consider C^2 Hénon-like families of diffeomorphisms of $\mathbb{R}^2$ and study the boundary of the region of parameter values for which the non-wandering set is uniformly hyperbolic. Assuming sufficient dissipativity, we show that the loss of hyperbolicity is caused by a first homoclinic or heteroclinic tangency and that uniform hyperbolicity estimates hold *uniformly in the parameter* up to the bifurcation parameter and even, to some extent, at the bifurcation parameter.

1. *Introduction and statement of results*
Our aim in this paper is to study the *boundary of hyperbolicity* of certain families of two-dimensional maps.

1.1. *Hénon-like families.* We say that a family of C^2 plane diffeomorphisms is called a *Hénon-like family* if it can be written in the form

$$f_{a,b,\eta}(x, y) = (1 - ax^2 + y, bx) + \varphi(x, y, a)$$

where $a \in \mathbb{R}$, $b \neq 0$ and $\varphi(x, y, a)$ is a C^2 'perturbation' of the standard *Hénon family* $h_{a,b}(x, y) = (1 - ax^2 + y, bx)$ [**Hén76**] satisfying

$$\|\varphi\|_{C^2(x,y,a)} \leq \eta.$$

In this paper we consider $|b|$, $\eta > 0$ fixed sufficiently small and investigate the dynamics as the parameter a is varied. For simplicity we shall therefore omit b and η from the notation

and denote a Hénon-like family by $\{f_a\}$. For future reference we remark that the inverse of f_a is given by an equation of a similar form:

$$f_a^{-1}(x, y) = (y/b, \, x - 1 + ay^2/b^2) + \tilde{\varphi}(x, y, a)$$

where $\|\tilde{\varphi}\|_{C^2(x,y,a)} \to 0$ as $\|\varphi\|_{C^2(x,y,a)} \to 0$. We shall suppose without loss of generality that

$$\|\tilde{\varphi}\|_{C^2(x,y,a)} \leq \eta.$$

1.2. *The boundary of hyperbolicity.*

1.2.1. *Basic background.* Hénon and Hénon-like families have been extensively studied over the past 30 years or so. One of the earliest rigorous results on the subject is [**DN79**], in which it was shown that the non-wandering set $\Omega_{a,b}$ is uniformly hyperbolic for all $b \geq 0$ and all sufficiently large a (depending on b). On the other hand, for small $b \neq 0$ and $a \lessgtr 2$ there exists positive probability of 'strange attractors' which contain tangencies between stable and unstable leaves. This was first proved in [**BC91**] for the Hénon family and later generalized in [**MV93**] to Hénon-like families, see also [**WY01, LV03**]. Owing to the presence of tangencies, these attractors cannot be uniformly hyperbolic, but do turn out to satisfy weaker *non-uniform* hyperbolicity conditions [**BY93, BY00, BV01**].

1.2.2. *Complex methods.* More recently Bedford and Smillie have described the transition between these two regimes for Hénon families by identifying and describing some of the properties of the *boundary of uniform hyperbolicity* [**BS06**]. In particular, they show that, for small $|b|$, the non-wandering set is uniformly hyperbolic up until the first parameter a at which a tangency occurs between certain stable and unstable manifolds. Combining this with the statements contained in [**BS02**], their results also imply uniform bounds on the Lyapunov exponents of all invariant probability measures at the bifurcation parameter [**Bed05**]. Their methods rely crucially on previous work [**BS04**], which in turn is based on the polynomial nature of the Hénon family, a feature that allowed Bedford and Smillie to consider the complexification of the family and to apply original and highly sophisticated arguments of holomorphic dynamics.

1.2.3. *Real methods.* In this paper we develop a new and completely different strategy of approaching the problem, based purely on geometric 'real' arguments, which have the advantage of applying to general C^2 *Hénon-like* families. We also obtain the analogous *uniformity* results by showing that the hyperbolicity expansion and contraction rates are uniform right up to the point of tangency, and that even *at* the point of tangency some strong version of non-uniform hyperbolicity continues to hold: all Lyapunov exponents of all invariant measures are uniformly bounded away from 0.

THEOREM 1. *For all $|b| > 0$ and $\eta > 0$ sufficiently small we have the following property. For every Hénon-like family $\{f_a\}_{a \in \mathbb{R}}$ of plane diffeomorphisms, there exists a unique a^* such that:*

(1) *for all $a > a^*$ the non-wandering set Ω_a is uniformly hyperbolic;*

(2) *for $a = a^*$ the non-wandering set Ω_{a^*} contains an orbit of tangency but is 'almost uniformly hyperbolic' in the sense that all Lyapunov exponents of all invariant probability measures supported on Ω are uniformly bounded away from 0.*

Moreover, the bounds on the expansion and contraction rates for all $a \geq a^*$ are independent of a and of the family.

1.2.4. *Singular perturbations.*

We remark that this is not the only existing definition of Hénon-like in the literature. One standard approach is to consider 'singular' perturbations of the limiting one-dimensional map corresponding to the case $b = 0$:

$$f_a(x, y) = (1 - ax^2, 0) + \varphi_a(x, y).$$

This formulation has some slight technical issues, however. For example, one cannot assume that $\|\varphi_a\|_{C^2}$ is small on all of $\mathbb{R}^2$ since that would violate the requirement that f_a be a global diffeomorphism of $\mathbb{R}^2$. This can be dealt with by restricting our attention to some compact region, say $[-2, 2] \times [-2, 2]$, and supposing only that $\|\varphi_a\|_{C^2} \leq \eta$ in this region. Our arguments apply in this case also and yield a more local result on the hyperbolicity of the non-wandering set restricted to $[-2, 2] \times [-2, 2]$.

1.3. *Basic definitions.*

1.3.1. *Non-wandering set.*

We recall that a point z belongs to the *non-wandering* set Ω of f if it has the property that for every neighbourhood $\mathcal{U}$ of z there exists some $n \geq 1$ such that $f^n(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$. The non-wandering set is always invariant and closed (and thus, if bounded, also compact).

1.3.2. *Uniform hyperbolicity.*

We say that a compact invariant set Ω is *uniformly hyperbolic* (with respect to f) if there exist constants $C^u, C^s > 0$, $\lambda^u > 0 > \lambda^s$ and a *continuous* decomposition $T\Omega = E^s \oplus E^u$ of the tangent bundle such that, for every $x \in \Omega$, any non-zero vectors $v^s \in E_z^s$ and $v^u \in E_z^u$ and every $n \geq 1$, we have

$$\|Df_z^n(v^s)\| \leq C^s e^{\lambda^s n} \quad \text{and} \quad \|Df_z^n(v^u)\| \geq C^u e^{\lambda^u n}. \tag{1}$$

By standard hyperbolic theory, the stable and unstable subspaces E_z^s, E_z^u are tangent everywhere to the stable and unstable manifolds. In particular, uniform hyperbolicity is incompatible with the presence of tangencies in Ω between any stable and any unstable invariant manifolds associated to points of Ω.

1.3.3. *Non-uniform hyperbolicity.*

A weaker notion of hyperbolicity can be formulated in terms of invariant measures. For simplicity, we restrict our discussion to the two-dimensional setting, as relevant to the situation we consider in this paper. Let μ be an f-invariant ergodic probability measure with support in some compact invariant set Ω. By Oseledec's ergodic theorem [**Ose68**], there exist constants $\lambda^u \geq \lambda^s$ and a measurable

decomposition $T\Omega = E^s \oplus E^u$ such that, for μ-almost every z and any non-zero vectors $v^s \in E^s_z$ and $v^u \in E^u_z$, we have

$$\lim_{n \to \infty} \frac{1}{n} \log \|Df^n_z(v^s)\| = \lambda^s \quad \text{and} \quad \lim_{n \to \infty} \frac{1}{n} \log \|Df^n_z(v^u)\| = \lambda^u. \qquad (2)$$

The constants λ^s and λ^u are called the *Lyapunov exponents* associated with the measure μ. We say that μ is *hyperbolic* [**YP76, Pes77**] if

$$\lambda^u > 0 > \lambda^s.$$

Clearly the estimates (1) imply the limits (2) for any μ. The converse, however, is false in general: the measurable decomposition may not extend to a continuous one on all of Ω, and the exponential expansion and contraction in (2) implies only a limited version of (1) in which the constants C^s, C^u are measurable functions of x and not bounded away from 0. This definition of hyperbolicity in terms of Lyapunov exponents is sometimes called *non-uniform hyperbolicity* and is consistent in principle with the existence of tangencies between stable and unstable manifolds.

1.3.4. *The boundary between uniform and non-uniform hyperbolicity.* In general, there may be many ergodic invariant probability measures supported in Ω, of which some may be hyperbolic and some not. Even if they are all hyperbolic, the corresponding Lyapunov exponents may not be uniformly bounded away from 0. The situation in which all Lyapunov exponents of all ergodic invariant measures are uniformly bounded away from zero is, in some sense, as 'uniformly hyperbolic' as one can get while admitting the existence of tangencies. This situation can indeed occur, for example in the present context of Hénon-like maps. A first example of a set satisfying this property was given in [**CLR06**].

1.4. *A one-dimensional version.* After completing the proof of the Theorem 1 we realized that much simpler versions of our arguments yield an analogous, new and non-trivial result in the context of one-dimensional maps. We explain and give a precise formulation of this result. We consider first the quadratic family

$$h_a(x) = 1 - ax^2.$$

We choose this particular parametrization for convenience and consistency with our two-dimensional results, but any choice of a smooth family of unimodal or even multimodal maps with negative Schwarzian derivative would work in exactly the same way. It is well known that for $a > 2$ the non-wandering set Ω_a is uniformly expanding, although we emphasize that this result depends crucially on the negative Schwarzian derivative property. As the negative Schwarzian property is not robust with respect to C^2 perturbations, standard methods do not yield this statement for such perturbations.

THEOREM 2. *There exists a constant $\eta > 0$ such that if a family $\{g_a\}$ of C^2 one-dimensional maps satisfies*

$$\|g_a - h_a\|_{C^2} \leq \eta,$$

then there exists a unique parameter value a^* such that:

(1)   *for all $a > a^*$ the non-wandering set Ω is uniformly hyperbolic;*
(2)   *for $a = a^*$ the Lyapunov exponents of all ergodic invariant probability measures are all positive and uniformly bounded away from* 0.

Moreover, the rates of expansion and the bound on the Lyapunov exponents are uniform, independent of the family and of the parameter.

The proof of this result is exactly the same as that of Theorem 1 but vastly simpler, as all the more geometrical arguments concerning curvature etc become essentially trivial.

We emphasize that the uniform expansivity of Ω_a for a particular parameter value $a > 2$ is of course robust under sufficiently small perturbations of f_a, by standard hyperbolic theory. However, this approach *requires the size of the perturbation to depend on the parameter a* and, in particular, to shrink to zero as a tends to 2. The crucial point of our approach, in both the one-dimensional and two-dimensional settings, is that the size of the perturbation *does not* depend on the parameter.

1.5.   *Overview of the paper.*   We have divided our argument into three main sections. In §2 we analyze the geometric structure of stable and unstable manifolds of the two fixed points and define the parameter a^* as the first value of a for which a tangency occurs between some compact parts of these manifolds. We also identify a region $\mathcal{D}$ which we show contains the non-wandering set. In §3 we define a 'critical neighbourhood' Δ_ε outside of which our maps are uniformly hyperbolic by simple perturbation arguments. However, Δ_ε does contain points of Ω and thus we cannot ignore this region. To control the hyperbolicity in Δ_ε, we introduce the notions of 'hyperbolic coordinates' and 'critical points' which form the key technical core of our approach. Finally, in §4 we apply these techniques to prove the required hyperbolicity properties.

2.   *The non-wandering set*

In this section we define the parameter a^* that appears in the statement of our main theorems, and show that for $a \geq a^*$ the non-wandering set is contained in the closure of the unstable manifold of a hyperbolic fixed point restricted to a certain compact region of $\mathbb{R}^2$.

2.1.   *The parameter a^*.*   Below we will define the bifurcation parameter a^* as the first value of a for which there is a tangency between certain compact parts of the stable and unstable manifolds of the fixed points. This does not immediately imply that a^* is a first parameter of tangency, although this will follow from our proof of the fact that the non-wandering set is uniformly hyperbolic for all $a > a^*$.

2.1.1.   *Fixed points and invariant manifolds for the one-dimensional limit.*   For the endomorphisms $h_a = h_{a,0}$ with $a \geq 2$, there are two fixed points

$$p_a = \frac{-1 + \sqrt{1 + 4a}}{2a} \quad \text{and} \quad q_a = \frac{-1 - \sqrt{1 + 4a}}{2a} \quad (p_a > q_a),$$
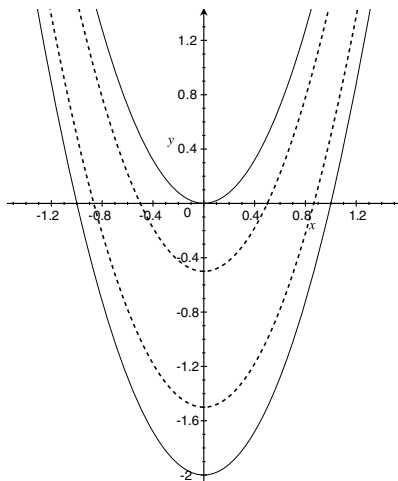
FIGURE 1. First two 'generations' of $W^s(q_*)$ and $W^s(p_*)$.

which are both hyperbolic. For the special parameter value $a = 2$, to simplify the notation we write $f_* = h_{2,0}$ and denote the two fixed points by $p_* = (1/2, 0)$, $q_* = (-1, 0)$.

Since q^* and p^* are repelling in the horizontal direction, their stable sets are simply their preimages:

$$W^s(q^*) = \bigcup_{n \geq 0} f_*^{-n}(q^*) \quad \text{and} \quad W^s(p^*) = \bigcup_{n \geq 0} f_*^{-n}(p^*).$$

In particular, these sets contain the following curves:

$$f_*^{-1}(q_*) = \{(x, y) : f_*(x, y) \equiv (1 - 2x^2 + y, 0) = (-1, 0)\} = \{y = 2x^2 - 2\}$$

and

$$f_*^{-2}(q_*) = \{(x, y) : f_*(x, y) \equiv (1 - 2x^2 + y, 0) = (1, 0)\} = \{y = 2x^2\}.$$

The first preimage of q^* is a parabola with a minimum at $(0, -2)$, intersecting the x-axis at $x = \pm 1$ and having slope equal to -4 at the point $q_* = (-1, 0)$; the second is a parabola with a minimum at $(0, 0)$. Similarly we can compute

$$f_*^{-1}(p_*) = \{(x, y) : f_*(x, y) \equiv (1 - 2x^2 + y, 0) = (1/2, 0)\} = \{y = 2x^2 - 1/2\}$$

which is a parabola with a minimum at $(0, -1/2)$, intersecting the x-axis at $x = \pm 1/2$ and having slope equal to 2 at the point $p_* = (1/2, 0)$; and

$$f_*^{-2}(p_*) = \{(x, y) : f_*(x, y) \equiv (1 - 2x^2 + y, 0) = (-1/2, 0)\} = \{y = 2x^2 - 3/2\}$$

which is a parabola with a minimum at $(0, -3/2)$. See Figure 1 for plots of $f_*^{-1}(q_*)$, $f_*^{-2}(q_*)$, $f_*^{-1}(p_*)$ and $f_*^{-2}(p_*)$.

The unstable manifolds $W^u(q_*)$ and $W^u(p_*)$ can be defined and computed in a similar way and are easily seen to be horizontal.
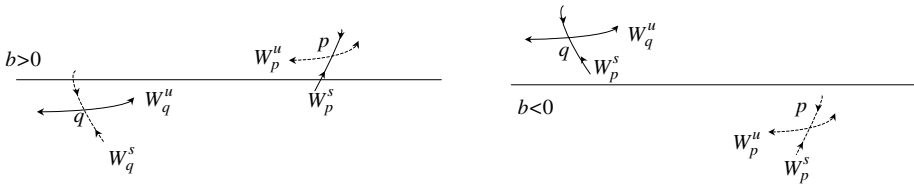
FIGURE 2. Fixed points and their local stable and unstable manifolds for the orientation-reversing ($b > 0$) and the orientation-preserving ($b < 0$) cases (dashed curves indicate negative eigenvalues).

2.1.2. *Fixed points for Hénon-like families.* Consider first the *Hénon family* $h_{a,b}(x, y) = (1 - ax^2 + y, bx)$. For $b \neq 0$, $h_{a,b}$ is a diffeomorphism. The hyperbolicity of the fixed points implies that there exists a neighbourhood of the set $\{(a, 0) : a \geq 2\}$ corresponding to pairs of parameters for which there are *analytic continuations* $q_{a,b}$, $p_{a,b}$ as hyperbolic fixed points of $h_{a,b}$. Considering η small, we also have that the analytic continuations q_{f_a} and p_{f_a} are also well defined and hyperbolic. For simplicity we shall often just refer to these two points as q and p, leaving implicit their dependence on f.

Explicit formulas for $q_{a,b}$, $p_{a,b}$ can be easily derived from the equation $(1 - ax^2 + y, bx) = (x, y)$ but these would not be particularly useful. Instead, we just observe that the fixed points must lie on the line $\{y = bx\}$ and so, for $a \approx 2$ and $b \gtrsim 0$, the vertical coordinates of $q_{a,b}$ and $p_{a,b}$ are negative and positive, respectively, and the opposite is true for $b \lesssim 0$. Clearly the same holds for $q = q_{f_a}$ and $p = p_{f_a}$ if η is sufficiently small. Moreover, the determinant of $h_{a,b}$ is given by

$$\det Dh_{a,b} = \det \begin{pmatrix} -2ax & 1 \\ b & 0 \end{pmatrix} = -b,$$

which is constant and negative if b is positive, positive if b is negative. We thus refer to the case $b > 0$ as the *orientation-reversing* case, and the case $b < 0$ as the *orientation-preserving* case. Recall that the determinant of a matrix is the product of the eigenvalues, and thus the sign of the determinant has implications for the signs of the eigenvalues which, as we shall see, in turn have implications for the geometry of the stable and unstable manifolds of the fixed points. For $b = 0$ the fixed points p_* and q_* have derivatives 4 and -2, respectively; thus, for $b \neq 0$ and η small, the expanding eigenvalues of p and q are ≈ 4 and ≈ -2, respectively. This implies that for the orientation-reversing case $b \gtrsim 0$, the contracting eigenvalues of q and p must be less than and greater than 0 respectively, while for the orientation-preserving case $b < 0$, they must be greater than and less than 0 respectively. The two situations are illustrated in Figure 2, with dashed lines showing the invariant manifolds corresponding to negative eigenvalues.

2.1.3. *Analytic continuation of stable and unstable manifolds.* By classical hyperbolic theory, compact parts of the stable manifolds depend continuously on the map (see, e.g., [**PdM82**]). Therefore, for small b and small η the analytic continuations q, p of the fixed points q_* and p_* have stable and unstable manifolds which are close to those computed above for the limiting case. Elementary calculations show that the actual geometrical relations between these continuations depend on whether we consider
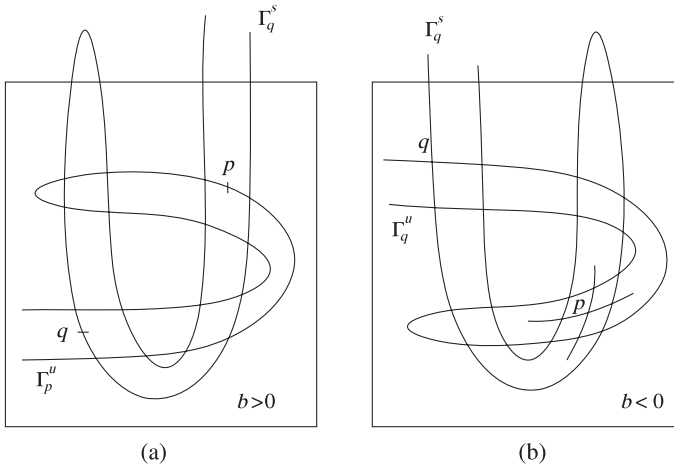
FIGURE 3. Invariant manifolds for $a > a^*$.

the orientation reversing ($b > 0$) or the orientation-preserving ($b < 0$) case, and are as illustrated in Figure 3. We let

$$\Gamma_a^u(p) \subset W_a^u(p), \quad \Gamma_a^s(q) \subset W_a^s(q), \quad \Gamma_a^u(q) \subset W_a^u(q)$$

denote the compact parts of the stable and unstable manifolds as shown in Figure 3 and note that since for $b = 0$ and $a > 2$ the unstable manifolds of p_a and q_a extend to the whole of the line, for each $a > 2$ and $b > 0$ sufficiently small we have that $W_{\mathrm{loc}}^u(p)$ crosses $W_{\mathrm{loc}}^s(p)$ four times, and for each $a > 2$ and $b < 0$ sufficiently small $W_{\mathrm{loc}}^u(q)$ crosses $W_{\mathrm{loc}}^s(q)$ four times; we can also ensure that the compact parts defined above and in Figure 3 intersect transversally. Again this continues to hold also for a Hénon-like family for sufficiently small η.

2.1.4. *Definition of a^*.* We are now ready to define the parameter a^*. Fix $b \neq 0$. For an *orientation-reversing* ($b > 0$) Hénon-like family f_a, let

$$a^* = \inf\{a : \Gamma_a^s(p) \text{ and } \Gamma_a^u(q) \text{ intersect transversally}\}.$$

For an *orientation-preserving* $b < 0$ Hénon-like family f_a, let

$$a^* = \inf\{a : \Gamma_a^s(q) \text{ and } \Gamma_a^u(q) \text{ intersect transversally}\}.$$

We also define a parameter $\hat{a}$ as the infimum of values a for which $W_{\mathrm{loc}}^u(p)$ crosses $W_{\mathrm{loc}}^s(p)$ four times (when $b > 0$) or $W_{\mathrm{loc}}^u(q)$ crosses $W_{\mathrm{loc}}^s(q)$ four times (when $b < 0$). Clearly these are weaker conditions than those defining a^* and thus $a^* \geq \hat{a}$. Notice that a^* and $\hat{a}$ both converge to $a = 2$ as b and η tend to 0.

2.2. *Localization of the non-wandering set.* In this section we carry out a detailed geometrical study aimed at showing that the non-wandering set is contained in a relatively
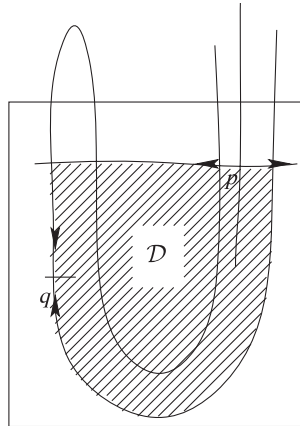
FIGURE 4. The region $\mathcal{D}$.

restricted region. To prove hyperbolicity we then only need to focus our efforts in this region. For the moment we restrict ourselves to the orientation-reversing case. At the end we shall remark how the orientation-preserving case follows by identical arguments with a few minor changes of notation. First of all we let $\mathcal{D}$ denote the closed topological disc bounded by compact pieces of $W^u(p)$ and $W^s(q)$ as shown in Figure 4. The main result of this section is the following.

PROPOSITION 1. *For all $a > \hat{a}$ we have*

$$\Omega \subset \overline{W^u(p)} \cap \mathcal{D} \cap \{[-2, 2] \times (-4b, 4b)\}.$$

Remark. In this paper we are interested in parameters $a \geq a^*(\geq \hat{a})$, but it is worth observing that from Proposition 1 it follows that for all $a \in (\hat{a}, a^*)$, and so in particular for a certain range of parameter values which may contain multiple tangencies, the recurrent dynamics is captured to some extent by the dynamics on $W^u(p)$. This includes all complex dynamical phenomena associated with the unfolding of the tangency at the parameter a^* (indeed, it includes the range of parameters considered by Benedicks and Carleson in [**BC91**]).

We split the proof of Proposition 1 into several lemmas. Once again we deal first with the case $b > 0$, and at the end of the proof indicate the modifications needed to deal with $b < 0$. We first define a relatively 'large' region R and show that $\Omega \subset R$; we then show in separate arguments that $\Omega \subset \mathcal{D}$ and $\Omega \subset \overline{W^u(p)}$, and finally refine our estimate to obtain the statement in Proposition 1. Let

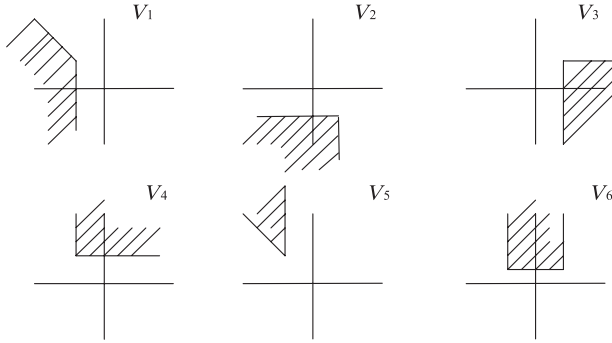$$R = (-2, 2) \times (-4, 4b) \subset \hat{R} = (-2, 2) \times (-4, 2).$$

FIGURE 5. Regions V_1 to V_6.

We also define the following six (overlapping) regions (see Figure 5):

$$V_1 = \{(x, y) : x \leq -2, \, y \leq |x|\},$$
$$V_2 = \{(x, y) : x \leq 2, \, y \leq -4\},$$
$$V_3 = \{(x, y) : x \geq 2, \, y \leq 2\},$$
$$V_4 = \{(x, y) : x \geq -2, \, y \geq 2\},$$
$$V_5 = \{(x, y) : x \leq -2, \, y \geq |x|\},$$
$$V_6 = \{(x, y) : |x| \leq 2, \, y \geq 4b\}.$$

Then

$$\hat{R} = \mathbb{R}^2 \setminus (V_1 \cup \cdots \cup V_5) \quad \text{and} \quad R = \mathbb{R}^2 \setminus (V_1 \cup \cdots \cup V_6)$$

and we prove the following lemmas.

LEMMA 1. $\Omega \subset R$.

Proof. We show that the orbit of every point $(x, y) \in V_i$, $i = 1, \ldots, 6$, is unbounded in either backward or forward time. This implies that no such point is non-wandering. For $n \in \mathbb{Z}$, let $(x_n, y_n) = f_a^n(x, y)$. We shall use repeatedly the fact that $a \approx 2$ and $b \approx 0$.

For $(x, y) \in V_1$ we have $x \leq -2$ and $y \leq |x|$, therefore $x_1 = 1 - ax^2 + y + \varphi_1(x, y, a) \leq 1 - ax^2 + |x| + \eta \leq -2$ and $y_1 = bx + \varphi_2(x, y, a) \leq -2b + \eta < |x_1|$, as long as η is sufficiently small. Thus $(x_1, y_1) \in V_1$, and $|x_1| \geq ax^2 - |x| - 1 - \eta \geq 2|x|$. Repeating the calculation we have $|x_n| \geq 2^n |x|$ and so $|x_n| \to \infty$.

For $(x, y) \in V_2$ we have $x_1 = 1 - ax^2 + y + \varphi_1(x, y, a) \leq -2$ and $y_1 = bx + \varphi_2(x, y, a) \leq 2b + \eta < |x_1|$. Thus $(x_1, y_1) \in V_1$ and so $|x_n| \to \infty$.

Similarly, for $(x, y) \in V_3$ we have $x_1 = 1 - ax^2 + y + \varphi_1(x, y, a) \leq 1 - 2 \cdot 2^2 + 2 + \eta \leq -2$ and $y_1 = bx + \varphi_2(x, y, a) \leq 2b + \eta < |x_1|$. Thus $(x_1, y_1) \in V_1$ and we argue as above.

For $(x, y) \in V_4$ we consider backward iterations of f_a. Note that $(x_{-1}, y_{-1}) = (y/b, x - 1 + ay^2/b^2) + \tilde{\varphi}(x, y, a)$. Then $x_{-1} \geq 2/b - \eta \geq -2$ and $y_1 \geq -2 + 4a/b^2 - \eta \geq 2$. Thus $f^{-1}(x, y) \in V_4$ and $y_1 \geq y/b$. Therefore $y_{-n} \geq y/b^n$ and so $|y_{-n}| \to \infty$.
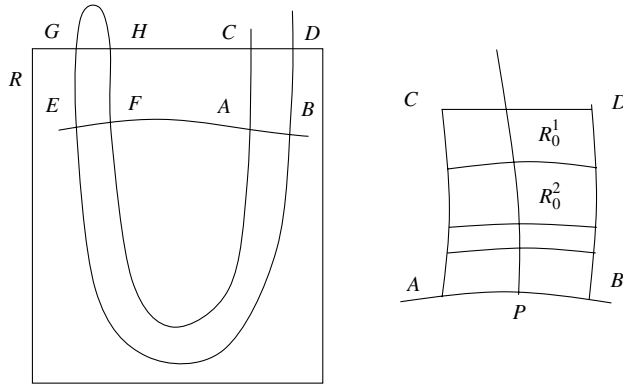
FIGURE 6. Regions R_0^i.

For $(x, y) \in V_5$ we have $y \geq |x| \geq 2$. Thus $x_1 \geq y/b - \eta \geq 2$ and $y_1 \geq y^2/b^2 \geq 2$. So $f^{-1}(x, y) \in V_4$, and we argue as above.

For $(x, y) \in V_6$ we have $x_{-1} = y/b + \tilde{\varphi}_2(x, y, a) \geq 2$ and $y_{-1} \geq 2$. Therefore $f_a(x, y) \in V_4$ and again we argue as above. □

LEMMA 2. $\mathcal{D} \subset \hat{R}$.

Proof. The arguments used above have implications for the locations of the stable and unstable manifolds of the fixed points. Indeed, the stable manifolds of the fixed points cannot intersect $V_1 \cup V_2 \cup V_3$ since all points in this region tend to infinity in forward time, whereas, by definition, points in the stable manifolds tend to the fixed points under forward iteration. Similarly, the unstable manifolds of the fixed points cannot intersect $V_4 \cup V_5$ since all points in this region tend to infinity in backward time. By definition, $\mathcal{D}$ is bounded by arcs of stable and unstable manifolds of the fixed point as in Figure 4, and therefore $\mathcal{D} \subset \mathbb{R}^2 \setminus (V_1 \cup \cdots \cup V_5) = (-2, 2) \times (-4, 2)$. □

LEMMA 3. $\Omega \subset \mathcal{D}$.

Proof. To show that $\Omega \subset \mathcal{D}$ we refine the strategy used in the proof of the previous lemma, and show that the orbits of all points outside $\mathcal{D}$ are unbounded in either backward or forward time. Since we have already shown that $\Omega \subset R$, we need only consider points in the region $R \setminus \mathcal{D}$.

Subdividing. We write

$$R \setminus \mathcal{D} = R_0 \cup R_1 \cup R_2 \cup R_3$$

where the regions R_0, R_1, R_2, R_3 are defined as follows. Consider the points A, B of intersection between $W^u(p)$ and $W^s(q)$ and the points C, D of intersection between $W^s(q)$ and $y = 4b$; see Figure 6. We let R_0 denote the closed region bounded by the arcs of manifolds AC, AB, BD and the line segment CD. Similarly, let R_1 denote the region bounded by the arcs of manifolds HF, FA, AC and the line segment HC. We let
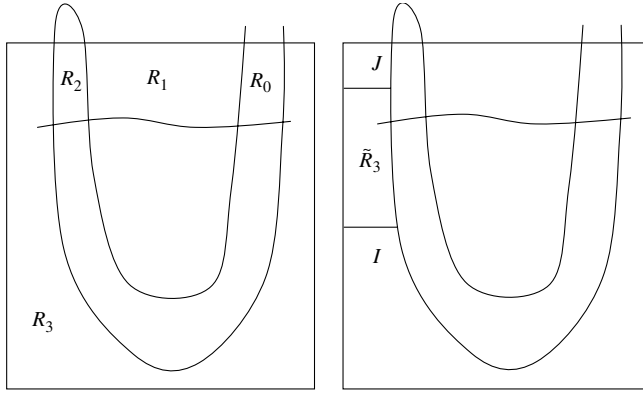
FIGURE 7. Regions R_1 to R_3 and $\tilde{R}_3$.

R_2 be the region bounded by the arcs of $W^u(p)$ and $W^s(q)$ between the points E and F, as in Figure 6; see also Figure 7. Then let $R_3 = R \setminus (\mathcal{D} \cup R_0 \cup R_1 \cup R_2)$. We also define

$$\tilde{R}_3 \subset R_3$$

as the region satisfying $-2b - \eta < y < 2b + \eta$ at the left-hand side of the arc of $W^s(q)$ between the points I and J of intersection of that manifold with the lines $y = -2b - \eta$ and $y = 2b + \eta$, as shown in Figure 7.

Points of R_0 escape in backward time. Since b is small, we have that all the points $(x, y) \in R_0$ satisfy $x > 0.2$. Notice that for the unperturbed Hénon map $h_{a,b}(x, y) = (1 - ax^2 + y, bx)$, any piece of curve γ with slope less than $1/10$ contained in the region where $|x| > 0.2$ is mapped to another curve with slope less than $1/10$. Indeed, letting (v_1, v_2) denote a tangent vector to γ with $|v_2|/|v_1| < 1/10$, we have $(v_1', v_2') = Dh_{a,b}(v_1, v_2) = (-2axv_1 + v_2, bv_1)$ whose slope is $|v_2'|/|v_1'| = |b/(-2ax + (v_2/v_1))| < 1/10$, provided b is small and a is close to 2. For future reference, note that if $|x| > 0.5$, we also have that the norm of (v_1, v_2) is uniformly expanded. So, since f_a is close to $h_{a,b}$ in the C^2 topology, we can assume that f_a also has this property in R_0.

Now denote by α_n the successive images of the segment CD intersected with R_0. Since these images cannot intersect each other, and CD has a point of the stable manifold of p, the curves α_n determine a system of 'fundamental domains' in R_0: they cross R_0 from one stable boundary to the other, and they converge to the arc of unstable manifold AB. Let R_0^i be the region of R_0 between α_{i-1} and α_i, with $\alpha_0 = CD$ (see Figure 6), and notice that $f^{-1}(R_0^i) \subset R_0^{i-1}$. We also have that $f^{-1}(R_0)$ falls outside R. This implies that $R_0 \setminus AB$ does not intersect Ω, and any point which has an iterate in $R_0 \setminus AB$ is not in Ω.

Points of R_1 map to R_3. We show that $f(R_1) \cap R \subset R_3$. Indeed, the unstable eigenvalue of p is positive and therefore $f(R_1)$ must remain on the same side of $W^s(q)$ as R_1. Also, since $f(R) \subset \mathbb{R} \times [-2b - \eta, 2b + \eta]$, we have that $f(R_1)$ does not intersect any of $\mathcal{D}, R_0, R_1, R_2$.

Points of R_3 map to $\tilde{R}_3$. We now show that $f(R_3) \subset \tilde{R}_3$. Again, we use the fact that $f(R) \subset \mathbb{R} \times [-2b - \eta, 2b + \eta]$. Then, since one of the components of the boundary of
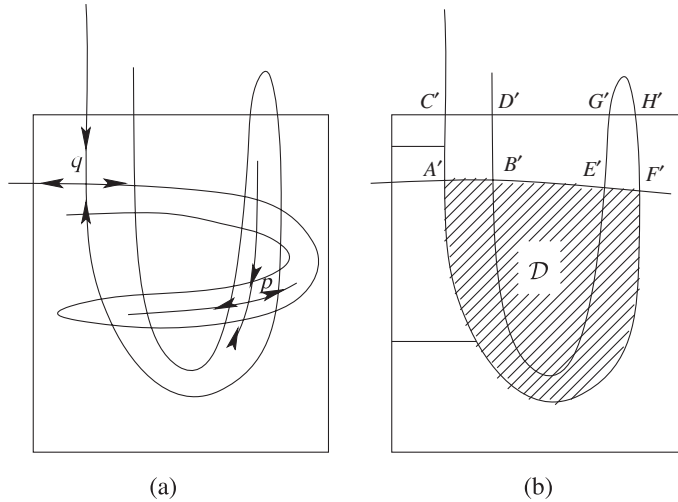
FIGURE 8. Invariant manifolds and the region $\mathcal{D}$ for $b < 0$.

R_3 is an arc of the stable manifold of q containing the fixed point q, and the unstable eigenvalue of q is positive, we conclude that the image of R_3 is contained $\tilde{R}_3$.

Points of $\tilde{R}_3$ escape in forward time. We can assume, if b is small, that all the points (x, y) in $\tilde{R}_3$ satisfy $x < -0.5$ (notice that, for $b = 0$, we have $q = (-1, 0)$). Take t to be a point in $\tilde{R}_3 \setminus W^s(q)$, and connect t to the boundary of $\tilde{R}_3$ by a horizontal line inside $\tilde{R}_3$; this determines a point $t' \in W^s(q)$. Again, by the proximity of f and $h_{a,b}$, and the fact that vectors with slope smaller than $1/10$ in $\tilde{R}_3 \cap R$ are sent by $Dh_{a,b}$ to vectors with slope smaller than $1/10$, and uniformly expanded, we have that the horizontal distance between $f(t)$ and $f(t')$ is uniformly expanded. Applying f repeatedly, as long as the image is inside $\tilde{R}_3 \cap R$, we have that the horizontal distance between the successive images of t and $W^s(q)$ increases exponentially. Hence the forward images of t leave R for some positive time.

Points of R_2 map to R_0 in backward time. Notice that $f^{-1}(R_2) \cap R \subset R_0$ since all the other regions in R outside $\mathcal{D}$ are mapped forward to the region $\tilde{R}_3$, and so do not contain points of the backward image of R_2. Moreover, the unstable boundary of R_2 belongs to $W^u(p)$ and approaches p as we apply f^{-1}, while the stable boundary cannot cross $W^s(q)$, thus $f^{-1}(R_2)$ does not intersect $\mathcal{D}$. Since $f^{-1}(R_2) \cap R \subset R_0$, the points in there that are not in $W^u(p)$ leave R for backward iterations. $\square$

LEMMA 4. $\Omega \subset \overline{W^u(p)}$.

Proof. Notice first of all that by the λ-lemma we have $q \in \overline{W^u(p)}$. Now suppose for contradiction that there exists $z = (x, y) \in \Omega$ with $z \notin \overline{W^u(p)}$. Then there exists ε and an ε-neighbourhood $B_\varepsilon(z)$ of z with $B_\varepsilon(z) \cap \overline{W^u(p)} = \emptyset$. Since Ω is f-invariant we have $f^{-n}(z) \in \Omega(f) \subset \mathcal{D}$ for all $n \in \mathbb{N}$ and therefore $z \in f^n(\mathcal{D})$ for all $n \in \mathbb{N}$. Notice that the boundary $\partial f^n(\mathcal{D})$ lies in $W^u(p) \cup f^n(EB^s)$, where EB^s denotes the piece of $W^s(q)$ between E and B, as in Figure 6. It is enough therefore to show that for large

n, $\partial f^n(\mathcal{D})$ is ε-dense in $f^n(\mathcal{D})$, as this will imply that $B_\varepsilon(z) \cap \overline{W^u(p)} \neq \emptyset$, contradicting the assumption. Because f is (strongly) area-contracting, the area of $f^n(\mathcal{D})$ tends to zero as $n \to \infty$. In particular, we must have that $B_\varepsilon(z) \cap \partial f^n(\mathcal{D}) \neq \emptyset$ for all $n \geq N$ sufficiently large. Moreover, the length of the part of the boundary which belongs to $W^s(q)$ also tends to zero. Thus most of the boundary belongs to $W^u(p)$ and so we must have $B_\varepsilon(z) \cap W^u(p) \neq \emptyset$ for all n sufficiently large. $\square$

2.2.1. *Completion of the proof of Proposition 1.*   Combining the results of the lemmas stated above we have that $\Omega \subset \overline{W^u(p)} \cap \mathcal{D}$. The statement in Proposition 1 now follows immediately by observing that $\Omega \subset \mathcal{D}$ implies $\Omega \subset f(\mathcal{D})$ and that $f(\mathcal{D}) \subset [-2, 2] \times [-4b, 4b]$ directly from the definition of f if η is sufficiently small.

Finally, in the case $b < 0$, we consider the stable and unstable manifolds of q crossing as in Figure 8 (the rectangle R is exactly the same), determining the region $\mathcal{D}$ in this case. The proof is entirely analogous, considering the points A', B' etc corresponding to the points A, B etc above. $\square$

3.   *Hyperbolic coordinates and critical points*

The key idea of our whole strategy is the notion of *dynamically defined critical point* which relies on the fundamental notion of *hyperbolic coordinates*. In this section we introduce these notions and develop the main technical ideas which we will use. In §3.1 we clarify the relations between various constants used in the argument and introduce some preliminary geometric constructions. In §3.2 we discuss the definition and basic theory of hyperbolic coordinates. In §3.3 we introduce the idea of *admissible curves* and prove certain estimates concerning the images of admissible curves. Finally, in §3.4 we introduce the notion of a dynamically defined critical point and prove that such critical points always exist in images of certain admissible curves.

3.1.   *Preliminary geometric definitions and fixing the constants.*

3.1.1.   *Fixing the constants.*   We now explain the required relations between the different constants used in the proof, and the order in which these constants are chosen. All constants are positive. First of all we fix the values of two constants

$$\delta = 1/10 \quad \text{and} \quad \alpha = 1/2.$$

The meanings of these constants will be introduced in §3.1.2 and §3.3 below. Even though we specify the actual numerical values of these constants, we shall continue to use them as general constants in the argument because they have specific geometric meanings and it is useful to keep track of their occurrence throughout the paper. We then fix a constant k_0 large enough so that

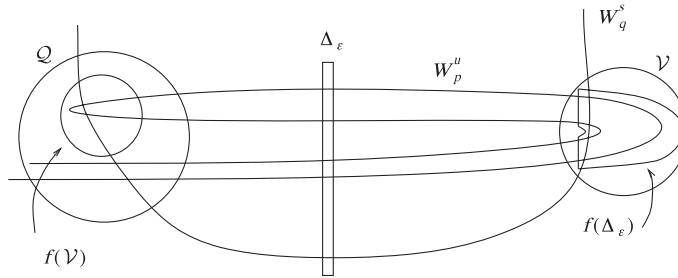$$\frac{\sqrt{\delta}}{2\sqrt{3}} \left(\sqrt{\frac{3}{\sqrt{5}}} \right)^{k_0 - 1} > 1. \tag{3}$$

FIGURE 9. The neighbourhoods $\mathcal{Q}$ and $\mathcal{V}$.

In §3.1.3 we fix a constant ε which will then remain unchanged. Finally, at certain places in the argument, we will require a to be sufficiently close to 2 and $|b|$ and η to be sufficiently small.

We remark that we can suppose that a is close to 2 without compromising the fact that hyperbolicity holds for all larger values of a. Indeed, once we fix a neighbourhood of 2 in the a parameter space, we can always guarantee uniform hyperbolicity for values of $a > 2$ outside this neighbourhood by taking $|b|$ and η sufficiently small (depending on the neighbourhood).

3.1.2. *The fixed point neighbourhoods.* Recall first of all that the map $f_* = h_{2,0}$ has two fixed points p_* and $q_* = (-1, 0)$ with $f_*(1, 0) = q_*$. Let $\mathcal{Q} = \mathcal{Q}_0 := B_\delta(q_*)$ be the open ball of radius δ centred at q_*, and let $\mathcal{V} = \mathcal{V}_0$ be the component of $f_*^{-1}(\mathcal{Q})$ not intersecting $\mathcal{Q}$; see Figure 9. The expanding eigenvalue at the point q_* is equal to 4 and so we can suppose that $|a - 2|, |b|, \eta$ are all small enough so that $\|Df_z\| > 3$ for all $z \in \mathcal{Q}$. Then, for $n \geq 0$, let

$$\mathcal{Q}_n(f) = \bigcap_{i=0}^{n} f^{-i}(\mathcal{Q}_0) \quad \text{and} \quad \mathcal{V}_n(f) = f^{-1}(\mathcal{Q}_n(f)) \cap \mathcal{V}.$$

Notice that $\mathcal{V}_n$ is just the component of $f^{-1}(\mathcal{Q}_n(f))$ containing $(1, 0)$. Since $\mathcal{Q}_n$ is a neighbourhood of q for every n, the set $\mathcal{V}_n \setminus f^{-1}(W_\delta^s(q))$, where $W_\delta^s(q)$ denotes the connected component of $W^s(q) \cap \mathcal{Q}_0$ containing q, has two components. Let

$$\mathcal{V}_n^- = V_n \cap \mathcal{D} \quad \text{and} \quad \mathcal{V}_n^+ = \mathcal{V}_n \setminus \mathcal{V}_n^-.$$

Notice that a piece of $W^s(q)$ forms the boundary between $\mathcal{V}_n^+$ and $\mathcal{V}_n^-$. We mention for future reference a simple estimate which we shall use below.

LEMMA 5. $d(z, f^{-1}(W_\delta^s(q))) \geq \delta/5^k$ *for all* $z \in \mathcal{V}_k \setminus \mathcal{V}_{k+1}$.

Proof. $z \in \mathcal{V}_k \setminus \mathcal{V}_{k+1}$ implies, by definition, $d(z_{k+1}, q) \geq \delta$. For points z close to $f^{-1}(W_\delta^s(q))$ this also means $d(z_{k+1}, W_\delta^s(q)) \geq \delta$ since such points come very close to the fixed point q and escape the δ-neighbourhood of q along the direction of $W^u(q)$. Thus, using the fact that the norm of the derivative Df in $\mathcal{D}$ is uniformly bounded above by 5, we obtain the result. $\square$

3.1.3. *The critical neighbourhood.* For $\varepsilon > 0$ we define a *critical neighbourhood*

$$\Delta_\varepsilon = (-\varepsilon, \varepsilon) \times (-4b, 4b).$$

Note that we can take ε sufficiently small so that $q_f \in f(\mathcal{V})$ and

$$f(\Delta_\varepsilon) \subset \mathcal{V}_{k_0}.$$

From now on we consider ε fixed. We also let

$$\Delta = \Delta_a = \{x \in \Delta_\varepsilon : f(x) \notin \mathcal{D}\}.$$

For a sufficiently close to 2 and $|b|$ and η sufficiently small we have uniform hyperbolicity outside Δ_ε. We state this fact more formally in the following.

LEMMA 6. *For every $\hat{\lambda} \in (0, \log 2)$ and $|a - 2|$, $|b|$, $\eta > 0$ sufficiently small, there exists a constant $C_\varepsilon > 0$ such that for all $k \geq 1$, any point z with $z, f(z), \ldots, f^{k-1}(z) \notin \Delta_\varepsilon$, and any vector v with slope less than α, we have*

$$slope\ Df_z^k(v) < \alpha, \tag{4}$$

$$\|Df_z^k(v)\| \geq C_\varepsilon e^{\hat{\lambda} k} \|v\|. \tag{5}$$

If, moreover, $f^k(z) \in \Delta$, then we have

$$\|Df_z^k(v)\| \geq e^{\hat{\lambda} k} \|v\|. \tag{6}$$

Proof. This is a standard result (see, for example, [**BC91**] or [**MV93**]) and so we omit the details. We just mention that it follows from the fact that the limiting one-dimensional map $h_{2,0}$ satisfies uniform expansivity estimates outside an arbitrary critical neighbourhood Δ_ε (with constant $\hat{\lambda}$ arbitrarily close to $\log 2$ but constant C_ε depending on ε and arbitrarily small for ε small); see, e.g., [**dMvS93**].

Considering this one-dimensional map as embedded in the space of two-dimensional maps and using the fact that uniform hyperbolicity is an open condition, we obtain the statement in the lemma for $|b|$, $\eta \neq 0$ sufficiently small. $\square$

3.2. *Hyperbolic coordinates.* The notion of 'hyperbolic coordinates' is inspired by some constructions in [**BC91, MV93**], developed in [**LV03**] and formalized in [**HL06**] as an alternative framework with which to approach the classical theory of invariant manifolds. Here we review the basic definitions and theory to the extent to which they will be required for our purposes.

3.2.1. *Hyperbolicity of compositions of linear maps.* We recall the notion of hyperbolic coordinates and give the basic definitions and properties in the general context of C^2 diffeomorphisms of a Riemannian surface M. For $z \in M$ and $k \geq 1$, let

$$F_k(z) = \|Df_z^k\| \quad \text{and} \quad E_k(z) = \|(Df_z^k)^{-1}\|^{-1}$$

denote the maximum expansion and the maximum contraction, respectively, of Df_z^k. Then we think of the quantity

$$H_k(z) = E_k(z)/F_k(z)$$

as the *hyperbolicity* of Df_z^k. Notice that $H_k \leq 1$ always. The condition $H_k = E_k/F_k < 1$ implies that the linear map Df^k maps the unit circle $\mathcal{S} \subset T_z M$ to an ellipse $\mathcal{S}_k = Df_z^k(\mathcal{S}) \subset T_{f^k(z)} M$ with well-defined major and minor axes. The unit vectors $e^{(k)}$, $f^{(k)}$ which are mapped, respectively, to the minor and major axis of the ellipse and are thus the *most contracted* and *most expanded* vectors, respectively, are given analytically as solutions to the differential equation $d\|Df_z^k(\cos\theta, \sin\theta)\|/d\theta = 0$ which can be solved to give the explicit formula

$$\tan 2\theta = \frac{2[(\partial_x f_1^k)(\partial_y f_1^k) + (\partial_x f_2^k)(\partial_y f_2^k)]}{(\partial_x f_1^k)^2 + (\partial_x f_2^k)^2 - (\partial_y f_1^k)^2 - (\partial_y f_2^k)^2}.$$

Here $f = (f_1, f_2)$ are the two coordinate functions of f. Notice that $e^{(k)}$ and $f^{(k)}$ are always *orthogonal* and *do not* in general correspond to the stable and unstable eigenspaces of Df^k.

3.2.2. *Hyperbolic coordinates and stable and unstable foliations.*

We define the *hyperbolic coordinates of order k* at the point z as the orthogonal coordinates $\mathcal{H}_k(z)$ given at z by the most contracted and most expanded directions for Df_z^k. If f is C^2 and $H_k(z) < 1$, then hyperbolic coordinates are defined in some neighbourhood of z and give two orthogonal C^1 vector fields. In particular, they are locally integrable and thus give rise to two orthogonal foliations. We let $\mathcal{E}^{(k)}$ denote the *stable foliation of order k* formed by the integral curves of the vector field $\{e^{(k)}\}$ and $\mathcal{F}^{(k)}$ denote the *unstable foliation of order k* formed by the integral curves of the vector field $\{f^{(k)}\}$.

3.2.3. *Hyperbolic coordinates for Hénon-like maps.*

A crucial property of hyperbolic coordinates and finite order stable and unstable foliations is that, under very mild assumptions, they *converge* in quite a strong sense as $k \to \infty$. We formulate a version of this property here in our specific context.

PROPOSITION 2. *For every $k \geq 1$, hyperbolic coordinates $\mathcal{H}_k$ and stable and unstable foliations $\mathcal{E}^{(k)}$ and $\mathcal{F}^{(k)}$ are defined in $\mathcal{V}^+ \cup \mathcal{V}_k^-$. Moreover,*

(1)   *the angle between each stable direction $e^{(k)}$ and the slope of $f^{-1}(W_\delta^s(q))(\approx 2)$;*

(2)   *the curvature of each stable leaf*

are both $\lesssim b$. Also, the C^2 distance between leaves of $\mathcal{E}^{(k)}$ and leaves of $\mathcal{E}^{(k+1)}$ is $\lesssim bk$.

Proof. Analogous convergence results are formulated and proved in great generality in [**HL06**] under weak (subexponential) growth of the derivative. Here we shall need only some very particular cases of these estimates and therefore we first describe the specific setting in which they will be applied here. The main ingredient for the proof is the fact that by our choice of δ and assuming that $|a - 2|$, $|b|$ and η are small enough, we have that $\|Df(z) - Df_*(q_*)\|$ is small for all $z \in \mathcal{Q}$ and thus

$$E_k(z_0) \leq b^k \quad \text{and} \quad F_k(z_0) \geq 3^k \quad \text{for all } z_0 \in \mathcal{V}_k. \tag{7}$$

It follows immediately that hyperbolic coordinates of order k and their associated foliations are well defined in $\mathcal{V}_k$. Points in $\mathcal{V}_k^-$ are then re-injected into $\mathcal{D} \setminus \mathcal{Q}$ and these hyperbolicity

estimates can no longer be guaranteed, a priori, for all time. Points in $\mathcal{V}_k^+$, however, are outside $\mathcal{D}$ and therefore, by the arguments of §2, eventually escape towards infinity. In particular, the required hyperbolicity conditions can be guaranteed to hold for all positive iterates. This implies that hyperbolic coordinates of order k are well defined in $\mathcal{V}^+ \cup \mathcal{V}_k^-$ as in the statement of the proposition.

The statements about the direction of the stable directions, the curvature of the leaves and the C^2 distance between stable leaves of different orders all follow directly from [**HL06**, Main theorem]. These calculations are purely technical and do not add to our geometrical understanding of this situation; we therefore omit the details and refer the reader to that paper.   □

3.3. Admissible curves.

3.3. *Admissible curves.*   Recall that the curvature $\kappa(s)$ of a parametrized curve $\gamma(s) = (x(s), y(s))$ is given by

$$\kappa(s) = \frac{|\dot{x}\ddot{y} - \dot{y}\ddot{x}|}{\|(\dot{x}, \dot{y})\|^3} = \frac{|\dot{\gamma} \times \ddot{\gamma}|}{|\dot{\gamma}|^3}.$$

The equivalence between the two formulas comes from the relation $(v_1, v_2) \times (w_1, w_2) = v_1 w_2 - v_2 w_1$.

Definition 1. For $\alpha > 0$, we say that a C^2 curve $\gamma = \gamma(s) = (x(s), y(s))$ is *admissible* if $|\dot{y}(s)|/|\dot{x}(s)| < \alpha$ and $|\kappa(s)| < \alpha$ for all s.

We remark that both the curvature and the slope of tangent vectors of a curve are independent of the parametrization, and thus so is the definition of admissibility. We shall want to compare the curvature at a point of a curve and at the corresponding point of its image; so, suppose $\gamma_{i-1}(s)$ is a parametrized C^2 curve and $\gamma_i(s) = f(\gamma_{i-1}(s))$. For simplicity we shall often omit the parameter s and simply write Df to denote the derivative at the point $\gamma_{i-1}(s)$.

PROPOSITION 3. *Let $\{\gamma_i\}_{i=0}^n$ be a sequence of C^2 curves with $\gamma_i = f^i(\gamma_0)$. Suppose that for some s, n is a 'hyperbolic time' in the sense that*

$$\|\dot{\gamma}_n(s)\| \geq C e^{\lambda j} \|\dot{\gamma}_{n-j}(s)\|$$

for all $j = 0, \ldots, n-1$. Then for $|b|$, η sufficiently small, $\kappa_0(s) < \alpha$ implies $\kappa_n(s) < \kappa_0(s) < \alpha$.

COROLLARY 1. *If $\gamma \subset \mathcal{D} \setminus \Delta_\varepsilon$ is admissible, then $f(\gamma)$ is also admissible.*

Proof. This follows from Proposition 3 and the hyperbolicity outside Δ_ε. Condition (4) implies that the slope of each tangent vector to $f(\gamma)$ is less than α and condition (5) together with Lemma 3 gives the curvature less than α.   □

To prove Proposition 3 we first prove a general curvature estimate. We fix some bounded neighbourhood $\hat{R}$ of R and, as above, suppose $\{\gamma_i\}_{i=0}^n$ is a sequence of C^2 (not necessarily admissible) curves with $\gamma_i = f^i(\gamma_0)$, all contained in $\hat{R}$.

LEMMA 7. *There exists $K > 0$ independent of a, b, η such that for all $i = 1, \ldots, n$ we have*

$$\kappa_i(s) \leq K(b + \eta) \frac{|\dot{\gamma}_{i-1}(s)|^3}{|\dot{\gamma}_i(s)|^3} \kappa_{i-1}(s) + K(b + \eta) \frac{|\dot{\gamma}_{i-1}(s)|^3}{|\dot{\gamma}_i(s)|^3}.$$

Proof. We use the formula $\kappa = |\dot{\gamma} \times \ddot{\gamma}|/|\dot{\gamma}|^3$ for the curvature. We have

$$\dot{\gamma}_i = (Df)\dot{\gamma}_{i-1} = \begin{pmatrix} f_{1,x} & f_{1,y} \\ f_{2,x} & f_{2,y} \end{pmatrix} \dot{\gamma}_{i-1} = \begin{pmatrix} -2ax_{i-1} + \varphi_{1,x} & 1 + \varphi_{1,y} \\ b + \varphi_{2,x} & \varphi_{2,y} \end{pmatrix} \dot{\gamma}_{i-1}$$

and

$$\ddot{\gamma}_i = \begin{pmatrix} \nabla f_{1,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{1,y} \cdot \dot{\gamma}_{i-1} \\ \nabla f_{2,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{2,y} \cdot \dot{\gamma}_{i-1} \end{pmatrix} \dot{\gamma}_{i-1} + (Df)\ddot{\gamma}_{i-1}.$$

Therefore $\dot{\gamma}_i \times \ddot{\gamma}_i$ is given by

$$(Df)\dot{\gamma}_{i-1} \times \begin{pmatrix} \nabla f_{1,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{1,y} \cdot \dot{\gamma}_{i-1} \\ \nabla f_{2,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{2,y} \cdot \dot{\gamma}_{i-1} \end{pmatrix} \dot{\gamma}_{i-1} + (Df)\dot{\gamma}_{i-1} \times (Df)\ddot{\gamma}_{i-1} \qquad (8)$$

where

$$\nabla f_{1,x} = \begin{pmatrix} -2a + \varphi_{1,xx} \\ \varphi_{1,xy} \end{pmatrix}, \qquad (9)$$

and

$$\nabla f_{1,y} = \begin{pmatrix} \varphi_{1,xy} \\ \varphi_{1,yy} \end{pmatrix}; \quad \nabla f_{2,x} = \begin{pmatrix} \varphi_{2,xx} \\ \varphi_{2,xy} \end{pmatrix}; \quad \nabla f_{2,y} = \begin{pmatrix} \varphi_{2,xy} \\ \varphi_{2,yy} \end{pmatrix}. \qquad (10)$$

We shall estimate the two terms of (8) separately. These will yield the two terms in the statement of the lemma. For the second term we have

$$|(Df)\dot{\gamma}_{i-1} \times (Df)\ddot{\gamma}_{i-1}| = |\det(Df)||\dot{\gamma}_{i-1} \times \ddot{\gamma}_{i-1}| = |\det(Df)|\kappa_{i-1}|\gamma_{i-1}|^3.$$

Indeed, for the first equality, $|\dot{\gamma}_{i-1} \times \ddot{\gamma}_{i-1}|$ is the area of the parallelogram defined by the two vectors $\dot{\gamma}_{i-1}$ and $\ddot{\gamma}_{i-1}$, and $|(Df)\dot{\gamma}_{i-1} \times (Df)\ddot{\gamma}_{i-1}|$ is the area of the parallelogram defined by the two vectors $(Df)\dot{\gamma}_{i-1}$ and $(Df)\ddot{\gamma}_{i-1}$ which of course is just the image of the first parallelogram under Df. The second equality follows immediately from the definition of κ_{i-1}. So it just remains to show that the value of $|\det(Df)|$ is bounded above by some multiple of b and η. Indeed, writing $f = h + \varphi$ we have, by the 'row-linearity' of the determinant,

$$\det(Df) = \det \begin{pmatrix} h_{1x} + \varphi_{1x} & h_{1y} + \varphi_{1y} \\ h_{2x} + \varphi_{2x} & h_{2x} + \varphi_{2y} \end{pmatrix}$$

$$= \det \begin{pmatrix} h_{1x} & h_{1y} \\ h_{2x} + \varphi_{2x} & h_{2x} + \varphi_{2y} \end{pmatrix} + \det \begin{pmatrix} \varphi_{1x} & \varphi_{1y} \\ h_{2x} + \varphi_{2x} & h_{2x} + \varphi_{2y} \end{pmatrix}$$

$$= \det \begin{pmatrix} h_{1x} & h_{1y} \\ h_{2x} & h_{2x} \end{pmatrix} + \det \begin{pmatrix} h_{1x} & h_{1y} \\ \varphi_{2x} & \varphi_{2y} \end{pmatrix} + \det \begin{pmatrix} \varphi_{1x} & \varphi_{1y} \\ h_{2x} & h_{2x} \end{pmatrix}$$

$$+ \det \begin{pmatrix} \varphi_{1x} & \varphi_{1y} \\ \varphi_{2x} & \varphi_{2y} \end{pmatrix}.$$

Using $h_{1x} = -2a, h_{1y} = 1, h_{2x} = b, h_{2y} = 0$ and $\|\varphi\|_{C^2} \leq \eta$, this gives

$$\det(Df) \leq b + (2a\eta + \eta) + (2a\eta + \eta) + \eta = b + 4a\eta + 3\eta \leq b + 12\eta$$

where in the last step we have used the fact that a is close to 2. Substituting this above gives the required bound for the second term of (8). To bound the first term we write

$$(Df)\dot{\gamma}_{i-1} \times \begin{pmatrix} \nabla f_{1,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{1,y} \cdot \dot{\gamma}_{i-1} \\ \nabla f_{2,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{2,y} \cdot \dot{\gamma}_{i-1} \end{pmatrix} \dot{\gamma}_{i-1} = \begin{pmatrix} a_1 & b_1 \\ c_1 & d_1 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \times \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}.$$

The norm of this expression is bounded above by

$$
\begin{aligned}
&|a_1 c_1 v_1^2 + a_1 d_2 v_1 v_2 + b_1 c_2 v_1 v_2 + b_1 d_2 v_2^2 - a_2 c_1 v_1^2 - a_2 d_1 v_1 v_2 - b_2 c_1 v_1 v_2 - d_1 b_2 v_2^2| \\
&\leq \max\{|a_1 c_2 - a_2 c_1|, |b_1 d_2 - d_1 b_2| + |a_1 d_2 - c_1 b_2| + |b_1 c_2 - a_2 d_1|\} \, (v_1^2 + v_2^2) \\
&\leq 4 \max\{|a_1 c_2 - a_2 c_1|, |b_1 d_2 - d_1 b_2|, |a_1 d_2 - c_1 b_2|, |b_1 c_2 - a_2 d_1|\} \, (v_1^2 + v_2^2) \\
&\leq 8 \max\{|a_1 c_2|, |a_2 c_1|, |b_1 d_2|, |d_1 b_2|, |a_1 d_2|, |c_1 b_2|, |b_1 c_2|, |a_2 d_1|\} \, |\dot{\gamma}_{i-1}|^2.
\end{aligned}
$$

All the terms contain a factor $\dot{\gamma}_{i-1}$. Each of the terms b_2, c_2, d_2, see (10), contains a bounded constant multiplied by the factor η; the term a_2, see (9), is of the order of $2a$ but here it is multiplied by either c_1 or d_1, each one of which contains a term that is bounded by η. Therefore, there exists a constant $K > 0$ such that

$$\left| (Df)\dot{\gamma}_{i-1} \times \begin{pmatrix} \nabla f_{1,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{1,y} \cdot \dot{\gamma}_{i-1} \\ \nabla f_{2,x} \cdot \dot{\gamma}_{i-1} & \nabla f_{2,y} \cdot \dot{\gamma}_{i-1} \end{pmatrix} \dot{\gamma}_{i-1} \right| \leq K \eta |\gamma_{i-1}|^3. \qquad \square$$

Proof of Proposition 3. Applying Lemma 7 recursively we get

$$
\begin{aligned}
\kappa_n(s) &\leq K(b+\eta)\kappa_{n-1}(s)\frac{|\dot{\gamma}_{n-1}|^3}{|\dot{\gamma}_n|^3} + K(b+\eta)\frac{|\dot{\gamma}_{n-1}|^3}{|\dot{\gamma}_n|^3} \\
&\leq (K(b+\eta))^2 \kappa_{n-2}\frac{|\dot{\gamma}_{n-2}|^3}{|\dot{\gamma}_n|^3} + (K(b+\eta))^2 \frac{|\dot{\gamma}_{n-2}|^3}{|\dot{\gamma}_n|^3} + K(b+\eta)\frac{|\dot{\gamma}_{n-1}|^3}{|\dot{\gamma}_n|^3} \\
&\leq \cdots .
\end{aligned}
$$

Using the expansivity assumption and b, η small, this gives

$$\kappa_n(s) \leq \frac{1}{C^3}(K(b+\eta)e^{-\lambda})^n \kappa_0(s) + \frac{1}{C^3}\frac{K(b+\eta)e^{-\lambda}}{1 - K(b+\eta)e^{-\lambda}} \leq \kappa_0(s) \leq \alpha. \qquad \square$$

3.4. *Critical points.* The next proposition makes precise the notion of a *critical point of order k*. We recall that γ is a *C^2 admissible curve* if all its tangent vectors have slope less than α and it has curvature less than α. We say that γ is a *long admissible curve* if it is an admissible curve which crosses the entire length of Δ_ε.

PROPOSITION 4. *Let $\gamma \subset \Delta_\varepsilon \cap \mathcal{D}$ be a long admissible curve. Then there exists a unique point $c^{(k)} \in \gamma$ such that $\gamma_0 = f(\gamma)$ has a (quadratic) tangency at $c_0^{(k)} = f(c^{(k)}) \in \mathcal{V}_k^- \cup \mathcal{V}^+$ with the stable foliation $\mathcal{E}^{(k)}$, for any $k \geq k_0$. Moreover, there exists a constant K, independent of b, η, such that $d(c_0^{(k)}, c_0^{(k+1)}) \leq K b^k$. In particular, the sequence $\{c_0^{(k)}\}$ is Cauchy.*

Definition 2. We call $c^{(k)}$ and $c_0^{(k)}$ respectively a *critical point* and *critical value* of order k, associated with the long admissible curve γ.

We remark that critical values $c_0^{(k)}$ of finite order are not guaranteed to be outside $\mathcal{D}$; however, we shall show below that their limit points as $k \to \infty$, i.e. the 'real' critical points, always fall strictly outside $\mathcal{D}$ for $a > a^*$.

Given a parametrized curve $\gamma_0 = \gamma_0(t)$ and its image $\gamma_1 = \gamma_1(t) = f(\gamma_0(t))$, we denote by $\kappa_0(t)$ the curvature of γ_0 at the point $\gamma_0(t)$ and by $\kappa_1(t)$ the curvature of γ_1 at the point $\gamma_1(t)$.

LEMMA 8. *Let $\gamma_0(t)$ be an admissible curve and let $\gamma_1(t) = f(\gamma(t)) = (\xi_1(t), \eta_1(t))$. Suppose that for some t we have $\dot{\eta}_1(t) \neq 0$ and $|\dot{\xi}_1(t)/\dot{\eta}_1(t)| < 1$. Then $|\kappa_1(t)| > a/b \gg 1$.*

Lemma 8 essentially says that if the tangent direction of the image of an admissible curve at a certain point is roughly vertical (or at least contained in the 'vertical' cone between the positive and the negative diagonals), then the curvature at this point is strictly bounded away from 0. This does not apply to admissible curves outside Δ_ε, since we have shown above (Corollary 1) that images of such curves are still admissible and therefore their tangent directions are roughly horizontal. We will instead apply it below to the images of admissible curves inside Δ_ε as a way of pinpointing the location of *folds*.

Proof. First recall that the curvature $\kappa_1(t)$ is independent of the choice of parametrization, and also that the condition $|\dot{\xi}_1(t)/\dot{\eta}_1(t)| < 1$ is independent of the parametrization since $|\dot{\xi}_1(t)/\dot{\eta}_1(t)|$ is just the slope of the tangent vector. We therefore choose the parametrization

$$\gamma_0(t) = (t, y(t)).$$

For simplicity we also omit the subscript 1 from the coordinate functions of γ_1 and just write $\gamma_1(t) = (\xi(t), \eta(t))$. From the definition of f we have

$$(\xi(t), \eta(t)) = (1 + at^2 + y(t) + \varphi_1(\gamma_0(t)), \, bt + \varphi_2(\gamma_0(t))),$$
$$(\dot{\xi}(t), \dot{\eta}(t)) = (-2at + \dot{y}(t) + \nabla\varphi_1(\gamma_0(t)) \cdot \dot{\gamma}_0(t), \, b + \nabla\varphi_2(\gamma_0(t)) \cdot \dot{\gamma}_0(t)),$$
$$(\ddot{\xi}(t), \ddot{\eta}(t)) = (-2a + \dot{y}(t) + D^2\varphi_1(\gamma_0(t)) \, [\dot{\gamma}_0(t)]^2, \, D^2\varphi_2(\gamma_0(t)) \, [\dot{\gamma}_0(t)]^2).$$

Choosing η sufficiently small, for example so that $4\|\nabla\varphi_2(\gamma_0(t))\|(1 + \alpha) < b$, this implies

$$3b/4 \leq |\dot{\eta}(t)| \leq 5b/4. \tag{11}$$

We can now compute the curvature $\kappa_1(t)$. First notice that the condition $|\dot{\xi}_1(t)/\dot{\eta}_1(t)| < 1$ implies $\|(\dot{\xi}(t), \dot{\eta}(t))\| \leq \sqrt{2}|\dot{\eta}(t)|$; thus we have

$$\kappa_1(t) = \frac{|\ddot{\xi}(t)\dot{\eta}(t) - \dot{\xi}(t)\ddot{\eta}(t)|}{\|(\dot{\xi}(t), \dot{\eta}(t))\|^3} \geq \frac{|\ddot{\xi}(t)\dot{\eta}(t) - \dot{\xi}(t)\ddot{\eta}(t)|}{4|\dot{\eta}(t)|^3}.$$

Dividing numerator and denominator by $|\dot{\eta}(t)|$ and using the conditions $|\dot{\xi}_1(t)/\dot{\eta}_1(t)| < 1$ and (11), we get

$$\kappa_1(t) \geq \frac{|\ddot{\xi}(t) - (\dot{\xi}(t)/\dot{\eta}(t))\ddot{\eta}(t)|}{4|(\dot{\eta}(t))|^2} \geq \frac{|\ddot{\xi}(t)| - |(\dot{\xi}(t)/\dot{\eta}(t))| \, |\ddot{\eta}(t)|}{4|(\dot{\eta}(t))|^2}$$
$$\geq \frac{|\ddot{\xi}(t)| - |\ddot{\eta}(t)|}{4|(\dot{\eta}(t))|^2} \geq \frac{|\ddot{\xi}(t)| - |\ddot{\eta}(t)|}{7b^2}.$$
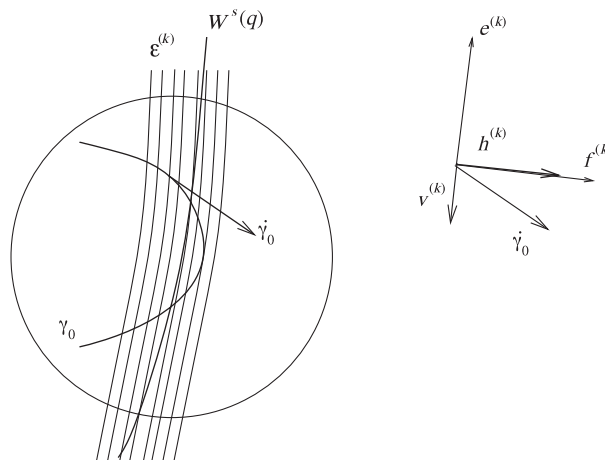
FIGURE 10. Hyperbolic coordinates.

Finally, from the formulas for $\ddot{\xi}(t)$ and $\ddot{\eta}(t)$ and the fact that $|\dot{y}(t)| \le \alpha$ by the admissibility of γ_0, we get

$$|\ddot{\xi}(t)| - |\ddot{\eta}(t)| \ge 2a - \alpha - 2\|\varphi\|_{C^2} \ge a$$

as long as η is sufficiently small. □

Proof of Proposition 4. The existence of a tangency between $f(\gamma)$ and the stable foliation $\mathcal{E}^{(k)}$ follows by the simple geometric observation that the image of a long admissible curve necessarily 'changes direction' between one endpoint and the other. Thus, by a simple intermediate value argument it follows that there is some point of tangency.

Now, Proposition 2 says that the leaves of the stable foliations $\mathcal{E}^{(k)}$ are close to the piece of stable manifold $f^{-1}(W_\delta^s(q))$ and thus have slope close to 2, and that their curvature is small. In particular, the point of tangency must occur at some point at which the tangent direction to $f(\gamma)$ is close to 2 and therefore Proposition 4 shows that at this point of tangency $f(\gamma)$ has strictly positive curvature. This implies that the tangency is quadratic as well as unique (see Figure 10). □

4. *Hyperbolicity estimates*

This is the final and main section of the paper. We apply the notion of hyperbolic coordinates and dynamically defined critical points to prove Theorem 1. In §4.1 we combine the hyperbolic coordinates and the curvature estimates to show that all components of the unstable manifold $W^u(p)$ in Δ_ε are almost horizontal curves with small curvature; in particular they all have well-defined critical points. In §4.2 we take advantage of the structure of critical points on such components to show that points in the critical region $\Delta_\varepsilon \setminus \Delta$ recover hyperbolicity after some bounded number of iterations depending only on the parameter a. In §4.3 we then extend these estimates to uniform expansion estimates on all of $W^u(p)$, with a hyperbolicity constant C_a depending only

on the parameter. In §4.4 we then show how to extend this hyperbolicity to the closure of $W^u(p)$ and thus to the whole non-wandering set Ω. Finally, in §4.5, we consider the bifurcation parameter value $a = a^*$ and show that all Lyapunov exponents are uniformly bounded away from 0.

4.1. *Shadowing.* Let

$$\lambda = \min\left\{\frac{1}{2}\ln\frac{3}{\sqrt{5}}, \hat{\lambda}\right\}. \tag{12}$$

PROPOSITION 5. *For all $a \geq a^*$, all components of $W^u(p) \cap \Delta_\varepsilon$ are long admissible curves. Moreover, for all $z \in W^u(p) \cap (\Delta_\varepsilon \setminus \Delta)$, any vector v tangent to $W^u(p)$ at z and $k \geq 1$ such that $f(z) \in \mathcal{V}_k \setminus \mathcal{V}_{k+1}$, we have*

$$\|Df_z^k(v)\| \geq e^{\lambda k}\|v\|.$$

We emphasize that Proposition 5 holds also for parameter values for which the first tangency occurs.

Proof. We first prove the expansivity statement and then the admissibility of leaves of $W^u(p)$ in Δ_ε.

4.1.1. *Expansion.* If $\gamma(s) = (x(s), y(s)) \subset \Delta_\varepsilon \cap \mathcal{D}$ is a long admissible curve, we consider the tangent vectors $\dot{\gamma}(s)$ and their images $\dot{\gamma}_0(s) = Df(\dot{\gamma}(s))$. By Proposition 4, $\dot{\gamma}_0$ is tangent to the stable direction $e^{(k)}$ at the point $c_0^{(k)}$. For this and other nearby points on γ we can write the tangent vector as

$$\dot{\gamma}_0 = h_0^{(k)} f^{(k)} + v_0^{(k)} e^{(k)},$$

where $(f^{(k)}, e^{(k)})$ is the orthogonal basis given by the most expanded and most contracted direction for Df^k, and $h_0^{(k)}$ and $v_0^{(k)}$ are the components of $\dot{\gamma}_0$ in this basis. Notice that the basis itself depends on the point. Proposition 2 implies that the basis varies *very slowly* with the base point, and Proposition 4 implies that the tangent vector $\dot{\gamma}_0$ is varying at *positive speed* with respect to this basis. We omit the calculations which are relatively standard; see, for example, [**LV03**]. Specifically, this implies that the component $h_0^{(k)}$ of the tangent vector $\dot{\gamma}_0$ at some point $z_0 = f(z) \in \gamma_0$ is proportional to the distance between z and the critical point $c^{(k)}$ of order k. In our setting, the constants actually give

$$|h_0^{(k)}(z_0)| \geq d(z, c^{(k)}). \tag{13}$$

We can now prove the following.

LEMMA 9. *Suppose $\gamma \subset \Delta_\varepsilon$ is an admissible curve, $z \in \gamma$, $z_0 = f(z) \in \mathcal{V}_k \setminus \mathcal{V}_{k+1}$ and $c^{(k)}$ is the critical point of order k in γ. Then for a vector w tangent to γ at z and all $j = 0, \ldots, k$, we have*

$$\|Df_z^{j+1}(w)\| \geq 3^j d(z, c^{(k)})\|w\|.$$

In particular,

$$\|Df_z^{k+1}(w)\| \geq e^{\lambda(k+1)}\|w\|.$$

Proof. The first estimate follows immediately from (13) and (7). To prove the second we need to find a bound for $d(z, c^{(k)})$ in terms of k. Using the quadratic nature of γ_0 and the proximity to the one-dimensional map $1 - ax^2$ with $a \approx 2$, we obtain

$$d(z, c^{(k)}) \geq \frac{1}{3}\sqrt{d(z_0, c_0^{(k)})}. \tag{14}$$

To estimate $d(z_0, c_0^{(k)})$ we use the observation that the 'real' critical value c_0 on γ_0, i.e. the point of tangency between γ_0 and the limiting stable foliation $\mathcal{E}^{(\infty)}$, lies necessarily either on $W^s(q)$ (this is only a possibility if $a = a^*$) or to the right of $W^s(q)$ in $\mathcal{Q}$. We write this as $\delta_0 = d(c_0, W^s(q)) \geq 0$. Combining this with Lemma 5 and the rate of convergence of critical points of finite order $d(c_0^k, c_0) \leq Kb^k$ as mentioned in Proposition 4, and taking b sufficiently small, we get

$$\begin{aligned} d(z_0, c_0^{(k)}) &\geq d(z_0, W^s(q)) + d(W^s(q), c_0) - d(c_0^{(k)}, c_0) \\ &\geq \frac{\delta}{2}5^{-k} + \delta_0 - Kb^k \geq \frac{\delta}{3}5^{-k}. \end{aligned}$$

Substituting this into (14) and using the fact that we can assume $k \geq k_0$ together with the definition of k_0 in (3) and of λ in (12), we have

$$3^k d(z, c^{(k)}) \geq \frac{\sqrt{\delta}}{2\sqrt{3}}\left(\frac{3}{\sqrt{5}}\right)^k \geq e^{\lambda(k+1)}. \qquad \square$$

4.1.2. *Admissibility.* Returning to the proof of Proposition 5, to obtain the statement about admissibility, notice first of all that by combining Lemma 9 with Lemma 3 we immediately obtain the statement that if $\gamma \subset W^u(p) \cap \Delta_\varepsilon$ is admissible and k is the first time that $f^k(\gamma) \subset \Delta_\varepsilon$, then $f^k(\gamma)$ is admissible. Now, by choosing $|b|$ and η small we can guarantee that $W^u_{\mathrm{loc}}(p) \cap \Delta_\varepsilon$ is a long admissible curve. Moreover, every piece of $W^u(p) \cap \Delta_\varepsilon$ is the image of some curve in $W^u_{\mathrm{loc}}(p) \cap \Delta_\varepsilon$ and is therefore admissible. $\square$

4.2. *Hyperbolicity after returns to Δ_ε.* Proposition 5 gives a pointwise recovery time for the hyperbolicity of points in the critical region, based on their position. The following proposition gives a key uniformity estimate in the phase space for each parameter $a > a^*$.

PROPOSITION 6. *For all $a > a^*$ there exists a constant N_a such that for $z \in W^u(p) \cap \Delta_\varepsilon \cap \Omega(f)$ and v a tangent vector to $W^u(p)$ at z, there exists $n(z) \leq N_a$ such that $Df_z^{n(z)}(v)$ is almost horizontal and*

$$\|Df_z^{n(z)}(v)\| \geq e^{\lambda n(z)}\|v\|.$$

We remark that the constant N is *not* uniformly bounded in a and in particular *does not* apply to $a = a^*$. However, it gives us a *uniformity statement* in z which will imply, as we shall see below, uniform hyperbolicity for each given parameter value $a > a^*$. For the proof we need to extend the definition of admissibility naturally to curves which are only differentiable of class C^{1+1} (Lipschitz continuous derivative).

Definition 3. We say that $\gamma(s) \subset \Delta_\varepsilon$ is a C^{1+1} admissible curve if $|\dot{y}|/|\dot{x}| < \alpha$ and $\dot{\gamma}(s)$ is Lipschitz with Lipschitz constant $\leq \alpha$.

We also give the formal definition of a 'real' critical point, which applies both to C^2 and to C^{1+1} admissible curves.

Definition 4. We say that $c \in \gamma$ is a critical point if $e^{(\infty)}$ is defined at $f(c) \in \gamma$ and coincides with $Df_c(\dot{\gamma}(c))$.

LEMMA 10. *For every $a > a^*$, every $z \in \overline{W^u(p)} \cap \Delta_\varepsilon \cap \Omega$ lies on a C^{1+1} admissible curve γ which is the limit of C^2 admissible curves in $W^u(p)$, and γ contains a unique critical point $c(\gamma)$ with $d(z, c) > 0$.*

Proof. We split the proof into two parts.

Every point lies on an admissible curve. We show first of all that every point $z \in \overline{W^u(p)} \cap \Delta_\varepsilon \cap \Omega$ lies on a C^{1+1} admissible curve which is the limit of C^2 admissible curves in $W^u(p)$. Let $z \in \overline{W^u(p)} \cap \Delta_\varepsilon \cap \Omega$ and let $z_n \to z$ be a sequence with $z_n \in W^u(p) \cap \Delta_\varepsilon \cap \Omega$. By Proposition 5, each z_n belongs to a long admissible curve $\gamma_n \subset W^u(p)$. We can write these as functions $\gamma_n : I \to \mathbb{R}$ with $I = [-\varepsilon, \varepsilon]$ and suppose that they converge pointwise to $\gamma : I \to \mathbb{R}$. Since I is compact and $\gamma_n, \dot{\gamma}_n$ are bounded and equicontinuous sequences, we have that γ is C^1 and $\gamma_n \to \gamma$ in the C^1 topology. To see that $\dot{\gamma}$ is Lipschitz, let $x, y \in I$ and observe that each $\dot{\gamma}_n$ is a Lipschitz function with uniformly bounded Lipschitz constant α. Then we have $|\dot{\gamma}_n(x) - \dot{\gamma}_n(y)| \le \alpha |x - y|$ and hence $|\dot{\gamma}_n(x) - \dot{\gamma}_n(y)| \le \alpha |x - y|$.

Every admissible curve contains a critical point. We now show that any such curve γ contains a unique critical point. We show first that it must contain at most one, and then argue that it must contain at least one. Let $\theta(\gamma_n(t))$ be the angle between the vectors $Df_{(t,\gamma_n(t))}(1, \gamma_n'(t))$ and $e^{(\infty)}(f(t, \gamma_n(t)))$. Since the image of each admissible curve is quadratic with respect to $\mathcal{E}^{(\infty)}$, we have that $\theta(\gamma_n(t))$ has a strictly non-zero derivative having at most one root corresponding to a point of tangency between $f(\gamma_n)$ and $\mathcal{E}^{(\infty)}$. Since $\gamma_n \to \gamma$ in the C^1 topology, $\theta(\gamma(t))$ also has strictly non-zero derivative having at most one root corresponding to a point of tangency between $f(\gamma)$ and $\mathcal{E}^{(\infty)}$. To see that such a point exists, observe that if $a > a^*$, then the graph of γ crosses the boundary of Δ twice and $f(\gamma \cap \Delta)$ is outside $\mathcal{D}$ where the foliation $\mathcal{E}^{(\infty)}$ is well defined, with the extreme points of $f(\gamma \cap \Delta)$ both lying on a piece of $W^s(q)$ which is a leaf of the foliation $\mathcal{E}^{(\infty)}$. This implies that there exists a point outside the interior of $\mathcal{D}$ where $f(\gamma)$ is tangent to $\mathcal{E}^{(\infty)}$. □

Lemma 10 allows us to define a canonical set $\mathcal{C}_a$ of *critical points* as the union of all critical points $c(\gamma)$ for C^{1+1} admissible curves γ which are C^1 limits of long admissible curves of $W^u \cap \Delta_\varepsilon$. In the next lemma we show that this set is bounded away from the set of non-wandering points.

LEMMA 11. *For all $a > a^*$ we have $d(\mathcal{C}_a, \Omega) > 0$.*

We emphasize that $d(\mathcal{C}_a, \Omega)$ is not uniformly bounded in the parameter. The constant N_a in Proposition 6 will be defined below in terms of $d(\mathcal{C}_a, \Omega)$.

Proof. Notice first of all that $\mathcal{C}_a \subset \Delta_\varepsilon$ and thus $\mathcal{C}_a$ is bounded. Let $c_k = c(\gamma_k)$ be a sequence converging to some point c; we need to show that $c \in \mathcal{C}_a$. Since each γ_k is the limit of long admissible curves, we can consider sequences $\gamma_k^{(n)} \to \gamma_k$ for each k. Using Lemma 10

and the fact that $\{\gamma_k^{(k)}\}$ converges pointwise to γ, we conclude that this convergence is in fact C^1. Since $\theta(\gamma_k^{(k)}(c_k)) \to 0$, we have that $\theta(\gamma(c)) = 0$ and this implies that c is a critical point as required.

We have therefore shown that the critical set $\mathcal{C}_a$ is compact. Since Ω is also compact, it is sufficient to show that $\mathcal{C}_a \cap \Omega = \emptyset$ to imply that they are at some positive distance apart. Disjointness follows from the observation that the image of a critical point is always outside $\mathcal{D}$, while Ω is an invariant set contained in $\mathcal{D}$. $\square$

Proof of Proposition 6. By Lemma 11 and the uniform approximation of the critical set $\mathcal{C}$ by the finite-order critical sets $\mathcal{C}^{(n)}$, there exists N_a sufficiently large so that the following two conditions hold (using also $\lambda < \log 3$):

$$d(\mathcal{C}_a^{(N_a)}, \mathcal{C}_a) < d(\mathcal{C}_a, \Omega)/2 \quad \text{and} \quad 3^{N_a} d(\mathcal{C}_a^{(N_a)}, \Omega) \geq e^{\lambda N_a}. \tag{15}$$

Now consider $z \in \Delta_\varepsilon \cap W^u(p) \cap \Omega$ and let $n \geq 1$ be such that $f(z) \in \mathcal{V}_n \setminus \mathcal{V}_{n+1}$. Recall that $f(\Delta_\varepsilon) \subset \mathcal{V}_{k_0}$, therefore such an n is always well defined except for those points which map exactly to the curve $f^{-1}(W_\delta^s(q))$ that forms the boundary between $\mathcal{V}^+$ and $\mathcal{V}^{-1}$. For these points we take $n = +\infty$, and then let

$$n(z) = \min\{n, N_a\}.$$

If $n \leq N_a$, the statement follows from Proposition 5; otherwise, our choice of N_a in (15) gives

$$\|Df^{N_a}(v)\| \geq 3^{N_a} d(z, \mathcal{C}_a^{(N_a)}) \|v\| \geq 3^{N_a} d(\Omega, \mathcal{C}_a^{(N_a)}) \|v\|$$
$$\geq 3^{N_a} d(\Omega, \mathcal{C}_a) \|v\|/2 \geq e^{\lambda N_a} \|v\|.$$

The first inequality follows from Lemma 9, the second one follows from $z \in \Omega$, the third one follows from the first part of (15), and the last one follows from the second part of (15).

Finally, considering the components of v in hyperbolic coordinates, we have $\|v_{N_a}^{(N_a)}\| \leq (b/3)^{N_a}$ and $\|h_{N_a}^{(N_a)}\| \geq e^{\lambda N_a}$, therefore $Df^{N_a}(v)$ is almost horizontal. $\square$

4.3. *Uniform hyperbolicity on $W^u(p)$.*
The following proposition is essentially a corollary of Proposition 6. However, we state it separately as it gives an explicit construction of the constant C_a of hyperbolicity for each $a > a^*$. Before stating the result, we define this constant.

Let $C_{N_a}^- = \min\{\|(Df_z^j)^{-1}\|^{-1} : x \in \mathcal{D}, 1 \leq j \leq N_a\}$ and $C_{N_a}^+ = \max\{\|Df_z^j\| : x \in \mathcal{D}, 1 \leq j \leq N_a\}$ denote the maximum possible contraction and the maximum possible expansion exhibited by any vector $v \in T_x \mathbb{R}^2$ for any point $x \in \mathcal{D}$ in at most N_a iterations. Letting C_ε denote the constant of hyperbolicity as in (5) on page 1064, we then let

$$C_a = \min\left\{\frac{C_\varepsilon}{C_N^+}, \frac{C_N^- e^{-\lambda N}}{C_N^+}\right\}.$$

PROPOSITION 7. *For all $a > a^*$, all $z \in W^u(p) \cap \Omega(f)$ and all vectors w tangent to $W^u(p)$ at z, we have*

$$\|Df_z^n(w)\| \geq C_a e^{\lambda n} \|w\|$$

for all $n \geq 1$.

Proof. Let $z \in W^u(p) \cap \Omega(f)$ and let w be tangent to $W^u(p)$ at z. Since we do not assume anything about the location of z, the vector w may or may not be almost horizontal. We distinguish these two possibilities.

Case 1: w is almost horizontal. Let $0 \leq k_1 < \cdots < k_s < n$ be the sequence of returns of the iterates of z to Δ_ε (with $k_1 = 0$ if $z \in \Delta_\varepsilon$ and $k_1 > 0$ otherwise). Then for each k_i we have an integer $n_i = n(z_{k_i}) \leq N_a$ given by Proposition 5. Hence we can write

$$k_{i+1} = k_i + n_i + q_i$$

where q_i is the number of iterates during which the point remains outside Δ_ε. From Proposition 5 and properties (4) and (6), the images of the vector at these iterates remain horizontal and we have

$$\|Df_z^{k_i}(w)\| \geq e^{\lambda k_i} \|w\|$$

for all $i = 1, \ldots, s$, and in particular for $i = s$. If $k_s + n_s \leq n$, applying (6) to the remaining iterates gives $\|Df_z^n(w)\| \geq C_\varepsilon e^{\lambda n} \|w\| \geq C_a e^{\lambda n} \|w\|$ as required.

If $k_s + n_s > n$, we have expansion for the first k_s iterates, giving $\|Df^{k_s}(w)\| \geq e^{\lambda k_s} \|w\|$. There follow $n - k_s \leq n_s \leq N_a$ iterates (since $n_s \leq N_a$) during which we can bound the contraction coarsely by the N_ath power of the maximum contraction in the region $\mathcal{D}$, which gives

$$\|Df^n(w)\| \geq C_N^- e^{\lambda k_s} \|w\| = C_N^- e^{-\lambda N} e^{\lambda n} \|w\|.$$

Case 2: w is not almost horizontal. We now suppose that w is not almost horizontal.

CLAIM 1. *There exists*

$$N_a \geq m > 0$$

such that $f^{-m}(z) \in \Delta_\varepsilon$ and $w_{-m} = Df^{-m}(w)$ is almost horizontal.

Proof. We show first of all that some preimage of z lies in Δ_ε. Indeed, $z \in W^u(p)$ implies that $z_{-n} \to p$ as $n \to \infty$ and therefore that w_{-n} is almost horizontal for sufficiently large n since the local unstable manifold of p is admissible. By the invariance of the unstable conefield outside Δ_ε, images of w_{-n} remain almost horizontal unless some return to Δ_ε takes place.

Now let $m > 0$ be the smallest integer such that $f^{-m} \in \Delta_\varepsilon$. Then w_{-m} is almost horizontal since every component of W^u in Δ_ε is almost horizontal. From Proposition 6 it follows that $Df_{z_{-m}}^{n(z_{-m})}(w_{-m})$ is almost horizontal and therefore $m \leq n(z_m) \leq N_a$; otherwise w will be almost horizontal. $\qquad\square$

Returning to the proof of Proposition 7, we can now argue as in case 1 to obtain exponential growth starting from time $-m$:

$$\|Df^n(w)\| = \|Df^{n+m}(w_{-m})\| \geq C' e^{\lambda(n+m)} \|w_{-m}\| \tag{16}$$

where $C' = \min\{C_\varepsilon, C_N^- e^{-\lambda N}\}$. Moreover,

$$\|w\| = \|Df^m(w_{-m})\| \le \|Df^m\| \, \|w_{-m}\| \le C_N^+ \|w_{-m}\|.$$

Substituting this back into (16) completes the proof. □

4.4. *Uniform hyperbolicity on Ω.* We have obtained uniform expansion estimates for vectors tangent to $W^u(p)$. In this section we show that these estimates can be extended to Ω. This part of the argument uses very little of the specific Hénon-like form of the map and therefore we state it in a more abstract and general context.

PROPOSITION 8. *Let $f : \mathbb{R}^2 \to \mathbb{R}^2$ be a C^1 diffeomorphism and Ω a compact invariant set with $|\det Df| < 1$ on Ω. Suppose that there exists some invariant submanifold W that is dense in Ω and for which there exist constants $C, \lambda > 0$ such that $\|Df_z(v)\| \ge Ce^{\lambda n}$ for all $z \in W \cap \Omega$ and v tangent to W. Then Ω is uniformly hyperbolic with hyperbolic constants C and λ.*

Proposition 8 completes the proof of part (1) of Theorem 1 and shows that the rates of expansion and contraction admit uniform bounds independent of the parameter.

Proof. We shall show that Ω is uniformly hyperbolic by constructing an invariant hyperbolic splitting $E_z^s \oplus E_z^u$ at every point of Ω and then showing that this splitting is continuous. We carry out this construction in several steps. The starting point is the observation that E_z^u is already given by the tangent direction to W for all points $z \in \Omega \cap W$.

LEMMA 12. *For any $z \in \Omega$ and any sequence $z_j \in W$ with $z_j \to z$, the sequence $E^u(z_j)$ converges to a (unique) limit direction $E^u(z)$. Each vector $v \in E^u(z)$ satisfies*

$$\|Df_z^n(v)\| \ge Ce^{\lambda n}\|v\| \quad and \quad \|Df_z^{-n}(v)\| \le C^{-1}e^{-\lambda n}\|v\|$$

for all $n \ge 1$.

Proof. Suppose $z \in \Omega$ and let $z_j \in W$ be a sequence with $z_j \to z$. Consider the sequence of corresponding directions $E^u(z_j)$. By compactness (of the space $\mathbb{S}^1$ of possible directions) there must exist some subsequence z_{j_i} such that the corresponding directions $E_{j_i}^u$ converge to some direction which we call $E^u(z)$. Notice that a priori this direction is not unique since it depends on the choice of subsequence. We shall show first that the forward expansion and backward contraction estimates hold, and then show that this actually implies uniqueness.

Let $v \in E_z^u$ and let $v_{j_i} \in E_{z_{j_i}}^u$ be a sequence with $v_{j_i} \to v$. Then for each $n \in \mathbb{N}$ we have, by the continuity of Df^n,

$$\|Df_{z_j}^n(v_j)\| \to \|Df_z^n(v)\|.$$

By assumption we know that $\|Df_{z_{j_i}}(v_j)\| \ge Ce^{\lambda n}\|v_j\|$ and therefore

$$\|Df_z^n(v)\| \ge Ce^{\lambda n} - \varepsilon$$

for any $\varepsilon > 0$. Therefore $\|Df_z^n(v)\| \ge Ce^{\lambda n}$ and, since this holds for every n, we have the required statement as far as the expansion in forward time is concerned. To prove contraction in backward time it is sufficient to prove it for points on W and then apply

exactly the same approximation argument. For $z \in W$ this follows immediately from the uniform expansivity assumption in forward time. Indeed, letting $v_{-n} = Df_z^{-n}(v)$, the expansivity assumption gives

$$\|v\| \geq \|Df_{z-n}^n(v_{-n})\| \geq Ce^{\lambda n}\|v_{-n}\|$$

which immediately implies $\|v_{-n}\| \leq C^{-1}e^{-\lambda n}\|v\|$.

It remains to show uniqueness of $E^u(z)$ for each $z \in \Omega$. Suppose, for contradiction, that we could find two sequences $z_j \to z$ and $\tilde{z}_j \to z$ with corresponding directions $E_{z_j}^u$ and $E_{\tilde{z}_j}^u$ converging to two different directions E_z^u and $\tilde{E}_z^u$. Let $v \in E_z^u$ and $\tilde{v} \in \tilde{E}_z^u$. Then $v, \tilde{v}$ must be linearly independent and thus every other vector $w \in T_z\mathbb{R}^2$ can be written as a linear combination $w = a_1 v + a_2 \tilde{v}$ for some $a_1, a_2 \in \mathbb{R}$. By linearity and the backward contraction estimates obtained above this implies that

$$\|w_{-n}\| = \|Df_z^{-n}(w)\| \to 0$$

as $n \to \infty$. Since w was arbitrary this implies that all vectors are shrinking to zero in backward time. But this is impossible since we have assumed that $|\det Df| < 1$ and thus $|\det Df^{-1}| > 1$ on Ω. □

COROLLARY 2. *At every point $z \in \Omega$ there exists a unique tangent space splitting $E_z^u \oplus E_z^s$ which is invariant by the derivative Df and which satisfies the standard uniform hyperbolicity expansion and contraction estimates.*

Proof. Lemma 12 gives the expanding direction E_z^u of the splitting with the required hyperbolic expansion estimates in forward time. The invariance for points in W is automatic (since tangent directions to W are mapped to tangent directions to W), and the invariance for general points follows immediately from the definition of $E_z^u = \lim E_{z_j}^u$, the invariance of $E_{z_j}^u$ for $z_j \in W$, and the continuity of Df.

The stable direction E_z^s is given immediately by the limit of the sequence $e^{(n)}$ of vectors most contracted by Df_z^n, as discussed in §4.1. This also automatically gives the uniqueness and invariance. □

To complete the proof of Proposition 8, we just need to show that the given tangent space splitting is continuous. This follows by standard arguments from the uniqueness proved in Corollary 2. Indeed, for any $z \in \Omega$ and any sequence $z_j \in \Omega$ with $z_j \to z$, every limit point of the corresponding sequence of splittings $E_{z_j}^u \oplus E_{z_j}^s$ must also be a splitting $\tilde{E}_z^u \oplus \tilde{E}_z^s$. By approximation arguments identical to those used above, it follows that this splitting must also satisfy the uniform hyperbolic contraction and expansion estimates. Therefore, by uniqueness, it must coincide with the existing splitting $E_z^u \oplus E_z^s$. This completes the proof that Ω is uniformly hyperbolic. □

4.5. *Lyapunov exponents for f_{a^*}.* Finally, it remains to consider the dynamics of f_{a^*}. Recall that a^* is defined on page 1056 as the first parameter for which a tangency occurs between the compact parts of $W^s(q)$ and $W^u(p)$; see Figure 11 for the pictures in the two cases $b > 0$ and $b < 0$.
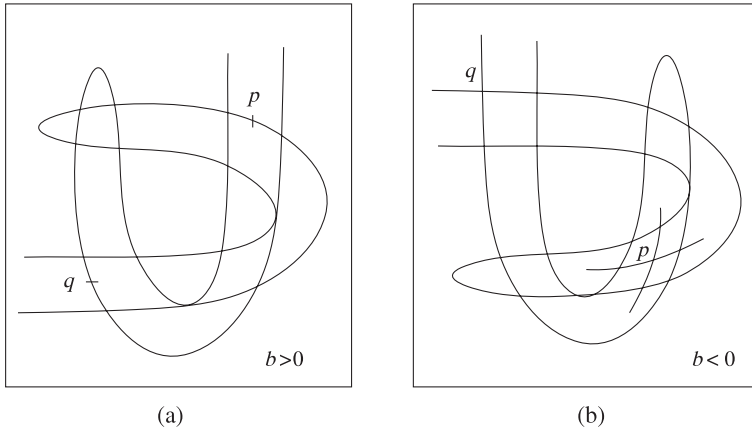
FIGURE 11. Invariant manifolds for $a = a^*$.

We need to show that, for $a = a^*$, all Lyapunov exponents are uniformly bounded away from 0. We show that for each point $z \in \Omega_{a^*}$ *not in the orbit of tangency* $\mathcal{T}$ (it is not necessary to consider the orbit of tangency since this is a countable set without recurrence and therefore cannot support any invariant probability measure) there exists a constant C_z, a vector v_z, and a sequence $\{n_i\}$ with $n_i \to \infty$ such that for all $i \geq 0$,

$$\|Df_z^{n_i}(v_z)\| \geq C_z e^{\lambda n_i} \|v_z\|.$$

This is obviously true if the orbit of z never enters Δ_ε in forward time or if it enters Δ_ε only a finite number of times. Indeed, suppose that there exists some k such that $f^i(z) \notin \Delta_\varepsilon$ for all $i \geq k$, and let w be a vector which is mapped to the horizontal vector $w_k = Df_z^k(w)$ after k iterations. By (5) we then have $\|Df_{z_k}^{k+n}(w)\| \geq C_\varepsilon e^{\lambda n} \|w_k\|$ for all $n \geq 1$. This implies that there exists a constant C_z such that $\|Df_z^{k+n}(w)\| \geq C_z e^{\lambda(k+n)} \|w\|$ for all $n \geq 1$.

Otherwise, there exists an infinite sequence $0 < m_1 < \cdots < m_k < \cdots$ such that $m_k \to \infty$ and $f^{m_k}(z) \in \Delta_\varepsilon$. By Lemma 10, $z_{m_i} = f^{m_i}(z)$ lies on either a C^2 long admissible curve or a C^{1+1} long admissible curve which is the C^1 limit of C^2 long admissible curves in $W^u(p)$. Since z has an infinite number of returns to Δ_ε, this implies in particular that $z \notin W^s(q)$, so $z_{m_i} \notin W^s(q)$, and hence there exists $n_i = n(z_{m_i})$ such that $f(z_{m_i}) \in \mathcal{V}_{n_i} \setminus \mathcal{V}_{n_i+1}$. Therefore *exactly* the same arguments as in Lemmas 9 and 12 show that for a vector v_i tangent to such an admissible curve γ at z_{m_i}, we have

$$\|Df_{z_{m_i}}^{n_i+1}(v_i)\| \geq e^{\lambda(n_i+1)} \|v_i\|. \tag{17}$$

Notice that since the C^1 limits of C^2 admissible curves are unique, as proved above, we have $v_{i+1} = Df^{m_{i+1}-m_i}(v_i)$. Then, by (5) and (17), we have

$$\|Df^{m_i+n_i+1-m_1}(v_1)\| \geq e^{\lambda(m_i+n_i+1-m_1)} \|v_1\|.$$

We can then define $v_z = Df^{-m_1}(v_1)$ and have $\|Df^{m_i+n_i+1}(v_z)\| \geq C_z e^{\lambda(m_i+n_i+1)} \|v_z\|$ where the constant C_z is required simply to compensate for the possible lack of expansion

for the first n_1 iterates. In particular, C_z can be chosen by considering the maximum possible contraction along the orbit of z for the first n_1 iterations:

$$C_z = \min_{\|v\|=1} \|Df_z^{n_1}(v)\|.$$

We have therefore shown that, for each $z \in \Omega$,

$$\limsup_{n\to\infty} \frac{1}{n} \ln \|Df_z^n\| \geq \lambda.$$

This clearly implies the same bound for the limit wherever it exists. In particular, any point which is typical for some ergodic invariant probability measure, and for which such a limit does therefore exist, will have a positive Lyapunov exponent at least λ. By dissipativity this immediately implies also that the other Lyapunov exponent is negative and uniformly bounded away from 0 both in the dynamical space and in the parameter space.

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