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Closed characteristics of second-order Lagrangians

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We study the existence of closed characteristics on three-dimensional energy manifolds of second-order Lagrangian systems. These manifolds are always non-compact, connected and not necessarily of contact type. Using the specific geometry of these manifolds, we prove that the number of closed characteristics on a prescribed energy manifold is bounded below by its second Betti number, which is easily computable from the Lagrangian.

1. Preface

Second-order Lagrangian systems are defined variationally by extremizing action functionals of the form $J[u] = \int_I L(u, u', u'')\,\mathrm{d}t$. The Lagrangian L depends not only on the state variable u and its first derivative, but also on its second derivative, which is not the usual situation for variational problems in classical mechanics. The Euler–Lagrange equations of such systems are given by

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}\frac{\partial L}{\partial u''} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u'} + \frac{\partial L}{\partial u} = 0 \tag{1.1}$$

and are, in essence, fourth-order differential equations. These systems have recently been used in many models in physics and engineering, and the literature pertaining to them is extensive. We refer the reader to [6–8] and the references therein for more information.

Under the natural hypothesis that L is convex in u'', a second-order Lagrangian system is equivalent to a two-degrees-of-freedom Hamiltonian system in $\mathbb{R}^4$ endowed with its standard symplectic form ω. The Hamiltonian is given by

$$H(u, u', u'', u''') = \left(\frac{\partial L}{\partial u'} - \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial u''}\right)u' + \frac{\partial L}{\partial u''}u'' - L(u, u', u''). \tag{1.2}$$

Introducing the symplectic coordinates $x = (u, v, p_u, p_v)$, the Hamiltonian becomes $H(x) = p_u v + L^*(u, v, p_v)$, where L^* is the Legendre transform of L with respect to u''. Hamilton's equations of motion are equivalent to (1.1) and yield a dynamical system ϕ^t on $\mathbb{R}^4$. The Hamiltonian H foliates $\mathbb{R}^4$ with three-dimensional energy manifolds $M_E = \{x \in \mathbb{R}^4 \mid H(x) = E\}$. These manifolds are invariant under the

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flow ϕ^t, and the dynamical behaviour of the system can be studied on an individual energy manifold. An energy manifold is *regular* if E is a regular value of H. In the context of second-order Lagrangians, this is equivalent to the condition that $\partial_{u''}L(u,0,0) \neq 0$ whenever $L(u,0,0)+E=0$. See § 4 for a discussion of the singular case. For more details on second-order Lagrangians, see [11].

Recently, the analysis of periodic orbits, or closed characteristics, on given energy manifolds has become an important issue in the study of general Hamiltonian systems [5, 9, 13]. In this paper, we study the geometric structure specific to second-order Lagrangian systems and place it in this broader context.

Given an arbitrary $(2n-1)$-dimensional manifold M embedded in $(\mathbb{R}^{2n},\omega)$, with ω the standard symplectic form, one can construct a Hamiltonian system for which M is the energy manifold for $E=0$. The particular choice of the Hamiltonian is not intrinsic to the problem of finding periodic orbits, which can be phrased in more geometric terms. The specific geometry of M and the 2-form ω define a characteristic line bundle,

$$\mathcal{E}_M = \{(x,\xi) \in T_xM \setminus \{0\} \mid \omega_x(\xi,\eta) = 0 \ \forall \eta \in T_xM\} \subset TM.$$

The vector field of any Hamiltonian H satisfying $M = H^{-1}(0)$ is a section of this bundle. The trajectory of a periodic orbit can thus be regarded as a closed characteristic of the line bundle, i.e. an embedding $\gamma : S^1 \to M$ of the circle into M for which

$$T\gamma = \mathcal{E}_M|_{\gamma}.$$

This formulation of the problem relates the existence of periodic orbits of the differential equation (1.1) to the geometric and topological properties of its energy manifolds.

Motivated by a novel result by Rabinowitz [9], Weinstein [13] conjectured in the 1970s that any compact hypersurface $M \in (\mathbb{R}^{2n},\omega)$, with the additional requirement that

$$\alpha(\xi) \neq 0, \qquad 0 \neq \xi \in \mathcal{E}_M,$$

for some 1-form α with $d\alpha = \omega$, i.e. M is of *contact type* relative to ω, has at least one closed characteristic. This conjecture was later proved by Viterbo [12].

Energy manifolds determined by second-order Lagrangians do not fit within this theory for two reasons, they are always non-compact and they are not necessarily of contact type in $(\mathbb{R}^4,\omega)$, as was proved in [2]. Even with a more general formulation via Reeb vector fields, the latter issue cannot necessarily be resolved (see [2]). However, in this paper, we show that these manifolds possess certain geometric and topological properties that guarantee the existence of closed characteristics.

In order to reduce the amount of technical detail, we restrict ourselves, for now, to Lagrangians that satisfy the following hypotheses.

(H1) $L(u,v,w) = \frac{1}{2}w^2 + K(u,v)$.

(H2) $K(u,v) \geqslant -C(|u|) - C(|u|)|v|^{\gamma}$, $\gamma < 4$, where $C(|u|)$ is locally bounded.

Note that (H2) is a lower bound on K; an upper bound is not necessary. These hypotheses can be weakened, as discussed in § 4. We now formulate the main result of this paper.

THEOREM 1.1. *Let M be a regular energy manifold of a second-order Lagrangian system with C^2 Lagrangian L satisfying hypotheses (H1) and (H2). Then the number of closed characteristics on M is bounded below by the second Betti number* $\dim H_2(M)$.

The proof of theorem 1.1 also provides a method for computing the second Betti number of M. Indeed, for any Lagrangian system with $\partial_w^2 L \geqslant \alpha > 0$, the homotopy type of M can be determined directly from the sign of its 'potential', $L(u,0,0)+E$ (see § 4 and [2]).

Theorem 1.1 is a generalization of the situation for first-order Lagrangian systems $L(u,u') = \frac{1}{2}(u')^2 + F(u)$ where $H = \frac{1}{2}(u')^2 - F(u)$. An energy manifold M is one dimensional, and each compact component of (regular) M consists of a single periodic solution. Thus the number of closed characteristics is exactly $\dim H_1(M)$. For second-order Lagrangians, $\dim H_2(M)$ is only a lower bound. One can easily give examples of systems with infinitely many different closed characteristics (see [11]). Note that, for Lagrangians of the special form $L = L(u,u'')$, hypothesis (H2) becomes void and the similarity between first- and second-order systems becomes even stronger.

In [11, theorem 12, p. 1408], a version of theorem 1.1 was proved under an additional hypothesis that the Lagrangian satisfies a twist property, defined in the next section. For some systems, this property can be verified (see [11, lemma 9, p. 1405]), but in many cases it cannot. Theorem 1.1 is an improvement of this previous result, removing the twist hypothesis. However, we do draw on the results for twist systems to prove theorem 1.1.

2. Geometry of second-order Lagrangians

To establish the existence of closed characteristics on energy manifolds of second-order Lagrangian systems, we use their variational structure. A closed characteristic is equivalent to a periodic solution u, which can be found as a critical point of the action, i.e.

$$\delta_{u,T} \int_0^T [L(u,u',u'') + E]\,\mathrm{d}t = 0,$$

where $T > 0$ is the period of u. Note that variations are taken in T as well as u.

We consider functions that have a simple profile consisting of two monotone laps: u_+, which increases from some minimal value u_1 to a maximal value u_2, and u_-, which decreases from u_2 back to u_1, with $u' = 0$ at u_1 and u_2. If u_+ and u_- are solutions, then their concatenation $u_+ \# u_-$ is called a 'broken geodesic', and the extrema u_1 and u_2 will be called concatenation points. Note that a broken geodesic need not be a solution to (1.1) at its concatenation points, since the third derivatives need not match (see [11]).

We obtain a periodic solution from the method of broken geodesics in two steps. First we must determine when monotone laps exist between given values of u, and this is accomplished in § 3 via minimization. Then it must be shown that there exists a broken geodesic that is a solution to (1.1), which follows from the geometric and topological properties of M, as we now explain.

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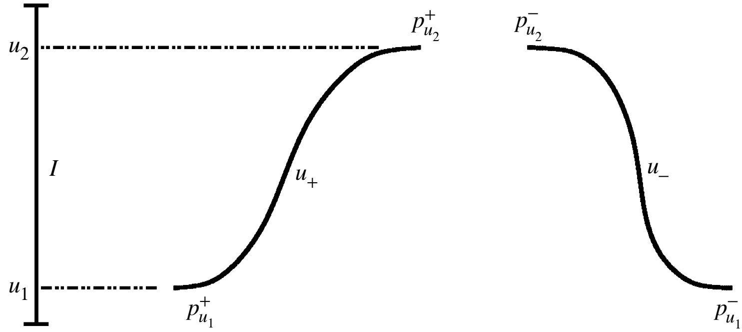

Figure 1. The increasing and decreasing laps u_+ and u_-, respectively.

From the Hamiltonian (1.2), solutions must satisfy

$$\frac{\partial L}{\partial u''}u'' - L(u,0,u'') = E$$

at points where $u' = 0$. We denote this level set in the (u,u'')-plane by N. Note that N is the section of M defined by $M \cap \{u' = 0\}$. Indeed, $M \cap \{u' = 0\}$ is the cylinder $N \times \mathbb{R}$, where the $\mathbb{R}$ variable is determined by the p_u coordinate. Due to the convexity of L with respect to u'', the manifold N consists of two graphs in the (u,u'')-plane. In particular, the projection π of N onto the u-axis can be characterized by $\pi N = \{u : L(u,0,0) + E \geqslant 0\}$, and the sets $N \cap \{(u,u'') \mid u \geqslant 0\}$ and $N \cap \{(u,u'') \mid u \leqslant 0\}$ are graphs over πN. A particular connected component of πN will be denoted by I, and will be referred to as an *interval component*.

We will consider broken geodesics whose values lie in a single interval component I. Given such a component I of πN, define $B = \{(u_1,u_2) \in I \times I : u_1 < u_2\}$. For given laps u_+ and u_- let $p^+_{u_1}$, $p^+_{u_2}$ and $p^-_{u_1}$, $p^-_{u_2}$ be the p_u values at the concatenation points. As shown in [11], if the condition

$$p^+_{u_1} - p^-_{u_1} = 0 \quad \text{and} \quad p^+_{u_2} - p^-_{u_2} = 0 \tag{2.1}$$

is satisfied at the concatenation points, then $u_+\#u_-$ is a periodic solution, and thus a closed characteristic.

Let $(u_1,u_2) \in B$ and $p^+_{u_1}, p^+_{u_2} \in \mathbb{R}$. Consider the trajectory

$$x(t) = \phi^t(u_1, 0, p^+_{u_1}, p_v(u_1))$$

of the Hamiltonian flow. Here, $p_v = u''$ is a function of u_1, since the initial point has $v = u' = 0$, and hence is in N. Thus there are two choices for $p_v(u_1)$, and we will choose $p_v(u_1) > 0$.

Define $f_+(u_1,p^+_{u_1})$ and $g_+(u_1,p^+_{u_1})$ to be the values of u and p_u at the first maximum of $\pi x(t)$, respectively (see figure 1). As $p^+_{u_1} \to \infty$, then $f_+(u_1,p^+_{u_1}) \to u_1$. The maps f_+ and g_+ are well defined for fixed u_1 with decreasing $p^+_{u_1}$ as long as $f_+(u_1,p^+_{u_1}) \leqslant \max I$. In addition, the f_+ and g_+ are smooth in $(u_1,p^+_{u_1})$ on the domain of definition P_+. We can define analogous maps $f_-(u_2,p^-_{u_2})$ and $g_-(u_2,p^-_{u_2})$ as the values of u and p_u at the first minimum of a decreasing lap (see figure 1),

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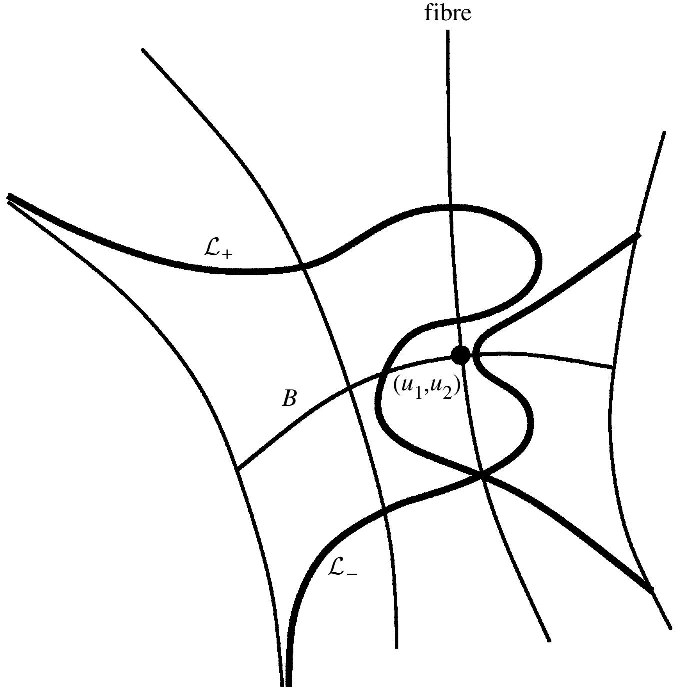

Figure 2. The cotangent bundle T^*B and the intersecting Lagrangian submanifolds $\mathcal{L}_+$ and $\mathcal{L}_-$.

with domain of definition P_-. Let

$$\mathcal{L}_+ = \{(u, f_+(u, p_u), p_u, g_+(u, p_u))\},$$
$$\mathcal{L}_- = \{(f_-(u, p_u), u, g_-(u, p_u), p_u)\}$$

be subsets of the cotangent bundle T^*B. Then $\mathcal{L}_\pm$ are two-dimensional submanifolds of T^*B given as graphs over the (u_1, p_{u_1})-plane and the (u_2, p_{u_2})-plane, respectively (see figure 2).

The submanifolds $\mathcal{L}_\pm$ are, in fact, Lagrangian submanifolds of T^*B. Condition (2.1) implies that intersection points of the manifolds $\mathcal{L}_\pm$ correspond to broken geodesics that are periodic solutions, i.e. $u_1 = f_-(u_2, p_{u_2})$, $f_+(u_1, p_{u_1}) = u_2$, $p_{u_1} = g_-(u_2, p_{u_2})$ and $g_+(u_1, p_{u_1}) = p_{u_2}$.

In the special case that there exist unique laps u_+ and u_- for all $(u_1, u_2) \in B$, then the system is a twist system, as mentioned in the introduction. In this case, $\mathcal{L}_\pm$ are exact Lagrangian submanifolds of T^*B, i.e. $\mathcal{L}_\pm$ are the graphs of exact 1-forms on B provided by generating functions for the laps [11]. In this case, a direct variational principle exists in terms of just the extrema u_1 and u_2. This case is analysed in detail in [11], and will be used here to prove the main result. If the Lagrangian system is not a twist system, a generating function can still be found by considering the full action J_E as in [1]. However, in this paper, we will use a continuation principle to study $\mathcal{L}_+ \cap \mathcal{L}_-$ via continuation to a twist system.

We denote by $\pi : T^*B \to B$ the canonical projection onto the base and by π^* the projection onto the (p_{u_1}, p_{u_2}) coordinates of a point in T^*B. The following lemma is proved by theorem 3.12 in § 3 and shows that, for any Lagrangian satisfying (H1) and (H2), the projections $\pi\mathcal{L}_\pm$ cover the base, which plays a crucial role in our intersection theory.

LEMMA 2.1. *Let L satisfy hypotheses (H1), (H2). Then, for each pair $(u_1, u_2) \in B$, there exist increasing and decreasing laps u_+ and u_- (in C^5), respectively. In particular, $\pi\mathcal{L}_\pm = B$.*

The next lemma establishes that the intersection set $\pi(\mathcal{L}_+ \cap \mathcal{L}_-)$ is always strictly contained in B for Lagrangians satisfying (H1) and (H2).

LEMMA 2.2. *Let L satisfy hypotheses (H1) and (H2). Then*

$$\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \cap \partial B = \emptyset,$$

where $\partial B = B \cap (\mathbb{R}^2 \setminus \mathrm{int}_{\mathbb{R}^2}\ B)$. Moreover, if $\mathrm{cl}(B)$ is compact, then there exists a compact set $\hat{B} \subset \mathrm{int}_{\mathbb{R}^2} B$ such that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \subset \hat{B}$.

Proof. Let $\pi^{-1}(u_1, u_2) = \zeta$ be a fibre in T^*B. For each point $(u_1, u_2) \in B$, lemma 2.1 implies that $\zeta \cap \mathcal{L}_\pm \neq \emptyset$. Take a point $(u_1, u_2) \in \partial B$ and consider the points $(p^+_{u_1}, p^+_{u_2}) \in \zeta \cap \mathcal{L}_+$ and $(p^-_{u_1}, p^-_{u_2}) \in \zeta \cap \mathcal{L}_-$. Lemma 7 of [11] implies that, for each pair $(p^+_{u_1}, p^+_{u_2})$ and $(p^-_{u_1}, p^-_{u_2})$, either $p^+_{u_1} - p^-_{u_1}$ or $p^+_{u_2} - p^-_{u_2}$ has a definite sign (strictly negative). Thus, for any boundary point $(u_1, u_2) \in \partial B$, we have

$$\pi^*(\zeta \cap \mathcal{L}_+) \neq \pi^*(\zeta \cap \mathcal{L}_-), \tag{2.2}$$

which implies that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \cap \partial B = \emptyset$.

Now assume that $\mathrm{cl}(B)$ is compact, and hence I is a compact interval component. Define

$$B_\delta = \{(u_1, u_2) \in B \mid u_1 \geqslant \min I + \delta,\ u_2 \leqslant \max I - \delta\}.$$

From lemma 8 of [11], there exists $\delta_0 > 0$ such that, for all $\delta \leqslant \delta_0$, the boundaries $\{u_1 = \min I + \delta\}$ and $\{u_2 = \max I - \delta\}$ satisfy equation (2.2). This proves that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \cap \partial B_\delta = \emptyset$, where ∂B_δ is defined in same way as ∂B. Define the diagonal $\triangle = \{(u_1, u_2) \in \mathrm{cl}(B) \mid u_1 = u_2\}$. Suppose now that there exists a sequence of points (u_1^n, u_2^n) accumulating at $\mathrm{cl}(B_\delta) \cap \triangle$. Then it follows from lemma 5 of [11] that

$$\|(p^+_{u_1^n} - p^-_{u_1^n}, p^+_{u_2^n} - p^-_{u_2^n})\| \to \infty \quad \text{as } n \to \infty$$

for any pair $(p^+_{u_1^n} - p^-_{u_1^n}, p^+_{u_2^n} - p^-_{u_2^n})$ in $(\zeta_n \cap \mathcal{L}_+) \times (\zeta_n \cap \mathcal{L}_-)$, where $\zeta_n = \pi^{-1}(u_1^n, u_2^n)$. The latter combined with the behaviour of $\mathcal{L}_+ \cap \mathcal{L}_-$ on ∂B_δ now implies that there exists a compact set $\hat{B} \subset \mathrm{int}_{\mathbb{R}^2} B_\delta \subset \mathrm{int}_{\mathbb{R}^2} B$ such that $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \subset \hat{B}$. □

To study $\mathcal{L}_+ \cap \mathcal{L}_-$, we use the intersection number $\iota(\mathcal{L}_+, \mathcal{L}_-)$. Our approach is to define $\iota(\mathcal{L}_+, \mathcal{L}_-)$ via the Brouwer degree by constructing proper equations on T^*B whose zero sets are $\mathcal{L}_\pm$. This can be done in many ways and the intersection number $\iota(\mathcal{L}_+, \mathcal{L}_-)$ does not depend on the particular choice of the defining equations.

Let

$$F_+(u_1, p_{u_1}, u_2, p_{u_2}) = [u_2 - f_+(u_1, p_{u_1}), p_{u_2} - g_+(u_1, p_{u_1})],$$
$$F_-(u_1, p_{u_1}, u_2, p_{u_2}) = [u_1 - f_-(u_2, p_{u_2}), p_{u_1} - g_-(u_2, p_{u_2})],$$

where the domain of definition of F_+ is $(u_1, p_{u_1}) \in P_+$, $(u_2, p_{u_2}) \in I \times \mathbb{R}$ and the domain of definition of F_- is $(u_2, p_{u_2}) \in P_-$, $(u_1, p_{u_1}) \in I \times \mathbb{R}$. Then $\mathcal{L}_\pm$ are the level sets $F_\pm^{-1}(0)$ in T^*B. Define $F(u_1, p_{u_1}, u_2, p_{u_2}) = [F_+, F_-]$ on $P_+ \times P_-$. Then the zero set of F is $F^{-1}(0) = \mathcal{L}_+ \cap \mathcal{L}_-$, which is bounded and contained in $\operatorname{int}(P_+ \times P_-)$. The latter follows from lemma 2.2. Indeed, for any intersection point, $(u_1, u_2) \in \hat{B}$, we have $u_1 \in \operatorname{int} I$. If $(u_1, p_{u_1}) \in \partial P_+$, then $u_1 \in \partial I$, a contradiction. Thus $(u_1, p_{u_1}) \in \operatorname{int} P_+$. Similarly, it follows that $(u_2, p_{u_2}) \in \operatorname{int} P_-$. Since $\pi(\mathcal{L}_+ \cap \mathcal{L}_-) \subset \hat{B}$, the boundedness of $F^{-1}(0)$ follows from continuity. These facts combined justify the definition

$$\iota(\mathcal{L}_+, \mathcal{L}_-) = \deg(F, P_+ \times P_-, 0)$$

(cf. [3]). Since $\dim \mathcal{L}_\pm = 2$, we have

$$\iota(\mathcal{L}_+, \mathcal{L}_-) = \iota(\mathcal{L}_-, \mathcal{L}_+).$$

We are now ready to prove the main result.

Proof of theorem 1.1. Let M be a regular energy manifold corresponding to $H(x) = E$. We compare the Lagrangian system determined by $L_0 = L = \frac{1}{2}w^2 + K(u, v)$ with the system determined by $L_1 = \frac{1}{2}w^2 + K(u, 0)$. The latter system is of Swift–Hohenberg type, which is shown to be a twist system in [11]. The two systems are related by continuation. Specifically, define $L_\lambda = (1-\lambda)L_0 + \lambda L_1$. Then the energy manifolds M_λ are regular for all $\lambda \in [0, 1]$. Hence each M_λ is homotopy equivalent to $M = M_0$. Moreover, from the definition of L_λ, it is clear that the sections $N_\lambda = N$ and the base manifolds $B_\lambda = B$ for all $\lambda \in [0, 1]$.

Since M_0 and M_1 are homotopy equivalent, the Betti numbers $\dim H_k(M_0)$ and $\dim H_k(M_1)$, $k \geqslant 0$, are equal. In [2, § 7], it was shown that $\dim H_2(M_1)$ is equal to the number of compact components of the section N, which can be computed directly from the graph of the potential $K(u, 0)$, i.e. the number of compact intervals on which $K(u, 0) + E \geqslant 0$.

Since L_1 defines a twist system, the results in [11], specifically lemma 8 illustrated in figure 2, imply via a Conley index argument that, for each compact component of N, the overall degree $\deg(F, P_+ \times P_-, 0)$ is ± 1, and hence $\iota(\mathcal{L}_+^1, \mathcal{L}_-^1) = \pm 1$. The sign of $\iota(\mathcal{L}_+^1, \mathcal{L}_-^1)$ depends on the orientations of $\mathcal{L}_\pm$ induced by their definition as level sets of $F_\pm$. Since L_λ satisfies hypotheses (H1) and (H2) and $\partial_w^2 L_\lambda = 1 > 0$ for all $\lambda \in [0, 1]$, the results of lemma 2.1 apply for all $\lambda \in [0, 1]$. Moreover, lemma 2.2 implies that $\pi(\mathcal{L}_+^\lambda \cap \mathcal{L}_-^\lambda) \subset \hat{B}$ for some compact set $\hat{B} \subset B$ uniformly for all $\lambda \in [0, 1]$.

Now, the continuation property of the degree can be used to show that the intersection number can be continued for all $\lambda \in [0, 1]$. This fact requires a little argument. For each $\lambda_0 \in [0, 1]$, there exists an $\epsilon(\lambda_0) > 0$ and compact $D_{\lambda_0} \subset P_+ \times P_-$ such that $\mathcal{L}_+^\lambda \cap L_-^\lambda \subset D_{\lambda_0}$ for all $\lambda \in (\lambda_0 - \epsilon(\lambda_0), \lambda_0 + \epsilon(\lambda_0)) \cap [0, 1]$. Therefore, $\deg(F, P_+^\lambda \times P_-^\lambda, 0) = \deg(F, D_{\lambda_0}, 0)$, i.e. the function $\iota(\mathcal{L}_+^\lambda, \mathcal{L}_-^\lambda)$ is locally constant on $[0, 1]$. Since $[0, 1]$ is connected, $\iota(\mathcal{L}_+^\lambda, \mathcal{L}_-^\lambda)$ is globally constant on $[0, 1]$. In particular, $\iota(\mathcal{L}_+^0, \mathcal{L}_-^0) = \iota(\mathcal{L}_+^1, \mathcal{L}_-^1) \neq 0$.

Hence, for each compact component of N, the energy manifold M contains a closed characteristic. Therefore, the number of closed characteristics is at least $\dim H_2(M)$. □

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3. Existence of laps in the regular case

3.1. Properties of the Lagrangian action

Fix $E \in \mathbb{R}$ and a compact component interval component I. Given

$$\boldsymbol{u} = (u_1, u_2) \in B,$$
$$\boldsymbol{b} \in \mathcal{K} = \{(b_1, b_2) \in \mathbb{R}^2 : b_1 b_2 \geqslant 0 \text{ and } \max\{|b_1|, |b_2|\} < \tfrac{1}{2}\},$$

define

$$X_\tau(\boldsymbol{u}, \boldsymbol{b}) = \{u \in H^2([0,\tau]) : u(0) = u_1,\ u(\tau) = u_2,\ u'(0) = b_1, \\ u'(\tau) = b_2 \text{ and } u'(t) \neq 0 \text{ for } t \in (0,\tau)\}$$

and

$$J_E[u] = \int_0^\tau [\tfrac{1}{2}|u''(t)|^2 + K(u(t), u'(t)) + E]\,\mathrm{d}t,$$

which is well defined on $X(\boldsymbol{u}, \boldsymbol{b}) = \bigcup_{\tau \in \mathbb{R}^+} X_\tau(\boldsymbol{u}, \boldsymbol{b})$. To prove lemma 2.1, we consider the following minimization problem,

$$\mathcal{J}_E(\boldsymbol{u}, \boldsymbol{b}) = \inf_{\substack{u \in X_\tau \\ \tau \in \mathbb{R}^+}} J_E[u],$$

and establish the existence of minimizers.

Minimization J_E requires a growth condition on the Lagrangian. Hypothesis (H2) implies the following property.

LEMMA 3.1. *If hypothesis (H2) holds, then, for every $\epsilon > 0$, there exists $C_\epsilon \geqslant 0$ such that $K(u, v) + E + \epsilon^{-1} v^4 \geqslant -C_\epsilon |v|$ for all $u \in I$ and $v \in \mathbb{R}$.*

Proof. Since u is bounded, we have $K(u, v) \geqslant -C - C|v|^\gamma$. Thus

$$K(u, v) + E + \epsilon^{-1} v^4 \geqslant -C + E - C|v|^\gamma + \epsilon^{-1} v^4 \geqslant -C^*_\epsilon$$

for all $u \in I$ and $v \in \mathbb{R}$. Since $K(u, 0) + E \geqslant 0$ and $\partial_v K(u, 0)$ is bounded for $u \in I$, there exists $C_\epsilon > 0$ such that $K(u, v) + E + \epsilon^{-1} v^4 \geqslant -C_\epsilon |v|$ for all $u \in I$ and $v \in \mathbb{R}$. □

LEMMA 3.2. *If $u \in X(\boldsymbol{u}, \boldsymbol{b})$, then*

$$\int_0^\tau |u''|^2\,\mathrm{d}t \geqslant \frac{4(1 - |\boldsymbol{b}|_\infty)}{9|u_2 - u_1|^2} \int_0^\tau |u'|^4\,\mathrm{d}t - \frac{4|\boldsymbol{b}|_\infty^2}{9|u_2 - u_1|}.$$

Proof. Since u is monotone, we can reparametrize by $u'(t) = v(u)$ and let $z(u) = v|v|^{1/2}(u)$. Transforming to (u, z) variables yields

$$J_E[u(t)] = J_E[z(u)] = \int_{u_1}^{u_2} \left[\tfrac{2}{9}|z'(u)|^2 + \frac{K(u, z^{2/3}(u)) + E}{z^{2/3}(u)}\right] \mathrm{d}u,$$

with $z \in \chi + H_0^1([u_1, u_2])$, where χ is a smooth function satisfying $z(u_1) = b_1^{3/2}$ and $z(u_2) = b_2^{3/2}$. Hence z is absolutely continuous with

$$z(u) - z(u_1) = \int_{u_1}^{u} z'(\mu)\,\mathrm{d}\mu$$

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for all $u \in [u_1, u_2]$, which implies

$$|z(u) - b_1^{3/2}|^2 \leqslant |u_2 - u_1| \int_{u_1}^{u_2} |z'|^2 \, \mathrm{d}u.$$

Note that, under this transformation,

$$\int_0^\tau |u''(t)|^2 \, \mathrm{d}t = \frac{4}{9} \int_{u_1}^{u_2} |z'(u)|^2 \, \mathrm{d}u \quad \text{and} \quad \int_0^\tau |u'(t)|^4 \, \mathrm{d}t = \int_{u_1}^{u_2} |z(u)|^2 \, \mathrm{d}u.$$

Therefore,

$$\begin{aligned}
\int_0^\tau |u''|^2 \, \mathrm{d}t \\
&= \frac{4}{9} \int_{u_1}^{u_2} |z'|^2 \, \mathrm{d}u \\
&\geqslant \frac{4}{9|u_2 - u_1|^2} \int_{u_1}^{u_2} |z - b_1^{3/2}|^2 \, \mathrm{d}u \\
&= \frac{4}{9|u_2 - u_1|^2} \biggl[\int_{u_1}^{u_2} z^2 \, \mathrm{d}u - 2b_1^{3/2} \int_{u_1}^{u_2} z \, \mathrm{d}u + b_1^3 |u_2 - u_1| \biggr] \\
&\geqslant \frac{4}{9|u_2 - u_1|^2} \biggl[\int_{u_1}^{u_2} z^2 \, \mathrm{d}u - 2b_1^{3/2} |u_2 - u_1|^{1/2} \biggl(\int_{u_1}^{u_2} z^2 \, \mathrm{d}u \biggr)^{1/2} + b_1^3 |u_2 - u_1| \biggr] \\
&\geqslant \frac{4(1 - b_1)}{9|u_2 - u_1|^2} \int_{u_1}^{u_2} z^2 \, \mathrm{d}u - \frac{4b_1^2}{9|u_2 - u_1|} \\
&\geqslant \frac{4(1 - |\boldsymbol{b}|_\infty)}{9|u_2 - u_1|^2} \int_0^\tau |u'|^4 \, \mathrm{d}t - \frac{4|\boldsymbol{b}|_\infty^2}{9|u_2 - u_1|}.
\end{aligned}$$

□

Now we use this inequality to prove that J_E is bounded below on $X(\boldsymbol{u}, \boldsymbol{b})$, so that the minimization problem is well posed, i.e. $\mathcal{J}_E > -\infty$.

LEMMA 3.3. *There exists a constant* $C(|u_2 - u_1|, |\boldsymbol{b}|_\infty) > 0$ *such that* $J_E[u] \geqslant -C$ *for all* $u \in X(\boldsymbol{u}, \boldsymbol{b})$.

Proof. Applying lemmas 3.1 and 3.2, we obtain

$$\begin{aligned}
J_E[u] &= \int_0^\tau [\tfrac{1}{2}|u''|^2 + K(u, u') + E] \, \mathrm{d}t \\
&\geqslant \frac{2(1 - |\boldsymbol{b}|_\infty)}{9|u_2 - u_1|^2} \int_0^\tau |u'|^4 \, \mathrm{d}t - \frac{2|\boldsymbol{b}|_\infty^2}{9|u_2 - u_1|} + \int_0^\tau [K(u, u') + E] \, \mathrm{d}t \\
&\geqslant \int_0^\tau \biggl[K(u, u') + E + \frac{1}{9|u_2 - u_1|^2} |u'|^4 \biggr] \, \mathrm{d}t - \frac{2|\boldsymbol{b}|_\infty^2}{9|u_2 - u_1|} \\
&\geqslant -\int_0^\tau C u' \, \mathrm{d}t - \frac{2|\boldsymbol{b}|_\infty^2}{9|u_2 - u_1|} \\
&\geqslant -C|u_2 - u_1| - \frac{2|\boldsymbol{b}|_\infty^2}{9|u_2 - u_1|},
\end{aligned}$$

which implies that $J_E[u]$ is bounded below on $X(\boldsymbol{u}, \boldsymbol{b})$. □

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Define the sublevel set $J_E^a(\boldsymbol{u},\boldsymbol{b}) = \{u \in X(\boldsymbol{u},\boldsymbol{b}) : J_E[u] \leqslant a\}$.

LEMMA 3.4. *There exist positive constants C_1, C_2 and T_1, depending on a, $|u_2-u_1|$ and $|\boldsymbol{b}|_\infty$, such that, for any $u \in J_E^a(\boldsymbol{u},\boldsymbol{b})$, we have $\tau \geqslant T_1$, $\|u''\|_{L^2([0,\tau])} \leqslant C_1$ and $\|u'\|_{L^4([0,\tau])} \leqslant C_2$.*

Proof. We have

$$\begin{aligned} a &\geqslant J_E[u] \\ &= \frac{1}{2}\int_0^\tau |u''|^2\,\mathrm{d}t + \int_0^\tau [K(u,u') + E]\,\mathrm{d}t \\ &\geqslant \biggl[\frac{2(1-|\boldsymbol{b}|_\infty)}{9|u_2-u_1|^2} - \frac{1}{9|u_2-u_1|^2}\biggr]\int_0^\tau |u'|^4\,\mathrm{d}t - \frac{2|\boldsymbol{b}|_\infty^2}{9|u_2-u_1|} - C|u_2-u_1|. \end{aligned}$$

Therefore,

$$\int_0^\tau |u'|^4\,\mathrm{d}t \leqslant C(a,|\boldsymbol{b}|_\infty,|u_2-u_1|),$$

which also implies

$$\int_0^\tau |u''|^2\,\mathrm{d}t \leqslant C(a,|\boldsymbol{b}|_\infty,|u_2-u_1|).$$

As for a lower bound on τ, we argue as follows. Integrating u' over $[0,\tau]$, we find that

$$|u_2-u_1| \leqslant \tau^{1/2}\|u'\|_{L^2} \leqslant \tau^{3/4}\|u'\|_{L^4} \leqslant C\tau^{3/4}.$$

□

LEMMA 3.5. *There exists $C(\tau,a,\boldsymbol{u},|\boldsymbol{b}|_\infty)$ such that $\|u\|_{H^2([0,\tau])} \leqslant C$ for all $u \in J_E^a(\boldsymbol{u},\boldsymbol{b})$.*

Proof. By Cauchy–Schwarz,

$$\int_0^\tau |u'|^2\,\mathrm{d}t \leqslant C(a,|\boldsymbol{b}|_\infty,|u_2-u_1|)\tau^{1/2},$$

which implies that

$$\|u\|_{L^\infty([0,\tau])} \leqslant C(a,|\boldsymbol{b}|_\infty,|u_2-u_1|)\tau^{3/4} + |u_1|$$

and

$$\int_0^\tau u^2\,\mathrm{d}t \leqslant (C(a,|\boldsymbol{b}|_\infty,|u_2-u_1|)\tau^{3/4} + |u_1|)^2\tau.$$

Therefore,

$$\|u\|_{H^2([0,\tau])} \leqslant C(a,|\boldsymbol{b}|_\infty,|u_2-u_1|,|u_1|,\tau).$$

□

To find a minimizer, we need to establish that J_E is coercive and weakly lower semicontinuous along a minimizing sequence. Lemma 3.5 implies coercivity provided that τ is uniformly bounded, which is proved in § 3.2 for the regular case. We now show that J_E is sequentially weakly lower semicontinuous along sequences for which τ is bounded.

LEMMA 3.6. *Suppose that $\boldsymbol{u}_n \in X(\boldsymbol{u}, \boldsymbol{b})$, with both $\|u_n\|_{H^2([0,\tau_n])}$ and τ_n uniformly bounded. Then*

$$\liminf_{n_k \to \infty} J_E[u_{n_k}] \geqslant J_E[u]$$

for some $u \in H^2([0,\tau])$.

Proof. We can rescale t to separate the dependence on τ from variations in u. Let

$$\hat{X}_\tau(\boldsymbol{u}, \boldsymbol{b}) = \{q \in H^2([0,1]) : q(0) = u_1,\ q(1) = u_2,\ q'(0) = b_1/\tau,\\ q'(1) = b_2/\tau \text{ and } q'(s) \neq 0 \text{ for } s \in (0,1)\}.$$

Let

$$\hat{X}(\boldsymbol{u}, \boldsymbol{b}) = \bigcup_{\tau \in \mathbb{R}^+} \hat{X}_\tau(\boldsymbol{u}, \boldsymbol{b}) \subset H^2([0,1]).$$

Then

$$J_E[q] = \int_0^1 \left[\frac{1}{2\tau^3}|q''(s)|^2 + \tau K\left(q(s), \frac{q'(s)}{\tau}\right) + \tau E\right] \mathrm{d}s$$

for $q \in \hat{X}(\boldsymbol{u}, \boldsymbol{b})$.

The functions $q_n(s) = u_n(\tau s)$ are uniformly bounded in $H^2([0,1])$, and hence we can extract a weakly convergent subsequence $q_n \rightharpoonup q$ with $\tau_n \to \tau$. Observe that the functional $\int_0^1 \tau[K(q, q'/\tau) + E]\, \mathrm{d}s$ is continuous in $\tau > 0$ and weakly continuous in $q \in H^2([0,1])$. The functional $(1/2\tau^3)\int_0^1 |q''|^2\, \mathrm{d}s$ separates the variables τ and q and is continuous in τ and sequentially weakly lower semicontinuous in q. Hence

$$\frac{1}{2\tau^3}\int_0^1 |q''|^2\, \mathrm{d}s \leqslant \liminf_{n\to\infty} \frac{1}{2\tau_n^3}\int_0^1 |q_n''|^2\, \mathrm{d}s.$$

Therefore, $J_E[q] \leqslant \liminf_{n\to\infty} J_E[q_n]$. □

LEMMA 3.7. *If $\mathcal{J}_E(\boldsymbol{u}, \boldsymbol{b}) = J_E[u]$ for some $u \in X(\boldsymbol{u}, \boldsymbol{b})$, then $u \in C^5([0,\tau])$ satisfies the Euler–Lagrange equation (1.1) and $H(u, u', u'', u''') = E$.*

Proof. This follows from standard regularity theory (cf. [7]). □

Lemmas 3.5 and 3.6 imply that a minimizer exists in $H^2([0,\tau])$, provided that τ is bounded along some minimizing sequence. Lemma 3.7 states that a minimizer belonging to $X(\boldsymbol{u}, \boldsymbol{b})$ is a solution to the Euler–Lagrange equations. Therefore, we must show that minimizing sequences exist for which τ is bounded and the weak limit belongs to $X(\boldsymbol{u}, \boldsymbol{b})$. This issue will be addressed in § 3.2 for the regular case. We conclude this subsection with a technical lemma concerning the continuity of the infima $\mathcal{J}_E(\boldsymbol{u}, \boldsymbol{b})$ with respect to the parameter $\boldsymbol{b}$.

LEMMA 3.8. *Suppose that $\boldsymbol{b}_n \to \boldsymbol{b} \in \mathcal{K}$ and $\boldsymbol{u}_n \in X(\boldsymbol{u}, \boldsymbol{b}_n)$, with $\tau_n \to \tau$, $J_E[u_n] = \mathcal{J}_E(\boldsymbol{u}, \boldsymbol{b}_n)$ and $u_n \to u$ in $H^2([0,\tau])$. Then $J_E[u] = \mathcal{J}_E(\boldsymbol{u}, \boldsymbol{b})$.*

Proof. Again we can rescale t to separate the dependence on τ from variations in u. Let $\chi[\tau, \boldsymbol{b})] : [0,1] \to \mathbb{R}$ be a smooth strictly monotone function satisfying $\chi(0) = u_1$, $\chi(1) = u_2$, $\chi'(0) = b_1/\tau$ and $\chi'(1) = b_2/\tau$, and define

$$J_E[q, \tau; \boldsymbol{b}] = \int_0^1 L\left(q + \chi, \frac{q' + \chi'}{\tau}, \frac{q'' + \chi''}{\tau^2}\right)\tau\, \mathrm{d}s$$

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for

$$q \in \{q \in H^2([0,1]) : q'(s) + \chi'(s) \neq 0 \text{ for } s \in (0,1)\}.$$

Then

$$\inf_{q,\tau} J_E[q,\tau;\boldsymbol{b}] = \mathcal{J}_E(\boldsymbol{u},\boldsymbol{b}).$$

The family $\chi[\tau,\boldsymbol{b}]$ can be chosen to vary continuously in $\boldsymbol{b}$, the family of functionals $J_E[q,\tau;\boldsymbol{b}]$ is continuous in $\boldsymbol{b}$ for each fixed q and τ. Therefore, the infimum $\mathcal{J}_E(\boldsymbol{u},\boldsymbol{b})$ is upper semicontinuous with respect to $\boldsymbol{b}$ (cf. [10]).

Let $q_n(s) = u_n(\tau_n s) - \chi[\tau_n, \boldsymbol{b}_n](s)$. Since $J_E[\cdot,\cdot;\boldsymbol{b}]$ is continuous, we have

$$J_E[q,\tau;\boldsymbol{b}] = \lim_{n\to\infty} J_E[q_n,\tau_n;\boldsymbol{b}_n] = \lim_{n\to\infty} \mathcal{J}_E(\boldsymbol{u},\boldsymbol{b}_n) \leqslant \mathcal{J}_E(\boldsymbol{u},\boldsymbol{b}) \leqslant J_E[q,\tau;\boldsymbol{b}].$$

Therefore, $J_E[u] = J_E[q,\tau;\boldsymbol{b}] = \mathcal{J}_E(\boldsymbol{u},\boldsymbol{b})$. □

3.2. The existence of minimizers

In this section, we prove the existence of minimizers when $\boldsymbol{u} = (u_1,u_2) \in \operatorname{int} B$ for a single interval component I, and hence we will assume that $[u_1,u_2]$ is regular. In this case, the following property is due to continuity.

(P3) There exist $\rho > 0$ and $\delta_0 > 0$ such that $K(u,v) + E \geqslant \rho > 0$ for all $(u,v) \in [u_1,u_2] \times [-\delta_0,\delta_0]$.

Lemma 3.9. *Under hypotheses (H1) and (H2), there exists a constant $T_2 > 0$, depending on a, $|\boldsymbol{b}|_\infty$, $|u_2 - u_1|$, $1/\delta_0$ and $1/\rho$, such that, for any $u \in J_E^a$, we have $\tau \leqslant T_2$.*

Proof. Let

$$S_{\delta_0} = \{t \in [0,\tau] : |u'(t)| \geqslant \delta_0\},$$

where δ_0 is chosen in (P3). Since

$$|S_{\delta_0}|\delta_0^4 \leqslant \int_0^\tau |u'|^4 \,\mathrm{d}t,$$

we have $|S_{\delta_0}| \leqslant C(a,\delta^2,|u_2-u_1|,1/\delta_0^4)$. Let $\epsilon > 0$. Then

$$\begin{aligned} a &\geqslant J_E[u] \\ &\geqslant \int_{S_{\delta_0}^c} [K(u,u') + E]\,\mathrm{d}t + \int_{S_{\delta_0}} [K(u,u') + E]\,\mathrm{d}t \\ &\geqslant \rho(\tau - |S_{\delta_0}|) - \epsilon^{-1}\int_{S_{\delta_0}} |u'|^4\,\mathrm{d}t - C_\epsilon|u_2 - u_1|, \end{aligned}$$

which implies that $\tau \leqslant T_2(a,\delta^2,|u_2-u_1|,1/\delta_0^4,1/\rho)$, by lemma 3.4. □

Lemmas 3.3 and 3.9 imply that action is bounded below on $X(\boldsymbol{u},\boldsymbol{b})$, and the time τ is bounded on sublevel sets of J_E. Therefore, lemma 3.6 implies that J_E is coercive and sequentially weakly lower semicontinuous along any sequence in $X(\boldsymbol{u},\boldsymbol{b})$ on which J_E is bounded. Let

$$\operatorname{cl} X_\tau(\boldsymbol{u},\boldsymbol{b}) = \{u \in H^2([0,\tau]) : u_n \rightharpoonup u \text{ for some sequence } u_n \in X(\boldsymbol{u},\boldsymbol{b})\}.$$

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Functions

$$u \in \operatorname{cl} X(\boldsymbol{u}, \boldsymbol{b}) = \bigcup_{\tau > 0} \operatorname{cl} X_\tau(\boldsymbol{u}, \boldsymbol{b})$$

are monotone, possibly with critical inflection points. We have shown that the minimization problem is well posed in the sense that a minimizer exists in $\operatorname{cl} X(\boldsymbol{u}, \boldsymbol{b})$. However, we must still show that this minimizer lies in $X(\boldsymbol{u}, \boldsymbol{b})$ in order to apply lemma 3.7.

Without loss of generality, we can assume that the following condition holds.

(P4) The constant $\delta_0 > 0$ in (P3) can be chosen such that $K(u, v) + E$ is non-increasing in v for all $(u, v) \in [u_1, u_2] \times [-\delta_0, \delta_0]$.

Property (P4) is not a restriction on K. Consider the family of Lagrangians $|u''|^2/2 + K(u, v) - \alpha v$. Then $J_{\alpha,E}[u] = J_E[u] - \alpha|u_2 - u_1|$ for all $\alpha \in \mathbb{R}$. Hence the minimizers of $J_{\alpha,E}$ are the same for all $\alpha \in \mathbb{R}$. Since $[u_1, u_2]$ is compact, we can choose $\alpha \geqslant 0$ such that $\partial_v K(u, 0) - \alpha$ is strictly negative for all $u \in [u_1, u_2]$. Then, replacing $K(u, v)$ by $K(u, v) - \alpha v$ will satisfy (P4) without changing the minimization problem, and since $\alpha \geqslant 0$, the growth condition (H2) is still satisfied. Furthermore, property (P3) still holds with possibly smaller values of ρ and δ_0. Property (P4) is used in the following lemma, which implies that a minimizer must lie in $X(\boldsymbol{u}, \boldsymbol{b})$.

Lemma 3.10. *Suppose $[w_1, w_2] \subset [u_1, u_2]$. Let $u \in \operatorname{cl} \hat{X}_\tau(\boldsymbol{w}, \boldsymbol{b}_*)$ for some $\boldsymbol{b}_* = (b, b)$ with $0 < |b| < \delta_0$. Define $\hat{\tau} = |w_2 - w_1|/b > 0$ and $w \in X_{\hat{\tau}}(\boldsymbol{w}, \boldsymbol{b}_*)$ by $w(t) = bt + w_1$. Then $J_E[w] \leqslant J_E[u]$ and $w'(t) \neq 0$. If $u'' \not\equiv 0$, then $J_E[w] < J_E[u]$.*

Proof. As in the proof of lemma 3.2, transforming $u(t)$ and $w(t)$ into (u, z) variables, we have

$$\begin{aligned} J_E[u] &= \frac{2}{9} \int_{w_1}^{w_2} |z'|^2 \, \mathrm{d}u + \int_{w_1}^{w_2} \left[\frac{K(u, z^{2/3}) + E}{z^{2/3}} \right] \mathrm{d}u \\ &\geqslant \int_{w_1}^{w_2} \left[\frac{K(u, |b|) + E}{|b|} \right] \mathrm{d}u \\ &= J_E[w]. \end{aligned} \tag{3.1}$$

Here we have used properties (P3) and (P4). □

Corollary 3.11. *If $u \in \operatorname{cl} X(\boldsymbol{u}, \boldsymbol{b})$ is a minimizer of J_E, then $\hat{u}' \neq 0$ on $[0, \tau]$, and hence $u \in X(\boldsymbol{u}, \boldsymbol{b})$.*

Proof. Suppose u has a critical point at t_0. Since u is monotone, t_0 is contained in some maximal compact interval of critical points I. By continuity, for any $b_* = (b, b)$ with $|b|$ sufficiently small, there is an interval $[t_1, t_2]$ containing I such that $u'(t_1) = u'(t_2) = b$. Let $w_1 = u(t_1)$ and $w_2 = u(t_2)$. Then, using lemma 3.10, we can construct a function $w \in X(\boldsymbol{w}, \boldsymbol{b}_*)$ such that $J_E[w] < J_E[u|_{[t_1,t_2]}]$. Replacing $u|_{[t_1,t_2]}$ by w yields a function $\hat{u} \in H^2([0, \hat{\tau}])$ such that $J_E[\hat{u}] < J_E[u]$, which contradicts the fact that u is a minimizer. □

We have proved the following theorem, which implies lemma 2.1.

THEOREM 3.12. *Suppose that L satisfies hypotheses (H1) and (H2) on an interval component I_E. If $\boldsymbol{u} \in B$ and $\boldsymbol{b} \in \mathcal{K}$, then there exists a strictly monotone minimizer $u \in X(\boldsymbol{u}, \boldsymbol{b}) \cap C^5([0, \tau])$ of J_E that satisfies the Euler–Lagrange equation (1.1).*

In fact, the above results prove theorem 3.12 for $\boldsymbol{u} \in \operatorname{int} B$. In order to include all of B, one can choose a sequence of minimizers $u_n \in X(\boldsymbol{u}_n, \boldsymbol{b})$ with $\boldsymbol{u}_n \to \boldsymbol{u} \in \partial B$. To obtain a limit in $X(\boldsymbol{u}, \boldsymbol{b})$, we need to argue that τ_n is uniformly bounded. Suppose not, i.e. $\tau_n \to \infty$. Since $\|u_n\|_{H^2([0,\tau_n])} \leqslant C$, we would obtain, after appropriate shifts, a solution asymptotic to either u_1 or u_2, or both, which is a contradiction.

4. Extensions and concluding remarks

4.1. More general Lagrangians

Essentially, the hypotheses (H1) and (H2) in § 1 are stated in the manner most convenient to implement the minimization in § 3 without too many technical details. The analysis in that section is needed to establish the surjectivity of the projection $\pi\mathcal{L}_\pm$ onto the base B, which ensures that the continuation to a twist system is well defined. The geometric and topological considerations in § 2, other than surjectivity, require merely the convexity of L in u''.

Thus hypotheses (H1) and (H2) can be weakened. For example, the conditions

(H1′) $0 < \alpha \leqslant \partial_w^2 L(u, v, w) \leqslant \alpha^{-1}$ for all (u, v, w);

(H2′) $L(u, v, w) \geqslant \frac{1}{2}\alpha w^2 - C(|u|) - C(|u|)|v|^\gamma$, $\gamma < 4$,

where $C(|u|)$ is locally bounded, would also be sufficient. Hypothesis (H1′) implies that the action J_E is well defined on the Sobolev space $H^2(0, T)$, and (H2′) implies that the action is bounded below. Moreover, the use of other function spaces would allow super-quadratic growth of L in u''.

4.2. Sharp lower bounds

Consider the Lagrangian $L(u, u', u'') = \frac{1}{2}(u'')^2 - \frac{1}{4}(u')^4$ and $E > 0$. The corresponding action is not bounded below. Indeed, if $u_A(t) = A\sin(\pi(t - \frac{1}{2}T)/T)$, then

$$J[u_A] = \int_0^T [L(u_A, u_A', u_A'') + E]\,\mathrm{d}t = \frac{CA^2}{T^3} - \frac{C'A^4}{T^3} + ET.$$

Thus, for A large enough, $J[u_A] \to -\infty$ as $T \to 0$. This example shows that the minimization procedure can fail when $\gamma \geqslant 4$ in hypothesis (H2).

This problem is not just a failure of a particular method. For $E = 0$ in the previous example, it is not difficult to show that there are values u_1 and u_2 for which no monotone laps exists. The growth condition (H2) is a geometric restriction. Since M is non-compact, it is inevitable that some such restriction is necessary.

4.3. The topology of energy manifolds

For Lagrangians that satisfy the convexity hypothesis $\partial_w^2 L \geqslant \alpha > 0$, the topology of the energy manifolds $M_E = H^{-1}(E)$ can be completely determined from the sign changes in the potential $L(u, 0, 0) + E$. Consider the homotopy

$$L_\lambda(u, v, w) = (1 - \lambda)L(u, v, w) + \lambda[\tfrac{1}{2}\alpha w^2 + L(u, 0, 0)]$$

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of Lagrangians with corresponding Hamiltonians $H_\lambda(x) = p_u v + L^*_\lambda(u, v, p_v)$. If M is regular, then it is immediately clear that M_λ is regular for all $\lambda \in [0, 1]$. So $H_\lambda(x)$ defines a cobordism between $M = M_0$ and $M_1 = \{\frac{1}{2}\alpha w^2 + p_u v - L(u, 0, 0) = E\}$. A straightforward calculation shows that the height function $(\lambda, x) \to \lambda$ has no critical points, and hence standard Morse theory implies that M is homotopy equivalent to M_λ for all $\lambda \in [0, 1]$. In fact, they are diffeomorphic.

In [2], the homotopy type of M_1 was computed for the regular case, which implies $L(u, 0, 0) + E$ has simple zeros. There is a deformation retraction of M_1 onto a bouquet of circles and 2-spheres. Consequently, the homology of M_1 is determined by its Betti numbers with $\beta_0 = 1$ and $\beta_n = 0$ for $n > 2$. The second Betti number β_2 is the number of compact components of N, i.e. the number of compact intervals in $\mathbb{R}$ on which $L(u, 0, 0) + E \geqslant 0$. The first Betti number is the number of compact intervals on which $L(u, 0, 0) + E \leqslant 0$, which depends on the behaviour of $L(u, 0, 0)$ as $|u| \to \infty$. In any case, $\beta_1 \in \{\beta_2 - 1, \beta_2, \beta_2 + 1\}$.

A simple example shows that the lower bound $\dim H_2(M)$ in theorem 1.1 is sharp. Let $L(u, u', u'') = \frac{1}{2}(u'')^2 + \frac{1}{2}u^2$ and $E > 0$. Then $M_E \approx S^1 \times \mathbb{R}^2$, with $\dim H_2(M) = 0$, and solving the (linear) Euler–Lagrange equation explicitly shows that there are no closed characteristics.

4.4. Singular manifolds

Singularities in M occur at critical points of $L(u, 0, 0) + E$. Depending on the eigenvalues of these points as equilibrium points of the flow φ^t, there are three types of singularities: saddle (four real eigenvalues), saddle focus (four complex eigenvalues) and centre (four imaginary eigenvalues).

Consider an energy manifold M with (isolated) singular points in the interior of a compact component I of πN. The techniques in this paper imply that, for each component of $I \setminus \{\text{singular points}\}$, there is a closed characteristic independent of the type of the singularities. Note that this is already different from the first-order Lagrangian case, where singular manifolds cannot contain closed characteristics.

However, in the second-order case depending on the type of the singularities, even more closed characteristics must exist. It is shown in [11] that, if the twist property holds on each component of $I \setminus \{\text{singular points}\}$ and the singular points are either of saddle-focus or centre type, then the twist property holds on all of I, and additional closed characteristics exist with non-zero intersection number.

The arguments of this paper should be applicable in this case by continuation of a singular manifold with saddle-focus or centre type singularities to a twist system. The main issue is whether the surjectivity criterion in lemma 2.1 holds over all of I. We leave the details for future work, but we do not foresee any major problems in applying the techniques of [6, 7], which provide exactly the tools required to minimize in the presence of a saddle-focus or centre equilibrium, to show, again by minimization as in § 3, that the surjectivity condition holds.

4.5. Forcing of additional closed characteristics

For twist systems, it is shown in [4] that the existence of certain closed characteristics can force the existence of a multitude of closed characteristics due to their braiding and knotting. The above continuation method does not always immediately apply because the intersection numbers corresponding to these additional closed characteristics can be trivial, but, in certain cases, the topological information obtained from the braid type will imply non-trivial intersection number. In those cases, the arguments of this paper will imply the existence of more closed characteristics. One might also attempt to prove the existence of multiple solutions by more carefully studying the intersections using the fact that they are intersections of Lagrangian manifolds, which we leave for future work.

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