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NONMEASURABLE SETS AND UNIONS WITH RESPECT TO TREE IDEALS

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Abstract. In this paper, we consider a notion of nonmeasurablity with respect to Marczewski and Marczewski-like tree ideals s_0, m_0, l_0, cl_0, h_0, and ch_0. We show that there exists a subset of the Baire space ω^ω, which is s-, l-, and m-nonmeasurable that forms a dominating m.e.d. family. We investigate a notion of $\mathbb{T}$-Bernstein sets—sets which intersect but do not contain any body of any tree from a given family of trees $\mathbb{T}$. We also obtain a result on $\mathcal{I}$-Luzin sets, namely, we prove that if $\mathfrak{c}$ is a regular cardinal, then the algebraic sum (considered on the real line $\mathbb{R}$) of a generalized Luzin set and a generalized Sierpiński set belongs to s_0, m_0, l_0, and cl_0.

§1. Introduction and preliminaries. We will use standard set-theoretic notation following, for example, [14] . For a set X, $P(X)$ denotes the power set of X and $|X|$ denotes the cardinality of X. If κ is a cardinal number, then we denote:

- $[X]^\kappa = \{A \subseteq X : |A| = \kappa\}$;
- $[X]^{<\kappa} = \{A \subseteq X : |A| < \kappa\}$;
- $[X]^{\leq\kappa} = \{A \subseteq X : |A| \leq \kappa\}$.

Let X be an uncountable Polish space and $\mathcal{I} \subseteq P(X)$ be a σ-ideal. Let us recall some cardinal coefficients from Cichoń's Diagram:

- $\text{add}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} \notin \mathcal{I}\}$;
- $\text{non}(\mathcal{I}) = \min\{|A| : A \subseteq X \wedge A \notin \mathcal{I}\}$;
- $\text{cov}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge \bigcup\mathcal{A} = X\}$;
- $\text{cof}(\mathcal{I}) = \min\{|\mathcal{A}| : \mathcal{A} \subseteq \mathcal{I} \wedge (\forall A \in \mathcal{I})(\exists B \in \mathcal{A})(A \subseteq B)\}$;
- $\mathfrak{b} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge (\forall x \in \omega^\omega)(\exists f \in \mathcal{F})(\exists^\infty n)(x(n) < f(n))\}$;
- $\mathfrak{d} = \min\{|\mathcal{F}| : \mathcal{F} \subseteq \omega^\omega \wedge (\forall x \in \omega^\omega)(\exists f \in \mathcal{F})(\forall^\infty n)(x(n) < f(n))\}$.

We call $\mathfrak{b}$ the *bounding number* and $\mathfrak{d}$ the *dominating number*. A family $\mathcal{F} \subseteq \omega^\omega$ is *dominating*, if $\mathcal{F}$ has the property described in the definition of the

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dominating number (it does not have to be of minimal cardinality). We say that T is a *tree* on a set A if $T\subseteq A^{<\omega}$ and whenever $\tau \in T$, then $\tau \restriction n \in T$ for each natural n.

DEFINITION 1. Let T be a tree on a set A. Then,

- for each $\tau \in T \mathrm{succ}(\tau) = \{a \in A : \tau^\frown a \in T\}$;
- $\mathrm{split}(T) = \{\tau \in T : |\mathrm{succ}(\tau)| \geq 2\}$;
- ω-$\mathrm{split}(T) = \{\tau \in T : |\mathrm{succ}(\tau)| = \aleph_0\}$;
- for $\sigma \in T \mathrm{Succ}_T(\sigma) = \{\tau \in \mathrm{split}(T) : \sigma \subsetneq \tau, (\forall \tau' \in T)(\sigma \subsetneq \tau' \subsetneq \tau \to \tau' \notin \mathrm{split}(T))\}$;
- for $\sigma \in T \omega$-$\mathrm{Succ}_T(\sigma) = \{\tau \in \omega\text{-}\mathrm{split}(T) : \sigma \subsetneq \tau, (\forall \tau' \in T)(\sigma \subsetneq \tau' \subsetneq \tau \to \tau' \notin \omega\text{-}\mathrm{split}(T))\}$;
- $\mathrm{stem}(T) \in T$ is the node τ such that for each $\sigma \subsetneq \tau |\mathrm{succ}(\sigma)| = 1$ and $|\mathrm{succ}(\tau)| > 1$.

Let us now recall definitions of families of trees.

DEFINITION 2. A tree T on ω is called a

- Sacks tree or perfect tree, denoted by $T \in \mathbb{S}$, if for each node $\sigma \in T$, there is $\tau \in T$ such that $\sigma \subseteq \tau$ and $|\mathrm{succ}(\tau)| \geq 2$;
- Miller tree or superperfect tree, denoted by $T \in \mathbb{M}$, if $T \in \mathbb{S}$ and $\mathrm{split}(T) = \omega$-$\mathrm{split}(T)$;
- Laver tree, denoted by $T \in \mathbb{L}$, if for each node $\tau \supseteq \mathrm{stem}(T)$, we have $\tau \in \omega$-$\mathrm{split}(T)$;
- complete Laver tree, denoted by $T \in \mathbb{CL}$, if T is Laver and $\mathrm{stem}(T) = \emptyset$;
- *Hechler tree*, denoted by $T \in \mathbb{H}$, if for each node $\tau \supseteq \mathrm{stem}(T)$, we have that the set $\{n \in \omega : \tau^\frown n \notin T\}$ is finite;
- complete Hechler tree, denoted by $T \in \mathbb{CH}$, if T is Hechler and $\mathrm{stem}(T) = \emptyset$.

The notion of complete Laver trees was defined and investigated in [7], although Miller in [6] defines Laver trees *de facto* as complete Laver trees and Hechler trees as complete Hechler trees.

For a tree $T \subseteq \omega^{<\omega}$, let $[T]$ be a body of T, that is, the set of all infinite branches of T:

$$[T] = \{x \in \omega^\omega : (\forall n \in \omega)\,(x \restriction n \in T)\}.$$

We use the same notation for basic clopen sets generated by $\tau \in \omega^{<\omega}$:

$$[\tau] = \{x \in \omega^\omega : x \restriction |\tau| = \tau\}.$$

It will be clear from the context whether we mean a body of a tree or a clopen set.

DEFINITION 3. Let $\mathbb{T}$ be a family of trees. We say that $A \in P(\omega^\omega)$ belongs to the tree ideal t_0, if

$$(\forall P \in \mathbb{T})(\exists Q \in \mathbb{T})\,(Q \subseteq P \wedge [Q] \cap A = \emptyset).$$

Definition 4. Let $\mathbb{T}$ be a family of trees. We say that $A \in P(\omega^\omega)$ is *t*-measurable, if

$$(\forall P \in \mathbb{T})(\exists Q \in \mathbb{T})\ (Q \subseteq P \wedge ([Q] \subseteq A \vee [Q] \cap A = \emptyset)).$$

s_0 tree ideal is simply the classic Marczewski ideal (see [5]).

It is well known due to Judah and coworkers (see [12]) and Repický (see [10]) that $add(s_0) \leq \text{cov}(s_0) \leq cof(\mathfrak{c}) \leq non(s_0) = \mathfrak{c} < cof(s_0) \leq 2^{\mathfrak{c}}$. Moreover, in [16] Brendle and coworkers have also shown that $\mathfrak{c} < cof(m_0)$ and $\mathfrak{c} < cof(l_0)$. Clearly, $\omega_1 \leq add(l_0) \leq \text{cov}(l_0) \leq \mathfrak{c}$ holds. In [13], Goldstern and coworkers showed that it is relatively consistent with *ZFC* that $add(l_0) < \text{cov}(l_0)$.

Let us notice that the families s_0, l_0, m_0 form σ-ideals. On the other hand, cl_0 is not a σ-ideal. To see this it is enough to consider sets of the form $C_n = \{x \in \omega^\omega : \ x(0) = n\}$. Then $C_n \in cl_0$ for each n, but $\bigcup_n C_n = \omega^\omega$. Using the fact that s_0 is a σ-ideal, we may give another proof of the following well known result.

Proposition 5. *$cf(\mathfrak{c}) > \aleph_0$.*

Proof. Suppose that $\text{cf}(\mathfrak{c}) = \aleph_0$ and let $\mathbb{R} = \bigcup_{n\in\omega} A_n$, $|A_n| < \mathfrak{c}$ for each $n \in \omega$. Sets of cardinality less than $\mathfrak{c}$ belong to s_0, so $\mathbb{R} = \bigcup_{n\in\omega} A_n \in s_0$, a contradiction. ⊣

§2. Tree ideals and measurability.

In [1] the following result was obtained.

Theorem 6 (Brendle). *If $i_0, j_0 \in \{s_0, l_0, m_0\}$, and $i_0 \neq j_0$, then $i_0 \not\subseteq j_0$.*

First, we will compare the ideal cl_0 with the ideals s_0, m_0, l_0.

Fact 7. *$cl_0 \not\subseteq (l_0 \cup m_0 \cup s_0)$.*

Proof. To show the assertion, let us take $C_0 = \{x \in \omega^\omega : \ x(0) = 0\}$. Clearly, $C_0 \in cl_0$, but $C_0 \notin l_0 \cup m_0 \cup s_0$ since C_0 is a body of a Laver tree. ⊣

Let us recall the notion of some special kind of trees used in [1].

- A Miller tree T is an apple tree

$$(\forall \sigma \in \text{split}(T))(\forall \tau \in \text{Succ}_T(\sigma))(\forall n,m \in \omega)$$
$$(n > m \wedge \sigma^\frown n, \sigma^\frown m \in T \wedge \sigma^\frown m \subseteq \tau \rightarrow (\forall k < |\tau|)(\tau(k) < n))$$
and
$$(\forall \sigma, \tau \in \text{split}(T))(\sigma \subseteq \tau \rightarrow |\tau| \geq |\sigma| + 2).$$

- A tree $T = \{\tau_\sigma : \sigma \in 2^{<\omega}\}$ is a pear subtree of a Laver tree T_L, if T is a subtree of T_L and
  1. $\tau_\emptyset = \text{stem}(T_L)$;
  2. for each $\tau_\sigma \in T_L$ nodes $\tau_{\sigma^\frown 0} = \tau_\sigma{}^\frown k$ and $\tau_{\sigma^\frown 1} = \tau_\sigma{}^\frown l$, where $l > k > \max\{\max \text{rng}(\tau_{\sigma'}) : |\sigma'| = |\sigma|\}$ and $\tau_\sigma{}^\frown k, \tau_\sigma{}^\frown l \in T_L$.

Each Miller tree contains an apple tree. Also, apple trees and pear trees are related in the following way [1, Theorem 2.1, Claim].

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Proposition 8 (Brendle). *$|[T_a]\cap[T_p]|\leq 1$ whenever T_a is an apple tree and T_p is a pear tree.*

Theorem 9. *The following statements are true:*

(i) $m_0 \not\subseteq cl_0$.
(ii) $s_0 \not\subseteq cl_0$.

Proof. To prove that $m_0\setminus cl_0\neq\emptyset$, we will slightly modify the proof of Theorem 2.1 from [1]. We will use the notions of apple trees and pear trees.

Let us now enumerate all apple trees $\{A_\alpha:\ \alpha<\mathfrak{c}\}$ and all complete Laver trees $\{C_\alpha:\ \alpha<\mathfrak{c}\}$. For each complete Laver tree C_α, denote its pear subtree by P_{C_α}.

We construct a sequence $(x_\alpha)_{\alpha<\mathfrak{c}}$ such that for every $\alpha<\mathfrak{c}$,

$$x_\alpha\in[P_{C_\alpha}]\setminus\bigcup_{\beta<\alpha}[A_\beta].$$

Thanks to Proposition 8 such a choice is possible. Finally, we set $X=\{x_\alpha:\ \alpha<\mathfrak{c}\}$. It is clear that $X\notin cl_0$. We will show that $X\in m_0$. Let T be a Miller tree. There exists $\xi<\mathfrak{c}$ for which $A_\xi\subseteq T$. We may find a family of Miller trees $\{T_\alpha:\alpha<\mathfrak{c}\}$ satisfying $T_\alpha\subseteq A_\xi$ for all $\alpha<\mathfrak{c}$ and $[T_\alpha]\cap[T_\beta]=\emptyset$ for distinct $\alpha,\beta<\mathfrak{c}$. Since $|X\cap[A_\xi]|\leq|\xi|<\mathfrak{c}$, there is $\eta<\mathfrak{c}$ with $[T_\eta]\cap X=\emptyset$. Therefore, $X\notin m_0$.

To prove that $s_0\setminus cl_0\neq\emptyset$, we use slight modification of the proof of Theorem 2.2 from [1], which fits a similar pattern from the first case. ⊣

The argument involving antichain of bodies of Miller trees in the above proof fits the general framework outlined in [1], Section 1.4.

As a consequence, we obtain the following result.

Corollary 10. *The following statements are true:*

(i) *There exists a cl-nonmeasurable set which is m-measurable.*
(ii) *There exists a cl-nonmeasurable set which is s-measurable.*

Proof. It is enough to notice that any set outside cl_0 contains a *cl*-nonmeasurable subset. ⊣

The proof of the following theorem is inspired by the proof of Lemma 6 from [7] by A. Miller.

Theorem 11. $l_0\subseteq cl_0$.

Proof. Let $A\in l_0$ and let T be a complete Laver tree. We will find a complete Laver tree $T_0\subseteq T$ such that $[T_0]\cap A=\emptyset$. We will define a function $\varphi:T\to \mathrm{ORD}\cup\{\infty\}$, where ORD stands for the class of ordinal numbers. We start with $\varphi^{-1}[\{0\}]$:

$$\varphi(\tau)=0\Longleftrightarrow(\exists T'\subseteq T)(T'\in\mathbb{L}\wedge\mathrm{stem}(T')=\tau\wedge[T']\cap A=\emptyset).$$

Then recursively for $\alpha>0$, we set

$$\varphi(\tau)\leq\alpha\Longleftrightarrow(\exists^\infty n\in\omega)(\varphi(\tau^\frown n)<\alpha).$$

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Finally for $\tau \in T \setminus \varphi^{-1}[\mathrm{ORD}]$, let $\varphi(\tau) = \infty$. Notice that for each $\tau \in T$,

$$\tau \in \varphi^{-1}[\mathrm{ORD}] \Longleftrightarrow (\exists T' \subseteq T)(T' \in \mathbb{L} \wedge \mathrm{stem}(T') = \tau \wedge [T'] \cap A = \emptyset),$$

which is equivalent to $\varphi(\tau) = 0$. We claim that $\varphi(\emptyset) \neq \infty$. Suppose otherwise. It implies that there are infinitely many (in fact—relatively cofinitely many) nodes in T of the form $\emptyset^\frown n$ for which $\varphi(\emptyset^\frown n) = \infty$. By simple induction, we will find a complete Laver tree $T' \subseteq T$ satisfying

$$(\forall \tau \in T')(\varphi(\tau) = \infty).$$

In particular, it means that

$$(\forall T'' \subseteq T')(T'' \in \mathbb{L} \Rightarrow [T''] \cap A \neq \emptyset),$$

contradicting the fact that $A \in l_0$.

Hence $\varphi(\emptyset) = 0$; therefore, there exists a complete Laver tree $T_0 \subseteq T$ satisfying $[T_0] \cap A = \emptyset$. ⊣

Let us notice that the above reasoning provides the following result, which one may find useful in itself.

Theorem 12. *Let $A \in l_0$. Then for every Laver tree T, there exists a Laver tree $T' \subseteq T$ such that stem(T') = stem(T) and $[T'] \cap A = \emptyset$.*

Let us introduce the notion of $\mathbb{T}$-Bernstein sets.

Definition 13. Let $\mathbb{T}$ be a family of trees. We say that a set B is a $\mathbb{T}$-Bernstein set if for every $T \in \mathbb{T}, B \cap [T] \neq \emptyset$ and $B^c \cap [T] \neq \emptyset$.

Observe that each classical Bernstein set is an $\mathbb{S}$-Bernstein set. If $\mathbb{T} \subseteq \mathbb{T}'$ are families of trees, then $\mathbb{T}'$-Bernstein sets are $\mathbb{T}$-Bernstein sets. No $\mathbb{T}$-Bernstein set is in t_0 (or t-measurable), and if $\mathbb{T} \subseteq \mathbb{T}'$, then $\mathbb{T}'$-Bernstein sets do not belong to t_0. Also note that if $\mathbb{T} \subsetneq \mathbb{T}'$, then a $\mathbb{T}$-Bernstein set may not be a $\mathbb{T}'$-Bernstein set (e.g., one may fix a tree from $\mathbb{T}' \setminus \mathbb{T}$ whose body will be always omitted).

The following theorem slightly generalizes Theorems 2.1 and 2.2 from [1].

Theorem 14. *The following statements are true:*

(i) *There exists an $\mathbb{L}$-Bernstein set which belongs to m_0.*
(ii) *There exists an $\mathbb{M}$-Bernstein set which belongs to s_0.*

Proof. As in in the proof of Theorem 9, we will use the notions established in [1]. To prove (i), let us enumerate all Laver trees $\{L_\alpha : \alpha < \mathfrak{c}\}$ and all apple trees $\{A_\alpha : \alpha < \mathfrak{c}\}$. Let us construct two sequences: $(b_\alpha)_{\alpha<\mathfrak{c}}$ and $(x_\alpha)_{\alpha<\mathfrak{c}}$ such that for each $\alpha < \mathfrak{c}$:

$$b_\alpha \in [L_\alpha] \setminus (\bigcup_{\beta<\alpha} [A_\beta] \cup \{x_\xi : \xi < \alpha\}),$$
$$x_\alpha \in [L_\alpha] \setminus (\{b_\beta : \beta \leq \alpha\} \cup \{x_\beta : \beta < \alpha\}).$$

It can be done, since for each Laver tree L_α, there is a pear tree P_{L_α} for which $|[P_{L_\alpha}] \cap [A]| \leq 1$ for every apple tree A, so the set $[L_\alpha] \setminus (\bigcup_{\beta<\alpha} [A_\beta] \cup \{x_\xi : \xi <$

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$\alpha\}$) is nonempty at each step α. We will show that $B = \{b_\alpha : \alpha < \mathfrak{c}\}$ is the desired set. Let T be a Laver tree. Then $T = L_\alpha$ for some $\alpha < \mathfrak{c}$. Notice that the $b_\alpha \in B \cap [T]$ and $x_\alpha \in [T] \setminus B$.

To prove (ii), we use a similar modification of Theorem 2.2 from [1]. ⊣

Also let us observe that Theorem 11 yields the following result.

REMARK 15. No $\mathbb{CL}$-Bernstein set belongs to l_0.

Let us invoke the theorem by Miller from [7, Theorem 3].

THEOREM 16 (Miller). *Let A be an analytic subset of ω^ω. Then either A contains body of some complete Laver tree or A^c contains a body of some complete Hechler tree.*

Let $\mathcal{B}$ denote the family of Borel subsets of ω^ω.

THEOREM 17. *Let $(\mathbb{T}, t_0) \in \{(\mathbb{S}, s_0), (\mathbb{M}, m_0), (\mathbb{L}, l_0), (\mathbb{CL}, cl_0)\}$. Then $\mathcal{B} \cap t_0$ is the family of Borel sets, which do not contain any body of any tree from $\mathbb{T}$.*

PROOF. Case of $t_0 = s_0$ is evident since Borel sets have the perfect set property. Let $t_0 = m_0$. Let B be a Borel set. If B contains a body of a Miller tree, then clearly it is not in m_0. On the other hand, if B does not contain a body of any Miller tree, then Saint-Raymond Theorem (see [2, Corollary 21.23]) implies that B is σ-bounded, hence $B \in m_0$.

Let $t_0 = l_0$. Let B be a Borel set. Similarly to the previous case, if B contains a body of some Laver tree, then $B \notin l_0$. Conversely, suppose a contrario that B does not contain any body of any Laver tree, but there is a Laver tree L such that $[L'] \cap B \neq \emptyset$ for every Laver tree $L' \subseteq L$. Let us trim B and L in the following way:

$$B' = \{x \in \omega^\omega : \operatorname{stem}(L)^\frown x \in B\},$$
$$L' = \{\tau \in \omega^{<\omega} : \operatorname{stem}(L)^\frown \tau \in L\}.$$

The function $f : \omega^\omega \to \omega^\omega$ given by the formula $f(x) = \operatorname{stem}(L)^\frown x$ is continuous. Clearly, $B' = f^{-1}[B]$, hence B' is Borel, and $[L'] = f^{-1}[[L]]$ is a body of a complete Laver tree L'. B' still does not contain any body of any Laver tree, so by Theorem 16, there is a Hechler tree H body of which is contained in B'^c. $H \cap L'$ contains (in fact—is) a Laver tree, body of which B' should intersect—a contradiction. The case of $t_0 = cl_0$ is almost identical to the previous one. ⊣

REMARK 18. h_0 and ch_0 lack such a characterization.

PROOF. For the proof of the ch_0 case, let T be a complete Laver tree which is not Hechler. Then $[T] \cap [T_{CH}]$ is a body of a complete Laver tree for every complete Hechler tree T_{CH}, hence $[T] \notin ch_0$. Clearly, $[T]$ does not contain any body of any complete Hechler tree.

For the proof of the h_0 case, let us define a sequence $(C_n : n \in \omega)$ of subsets of ω^ω in the following way

$$C_n = \{x \in \omega^\omega : (\forall k \geq n)(x(k) \in 2\mathbb{N})\}.$$

For each $n \in \omega$, the set C_n is a body of a complete Laver tree. Let $C = \bigcup_{n\in\omega} C_n$. We claim that $[H] \nsubseteq C$ for any Hechler tree H. Consider a set

$$C' = \{x \in \omega^\omega : x \restriction |\text{stem}(H)| = \text{stem}(H) \wedge (\forall k \geq |\text{stem}(H)|)(x(k) \in 2\mathbb{N}+1)\}.$$

$C' \cap C = \emptyset$ and $C' \cap [H]$ is a body of a Laver tree, hence $[H] \nsubseteq C$. Furthermore, $C \notin h_0$. Indeed, let H be a Hechler tree satisfying $[H] \cap C = \emptyset$. Then $[H] \cap C_n = \emptyset$ for every $n \in \omega$, which implies that for each natural n, we have $\text{stem}(H) > n$, a contradiction. ⊣

There is a relation between $\mathbb{T}$-Bernstein sets and the trace of t_0 on $\mathcal{B}$. Before we discuss it, let us recall some notions. Let $\mathcal{I} \subseteq P(\omega^\omega)$ be a σ-ideal with a Borel base, that is, for every set $A \in \mathcal{I}$, there exists a Borel set $B \in \mathcal{B} \cap \mathcal{I}$ containing A, and let $\sigma(\mathcal{B} \cup \mathcal{I}) = \{B \triangle A : B \in \mathcal{B} \wedge A \in \mathcal{I}\}$ denote the σ-field generated by Borel sets and sets from $\mathcal{I}$.

Definition 19. We say that a set A is

- $\mathcal{I}$-nonmeasurable if $A \notin \sigma(\mathcal{B} \cup \mathcal{I})$;
- completely $\mathcal{I}$-nonmeasurable if $A \cap B$ is $\mathcal{I}$-nonmeasurable for each Borel set $B \notin \mathcal{I}$.

The equivalent (and more useful) formulation of the complete $\mathcal{I}$-nonmeasurability is this: A intersects each, but does not contain any, $\mathcal{I}$-positive Borel set B. Clearly, if A is completely $\mathcal{I}$-nonmeasurable and $B \in \mathcal{B} \backslash \mathcal{I}$, then $A \cap B \neq \emptyset$ and $B \nsubseteq A$. Conversely, if A is not completely $\mathcal{I}$-nonmeasurable, then there exists an $\mathcal{I}$-positive Borel set B such that $A \cap B$ is $\mathcal{I}$-measurable. It implies that there is a Borel $\mathcal{I}$-positive set $B' \subseteq B$ such that $B' \subseteq A$ or $B' \cap A = \emptyset$.

Corollary 20. *Let $(\mathbb{T}, t_0) \in \{(\mathbb{S}, s_0), (\mathbb{M}, m_0), (\mathbb{L}, l_0), (\mathbb{CL}, cl_0)\}$. Then a set B is $\mathbb{T}$-Bernstein if and only if it is completely $t_0 \restriction \mathcal{B}$-nonmeasurable, where $t_0 \restriction \mathcal{B}$ is a σ-ideal generated by $t_0 \cap \mathcal{B}$.*

Proof. By Theorem 17, a set A is $t_0 \restriction \mathcal{B}$-positive Borel set if and only if it contains a body of a tree from $\mathbb{T}$. Hence, B is $\mathbb{T}$-Bernstein if and only if it intersects each, but does not contain any, $t_0 \restriction \mathcal{B}$-positive Borel set. ⊣

§3. $\mathcal{I}$-Luzin sets and algebraic properties.

Let us recall the notion of $\mathcal{I}$-Luzin sets (see [6]). Let X be a Polish space and $\mathcal{I}$ be an ideal.

Definition 21. We say that a set L is an $\mathcal{I}$-Luzin set, if $(\forall A \in \mathcal{I})(|A \cap L| < |L|)$.

For the classic ideals of Lebesgue measure zero sets $\mathcal{N}$ and meager sets $\mathcal{M}$, we will call $\mathcal{M}$-Luzin sets generalized Luzin sets and $\mathcal{N}$-Luzin sets generalized Sierpiński sets.

We will consider $\mathcal{I}$-Luzin sets in the context of algebraic properties and tree ideals. We will work on the real line $\mathbb{R}$ with the standard addition. Since $\mathbb{R}$ is σ-compact, it does not contain even superperfect sets. We will tweak the definition a bit by saying that $A \subseteq \mathbb{R}$ belongs to t_0 if $h^{-1}[A]$ belongs to t_0 in ω^ω, where h is a homeomorphism between ω^ω and the subspace of irrational numbers (see [15] for the similar modification in the case of 2^ω). Having this in mind, we will usually mean by $[\tau]$, $\tau \in \omega^{<\omega}$, an open interval with rational endpoints on $\mathbb{R}$.

Before we proceed, let us define a nonstandard kind of fusion of Miller and Laver trees that we will use later. Let T be a Miller tree. Let $\tau_\emptyset \in \omega\text{-split}(T)$ and let T_0 be any Miller subtree of T such that $\tau_\emptyset$ remains an infinitely splitting node in T_0. Suppose we have a Miller subtree T_n and a set of nodes $B_n = \{\tau_\sigma : \sigma \in n^{\leq n}\}$ such that

(i) $\tau_\sigma \in \omega\text{-split}(T_n)$ for every $\sigma \in n^{\leq n}$;
(ii) $\tau_{\sigma^\frown k} \supseteq \tau_\sigma$ for every $k < n$ and $\sigma \in n^{<n}$;
(iii) $\tau_{\sigma^\frown k} \cap \tau_{\sigma^\frown j} = \tau_\sigma$ for every $\sigma \in n^{<n}$ and distinct $k, j < n$.

We extend the set of nodes B_n to $B_{n+1} = \{\tau_\sigma : \sigma \in (n+1)^{\leq n+1}\}$ in a way that preserves above conditions, so we will have $n+1$ levels of infinitely splitting nodes with fixed $n+1$ splits. The only $\sigma \in (n+1)^0$ is $\emptyset$, and $\tau_\emptyset$ is an old node. It is ω-splitting in T_n and T_n is a Miller tree, so we may find $\tau_n \supseteq \tau_\emptyset$, which is ω-splitting and $\tau_n \cap \tau_j = \tau_\emptyset$ for $j < n$. If we already have nodes τ_σ with desired properties for $\sigma \in (n+1)^{\leq k}$, $k < n+1$, then for τ_σ, $\sigma \in n^k$ (old node), we add $\tau_{\sigma^\frown n}$ such that conditions (i)–(iii) are still met. For a new node τ_σ, $\sigma \in (n+1)^k \setminus n^k$, we find $\tau_{\sigma^\frown j}$ for each $j < n+1$ such that conditions (i)–(iii) are satisfied too. Then let T_{n+1} be any Miller subtree of T_n for which nodes from B_{n+1} are still infinitely splitting.

We will call a sequence of trees $(T_n)_{n\in\omega}$ (or, interchangeably, their bodies $[T_n]$) derived that way a Miller fusion sequence. Similarly, we define a Laver fusion sequence. The only difference would be that if $\tau_\sigma \subseteq \tau_{\sigma^\frown k}$, then actually $\tau_{\sigma^\frown k} = \tau_\sigma{}^\frown j$ for some $j \in \omega$.

We have the following fact regarding fusion sequences of Miller or Laver trees.

Proposition 22. *For every Miller (resp. Laver) fusion sequence $(T_n)_{n\in\omega}$, the set $\bigcap_{n\in\omega} T_n$ is a Miller (resp. Laver) tree.*

Lemma 23. *For every sequence of intervals $(I_n)_{n\in\omega}$ and a Miller (resp. Laver) tree T, there is a Miller (resp. Laver) fusion sequence $(T_n)_{n\in\omega}$ such that for all $n > 0$*

$$\lambda([T_n] + I_n) < (1 + \sum_{k=0}^{n-1} (n-1)^k)\lambda(I_n).$$

Proof. Let us focus on the slightly more complicated "Miller" case. Let I_0 be an interval, $\varepsilon_0 = \lambda(I_0)$, and let T be a Miller tree. We proceed by induction on n. Let $\tau_\emptyset \in \omega\text{-split}(T)$ such that $\lambda([\tau_\emptyset]) < \varepsilon_0$. Then $\lambda([\tau_\emptyset] + I_0) = \lambda([\tau_\emptyset]) + \lambda(I_0) < 2\varepsilon_0$. Let T_0 be a Miller subtree of T such that $\tau_\emptyset = \operatorname{stem}(T_0)$ and $\tau_\emptyset \in \omega\text{-split}(T_0)$. Clearly, we have $\lambda([T_0] + I_0) < 2\varepsilon_0$.

Now assume that we have a tree T_n that is a member of the emerging Miller fusion sequence. Denote by B_n, associated with T_n set of nodes satisfying conditions (i)–(iii). Let $\varepsilon_{n+1} = \lambda(I_{n+1})$. Let us define for each $\sigma \in \omega^{<\omega}$ and

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interval I a set

$$N_\sigma(I) = \{\tau_\sigma{}^\frown k \in T_n : [\tau_\sigma{}^\frown k] \subseteq I \wedge (\forall j < n)(\tau_{\sigma^\frown j} \not\supseteq \tau_\sigma{}^\frown k)\}.$$

Observe that for each $\sigma \neq \emptyset$ and $d > 0$, there is an interval I satisfying $\lambda(I) < d$ and $|N_\sigma(I)| = \aleph_0$ since $\tau_\sigma \in \omega\text{-split}(T_n)$ and $[\tau_\sigma]$ is a bounded interval which contains $[\tau_\sigma{}^\frown k]$ for infinitely many $k \in \omega$. At each level $k < n$ for every $\sigma \in n^k$, let I_σ be an interval with $\lambda(I_\sigma) < \frac{\varepsilon_{n+1}}{(n+1)^n}$ such that the set $N_\sigma(I_\sigma)$ is infinite and choose $\tau_{\sigma^\frown n} \in \omega\text{-split}(T_n)$ such that $\tau_{\sigma^\frown n} \supseteq \tau_\sigma{}^\frown l$ for some $\tau_\sigma{}^\frown l \in N_\sigma(I_\sigma)$. At the level n, let us fix for every $\sigma \in n^n$ an interval I_σ satisfying $\lambda(I_\sigma) < \frac{\varepsilon_{n+1}}{(n+1)^n}$ such that the set $N_\sigma(I_\sigma)$ is infinite and pick $\tau_{\sigma^\frown 0}, \tau_{\sigma^\frown 1}, \dots, \tau_{\sigma^\frown n}$, which are extensions of some nodes $\tau_\sigma{}^\frown k_0, \tau_\sigma{}^\frown k_1, \dots, \tau_\sigma{}^\frown k_n \in N_\sigma(I_\sigma)$, respectively. Finally, we pick the remaining nodes to complete the set B_{n+1} according to the definition of a Miller fusion sequence however we like. We take as T_{n+1} any Miller subtree of T_n whose nodes from B_{n+1} are infinitely splitting and whose body is covered by intervals I_σ, $\sigma \in n^{\leq n}$ (which is possible since each $N_\sigma(I_\sigma)$ is infinite). Let us approximate $\lambda([T_{n+1}] + I_{n+1})$:

$$\lambda([T_{n+1}] + I_{n+1}) \leq \lambda\left(\bigcup\{I_\sigma + I_{n+1} : \sigma \in n^{\leq n}\}\right) \leq \sum_{\sigma \in n^{\leq n}} (\lambda(I_\sigma) + \lambda(I_{n+1}))$$
$$< \sum_{\sigma \in n^{\leq n}} (\varepsilon_{n+1}(n+1)^n + \varepsilon_{n+1}).$$

Since the count of intervals I_σ is $|n^{\leq n}| = \sum_{k=0}^{n} n^k \leq (n+1)^n$, we have

$$\lambda([T_{n+1}] + I_{n+1}) \leq \sum_{k=0}^{n} n^k(\varepsilon_{n+1}(n+1)^n + \varepsilon_{n+1}) \leq (n+1)^n \varepsilon_{n+1}(n+1)^n$$
$$+ \sum_{k=0}^{n} n^k \varepsilon_{n+1} = \varepsilon_{n+1} + \sum_{k=0}^{n} n^k \varepsilon_{n+1} = \left(1 + \sum_{k=0}^{n} n^k\right)\varepsilon_{n+1}. \quad \dashv$$

REMARK 24. In the above lemma in the case of a Laver tree, we may demand that $\mathrm{stem}(T) = \mathrm{stem}(\bigcap_{n\in\omega} T_n)$, if $\mathrm{stem}(T)$ is nonempty.

PROOF. The major difference is at the first step of the induction. Instead of picking a suitable "far enough" node $\tau_\emptyset \in T$ such that $\lambda([\tau_\emptyset] + I_0) < 2\lambda(I_0)$, we already restrict the choice of nodes at the stem level by picking an interval $I_\emptyset$ of measure $\lambda(I_\emptyset) < \lambda(I_0)$ such that a set

$$N_{\mathrm{stem}(T)}(I_\emptyset) = \{\mathrm{stem}(T)^\frown k \in T : [\mathrm{stem}(T)^\frown k] \subseteq I_\emptyset\}$$

is infinite. It can be done since $\mathrm{stem}(T) \neq \emptyset$, so all clopen sets $[\mathrm{stem}(T)^\frown k]$, $k \in \omega$, are contained in an interval. We take a Laver subtree T_0 of T for which $[T] \subseteq I_\emptyset$ and $\mathrm{stem}(T) = \mathrm{stem}(T_0)$ (so all nodes extending $\mathrm{stem}(T_0)$ come from $I_\emptyset$). Then we continue analogously to the proof of Lemma 23. $\dashv$

LEMMA 25. *There exists a dense G_δ set G such that for each Miller (resp. Laver or complete Laver) tree T, there exists a Miller (resp. Laver or complete Laver) subtree $T' \subseteq T$ such that $G + [T'] \in \mathcal{N}$.*

PROOF. Let $D = \{d_n : n \in \omega\}$ be a countable dense set, $G = \bigcap_{n\in\omega}\bigcup_{k>n} I_k$, where I_k is an interval with a center d_k and $\lambda(I_k) < 1(k)^{k-1}2^k$. The proofs are almost identical in the cases of Miller and Laver trees, so without loss of generality let us focus on the "Miller" case. Let T be a Miller tree. By Lemma 23, there is a Miller fusion sequence $(T_n)_{n\in\omega}$ such that

$$\lambda([T_n]+I_n) < (1+\sum_{k=0}^{n-1}(n-1)^k)\lambda(I_n) \leq n^{n-1}\frac{1}{n^{n-1}2^n} = \frac{1}{2^n}.$$

$T' = \bigcap_{n\in\omega} T_n$ is a Miller tree contained in all T_n's, so we may replace $[T_n]$ with $[T']$ in the above formula and it still holds. Then for a fixed $n \in \omega$,

$$\lambda(\bigcup_{k>n} I_k + [T']) = \lambda(\bigcup_{k>n}([T']+I_k)) \leq \sum_{k>n}\lambda([T']+I_k) \leq \sum_{k>n}\frac{1}{2^k} = \frac{1}{2^n},$$

so, given that $[T'] + \bigcap_{n\in\omega}\bigcup_{k>n} I_k \subseteq \bigcap_{n\in\omega}\bigcup_{k>n}([T']+I_k)$, we have

$$\lambda(G+[T']) \leq \lambda(\bigcap_{n\in\omega}\bigcup_{k>n}([T']+I_k)) \leq \lim_{n\to\infty}\frac{1}{2^n} = 0.$$

In the case of a complete Laver tree T, let us observe that $T = \bigcup_{n\in\omega} T_n$, where for each $n \in \omega$, the set $T_n = \{\sigma \in T : (n) \subseteq \sigma \vee \sigma \subseteq (n)\}$ is a Laver tree with a nonempty stem. Let us notice that $[T] = \bigcup_{n\in\omega}[T_n]$. By Lemma 23, Remark 24, and the first part of the proof, we find for each (nonempty) T_n, a Laver subtree T_n' which shares the stem with T_n, for which we have

$$[T_n'] + G \in \mathcal{N}.$$

Then $T' = \bigcup_{n\in\omega} T_n'$ is a complete Laver subtree of T and

$$[T'] + G = [\bigcup_{n\in\omega} T_n'] + G = \bigcup_{n\in\omega}[T_n'] + G = \bigcup_{n\in\omega}([T_n'] + G) \in \mathcal{N}$$

as a countable union of null sets. ⊣

Before we proceed to the main theorem of this section, let us recall a generalized version of Rothberger's theorem (see [11]).

THEOREM 26 (Essentially Rothberger). *Assume that L is a generalized Luzin set, S is a generalized Sierpiński set, and $\kappa = \max\{|L|,|S|\}$ is a regular cardinal. Then $|L| = |S| = \kappa$.*

PROOF. Assume that $\kappa = |L| > |S|$ and κ is a regular cardinal. Let M be a meager set of full measure (the Marczewski decomposition, see [4]). Then

$$\kappa = |L\cap\mathbb{R}| = |L\cap(M+S)| = |\bigcup_{s\in S}(L\cap(M+s))| < \kappa,$$

by regularity of κ. In the case of $\kappa = |S| > |L|$, the proof is almost the same. ⊣

The following theorem extends the result obtained in [6, Theorem 2.12].

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Theorem 27. *Let $\mathfrak{c}$ be a regular cardinal and $t_0 \in \{s_0, m_0, l_0, cl_0\}$. Then for every generalized Luzin set L and generalized Sierpiński set S, we have $L + S \in t_0$.*

Proof. Let L and S be a generalized Luzin set and generalized Sierpiński set, respectively. If $|L| < \mathfrak{c}$ and $|S| < \mathfrak{c}$, then $L + S \in t_0$ because every set of cardinality less than $\mathfrak{c}$ belongs to t_0. Hence, without loss of generality (Theorem 26), let us assume that $|L| = |S| = \mathfrak{c}$.

We will proceed with the proof in the case $t_0 = m_0$, the other cases are almost identical. Let T be a Miller tree. By the virtue of Lemma 25, let G be a dense G_δ set and $T' \subseteq T$ a Miller tree such that $[T'] + G \in \mathcal{N}$. Let $A = -G$ and $B = ([T'] + G)^c$. Then $[T'] \subseteq (A + B)^c$. We will show that there is a Miller tree $T'' \subseteq T'$ body of which is contained in $(L + S)^c$. We have

$$\begin{aligned} L + S &= ((L \cap A) \cup (L \cap A^c)) + ((S \cap B) \cup (S \cap B^c)) \\ &= ((L \cap A) + (S \cap B)) \cup ((L \cap A) + (S \cap B^c)) \\ &\quad \cup ((L \cap A^c) + (S \cap B)) \cup ((L \cap A^c) + (S \cap B^c)). \end{aligned}$$

$(L \cap A) + (S \cap B) \subseteq A + B$ and sets $(L \cap A) + (S \cap B^c)$, $(L \cap A^c) + (S \cap B)$, and $(L \cap A^c) + (S \cap B^c)$ are generalized Luzin, generalized Sierpiński and of cardinality less than $\mathfrak{c}$; therefore, their intersection with $[T']$ has cardinality less than $\mathfrak{c}$. It follows that indeed there exists a Miller tree $T'' \subseteq T'$ such that $(L + S) \cap [T''] = \emptyset$, hence $L + S$ belongs to m_0. $\dashv$

Let us remark that the assumption that $\mathfrak{c}$ is regular cannot be omitted due to the following result [6, Theorem 2.13].

Theorem 28. *It is consistent that there exist generalized Luzin set L and generalized Sierpiński set S such that $L + S = \mathbb{R}^n$, and $\mathfrak{c} = \aleph_{\omega_1}$.*

§4. Eventually different families and t-measurablity. Two members $f, g \in \omega^\omega$ of the Baire space are *eventually different* (briefly: e.d.), if $f \cap g$ is a finite subset of $\omega \times \omega$. Maximal eventually different families with respect to inclusion are called *m.e.d. families*.

Every e.d. family is a meager subset of the Baire space. It is natural to ask whether the existence of m.e.d. families that are either s-measurable or s-nonmeasurable can be proven in ZFC. It is relatively consistent with ZFC that there is a m.e.d. family $\mathcal{A}$ of cardinality less than $\mathfrak{c}$ (see [3]). In such a case, $\mathcal{A} \in s_0$. On the other hand, there exists a perfect e.d. family; therefore, not all m.e.d. families are in s_0. The following two theorems answer this question positively.

Theorem 29. *There exists an s-nonmeasurable m.e.d. family.*

Proof. Let us fix a perfect tree $T \subseteq \omega^{<\omega}$ such that $[T]$ is e.d. in ω^ω. Let $\{T_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathbb{S}(T)$ of all perfect subtrees of T. By the transfinite recursion, we define

$$\{(a_\alpha, d_\alpha, x_\alpha) \in [T] \times [T] \times \omega^\omega : \alpha < \mathfrak{c}\}$$

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such that for any $\alpha < \mathfrak{c}$, we have:

(1) $a_\alpha, d_\alpha \in [T_\alpha]$;
(2) $\{a_\xi : \xi < \alpha\} \cap \{d_\xi : \xi < \alpha\} = \emptyset$;
(3) $\{a_\xi : \xi < \alpha\} \cup \{x_\xi : \xi < \alpha\}$ is e.d.;
(4) $\forall^\infty n\, x_\alpha(n) = d_\alpha(n)$ but $x_\alpha \neq d_\alpha$.

Assume that we are at the step $\alpha < \mathfrak{c}$ of the construction and we have already defined the sequence

$$\{(a_\xi, d_\xi, x_\xi) \in [T]^2 \times \omega^\omega : \xi < \alpha\}.$$

We can choose $a_\alpha, d_\alpha \in [T_\alpha]$ ($[T_\alpha]$ has cardinality $\mathfrak{c}$) which fulfills conditions (1), (2). Then choose any $x_\alpha \in \omega^\omega$ distinct from d_α but $(\forall^\infty n) d_\alpha(n) = x_\alpha(n)$. Then $x_\alpha \in \omega^\omega \setminus [T]$ and

$$\{a_\xi : \xi \leq \alpha\} \cup \{x_\xi : \xi \leq \alpha\}$$

forms an e.d. family in ω^ω. This completes the construction.

Now, let us extend the set $\{a_\alpha : \alpha < \mathfrak{c}\} \cup \{x_\alpha : \alpha < \mathfrak{c}\}$ to m.e.d. family A. It is easy to check that A is the desired s-nonmeasurable m.e.d. family. $\dashv$

In [9, Theorem 2.2] it was shown that if $\mathfrak{d} = \omega_1$, then there exists a s-nonmeasurable m.e.d. family $\mathcal{A}$ with a dominating subfamily $\mathcal{A}' \in [\mathcal{A}]^{\omega_1}$. Here, s-nonmeasurability can be replaced by l-, m-, or cl-nonmeasurability.

In the same paper, it was proved that the following statement is relatively consistent with ZFC: "$\omega_1 < \mathfrak{d}$ and there exists cl-nonmeasurable m.e.d. family $\mathcal{A}$ with a dominating subfamily $\mathcal{A}' \subseteq \mathcal{A}$ of cardinality $\mathfrak{d}$."

The next theorem generalizes the result obtained in [8, Theorem 4.2].

THEOREM 30. *There exists a m.e.d. family $\mathcal{A} \subseteq \omega^\omega$ such that $\mathcal{A}$ is not s-, l-, and m-measurable, with a dominating subfamily $\mathcal{D} \in [\mathcal{A}]^{\leq \mathfrak{d}}$.*

PROOF. Let $\mathcal{D}_0$ be a dominating family of cardinality $\mathfrak{d}$. We will show that there is an e.d. dominating family $\mathcal{D}$ of the same cardinality. Let $\mathcal{P} = \{A_m \in [\omega]^\omega : m \in \omega\}$ be a partition of ω into infinite subsets with

$$A_m = \{k_{m,i} : i \in \omega\}, \qquad k_{m,0} < k_{m,1} < \cdots$$

Let us construct a tree T in the following way. Set

$$T_0 = \{\emptyset\},$$
$$T_1 = \{(0, n) : n \in \omega\}.$$

Fix $n \in \omega$ and assume that we have defined $T_n \subseteq \omega^n$. Let $T_n = \{\sigma_m^n : m \in \omega\}$. Define

$$T_{n+1,m} = \{\sigma_m^n \cup \{(n, k_{m,i})\} : i \in \omega\}$$

and $T_{n+1} = \bigcup_{m \in \omega} T_{n+1,m}$. Finally, set $T = \bigcup_{n \in \omega} T_n$. Clearly, $[T]$ is an e.d. family.

Now let us define an embedding $f : \mathcal{D}_0 \to [T]$ as follows. Fix $d \in \mathcal{D}_0$. Define

$$f(d)(0) = d(0),$$

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$$f(d)(n) = k_{m,d(n)}, \text{ where } f(d) \restriction n = \sigma_m^n.$$

Clearly, f is well defined. Indeed, $f(d) \restriction 1 \in T_1$ and if $f(d) \restriction n = \sigma_m^n \in T_n$ then $f(d) \restriction n+1 = f(d) \restriction n^\frown f(d)(n) = \sigma_m^n \cup \{(n, k_{m,d(n)})\} \in T_{n+1,m} \subseteq T_{n+1}$.

Notice that f is injective and $d \leq f(d)$ for every $d \in \mathcal{D}_0$. Now set

$$\mathcal{D} = \{4f(d) : d \in \mathcal{D}_0\} \subseteq (4\mathbb{N})^\omega.$$

It is a dominating family in ω^ω of cardinality $|\mathcal{D}_0| = \mathfrak{d}$.

Now let us choose e.d. trees $S \subseteq (4\mathbb{N}+1)^{<\omega}$, $M \subseteq (4\mathbb{N}+2)^{<\omega}$, and $L \subseteq (4\mathbb{N}+3)^{<\omega}$, where S is a perfect tree, M is Miller, and L is Laver.

Let $\{S_\alpha : \alpha < \mathfrak{c}\}$ be an enumeration of the family $\mathbb{S}(S)$ of all perfect subtrees of S. Analogously, let $\mathbb{M}(M) = \{M_\alpha : \alpha < \mathfrak{c}\}$ and $\mathbb{L}(L) = \{L_\alpha : \alpha < \mathfrak{c}\}$. By the transfinite recursion, let us define

$$\{w_\alpha \in [[S]]^2 \times \omega^\omega \times [[M]]^2 \times \omega^\omega \times [[L]]^2 \times \omega^\omega : \alpha < \mathfrak{c}\},$$

where $w_\alpha = (a_\alpha^s, d_\alpha^s, x_\alpha^s, a_\alpha^m, d_\alpha^m, x_\alpha^m, a_\alpha^l, d_\alpha^l, x_\alpha^l)$ and for every $\alpha < \mathfrak{c}$:

1. $a_\alpha^s, d_\alpha^s \in [S_\alpha]$;
2. $\{a_\xi^s : \xi < \alpha\} \cap \{d_\xi^s : \xi < \alpha\} = \emptyset$;
3. $\{a_\xi^s : \xi < \alpha\} \cup \{x_\xi^s : \xi < \alpha\}$ is e.d.;
4. $\forall^\infty n\ x_\alpha^s(n) = d_\alpha^s(n)$ but $x_\alpha^s \neq d_\alpha^s$;
5. $a_\alpha^m, d_\alpha^m \in [M_\alpha]$;
6. $\{a_\xi^m : \xi < \alpha\} \cap \{d_\xi^m : \xi < \alpha\} = \emptyset$;
7. $\{a_\xi^m : \xi < \alpha\} \cup \{x_\xi^m : \xi < \alpha\}$ is e.d.;
8. $\forall^\infty n\ x_\alpha^m(n) = d_\alpha^m(n)$ but $x_\alpha^m \neq d_\alpha^m$;
9. $a_\alpha^l, d_\alpha^l \in [L_\alpha]$;
10. $\{a_\xi^l : \xi < \alpha\} \cap \{d_\xi^l : \xi < \alpha\} = \emptyset$;
11. $\{a_\xi^l : \xi < \alpha\} \cup \{x_\xi^l : \xi < \alpha\}$ is e.d.;
12. $\forall^\infty n\ x_\alpha^l(n) = d_\alpha^l(n)$ but $x_\alpha^l \neq d_\alpha^l$.

Now assume that we are at the step $\alpha < \mathfrak{c}$ of the construction and we have a partial sequence

$$(w_\xi : \xi < \alpha),$$

which has a length at most $\omega \cdot |\alpha| < \mathfrak{c}$. The construction of w_α is similar to the construction of $(a_\alpha, d_\alpha, x_\alpha)$ in Theorem 29.

Now let us set:

$$\mathcal{A}_s = \{a_\alpha^s : \alpha < \mathfrak{c}\} \cup \{x_\alpha^s : \alpha < \mathfrak{c}\},$$
$$\mathcal{A}_m = \{a_\alpha^m : \alpha < \mathfrak{c}\} \cup \{x_\alpha^m : \alpha < \mathfrak{c}\},$$
$$\mathcal{A}_l = \{a_\alpha^l : \alpha < \mathfrak{c}\} \cup \{x_\alpha^l : \alpha < \mathfrak{c}\}.$$

Notice that $\mathcal{D} \cup \mathcal{A}_s \cup \mathcal{A}_m \cup \mathcal{A}_l$ forms an e.d. family. Let $\mathcal{A}$ be any m.e.d. family containing $\mathcal{D} \cup \mathcal{A}_s \cup \mathcal{A}_m \cup \mathcal{A}_l$.

Clearly, $\mathcal{A}$ contains $\mathcal{D}$, which is a dominating family of cardinality $\mathfrak{d}$.

Notice that $\mathcal{A}$ is s-nonmeasurable. Indeed, every perfect subset of $[S]$ is of the form $[S_\alpha]$ for some $\alpha < \mathfrak{c}$. By condition (1) of construction $a_\alpha^s \in \mathcal{A} \cap [S_\alpha]$. On the other hand (by conditions (2) and (4)), $d_\alpha^s \in [S_\alpha] \setminus \mathcal{A}$.

Similarly, we prove that $\mathcal{A}$ is m- and l-nonmeasurable, which completes the proof. $\dashv$

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