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ADDING HANDLES TO RIEMANN'S MINIMAL SURFACES

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Abstract We construct embedded periodic minimal surfaces with planar ends. We use the Weierstrass representation and solve the period problem using the implicit function theorem at a singular point.

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1. Introduction

In this paper we consider embedded, complete, simply periodic minimal surfaces in Euclidean space $\mathbb{R}^3$, with horizontal planar ends and finite topology in the quotient by the period. We call them simply periodic minimal surfaces with horizontal planar ends.

The classical example is the family of Riemann examples. These surfaces may be imagined as a periodic set of horizontal equidistant planes with one neck between each plane. Wei constructed a similar family where the number of necks is alternately 1 and 2. The parameter for these families is the period $\mathcal{T}$, a non-horizontal vector. When the period $\mathcal{T}$ becomes horizontal, these surfaces degenerate. The degenerate surface may be seen as horizontal planes with infinitesimally small necks between them. The goal in this paper is to start from such a degenerate situation and recover the family of minimal surfaces. A necessary condition for the existence of the family is that the infinitesimal necks satisfy a *balancing* condition (see Theorem 1.3). This is also sufficient up to a non-degeneracy hypothesis (see Theorem 1.4).

To state our results we need some definitions. Let $\{M_t\}$, $t > 0$, be a family of simply periodic minimal surfaces with horizontal planar ends and period $\mathcal{T}_t$. We may order the ends of M_t by their height and label them ∞_k, $k \in \mathbb{Z}$. Our hypotheses are as follows.

Hypothesis 1.1 (planar domains and necks). *The number N of ends of the quotient $M_t/\mathcal{T}_t$ does not depend on t (N is even). There exists positive integers n_k, $k \in \mathbb{Z}$, such that $n_{k+N} = n_k$, and a covering of M_t by domains $\Omega_{k,t}$ and $U_{k,i,t}$, $k \in \mathbb{Z}$, $i = 1, \dots, n_k$, such that $\Omega_{k+N,t} = \Omega_{k,t} + \mathcal{T}_t$, $U_{k+N,i,t} = U_{k,i,t} + \mathcal{T}_t$, and*

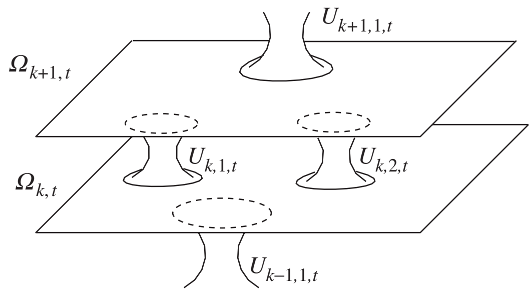

Figure 1. Planar domains and necks.

(a) *for all k, $\Omega_{k,t}$ is a graph over a domain in the horizontal plane, and contains the end ∞_k; and*

(b) *for all k, i, $U_{k,i,t}$ is conformally an annulus whose two boundary components lie in $\Omega_{k,t}$ and $\Omega_{k+1,t}$. The Gauss map is one to one in $U_{k,i,t}$.*

We call $\Omega_{k,t}$ a planar domain and $U_{k,i,t}$ a neck between $\Omega_{k,t}$ and $\Omega_{k+1,t}$. n_k is the number of necks between $\Omega_{k,t}$ and $\Omega_{k+1,t}$.

Hypothesis 1.2 (asymptotic behaviour when $t \to 0$).

(a) *For all k, the Gauss map converges on $\Omega_{k,t}$ to a vertical vector when $t \to 0$.*

(b) *For all k, i, $U_{k,i,t}$ is contained in a Euclidean ball whose radius goes to zero and whose centre converges to a point $p_{k,i}$ in the horizontal plane $x_3 = 0$. This implies that $T = \lim \mathcal{T}_t$ exists, T is a horizontal vector and $p_{k+N,i} = p_{k,i} + T$. Moreover, for each k, we assume that the points $p_{k,i}$, $p_{k+1,j}$, $i = 1, \dots, n_k$, $j = 1, \dots, n_{k+1}$, are distinct.*

(c) *We may rescale M_t so that for any k, i, the* necksize *of $U_{k,i,t}$ has a non-zero finite limit when $t \to 0$, where the necksize of $U_{k,i,t}$ is the vertical component of the flux of $U_{k,i,t}$.*

Recall that the flux of $U_{k,i,t}$ is the integral of the conormal on a circle going around the neck (there is a matter of orientation which is clearly irrelevant for this hypothesis). For a catenoid the necksize is the length of the waist. It is known (although we will not use it) that under these hypotheses the necks converge (after suitable rescaling) to catenoids, so the necksize is essentially a way to measure the length of the waist of the neck. By Hypothesis 1.2 (b), all necksizes go to 0 when $t \to 0$, so Hypothesis 1.2 (c) is about how fast they go to 0 relative to each other.

1.1. Forces

As we will see, $\{p_{k,i}\}$ must satisfy a balancing condition, which is best explained using forces. Let $\{p_{k,i}\}$, $k \in \mathbb{Z}$, $i = 1, \dots, n_k$, be a periodic set of points in the plane,

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i.e. $p_{k+N,i} = p_{k,i} + T$. Consider the points $p_{k,i}$ as particles in the plane, with charge

$$Q(p_{k,i}) = \frac{(-1)^k}{n_k}.$$

Let $f(p,p')$ be the two-dimensional electrostatic force exerted by p' on p:

$$f(p,p') = Q(p)Q(p')\frac{p-p'}{\|p-p'\|^2}.$$

The force exerted by all other particles on $p_{k,i}$ is defined as

$$F_{k,i} = 2\sum_{j\neq i} f(p_{k,i},p_{k,j}) + \sum_{j=1}^{n_{k+1}} f(p_{k,i},p_{k+1,j}) + \sum_{j=1}^{n_{k-1}} f(p_{k,i},p_{k-1,j}).$$

So $p_{k,i}$ interacts repulsively with the particles $p_{k,j}$—mind the factor 2—and attractively with the particles $p_{k-1,j}$ and $p_{k+1,j}$. We say that the configuration $\{p_{k,i}\}$ is *balanced* if all forces are zero.

Theorem 1.3. *Let $\{M_t\}$, $t>0$ be a family of simply periodic minimal surfaces satisfying Hypotheses 1.1 and 1.2. Then the configuration $\{p_{k,i}\}$ is balanced. Moreover, we have the following geometric information: we may rescale M_t (by a factor going to infinity) so that for all k, i, the necksize of $U_{k,i,t}$ converges to $1/n_k$ when $t\to 0$. We may rescale M_t (by another factor going to infinity) so that for all k, the distance between the asymptotic planes of the ends ∞_k and ∞_{k+1} converges to $1/n_k$.*

Theorem 1.4. *Let $\{p_{k,i}\}$ be a* non-degenerate *balanced configuration. Then for $t>0$ small enough there exists a smooth family M_t of embedded simply periodic minimal surfaces with horizontal planar ends satisfying Hypotheses 1.1 and 1.2. Moreover, this family is unique in the following sense: if M'_t is another family of simply periodic minimal surfaces with the same period $\mathcal{T}_t$ and satisfying Hypotheses 1.1 and 1.2 (with the same numbers n_k and points $p_{k,i}$), then up to a translation, $M'_t = M_t$ for $t>0$ small enough (this may be used to detect symmetries of M_t).*

Here non-degenerate means the following. Let $m = n_1 + \ldots n_N$. Let F (respectively, p) be the vector in $\mathbb{R}^{2m}$ whose components are the $F_{k,i}$ (respectively, $p_{k,i}$) for $k=1,\ldots,N$ and $i=1,\ldots,n_k$. We say that the balanced configuration $\{p_{k,i}\}$ is *non-degenerate* if the differential of the map $p\mapsto F:\mathbb{R}^{2m}\to\mathbb{R}^{2m}$ has rank $2(m-1)$. It cannot have rank $2m$ because from $f(p',p) = -f(p,p')$, one has

$$\forall p,\quad \sum_{k=1}^{N}\sum_{i=1}^{n_k} F_{k,i} = 0.$$

So non-degenerate means that the differential has maximal possible rank. Note that the period T is fixed in this definition.

From the kernel point of view, the forces are clearly invariant under translation of all particles, so the kernel of the differential has dimension at least two. So non-degenerate means that translations are the only infinitesimal deformations of the configuration.

We identify $\mathbb{R}^2$ with $\mathbb{C}$. Then

$$f(p,p') = \frac{Q(p)Q(p')}{\bar{p} - \bar{p}'},$$

so $p \mapsto F : \mathbb{C}^m \to \mathbb{C}^m$ is antiholomorphic. Non-degenerate means that the $m \times m$ complex matrix $\partial F_{k,i}/\partial \bar{p}_{\ell,j}$ has complex rank $m-1$.

1.2. Overview of the paper

In § 2 we give examples and classification results. We prove Theorem 1.4 in §§ 3–7. We use the Weierstrass representation. Recall that given a Riemann surface Σ, a meromorphic function $g : \Sigma \to \mathbb{C} \cup \infty$ (the Gauss map) and a holomorphic differential η on Σ (the height differential), the Weierstrass representation formulae are

$$\phi = (\phi_1, \phi_2, \phi_3) = (\tfrac{1}{2}(g^{-1} - g)\eta, \tfrac{1}{2}\mathrm{i}(g^{-1} + g)\eta, \eta), \tag{1.1}$$

$$\psi(z) = \left(\mathrm{Re}\int_{z_0}^{z} \phi_1,\ \mathrm{Re}\int_{z_0}^{z} \phi_2,\ \mathrm{Re}\int_{z_0}^{z} \phi_3\right), \tag{1.2}$$

where $z \in \Sigma$ and $z_0 \in \Sigma$ is a base point. The problem is that $\psi(z)$ depends on the path of integration; this is usually called the *period problem*.

We define all possible reasonable candidates for the Weierstrass data (Σ, g, η) of the minimal surface we want to construct, depending on some parameters, and then adjust the parameters to solve the period problem. The most important parameter for our construction is a small non-zero real number r. Σ is a sum of Riemann spheres connected by small necks whose 'size' is controlled by the parameter r. We write X for the collection of all other parameters. We write the period problem as a finite set of equations $\mathcal{F}(r, X) = 0$.

When $r = 0$, Σ degenerates into a sum of disjoint Riemann spheres, so the Weierstrass data degenerate into the Weierstrass data of disjoint minimal surfaces. The key point is that the map $\mathcal{F}(r, X)$ extends smoothly to $r = 0$. Moreover, the limit $\mathcal{F}(0, X)$ can be *explicitly* computed.

The equation $\mathcal{F}(0, X) = 0$ boils down to the balancing condition. More specifically, the forces come from the horizontal periods of the Weierstrass data around the necks.

Since we have an explicit formula for $\mathcal{F}(0, X)$, we can compute explicitly $D_2\mathcal{F}(0, X)$. The non-degeneracy condition gives that $D_2\mathcal{F}(0, X)$ is invertible.

The implicit function theorem (in finite dimension) says that for r small enough, there exists a unique $X(r)$ such that $\mathcal{F}(r, X(r)) = 0$. This proves the existence of the family of minimal surfaces. It degenerates when $r = 0$.

Finally, we prove in § 7 that the surfaces are embedded. It is usually not easy to prove that a minimal surface given in terms of its Weierstrass data is embedded. In our case, we have explicit asymptotic formulae for the Weierstrass data when $r \to 0$. Using this, we can decompose the surfaces into pieces which are either graphs or converge to catenoids, and prove that it is embedded.

We prove Theorem 1.3 in § 8. We prove that if a family of minimal surfaces satisfy our hypotheses, then its Weierstrass data are some of the candidates introduced above,

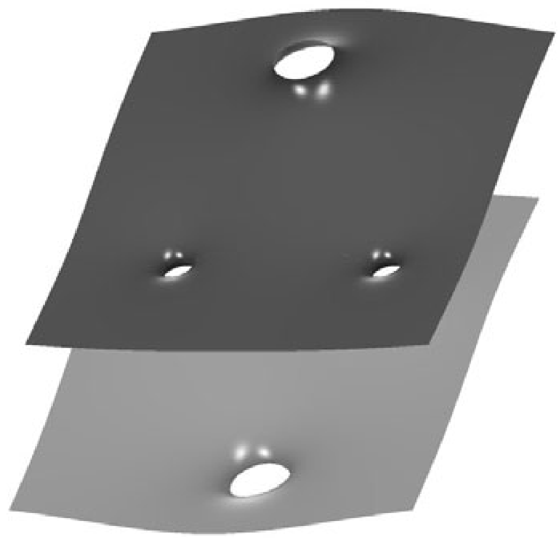

Figure 2. One of Wei's examples. Computer image by J. Hoffman and F. Wei.

so it has to satisfy the equation $\mathcal{F}(0, X) = 0$, which implies the balancing condition. This proof does not give any geometrical interpretation of these forces. It would be very interesting to have a more geometric proof of Theorem 1.3.

2. Examples

In this section, $F_{k,i}$ is the *conjugate* of the force. This is more convenient for computations.

2.1. Generalization of the Riemann and Wei examples

Proposition 2.1. *Let $n \in \mathbb{N}^*$. Let $\theta_i = (i\pi/(n+1))$. The following configuration is balanced and non-degenerate: $N = 2$, $n_1 = n$, $n_2 = 1$, $\forall i = 1, \ldots, n$, $p_{1,i} = \cot\theta_i$, $p_{2,1} = \sqrt{-1}$ and $T = 2\sqrt{-1}$. We use the notation $\sqrt{-1}$ to avoid confusion with the index i. $n = 1$ gives the Riemann example; $n = 2$ gives the Wei example.*

Proof. The proof is an elementary computation

$$\begin{aligned} F_{1,i} &= -\frac{1/n}{p_{1,i} - p_{2,1}} - \frac{1/n}{p_{1,i} - p_{2,1} + T} + \sum_{j \neq i} \frac{2/n^2}{p_{1,i} - p_{1,j}} \\ &= \frac{2}{n^2}\left(-n\frac{\cot\theta_i}{1 + \cot^2\theta_i} + \sum_{j \neq i} \frac{1}{\cot\theta_i - \cot\theta_j}\right), \end{aligned}$$

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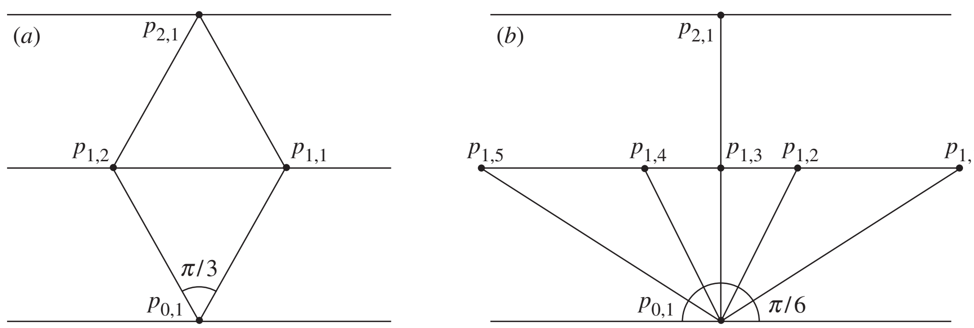

Figure 3. (a) $n=2$ (Wei example). (b) $n=5$.

$$\frac{1}{\cot\theta_i-\cot\theta_j}-\frac{\cot\theta_i}{1+\cot^2\theta_i}=\frac{1+\cot\theta_i\cot\theta_j}{(\cot\theta_i-\cot\theta_j)(1+\cot^2\theta_i)}=-\frac{\cot(\theta_i-\theta_j)}{1+\cot^2\theta_i},$$
$$F_{1,i}=\frac{2}{n^2(1+\cot^2\theta_i)}\biggl(-\cot\theta_i-\sum_{j\neq i}\cot(\theta_i-\theta_j)\biggr).$$

It is easy to see from this formula that $F_{1,i}=0$. By symmetry, $F_{2,1}=0$. This proves that the configuration is balanced. We now prove it is non-degenerate. Using that

$$\frac{\mathrm{d}p_{1,j}}{\mathrm{d}\theta_j}=-(1+\cot^2\theta_j),$$

we find that

$$\frac{\partial F_{1,i}}{\partial p_{1,j}}=\frac{-2}{n^2(1+\cot^2\theta_i)(1+\cot^2\theta_j)}M_{i,j},$$

with

$$M_{i,i}=1+\cot^2\theta_i+\sum_{j\neq i}1+\cot^2(\theta_i-\theta_j),$$
$$M_{i,j}=-1-\cot^2(\theta_i-\theta_j)\quad\text{if } j\neq i.$$

Since $M_{i,i}>\sum_{j\neq i}|M_{i,j}|$, the matrix M is invertible by standard linear algebra. Hence the matrix $\partial F_{1,i}/\partial p_{1,j}$ is invertible, which implies that the differential of F has complex rank at least n. Hence the configuration is non-degenerate. $\square$

2.2. Uniqueness of the examples of § 2.1

Proposition 2.2. *Let $n\in\mathbb{N}^*$. Let $p_{k,i}$ be a balanced configuration such that $N=2$, $n_1=n$, $n_2=1$ and $T=2\sqrt{-1}$. Then up to translation and permutation, $p_{k,i}$ is the configuration of Proposition 2.1.*

We will see in Proposition 2.4 that T cannot be zero for a balanced configuration. Hence $T=2\sqrt{-1}$ can always be achieved by scaling and rotation.

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Proof of the proposition. Without loss of generality we may assume that $p_{2,1} = \sqrt{-1}$. Write $p_{1,i} = z_i$. Then

$$F_{1,i} = \frac{2}{n^2}\left(\sum_{j\neq i}\frac{1}{z_i - z_j} - n\frac{z_i}{z_i^2+1}\right).$$

Since the sum of all forces is zero we may discard the equation $F_{2,1} = 0$. Therefore the configuration is balanced if and only if $z_1, \dots, z_n$ satisfy the n equations $F_{1,i} = 0$. Note that if $(z_1, \dots, z_n)$ is a solution then for any permutation σ, $(z_{\sigma(1)}, \dots, z_{\sigma(n)})$ is also a solution. We shall prove that $z_1, \dots, z_n$ are the roots of a one variable polynomial of degree n, which proves that $(z_1, \dots, z_n)$ is unique up to permutation.

Let $(z_1, \dots, z_n)$ be a solution. Given $J \subset \{1, \dots, n\}$ let σ_k^J be the kth elementary symmetric function of the variables z_i, $i \in \{1, \dots, n\} \setminus J$, namely

$$\sigma_k^J = \sum_{\substack{i_1<\dots<i_k\\ i_j\in\{1,\dots,n\}\setminus J}} z_{i_1}\dots z_{i_k}.$$

If $J = \emptyset$ we simply write σ_k. Let

$$E_i = F_{1,i} \times \frac{n^2}{2}(1+z_i^2)\prod_{j\neq i}(z_i - z_j).$$

The goal is to write E_i as a polynomial in the variable z_i with coefficients depending only on $\sigma_1, \dots, \sigma_n$. This is quite computational, we give the main steps of the computation below:

$$\begin{aligned} E_i &= (1+z_i^2)\sum_{j\neq i}\prod_{k\neq i,j}(z_i - z_k) - nz_i\prod_{j\neq i}(z_i - z_j)\\ &= (1+z_i^2)\sum_{j\neq i}\sum_{k=0}^{n-2}\sigma_k^{\{i,j\}}(-1)^k z_i^{n-2-k} - nz_i\sum_{k=0}^{n-1}\sigma_k^{\{i\}}(-1)^k z_i^{n-1-k}, \end{aligned}$$

$$\sum_{j\neq i}\sigma_k^{\{i,j\}} = (n-1-k)\sigma_k^{\{i\}},$$

$$\sigma_k^{\{i\}} = \sum_{j=0}^{k}\sigma_j(-1)^{k-j}z_i^{k-j},$$

$$E_i = \sum_{j=2}^{n}\sigma_{j-2}(-1)^j z_i^{n-j}\sum_{k=j-2}^{n-1}(n-1-k) - \sum_{j=0}^{n-1}\sigma_j(-1)^j z_i^{n-j}\sum_{k=j}^{n-1}(1+k).$$

Hence z_i is a root of the polynomial

$$P(z) = \sum_{j=2}^{n}\sigma_{j-2}(-1)^j z^{n-j}\frac{(n-j+1)(n-j+2)}{2} - \sum_{j=0}^{n-1}\sigma_j(-1)^j z^{n-j}\frac{(n-j)(n+1+j)}{2}.$$

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Now the key point is that $z_1, \dots, z_n$ are distinct so they are all roots of P. Looking at the highest order term we find that

$$P(z) = -\frac{n(n+1)}{2}\prod_{j=1}^{n}(z - z_j) = -\frac{n(n+1)}{2}\sum_{j=0}^{n}\sigma_j(-1)^j z^{n-j}.$$

Comparing the two formulae for P we find that $\sigma_1 = 0$ and if $2 \leqslant j \leqslant n$,

$$\sigma_j = -\frac{(n+2-j)(n+1-j)}{(j+1)j}\sigma_{j-2}.$$

This determines P and shows that $\{z_1, \dots, z_n\}$ is unique. Explicitly,

$$P(z) = -\frac{n(n+1)}{2}\sum_{0 \leqslant k \leqslant (n/2)}\frac{(-1)^k n!}{(n-2k)!(2k+1)!}z^{n-2k}.$$

The roots of this polynomial are of course $\cot(i\pi/(n+1))$. □

2.3. Inductive construction of more complicated examples

Let $F^+_{k,i}$ (respectively, $F^-_{k,i}$) be the sum of the forces exerted by the particles $p_{k+1,j}$ (respectively, $p_{k-1,j}$) on $p_{k,i}$, namely,

$$F^+_{k,i} = \sum_{j=1}^{n_{k+1}} f(p_{k,i}, p_{k+1,j}).$$

Proposition 2.3. *Let $p_{k,i}$ and $p'_{k,i}$ be two balanced configurations. We use primes for all quantities associated to the configuration $p'_{k,i}$, e.g. $p'_{k+N',i} = p'_{k,i} + T'$. Assume that*

(1) $n_1 = n'_1 = 1$,

(2) $p_{1,1} = p'_{1,1} = 0$, *and*

(3) $F^+_{1,1} = F'^+_{1,1} \neq 0$.

Define the configuration $p''_{k,i}$ as follows:

$$\begin{aligned}
&\forall k \in \{1, \dots, N\}, && n''_k = n_k \text{ and } p''_{k,i} = p_{k,i},\\
&\forall k \in \{1, \dots, N'\}, && n''_{k+N} = n'_k \text{ and } p''_{k+N,i} = p'_{k,i} + T,\\
&\forall k \in \mathbb{Z}, && p''_{k+N+N',i} = p''_{k,i} + T + T'.
\end{aligned}$$

The configuration $p''_{k,i}$ is periodic with $N'' = N + N'$ and $T'' = T + T'$. Then we have the following conclusions.

(1) *The configuration $p''_{k,i}$ is balanced.*

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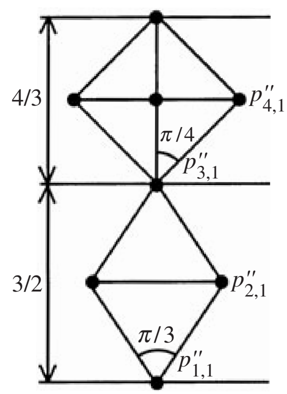

Figure 4. $n = 2$ and $n' = 3$.

(2) *Assume that the configurations* $p_{k,i}$ *and* $p'_{k,i}$ *are unique up to translation. Then* $p''_{k,i}$ *is also unique up to translation, in the sense that if* $\tilde{p}''_{k,i}$ *is another balanced configuration, with* $\tilde{N}'' = N''$, $\tilde{n}''_k = n''_k$ *and* $\tilde{T}'' = T''$, *then up to translation and permutation,* $\tilde{p}''_{k,i} = p''_{k,i}$.

(3) *Assume that* $p_{k,i}$ *and* $p'_{k,i}$ *are non-degenerate. Then so is* $p''_{k,i}$.

Note that condition (2) may always be achieved by translation, and condition (3) may always be achieved by rotation and scaling, provided $F^+_{1,1} \neq 0$ and $F'^+_{1,1} \neq 0$.

Before proving the proposition, let us give an example (see figure 4). Consider the configuration of Proposition 2.1, scaled by $(n+1)/2n$, and with the indices 1 and 2 exchanged so that $n_1 = 1$ and $n_2 = n$. An elementary computation gives $F^+_{1,1} = -\sqrt{-1}$. Using Proposition 2.3 and induction, we can construct balanced configurations such that n_k is any periodic sequence of positive integers satisfying $n_k = 1$ for odd k. All these configurations are non-degenerate and unique up to translation.

Proof of Proposition 2.3. (1) From the definition, one can deduce that $p''_{N+1,1} = p_{N+1,1}$. Hence $F''_{k,i} = 0$ for $k = 2, \dots, N$. Also $p''_{N+N'+1,1} = p'_{N'+1,1} + T$, so $F''_{k+N,i} = 0$ for $k = 2, \dots, N'$. Moreover,

$$F''_{N+1,1} = F''^+_{N+1,1} + F''^-_{N+1,1} = F'^+_{1,1} + F^-_{1,1} = F'^+_{1,1} - F^+_{1,1} = 0.$$

Since the sum of the forces is zero, also $F''_{1,1} = 0$. This proves (1).

(2) Let $\tilde{p}''_{k,i}$ be another balanced configuration with $\tilde{T}'' = T''$. Without loss of generality we may assume that $\tilde{p}''_{1,1} = p''_{1,1} = 0$.

Let $\tilde{T} = \tilde{p}''_{N+1,1} - \tilde{p}''_{1,1}$. Consider the configuration $\tilde{p}_{k,i}$ defined by $\tilde{p}_{k,i} = \tilde{p}''_{k,i}$ for $k = 1, \dots, N$ and $\tilde{p}_{k+N,i} = \tilde{p}_{k,i} + \tilde{T}$ for $k \in \mathbb{Z}$. Then $\tilde{p}_{N+1,1} = \tilde{p}''_{N+1,1}$ so $\tilde{F}_{k,i} = 0$ for $k = 2, \dots, N$. Since the sum of the forces is zero, $\tilde{F}_{1,1} = 0$ as well. Hence the configuration $\tilde{p}_{k,i}$ is balanced. Let $\lambda = \tilde{T}/T$. Since the configurations $\tilde{p}_{k,i}$ and $\lambda p_{k,i}$ have the same period, they differ by a translation. Since $\tilde{p}_{1,1} = p_{1,1} = 0$, we have

$$\tilde{p}_{k,i} = \lambda p_{k,i}.$$

Let $\tilde{T}' = \tilde{p}''_{N+N'+1,1} - \tilde{p}''_{N+1,1}$. Consider the configuration $\tilde{p}'_{k,i}$ defined by $\tilde{p}'_{k,i} = \tilde{p}''_{k+N,i} - \tilde{T}$ for $k = 1, \ldots, N'$ and $\tilde{p}'_{k+N',i} = \tilde{p}'_{k,i} + \tilde{T}'$ for $k \in \mathbb{Z}$. Then as above, $\tilde{p}'_{k,i}$ is balanced. Let $\lambda' = \tilde{T}'/T'$. Then

$$\tilde{p}'_{k,i} = \lambda' p'_{k,i},$$
$$0 = \tilde{F}''_{N+1,1} = \tilde{F}''^{+}_{N+1,1} + \tilde{F}''^{-}_{N+1,1} = \tilde{F}'^{+}_{1,1} + \tilde{F}^{-}_{1,1} = \frac{1}{\lambda'} F'^{+}_{1,1} - \frac{1}{\lambda} F^{+}_{1,1}.$$

Hence $\lambda = \lambda'$. Also
$$\tilde{T}'' = \tilde{T} + \tilde{T}' = \lambda T + \lambda T' = \lambda T'',$$

which implies that $\lambda = 1$. Hence $\tilde{p}''_{k,i} = p''_{k,i}$, which proves (2).

(3) Recall that $p_{k,i}$ non-degenerate means that if $\tilde{p}_{k,i}(t)$ is a deformation of $p_{k,i}$, such that $\tilde{T}(t) = T$ and $\tilde{F}_{k,i}(t) = o(t)$, then, up to a translation, $\tilde{p}_{k,i}(t) = p_{k,i} + o(t)$. So we see that the proof of (3) is essentially the same as the proof of (2), although of course non-degenerate and unique up to translation are not equivalent. $\square$

2.4. Further results

Let me state the following.

- If $N = 2$ and $n_1 = n_2 = 2$, then there are no balanced configurations.
- If $N = 2$, $n_1 = 3$ and $n_2 = 2$, then there are at least two non-degenerated balanced configurations. For one of them, the points $p_{1,1}$, $p_{1,2}$ and $p_{1,3}$ are not on a line (communicated by M. Weber at MSRI).

These examples show that given a periodic sequence of integers n_k, one cannot hope for existence nor uniqueness in general. Let me conclude this section with a simple observation.

Proposition 2.4. *There is no balanced configuration with $T = 0$.*

Proof. When $T = 0$, a straightforward computation gives

$$\sum_{k=1}^{N} \sum_{i=1}^{n_k} p_{k,i} F_{k,i} = 2 \sum_{k=1}^{N} \sum_{i=1}^{n_k} \sum_{j=i+1}^{n_k} \frac{1}{n_k^2} - \sum_{k=1}^{N} \sum_{i=1}^{n_k} \sum_{j=1}^{n_{k+1}} \frac{1}{n_k n_{k+1}} = -\sum_{k=1}^{N} \frac{1}{n_k}.$$

Hence the forces cannot all vanish. $\square$

3. The Weierstrass data

We now begin the proof of Theorem 1.4. In this section we define all possible candidates for the Weierstrass data of the family of minimal surfaces we want to construct, depending on certain parameters.

It is well known [**6**] that a simply periodic minimal surface with finite total curvature in the quotient may be conformally parametrized on a compact Riemann surface Σ minus a finite number of points corresponding to the ends. Moreover, the Gauss map g

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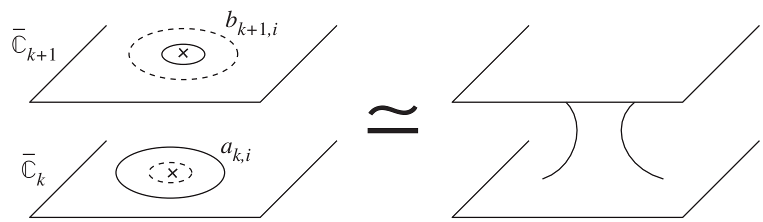

Figure 5. Creating necks.

is meromorphic on Σ and in the case of horizontal planar ends, the height differential η is holomorphic on Σ.

We define Σ and g explicitly and we define η by prescribing its periods.

3.1. The Riemann surface and the Gauss map

Consider N copies of the Riemann sphere $\overline{\mathbb{C}} = \mathbb{C} \cup \infty$, labelled $\overline{\mathbb{C}}_1, \dots, \overline{\mathbb{C}}_N$. On each $\overline{\mathbb{C}}_k$, $k = 1, \dots, N$, consider the meromorphic function

$$g_k(z) = \sum_{i=1}^{n_k} \frac{\alpha_{k,i}}{z - a_{k,i}} - \sum_{i=1}^{n_{k-1}} \frac{\beta_{k,i}}{z - b_{k,i}},$$

where the poles $a_{k,i}$, $b_{k,i}$ are distinct complex numbers, and $\alpha_{k,i}$, $\beta_{k,i}$ are non-zero complex numbers such that

$$\sum_{i=1}^{n_k} \alpha_{k,i} = \sum_{i=1}^{n_{k-1}} \beta_{k,i} = 1. \tag{3.1}$$

These are parameters of the construction. The first equality in (3.1) implies that g_k has a zero of order at least 2 at infinity: this will be needed later. The second equality is a normalization. For each $k = 1, \dots, N$, and $i = 1, \dots, n_k$, we identify a small annulus around $a_{k,i}$ in $\overline{\mathbb{C}}_k$ with a small annulus around $b_{k+1,i}$ in $\overline{\mathbb{C}}_{k+1}$, thus creating n_k small necks between $\overline{\mathbb{C}}_k$ and $\overline{\mathbb{C}}_{k+1}$. We do this as follows. Let $v_{k,i} = 1/g_k$. This function has a simple zero at $a_{k,i}$, and hence is one to one in a neighbourhood of $a_{k,i}$. There exists $\varepsilon > 0$ such that $v_{k,i}$ is biholomorphic from a neighbourhood of $a_{k,i}$ to the disk $D(0, \varepsilon)$. We think of $v_{k,i}$ as a complex coordinate in a neighbourhood of $a_{k,i}$, and really forget that it is defined everywhere on $\overline{\mathbb{C}}_k$. When there is no possible confusion, we will write $v = v_{k,i}$. In the same way, $w_{k+1,i} = 1/g_{k+1}$ is biholomorphic from a neighbourhood of $b_{k+1,i}$ in $\overline{\mathbb{C}}_{k+1}$ to the disk $D(0, \varepsilon)$.

Consider a positive number r such that $0 < r < \varepsilon^2$. Remove the disk $|v_{k,i}| \leqslant (r/\varepsilon)$ from $\overline{\mathbb{C}}_k$ and $|w_{k+1,i}| \leqslant (r/\varepsilon)$ from $\overline{\mathbb{C}}_{k+1}$. Identify the points in $\overline{\mathbb{C}}_k$ and $\overline{\mathbb{C}}_{k+1}$ whose respective coordinates $v = v_{k,i}$ and $w = w_{k+1,i}$ satisfy

$$\frac{r}{\varepsilon} < |v| < \varepsilon, \quad \frac{r}{\varepsilon} < |w| < \varepsilon, \quad vw = r.$$

Doing this for all $k = 1, \dots, N$ and $i = 1, \dots, n_k$ defines a compact Riemann surface we call Σ (r is the same for all necks and when $k = N$, $k + 1$ should be understood as 1).

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From the topological point of view, the genus of Σ is

$$G(\Sigma) = 1 + \sum_{k=1}^{N}(n_k - 1).$$

We define the Gauss map $g : \Sigma \to \mathbb{C} \cup \infty$ by

$$g(z) = \begin{cases} \sqrt{r} g_k(z) & \text{if } z \in \overline{\mathbb{C}}_k, \quad k \text{ even}, \\ \dfrac{1}{\sqrt{r} g_k(z)} & \text{if } z \in \overline{\mathbb{C}}_k, \quad k \text{ odd}. \end{cases}$$

To see that g is well defined on Σ, consider the coordinates $v = v_{k,i}$ and $w = w_{k+1,i}$. If k is even, then $g = \sqrt{r}/v$ on $\overline{\mathbb{C}}_k$ and $g = w/\sqrt{r}$ on $\overline{\mathbb{C}}_{k+1}$. Both values of g agree when $vw = r$. This proves that g has the same value at the two points that are identified when defining Σ. The case k odd is similar. This proves that g is a well-defined meromorphic function on Σ.

Remark 3.1. A more natural way to define Σ would be to use z as a local coordinate instead of g_k. Consider N copies of the Riemann sphere, and points $a_{k,i}$, $i = 1, \dots, n_k$, and $b_{k,i}$, $i = 1, \dots, n_{k-1}$, in each sphere. Let $v = z - a_{k,i}$ and $w = z - b_{k+1,i}$. Identify points using the rule $vw = r_{k,i}$, where $r_{k,i}$ is a small complex number depending on the neck. This defines a compact Riemann surface Σ. The problem is that this does not define a meromorphic function. The natural way to define g is to prescribe its zeroes and poles, but then we have to check the conditions of Abel's Theorem, which means more equations to solve. When Abel's conditions are not satisfied, g only exists as a multi-valued function. Since we solve all equations at the same time (using the implicit function theorem), we have to compute the periods of the Weierstrass data when Abel's conditions are not yet satisfied, which means that we have to compute the integrals of multi-valued differentials.

So instead of defining the Riemann surface and then the Gauss map, we define both at the same time. Instead of gluing Riemann spheres, we glue couples $(\overline{\mathbb{C}}_k, g_k)$. In fact, this construction gives all possible candidates for the Weierstrass data of a minimal surface satisfying Hypotheses 1.1 and 1.2. We will see this in § 8 when we prove Theorem 1.3.

3.2. The height differential

By standard Riemann surface theory [**3**, p. 228], the space of holomorphic differentials on Σ is isomorphic to $\mathbb{C}^G$ (G is the genus of Σ). The isomorphism is given by integration on the G curves $A_1, \dots, A_G$ of a 'canonical basis' of the homology of Σ. Recall that a canonical basis is a set of $2G$ closed curves $A_1, \dots, A_G$, $B_1, \dots, B_G$ such that the intersection numbers satisfy $A_i \cdot B_i = 1$ and all other intersection numbers are zero. We define a canonical basis as follows. Let $A_{k,i}$ be the circle $|v_{k,i}| = \varepsilon$ in $\overline{\mathbb{C}}_k$ oriented positively (i.e. anticlockwise). Note that $A_{k,i}$ is homotopic to the circle $|w_{k+1,i}| = \varepsilon$ in $\overline{\mathbb{C}}_{k+1}$, oriented negatively, because $v = \varepsilon e^{i\theta}$ gives $w = (r/\varepsilon)e^{-i\theta}$. For $i \geqslant 2$, let $B_{k,i}$ be a closed curve in Σ which intersects $A_{k,1}$ with intersection number -1 and $A_{k,i}$ with

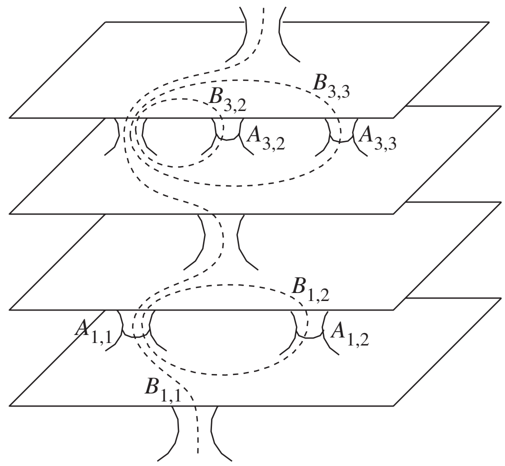

Figure 6. The Riemann surface in the case $N = 4$, $n_1 = 2$, $n_2 = 1$, $n_3 = 3$, $n_4 = 1$. The top and bottom necks have to be identified. This surface has genus 4. We have represented the Riemann spheres as planes so that the picture looks like the minimal surface we want to construct.

intersection number $+1$, and does not intersect any other A-curve or B-curve. Let $B_{1,1}$ be a closed curve in Σ which intersects all curves $A_{k,1}$ with intersection number $+1$, and does not intersect any other A-curve or B-curve. We will define these B-curves more precisely in § 5.2 when we compute the periods of the Weierstrass data. The following set of curves: $A_{1,1}$, $B_{1,1}$, $A_{k,i}$, $B_{k,i}$ for $k = 1, \dots, N$ and $i = 2, \dots, n_k$ form a basis of the homology of Σ. Note that the number of these curves is $2 + 2\sum_{k=1}^{N}(n_k - 1) = 2G(\Sigma)$. This is not a canonical basis because the intersection numbers $A_{1,1} \cdot B_{1,i}$ are not right, but replacing $B_{1,i}$ by $B_{1,1} + B_{1,i}$ gives a canonical basis.

Proposition 3.2. *Consider some numbers $\gamma_{k,i}$, $k = 1, \dots, N$, $i = 1, \dots, n_k$, such that for any k,*

$$\sum_{i=1}^{n_k} \gamma_{k,i} = 1. \tag{3.2}$$

These are the remaining parameters of the construction. There exists a unique holomorphic differential η on Σ such that for any $k = 1, \dots, N$ and $i = 1, \dots, n_k$, one has

$$\int_{A_{k,i}} \eta = 2\pi \mathrm{i} \gamma_{k,i}. \tag{3.3}$$

Proof. There exists a unique holomorphic differential η on Σ such that (3.3) holds for all curves $A_{k,i}$ of the canonical basis. It remains to prove that (3.3) holds for the remaining A-curves, namely $A_{k,1}$, $k \geqslant 2$. Consider the domain in $\overline{\mathbb{C}}_k$ bounded by the curves $A_{k,i}$, $i = 1, \dots, n_k$, and $A_{k-1,i}$, $i = 1, \dots, n_{k-1}$. Recall that $A_{k,i}$ is a small circle around $a_{k,i}$, oriented positively, while $A_{k-1,i}$ is a small circle around $b_{k,i}$, oriented negatively. By the Cauchy Theorem,

$$\sum_{i=1}^{n_{k-1}} \int_{A_{k-1,i}} \eta = \sum_{i=1}^{n_k} \int_{A_{k,i}} \eta.$$

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The result follows by induction on k using that $\sum \gamma_{k,i}$ does not depend on k. The fact that it is equal to 1 is a normalization. $\square$

3.3. Parameters of the construction

We write $\alpha_k = (\alpha_{k,1}, \ldots, \alpha_{k,n_k})$ and $\alpha = (\alpha_1, \ldots, \alpha_N)$. We define similarly β, γ, a and b. Let $X = (\alpha, \beta, \gamma, a, b)$. The parameters of the construction are (r, X). We summarize our hypotheses on the parameters: for each k, the numbers $a_{k,i}$, $b_{k,i}$ are distinct; the numbers $\alpha_{k,i}$, $\beta_{k,i}$ are non-zero and

$$\sum_{i=1}^{n_k} \alpha_{k,i} = \sum_{i=1}^{n_{k-1}} \beta_{k,i} = \sum_{i=1}^{n_k} \gamma_{k,i} = 1. \tag{3.4}$$

3.4. The equations

Let ∞_k be the point $z = \infty$ in $\overline{\mathbb{C}}_k$. The points $\infty_1, \ldots, \infty_N$ will be the N punctures on Σ, i.e the points corresponding to the planar ends. Note that thanks to (3.1), the Gauss map has multiplicity at least two at ∞_k, which is needed for a planar end. We recall the conditions so that (Σ, g, η) are the Weierstrass data for a complete simply periodic minimal surface with horizontal *embedded* planar ends.

(1) For any $p \in \Sigma$, not a puncture (i.e $p \neq \infty_k$ in our case) η has a zero at p if and only if g has either a zero or a pole, with the same multiplicity. For each puncture $p \in \Sigma$ (i.e $p = \infty_k$), g has a zero or a pole of multiplicity $m \geqslant 2$ at p; η has a zero of multiplicity $m - 2$.

(2) For any closed curve c on Σ, $\mathrm{Re} \int_c \phi_j = 0 \bmod \mathcal{T}_j$, $j = 1, 2, 3$, where $\mathcal{T} = (\mathcal{T}_1, \mathcal{T}_2, \mathcal{T}_3)$ is the period of the surface.

The zeros and poles of g are the zeros of all g_k. Recalling that $\mathrm{d}z$ has a double pole at ∞, the first condition may be written as follows.

(1′) The zeros of η are the zeros of $g_k \, \mathrm{d}z$, $k = 1, \ldots, N$, with the same multiplicity. In other words,

$$\mathrm{div}_0(\eta) = \sum_{k=1}^{N} \mathrm{div}_0(g_k \, \mathrm{d}z), \tag{3.5}$$

where div_0 means the formal sum of the zeros.

Remark 3.3. Note that by standard Riemann surface theory, the number of zeros of a holomorphic differential is $2G(\Sigma) - 2$, which is equal to the degree of the right-hand side of (3.5). Hence an inequality in (3.5) implies equality.

Provided condition (1) is satisfied, ϕ_1 and ϕ_2 only have poles at the punctures ∞_k. Therefore condition (2) needs only be checked for the curves of the canonical basis and for small circles around the punctures. Using the Residue Theorem as in the proof of Proposition 3.2, condition (2) may be written as follows.

(2′) For any $j = 1, 2, 3$, any $k = 1, \dots, N$ and $i = 1, \dots, n_k$,

$$\operatorname{Re} \int_{A_{k,i}} \phi_j = 0. \tag{3.6}$$

For any $j = 1, 2, 3$, any $k = 1, \dots, N$ and $i = 2, \dots, n_k$,

$$\operatorname{Re} \int_{B_{k,i}} \phi_j = 0, \tag{3.7}$$

$$\operatorname{Re} \int_{B_{1,1}} \phi_j = \mathcal{T}_j. \tag{3.8}$$

Note that condition (2) only asks that these periods are zero modulo $\mathcal{T}_j$. The above choices are motivated by our picture *a priori* of the surface we want to construct. The equations we have to solve are (3.5), (3.6) and (3.7). Equation (3.8) gives the period of the minimal surface.

4. Holomorphic extension to $r = 0$

The definitions of Σ and g are very explicit, but the definition of η is not. One question we need to answer is: where are the zeros of η? The key point to answer this question is that when $r \to 0$, the Riemann surface Σ degenerates. This allows us to compute the limit of η when $r \to 0$. When $r = 0$, we define Σ as the disjoint union $\overline{\mathbb{C}}_1 \cup \dots \overline{\mathbb{C}}_N$ and η by $\eta = \eta_k$ on $\overline{\mathbb{C}}_k$ where η_k is the unique meromorphic differential on $\overline{\mathbb{C}}_k$ with simple poles at $a_{k,i}$, $b_{k,i}$ with residues $\gamma_{k,i}$ and $-\gamma_{k-1,i}$. Explicitly,

$$\eta_k = \left(\sum_{i=1}^{n_k} \frac{\gamma_{k,i}}{z - a_{k,i}} - \sum_{i=1}^{n_{k-1}} \frac{\gamma_{k-1,i}}{z - b_{k,i}} \right) \mathrm{d}z.$$

The problem is to prove that $r \mapsto \eta$ is continuous at $r = 0$. This is true, but even better: continuous may be replaced by holomorphic. For this we need to think of r as a complex number. This does not change anything to the definition of Σ and η (this introduces some multi-valuation in the definition of g but we will not consider g in this section). We also fix the value of the parameter X.

Proposition 4.1. *Let $z \in \overline{\mathbb{C}}_k$, $z \neq a_{k,i}$, $z \neq b_{k,i}$. Then $r \mapsto \eta(z)$ is holomorphic in a neighbourhood of* 0.

It is important to realize that if $z \neq a_{k,i}$ and $z \neq b_{k,i}$ for all i, then for r small enough, z is outside of the disks that were removed when constructing Σ, so z may be seen as a point on Σ. Hence $\eta(z)$ makes sense.

Proof of the proposition. This result is essentially proven in Fay [**2**, Proposition 3.7, p. 51]. The difference is that in Fay, only one neck degenerates, whereas in our case, all necks degenerate at the same time. To be able to use the result of Fay, we introduce one parameter $r_{k,i}$ per neck. We define Σ as in § 3.1, identifying the points such that

$v_{k,i}w_{k+1,i} = r_{k,i}$ when $r_{k,i} \neq 0$. When $r_{k,i} = 0$ we do not identify points. Thus we have a compact Riemann surface Σ depending on complex parameters $r_{k,i}$. We first prove that η depends holomorphically on one $r_{k,i}$ (in the sense of the proposition) when all other parameters $r_{m,j}$, $(m,j) \neq (k,i)$ have fixed value, $r_{m,j} \neq 0$. We write $r = r_{k,i}$, $v = v_{k,i}$, $w = w_{k+1,i}$, $a = a_{k,i}$, $b = b_{k+1,i}$, $\Sigma_r = \Sigma$ and $\eta_r = \eta$. Fay defines a complex analytic 2-manifold $\mathcal{C}$ together with a holomorphic function $\rho : \mathcal{C} \to \mathbb{C}$ whose fibre $\mathcal{C}_r = \rho^{-1}(\{r\})$ is Σ_r if $r \neq 0$, and the fibre $\mathcal{C}_0$ is Σ_0 with the points a and b identified: a degenerate Riemann surface with a node (see the details on p. 50 of [**2**]). (The notation of Fay is $C = \Sigma_0$, $z_a = v$, $z_b = w$ and $t = r$.) He then proves (see [**2**, Proposition 3.7, p. 51], quoted with our notation) that

> there exists G linearly independent holomorphic 2-forms $\omega_{m,j}$ on $\mathcal{C}$ whose residues $u_{m,j,r}$ along $\mathcal{C}_r$ for r in a sufficiently small disk about $r = 0$ are a normalized basis for the holomorphic differentials on Σ_r if $r \neq 0$; while, for $r = 0$, the $G-1$ differentials $u_{m,j,0}$, $(m,j) \neq (k,i)$, are a normalized basis for the holomorphic differentials on Σ_0 and $u_{k,i,0}$ is the normalized differential of the third kind on Σ_0 with simple poles of residue $+1$, -1 at a, b.

What Fay means by the residue along $\mathcal{C}_r$ of a holomorphic 2-form ω, is the Poincaré residue of $\omega/(\rho - r)$. Namely, if z_1, z_2 are local coordinates on $\mathcal{C}$, and $\omega = f(z_1,z_2)\,\mathrm{d}z_1 \wedge \mathrm{d}z_2$, the Poincaré residue of $\omega/(\rho - r)$ is given by (see [**3**, p. 147]):

$$\left.\frac{f(z_1,z_2)\,\mathrm{d}z_1}{\partial\rho/\partial z_2}\right|_{\rho=r} = -\left.\frac{f(z_1,z_2)\,\mathrm{d}z_2}{\partial\rho/\partial z_1}\right|_{\rho=r}.$$

When $r \neq 0$, we may decompose

$$\eta_r = 2\pi\mathrm{i}\sum_{m,j}\gamma_{m,j}u_{m,j,r},$$

where the summation is on the indices m,j such that $A_{m,j}$ is a curve of the canonical basis. So η_r is the residue on $\mathcal{C}_r$ of the holomorphic 2-form

$$\omega = 2\pi\mathrm{i}\sum_{m,j}\gamma_{m,j}\omega_{m,j}.$$

From this we see that η_r depends holomorphically on r, and η_0 is the meromorphic differential on Σ_0 with simple poles of residue $+\gamma_{k,i}$, $-\gamma_{k,i}$ at $a_{k,i}$ and $b_{k+1,i}$, and whose integral on all curves $A_{l,j}$ is $2\pi\mathrm{i}\gamma_{l,j}$. So we have proven that for each (k,i), η depends holomorphically on $r_{k,i}$ in a neighbourhood of 0, when all other $r_{m,j}$ have fixed non-zero value. By Lemma 4.2 below, η depends holomorphically on all $r_{k,i}$ as a function of several complex variables. In particular when all $r_{k,i}$ are equal, we have proven the proposition. □

Lemma 4.2. *Let D be the unit disk in $\mathbb{C}$ and $D^* = D \setminus \{0\}$. Let $f : (D^*)^n \to \mathbb{C}$ be a holomorphic function of n variables $z = (z_1,\dots,z_n)$ such that for each i, for any $z_j \in D^*$, $j \neq i$, the function $z_i \mapsto f(z_1,\dots,z_n)$ extends holomorphically to D. Then f extends holomorphically to D^n.*

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Proof. Let $0 < r < 1$ and $D(r)$ be the disk of radius r. Let $C < \infty$ be the supremum of $|f(z)|$ on $(\partial D(r))^n$. Let $z \in (D^*(r))^n$. The function $z_1 \mapsto f(z_1, \dots, z_n)$ extends holomorphically to $D(r)$ so its maximum is on the boundary:

$$|f(z)| \leqslant \sup_{z_1 \in \partial D(r)} |f(z_1, \dots, z_n)|.$$

Repeating the process for each variable we find that $|f(z)| \leqslant C$ so f is bounded on $(D^*(r))^n$. By the Riemann extension theorem [**3**, p. 9] f extends holomorphically to $D(r)^n$. $\square$

Proposition 4.1 gives the limit of η away from the necks. The following proposition gives the behaviour of η on the necks.

Proposition 4.3. *Let $v = v_{k,i}$. On the domain $(|r|/\varepsilon) < |v| < \varepsilon$ of Σ, we have the formula*

$$\eta = f\left(v, \frac{r}{v}\right)\frac{\mathrm{d}v}{v} = -f\left(\frac{r}{w}, w\right)\frac{\mathrm{d}w}{w},$$

where f is a holomorphic function of two complex variables defined in a neighbourhood of $(0,0)$.

Proof. We continue with the notation of the previous proposition. All parameters are fixed except $r = r_{k,i}$. We use (v, w) as local coordinates on $\mathcal{C}$ and write $\omega = f(v, w)\,\mathrm{d}v \wedge \mathrm{d}w$. The Poincaré residue is

$$\eta_r = \left.\frac{f(v, w)\,\mathrm{d}v}{(\partial/\partial w)(vw - r)}\right|_{vw=r} = f\left(v, \frac{r}{v}\right)\frac{\mathrm{d}v}{v}.$$

This proves the formula of the proposition. $\square$

5. Estimation of the periods

We use Propositions 4.1 and 4.3 to estimate the periods of η, $g\eta$ and $g^{-1}\eta$ on the curves $A_{k,i}$, $B_{k,i}$. The following proposition gives the leading term of each period when $r \to 0$. We obtain formulae involving g_k and η_k, for which we have explicit formulae. In this section we think of r as a real number. The reason for this is that the B-periods are multi-valued functions of r when r is complex. This comes from the fact that one cannot define $B_{k,i}$ in a continuous way when r is complex. This multi-valuation is clear in our formulae: we get $\log r$ terms.

Proposition 5.1. *Let $r > 0$. Then*

$$\int_{A_{k,i}} g^{(-1)^k}\eta = \sqrt{r}(2\pi \mathrm{i}\, \mathrm{Res}_{a_{k,i}}\, g_k\eta_k + r\,\mathrm{holo}(r, X)),$$

$$\int_{A_{k,i}} g^{(-1)^{k+1}}\eta = \sqrt{r}(-2\pi \mathrm{i}\, \mathrm{Res}_{b_{k+1,i}}\, g_{k+1}\eta_{k+1} + r\,\mathrm{holo}(r, X)),$$

$$\int_{B_{k,i}} \eta = (\gamma_{k,i} - \gamma_{k,1})\log r + \mathrm{holo}(r, X),$$

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$$\int_{B_{k,i}} g^{(-1)^k}\eta = \frac{1}{\sqrt{r}}\left(\int_{b_{k+1,i}}^{b_{k+1,1}} g_{k+1}^{-1}\eta_{k+1} + r\log r\,\mathrm{holo}(r,X) + r\,\mathrm{holo}(r,X)\right),$$
$$\int_{B_{k,i}} g^{(-1)^{k+1}}\eta = \frac{1}{\sqrt{r}}\left(\int_{a_{k,1}}^{a_{k,i}} g_k^{-1}\eta_k + r\log r\,\mathrm{holo}(r,X) + r\,\mathrm{holo}(r,X)\right).$$

In this proposition $\mathrm{holo}(r,X)$ *means a holomorphic function of the complex variables* (r,X) *in a neighbourhood of* $(0,X_0)$*, where* X_0 *is any value of the parameters satisfying the conditions of § 3.3. In general the B-periods are multi-valued functions of the parameter* X*. They are only locally well defined.*

We prove this proposition in the following three sections.

5.1. The A-periods of $g^{\pm 1}\eta$

For the first formula we see $A_{k,i}$ as the circle $|v_{k,i}| = \varepsilon$ in $\overline{\mathbb{C}}_k$. By definition, $g^{(-1)^k} = \sqrt{r}g_k$. By Proposition 4.1, $\eta = \eta_k + r\,\mathrm{holo}(r,X,v)\,\mathrm{d}v$ on $A_{k,i}$. Hence the first formula follows from the residue theorem. For the second formula we see $A_{k,i}$ as the circle $|w_{k+1,i}| = \varepsilon$ in $\overline{\mathbb{C}}_{k+1}$, oriented negatively. The second formula comes from $g^{(-1)^{k+1}} = \sqrt{r}g_{k+1}$ and $\eta = \eta_{k+1} + r\,\mathrm{holo}(r,X,w)\,\mathrm{d}w$ on $A_{k,i}$.

5.2. The B-periods of η

Let $B_{k,i}$ be the union of the following four paths c_1, c_2, c_3, c_4.

- c_1 is a curve in $\overline{\mathbb{C}}_k$ which goes from the point $v_{k,1} = \varepsilon$ to the point $v_{k,i} = \varepsilon$. It does not depend on r, and we may choose it so that it depends continuously on X (if X is in a neighbourhood of X_0).
- c_2 is the curve parametrized by $v_{k,i} = r/t$ for $t \in [r/\varepsilon, \varepsilon]$. It goes from the point $v_{k,i} = \varepsilon$ to the point $w_{k+1,i} = \varepsilon$.
- c_3 is a curve in $\overline{\mathbb{C}}_{k+1}$ which goes from the point $w_{k+1,i} = \varepsilon$ to the point $w_{k+1,1} = \varepsilon$.
- c_4 is the curve parametrized by $w_{k+1,1} = r/t$ for $t \in [r/\varepsilon, \varepsilon]$. It goes from the point $w_{k+1,1} = \varepsilon$ to the point $v_{k,1} = \varepsilon$.

The integrals of η on c_1 and c_3 are holomorphic functions of (r,X) in a neighbourhood of $r = 0$ because η depends holomorphically on (r,X) on these paths. To compute the integral of η on c_2 we use Proposition 4.3. We expand the function f of this proposition

$$f(v,w) = \sum_{n\geqslant 0, m\geqslant 0} a_{n,m}v^n w^m.$$

We may assume that this series converges on $|v| \leqslant \varepsilon$, $|w| \leqslant \varepsilon$. Since $vw = r$ this gives

$$\eta = \sum a_{n,m}v^{n-1-m}r^m\,\mathrm{d}v,$$
$$\int_{A_{k,i}} \eta = 2\pi\mathrm{i}\,\mathrm{Res}_{v=0}\sum a_{n,m}v^{n-1-m}r^m = 2\pi\mathrm{i}\sum_n a_{n,n}r^n.$$

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Hence

$$\sum_n a_{n,n} r^n = \gamma_{k,i},$$

$$\begin{aligned}\int_{c_2} \eta &= \int_{v=\varepsilon}^{r/\varepsilon} \sum a_{n,m} v^{n-1-m} r^m \, \mathrm{d}v \\ &= \sum_n a_{n,n} r^n \log \frac{r}{\varepsilon^2} + \sum_{n \neq m} \frac{a_{n,m}}{n-m} (r^n \varepsilon^{m-n} - r^m \varepsilon^{n-m}) \\ &= \gamma_{k,i} \log r + \mathrm{holo}(r, X).\end{aligned}$$

When $i = 1$ this formula gives the integral of η on c_4, with a minus sign because c_4 is oriented the other way. This proves the third formula of the proposition.

5.3. The B-periods of $g^{\pm 1}\eta$

We start with the integral of $g^{(-1)^k}\eta$. For the paths c_1 and c_3 we only need Proposition 4.1.

$$\begin{aligned}\int_{c_1} g^{(-1)^k} \eta &= \sqrt{r} \int_{c_1} g_k \eta = \sqrt{r}\, \mathrm{holo}(r, X), \\ \int_{c_3} g^{(-1)^k} \eta &= \frac{1}{\sqrt{r}} \int_{c_3} g_{k+1}^{-1} \eta = \frac{1}{\sqrt{r}} \left(\int_{w_{k+1,i}=\varepsilon}^{w_{k+1,1}=\varepsilon} g_{k+1}^{-1} \eta_{k+1} + r\, \mathrm{holo}(r, X) \right).\end{aligned}$$

For the paths c_2 and c_4 we use Proposition 4.3 as in the previous section.

$$\begin{aligned}\int_{c_2} g^{(-1)^k} \eta &= \sqrt{r} \int_{c_2} g_k \eta = \sqrt{r} \int_{v=\varepsilon}^{r/\varepsilon} \sum_{n,m} a_{n,m} v^{n-m-2} r^m \, \mathrm{d}v \\ &= \sqrt{r} \left(\sum_m a_{m+1,m} r^m \log \frac{r}{\varepsilon^2} + \sum_{n \neq m+1} \frac{a_{n,m}}{n-m-1} \left(\frac{r^{n-1}}{\varepsilon^{n-m-1}} - \frac{r^m}{\varepsilon^{m+1-n}} \right) \right) \\ &= \frac{1}{\sqrt{r}} \left(r \log r\, \mathrm{holo}(r, X) + r\, \mathrm{holo}(r, X) + \sum_m \frac{a_{0,m}}{-m-1} \varepsilon^{m+1} \right).\end{aligned}$$

The leading (i.e last) term is equal to

$$-\frac{1}{\sqrt{r}} \int_{w=0}^{\varepsilon} \sum_m a_{0,m} w^m \, \mathrm{d}w = -\frac{1}{\sqrt{r}} \int_{w=0}^{\varepsilon} f(0, w) \, \mathrm{d}w = \frac{1}{\sqrt{r}} \int_{w_{k+1,i}=0}^{\varepsilon} g_{k+1}^{-1} \eta_{k+1}.$$

The integral on c_4 gives the same result with the leading term equal to

$$\frac{1}{\sqrt{r}} \int_{w_{k+1,1}=\varepsilon}^{0} g_{k+1}^{-1} \eta_{k+1}.$$

Collecting the four terms gives the fourth formula of the proposition. The proof of the fifth formula is entirely similar. □

6. Implicit function theorem

In this section, we prove Theorem 1.4 assuming that for each k, the zeros of $g_k\,\mathrm{d}z$ are simple. In § 9 we will see how to adapt the proof when $g_k\,\mathrm{d}z$ is allowed to have multiple zeros.

6.1. The map $\mathcal{F}$

Let $\zeta_{k,i}$ be the zeros of $g_k\,\mathrm{d}z$ in $\overline{\mathbb{C}}_k$, $i = 1,\dots,n_k + n_{k-1} - 2$. We define:

$$\mathcal{F}_{1,k,i} = \eta(\zeta_{k,i}).$$

As usual we write $\mathcal{F}_{1,k} = (\mathcal{F}_{1,k,1},\dots,\mathcal{F}_{1,k,n_k+n_{k-1}-2})$ and $\mathcal{F}_1 = (\mathcal{F}_{1,1},\dots,\mathcal{F}_{1,N})$. Since the simple zeros of a polynomial depend analytically on its coefficients, $\mathcal{F}_1$ depends analytically on (r, X) by Proposition 4.1. Note that $\mathcal{F}_1 = 0$ only says that η has at least a zero at each zero of g_k. By Remark 3.3, each zero is simple and η has no other zero.

We now look at the period problem. By definition of η, the equation $\operatorname{Re}\int_{A_{k,i}} \eta = 0$ is equivalent to $\gamma_{k,i} \in \mathbb{R}$, which we assume from now on. A straightforward computation gives

$$\operatorname{Re}\int \phi_1 + \mathrm{i}\operatorname{Re}\int \phi_2 = \frac{1}{2}\biggl(\overline{\int g^{-1}\eta} - \int g\eta\biggr).$$

In view of Proposition 5.1 we define:

$$\begin{aligned}
\mathcal{F}_{2,k,i} &= \frac{1}{\log r}\operatorname{Re}\int_{B_{k,i}} \eta, && i = 2,\dots,n_k,\\
\mathcal{F}_{3,k,i} &= \sqrt{r}\biggl(\overline{\int_{B_{k,i}} g^{-1}\eta} - \int_{B_{k,i}} g\eta\biggr), && i = 2,\dots,n_k,\\
\mathcal{F}_{4,k,i} &= \frac{(-1)^k}{\sqrt{r}}\biggl(\overline{\int_{A_{k,i}} g^{-1}\eta} - \int_{A_{k,i}} g\eta\biggr), && i = 1,\dots,n_k.
\end{aligned}$$

The reason for the $(-1)^k$ in the definition of $\mathcal{F}_{4,k,i}$ will be seen in Proposition 6.4. We define the vectors $\mathcal{F}_2$, $\mathcal{F}_3$ and $\mathcal{F}_4$ in the obvious way and $\mathcal{F} = (\mathcal{F}_1,\mathcal{F}_2,\mathcal{F}_3,\mathcal{F}_4)$. The equations of § 3.4 are equivalent to $\mathcal{F} = 0$. What we have done is rescale the periods by a suitable function of r so that by Proposition 5.1, $\mathcal{F}$ has a limit when $r \to 0$. The problem is that $\mathcal{F}_2$ and $\mathcal{F}_3$ are not differentiable with respect to r at $r = 0$. The problem comes from the $\log r$ terms. We solve this problem by writing

$$r = r(t) = \mathrm{e}^{-1/t^2}, \quad r(0) = 0, \quad t \in \mathbb{R}.$$

By Proposition 5.1, $(t, X) \mapsto \mathcal{F}$ is smooth in a neighbourhood of $t = 0$. Moreover, $\mathcal{F}(0, X)$ is given explicitly by

$$\begin{aligned}
\mathcal{F}_{1,k,i} &= \eta_k(\zeta_{k,i}),\\
\mathcal{F}_{2,k,i} &= \gamma_{k,i} - \gamma_{k,1},\\
\mathcal{F}_{3,k,i} &= (-1)^k \operatorname{conj}^{k+1}\biggl(\int_{a_{k,1}}^{a_{k,i}} g_k^{-1}\eta_k\biggr) + (-1)^k \operatorname{conj}^k\biggl(\int_{b_{k+1,1}}^{b_{k+1,i}} g_{k+1}^{-1}\eta_{k+1}\biggr),\\
\mathcal{F}_{4,k,i} &= 2\pi\mathrm{i}(-1)^k(\operatorname{conj}^{k+1}(\operatorname{Res}_{b_{k+1,i}} g_{k+1}\eta_{k+1}) - \operatorname{conj}^k(\operatorname{Res}_{a_{k,i}} g_k\eta_k)),
\end{aligned}$$

where conj is the conjugation in $\mathbb{C}$, i.e $\mathrm{conj}^k(z) = z$ if k is even and $\mathrm{conj}^k(z) = \bar{z}$ if k is odd.

Proposition 6.1. *Let $\{p_{k,i}\}$ be a balanced configuration. Define X_0 by*

$$\begin{aligned} \alpha_{k,i} &= \gamma_{k,i} = \beta_{k+1,i} = 1/n_k, \\ a_{k,i} &= (-1)^k \,\mathrm{conj}^{k+1}(p_{k,i}), \\ b_{k,i} &= (-1)^k \,\mathrm{conj}^{k+1}(p_{k-1,i}). \end{aligned}$$

Then $\mathcal{F}(0, X_0) = 0$. Conversely, if X is a solution to $\mathcal{F}(0, X) = 0$, then $X = X_0$ for some balanced configuration $\{p_{k,i}\}$ (up to some identifications to be explained in the proof). Moreover, if $\{p_{k,i}\}$ is a non-degenerate balanced configuration, then $D_2\mathcal{F}(0, X_0)$ is an isomorphism (again, up to some identifications). By the implicit function theorem, for t in a neighbourhood of 0, there exists a unique $X(t)$ in a neighbourhood of X_0 such that $\mathcal{F}(t, X(t)) = 0$.

The corresponding Weierstrass data give an immersed simply periodic minimal surface with embedded planar ends. We will see in § 7 that it is embedded. We prove the proposition in the next four sections.

Remark 6.2. Since $r(-t) = r(t)$ we have $\mathcal{F}(t, X(-t)) = \mathcal{F}(-t, X(-t)) = 0$. By uniqueness in the implicit function theorem, $X(-t) = X(t)$. Hence t and $-t$ give the same minimal surface. Moreover, $(\mathrm{d}X/\mathrm{d}t)(0) = 0$ so $X(t) = X_0 + O(t^2) = X_0 + O(1/\log r)$. This will be useful in § 7.

6.2. The equation $\mathcal{F}_1 = 0$ (zeros of η)

In the following sections we assume that $r = 0$. $\mathcal{F}_{1,k} = 0$ is equivalent to: $g_k\,\mathrm{d}z$ and η_k have the same zeros on $\overline{\mathbb{C}}_k$. Since they already have the same poles, they are proportional. By normalization (3.4), they are equal. Thus $\mathcal{F}_1 = 0$ is equivalent to $\alpha_{k,i} = \gamma_{k,i}$ and $\beta_{k,i} = \gamma_{k-1,i}$.

Proposition 6.3. *Let*

$$E = \left\{ (\alpha_k, \beta_k) \in \mathbb{C}^{n_k + n_{k-1}} \;\middle|\; \sum \alpha_{k,i} = \sum \beta_{k,i} = 0 \right\}.$$

The partial differential of $\mathcal{F}_{1,k}$ with respect to (α_k, β_k) is an isomorphism:

$$E \to \mathbb{C}^{n_k + n_{k-1} - 2}.$$

Proof. Since $\mathcal{F}_{1,k}$ is zero when $\alpha_k = \gamma_k$ and $\beta_k = \gamma_{k-1}$,

$$\begin{aligned} \frac{\partial}{\partial \alpha_{k,i}} \mathcal{F}_{1,k,j} &= -\frac{\partial}{\partial \gamma_{k,i}} \mathcal{F}_{1,k,j} = \frac{-1}{\zeta_{k,j} - a_{k,i}}, \\ \frac{\partial}{\partial \beta_{k,i}} \mathcal{F}_{1,k,j} &= -\frac{\partial}{\partial \gamma_{k-1,i}} \mathcal{F}_{1,k,j} = \frac{1}{\zeta_{k,j} - b_{k,i}}. \end{aligned}$$

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Let L be the partial differential of $\mathcal{F}_{1,k}$ at the point (α_k, β_k). We prove L is one to one. Assume that there exists $(\dot\alpha_k, \dot\beta_k) \in E$ such that $L(\dot\alpha_k, \dot\beta_k) = 0$. We use dots to distinguish between the point (α_k, β_k) where we compute the differential and the tangent vector $(\dot\alpha_k, \dot\beta_k)$. Let

$$f(z) = \sum_{i=1}^{n_k} \frac{\dot\alpha_{k,i}}{z - a_{k,i}} - \sum_{i=1}^{n_{k-1}} \frac{\dot\beta_{k,i}}{z - b_{k,i}}.$$

Then $L(\dot\alpha_k, \dot\beta_k) = 0$ gives $f(\zeta_{k,j}) = 0$. Since $\sum \dot\alpha_{k,i} = \sum \dot\beta_{k,i}$, f has at least a double zero at ∞. Hence f and g_k have the same zeros and poles so we may write $f = \lambda g_k$. This gives $\dot\alpha_{k,i} = \lambda \alpha_{k,i}$ and $\dot\beta_{k,i} = \lambda \beta_{k,i}$. Since $\sum \dot\alpha_{k,i} = 0$ and $\sum \alpha_{k,i} = 1$, we get $\lambda = 0$. □

6.3. The equation $\mathcal{F}_2 = 0$ (B-periods of η)

From the normalization $\sum \gamma_{k,i} = 1$, we see that $\mathcal{F}_{2,k} = 0$ is equivalent to

$$\gamma_{k,i} = 1/n_k.$$

6.4. The equation $\mathcal{F}_3 = 0$ (B-periods of ϕ_1, ϕ_2)

In this section we assume that $\mathcal{F}_1 = 0$ so $\eta_k = g_k \, \mathrm{d}z$. This gives

$$\mathcal{F}_{3,k,i} = (-1)^k \operatorname{conj}^{k+1}(a_{k,i} - a_{k,1}) + (-1)^k \operatorname{conj}^k(b_{k+1,i} - b_{k+1,1}).$$

So $\mathcal{F}_3 = 0$ is equivalent to

$$b_{k+1,i} - b_{k+1,1} = -\operatorname{conj}(a_{k,i} - a_{k,1}). \tag{6.1}$$

6.5. The equation $\mathcal{F}_4 = 0$ (A-periods of ϕ_1, ϕ_2)

In this section we assume that $\mathcal{F}_1 = 0$. Then $\eta_k = g_k \, \mathrm{d}z$ gives

$$\mathcal{F}_{4,k,i} = 2\pi \mathrm{i}(-1)^{k+1} \operatorname{conj}^k(\operatorname{Res}_{a_{k,i}} g_k^2) + 2\pi \mathrm{i}(-1)^k \operatorname{conj}^{k+1}(\operatorname{Res}_{b_{k+1,i}} g_{k+1}^2).$$

Expanding the squares and taking residues gives

$$\begin{aligned}\mathcal{F}_{4,k,i} = 4\pi \mathrm{i}(-1)^{k+1} \operatorname{conj}^k &\left(\sum_{j \neq i} \frac{\alpha_{k,i}\alpha_{k,j}}{a_{k,i} - a_{k,j}} - \sum_j \frac{\alpha_{k,i}\beta_{k,j}}{a_{k,i} - b_{k,j}}\right) \\ &+ 4\pi \mathrm{i}(-1)^k \operatorname{conj}^{k+1}\left(\sum_{j \neq i} \frac{\beta_{k+1,i}\beta_{k+1,j}}{b_{k+1,i} - b_{k+1,j}} - \sum_j \frac{\beta_{k+1,i}\alpha_{k+1,j}}{b_{k+1,i} - a_{k+1,j}}\right).\end{aligned}$$

The balancing condition of the introduction is hiding in this formula. To see it we need to introduce the parameters $p_{k,i}$. Let $m = n_1 + \dots n_N$. Given some complex numbers $p_{k,i}$, $k = 1, \dots, N$, $i = 1, \dots, n_k$, let $p \in \mathbb{C}^m$ be the vector whose components are $p_{k,i}$. Given $(T, p, q) \in \mathbb{C} \times \mathbb{C}^m \times \mathbb{C}^m$, define (a, b) by

$$\begin{aligned} a_{k,i} &= (-1)^k \operatorname{conj}^{k+1}(p_{k,i} + q_{k,1}), \\ b_{k,i} &= (-1)^k \operatorname{conj}^{k+1}(p_{k-1,i} + q_{k,i}), \end{aligned}$$

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where $p_{k+N,i} = p_{k,i} + T$ and $q_{k+N,i} = q_{k,i} + T$. Then

$$\mathcal{F}_{3,k,i} = -q_{k+1,i} + q_{k+1,1}.$$

Assuming that $\mathcal{F}_3 = 0$, we get

$$\mathcal{F}_{4,k,i} = -4\pi\mathrm{i}\left(2\sum_{j\neq i}\frac{\gamma_{k,i}\gamma_{k,j}}{\bar{p}_{k,i}-\bar{p}_{k,j}} - \sum_{j}\frac{\gamma_{k,i}\gamma_{k-1,j}}{\bar{p}_{k,i}-\bar{p}_{k-1,j}} - \sum_{j}\frac{\gamma_{k,i}\gamma_{k+1,j}}{\bar{p}_{k,i}-\bar{p}_{k+1,j}}\right).$$

Assuming that $\mathcal{F}_2 = 0$, we get

$$\mathcal{F}_{4,k,i} = -4\pi\mathrm{i}F_{k,i},$$

where $F_{k,i}$ is the force introduced in § 1.1. This proves the first statement of Proposition 6.1.

To prove the converse we need to do some identifications because $\mathcal{F}_3 = 0$ does not imply $q = 0$. We first remark that $\mathcal{F}_3$ and $\mathcal{F}_4$ are not changed if we translate all $a_{k,i}$, $b_{k,i}$ (k fixed, i varying) by the same amount. In fact, the Weierstrass data themselves are not affected by such a translation. Indeed, given some numbers λ_k, let $\tilde{a}_{k,i} = a_{k,i} + \lambda_k$ and $\tilde{b}_{k,i} = b_{k,i} + \lambda_k$. Let $(\tilde{\Sigma}, \tilde{g}, \tilde{\eta})$ be the corresponding Weierstrass data. Then it is straightforward to check that the map $\varphi : \Sigma \to \tilde{\Sigma}$, $z \in \overline{\mathbb{C}}_k \mapsto z + \lambda_k$ is an isomorphism. Moreover, $\varphi^*\tilde{g} = g$ and $\varphi^*\tilde{\eta} = \eta$. Hence the two Weierstrass data are isomorphic, so define the same minimal surface. So we make the following identification:

$$(a,b) \sim (a',b') \iff \forall k, \quad \exists\lambda_k, \quad \forall i, a'_{k,i} = a_{k,i} + \lambda_k, \quad b'_{k,i} = b_{k,i} + \lambda_k.$$

Concerning p and q, we make the following identifications:

$$\begin{aligned} p \sim p' &\iff \exists\lambda, \quad \forall k, \quad \forall i, p'_{k,i} = p_{k,i} + \lambda, \\ q \sim q' &\iff \forall k, \quad \exists\lambda_k, \quad \forall i, q'_{k,i} = q_{k,i} + \lambda_k. \end{aligned}$$

Then the map

$$(T,p,q) \mapsto (a,b)$$

is well defined and it is easy to see that it is an isomorphism, both spaces having the same dimension $\sum(2n_k - 1)$. With these identifications, $\mathcal{F}_3 = 0$ gives $q \sim 0$, which proves the second statement of Proposition 6.1.

I claim that the partial differential of $\mathcal{F}$ with respect to the variables (α,β), γ, q, p has the form

$$\begin{pmatrix} \mathcal{I}_1 & \cdot & 0 & 0 \\ 0 & \mathcal{I}_2 & 0 & 0 \\ \cdot & \cdot & \mathcal{I}_3 & 0 \\ \cdot & \cdot & \cdot & \mathcal{I}_4 \end{pmatrix},$$

where $\mathcal{I}_1, \dots, \mathcal{I}_4$ are invertible linear operators, so it is invertible.

Let me first explain the zeros in this matrix. If $\alpha_k = \gamma_k$ and $\beta_k = \gamma_{k-1}$, then $\eta_k = g_k\,\mathrm{d}z$ whatever the values of a and b, hence $\mathcal{F}_1 = 0$. This explains the zeros in the first line. The other zeros are clear.

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The fact that $\mathcal{I}_1$ is invertible is Proposition 6.3. $\mathcal{I}_2$ is clearly invertible, and so is $\mathcal{I}_3$ thanks to our identification on q. Up to a constant, $\mathcal{I}_4$ is the differential of F with respect to p. The problem is that it is not onto because the sum of the forces is always zero. We are saved by the following proposition which says that the same is true for $\mathcal{F}_4$.

Proposition 6.4.

$$\forall (t,X), \quad \sum_{k=1}^{N}\sum_{i=1}^{n_k} \mathcal{F}_{4,k,i}(t,X) = 0.$$

Proof. Consider the domain in $\overline{\mathbb{C}}_k$ bounded by the curves $A_{k,i}$, $i = 1,\dots,n_k$, and $A_{k-1,i}$, $i = 1,\dots,n_{k-1}$. If k is even, $g\eta$ is holomorphic in this domain so by the Cauchy Theorem

$$\sum_{i=1}^{n_{k-1}} \int_{A_{k-1,i}} g\eta = \sum_{i=1}^{n_k} \int_{A_{k,i}} g\eta.$$

Hence

$$\sum_{k=1}^{N}\sum_{i=1}^{n_k} (-1)^k \int_{A_{k,i}} g\eta = 0.$$

In the same way, when k is odd, $g^{-1}\eta$ is holomorphic in this domain, which gives

$$\sum_{k=1}^{N}\sum_{i=1}^{n_k} (-1)^k \int_{A_{k,i}} g^{-1}\eta = 0.$$

□

Hence we may see $\mathcal{F}_4$ as taking values in the subspace $\sum \mathcal{F}_{4,k,i} = 0$. The non-degeneracy condition gives that $\mathcal{I}_4$ is onto. Our identification on p gives that it is invertible. This proves the claim and Proposition 6.1. □

Remark 6.5. At this point we have two free parameters t and T. So the implicit function theorem gives a family of solutions depending on (t,T). I claim, however, that varying T does not give any new solution. To see this, let (Σ, g, η) be the Weierstrass data associated to some value of the parameters r, α, β, γ, T, p and q. Let λ be a positive real number. Let $(\tilde{\Sigma}, \tilde{g}, \tilde{\eta})$ be the Weierstrass data associated to the parameters $\tilde{r} = \lambda^2 r$, $\tilde{T} = \lambda T$, $\tilde{p} = \lambda p$, $\tilde{q} = \lambda q$, all other parameters having the same value. It is easy to check that $\varphi : \Sigma \to \tilde{\Sigma}$, $z \mapsto \lambda z$ is an isomorphism. Moreover, $\varphi^*\tilde{g} = g$ and $\varphi^*\tilde{\eta} = \eta$, so the two Weierstrass data are isomorphic.

In the same way, let $\lambda \in \mathbb{C}$ such that $|\lambda| = 1$. Let $\tilde{T} = \lambda T$, $\tilde{p} = \lambda p$, $\tilde{q} = \lambda q$, $\tilde{r} = r$. Then $\varphi : \Sigma \to \tilde{\Sigma}$, $z \in \overline{\mathbb{C}}_k \mapsto \mathrm{conj}^{k+1}(\lambda) z$ is an isomorphism, $\varphi^*\tilde{g} = \lambda g$ and $\varphi^*\tilde{\eta} = \eta$. So up to a rotation of angle $\arg\lambda$, the two minimal surfaces are the same. As a conclusion we may as well fix the value of T (equal to the period of the given balanced configuration).

7. Embeddedness

In this section, we prove that the minimal surface we obtained in the previous section is embedded. This will conclude the proof of Theorem 1.4. Given $t > 0$, let (Σ, g, η) be the Weierstrass data given by Proposition 6.1 and $\psi : \Sigma \to \mathbb{R}^3$ be the corresponding immersion. Recall that $r = \mathrm{e}^{-1/t^2}$. We write

$$\psi(z) = (\mathrm{horiz}(z), \mathrm{height}(z)) \in \mathbb{C} \times \mathbb{R} \sim \mathbb{R}^3.$$

Proposition 7.1. *There exists a constant C, not depending on t, such that the following statements are true.*

(1) *For any point z in $\overline{\mathbb{C}}_k$ such that $\forall i$, $|v_{k,i}| > \varepsilon$, $|w_{k,i}| > \varepsilon$,*

$$|\,\mathrm{height}(z) - \mathrm{height}(\infty_k)| \leqslant C.$$

(2) *For any point z in $\overline{\mathbb{C}}_k$ such that $(r/\varepsilon) < |v_{k,i}(z)| < \varepsilon$,*

$$\left|\mathrm{height}(z) - \mathrm{height}(\infty_k) - \frac{1}{n_k}\log|v_{k,i}(z)|\right| \leqslant C.$$

(3)
$$\left|\mathrm{height}(\infty_{k+1}) - \mathrm{height}(\infty_k) - \frac{1}{n_k}\log r\right| \leqslant C.$$

(4) *Let $P_{k,i} \in \Sigma$ be the point such that $v_{k,i} = \sqrt{r}$. (This is the point on the neck where $g = 1$.) Then*

$$2\sqrt{r}(\mathrm{horiz}(P_{k,j}) - \mathrm{horiz}(P_{k,i})) \to (-1)^k \,\mathrm{conj}^{k+1}(a_{k,j} - a_{k,i}) = p_{k,j} - p_{k,i},$$
$$2\sqrt{r}(\mathrm{horiz}(P_{k,j}) - \mathrm{horiz}(P_{k-1,i})) \to (-1)^k \,\mathrm{conj}^{k+1}(a_{k,j} - b_{k,i}) = p_{k,j} - p_{k-1,i}.$$

Hence we may translate the surface so that,

$$\forall k, \quad \forall i, \quad 2\sqrt{r}\,\mathrm{horiz}(P_{k,i}) \to p_{k,i}.$$

(5) *Let $0 < \sigma < \frac{1}{2}$. The image of the domain $r^{1-\sigma} < |v_{k,i}| < r^{\sigma}$ converges (up to translation) to a catenoid with necksize $2\pi/n_k$. Moreover, it is included in a vertical cylinder with radius $r^{\sigma-(1/2)}/n_k$.*

(6) *The period of ψ is*

$$\mathcal{T} = \mathrm{Re}\int_{B_{1,1}} \phi \simeq \left(\frac{T}{2\sqrt{r}}, \left(\sum_{k=1}^{N} \frac{1}{n_k}\right)\log r\right).$$

Proof. The proof of this proposition is straightforward computations similar to those of § 5. We omit the details. □

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End of proof of Theorem 1.4. It is now easy to prove embeddedness. Let $\sigma > 0$ be a small number. Consider the horizontal slab of $\mathbb{R}^3$:

$$\text{height}(\infty_{k+1}) + \frac{\sigma}{n_k}|\log r| \leqslant x_3 \leqslant \text{height}(\infty_k) - \frac{\sigma}{n_k}|\log r|.$$

By point (3) these slabs (for varying k) are disjoint. Let $z \in \Sigma$ such that $\psi(z)$ is in this slab. If r is small enough, z has to be in the domain of point (2) for some i. Moreover (up to some bounded terms that we can safely neglect),

$$\text{height}(\infty_k) + \frac{1-\sigma}{n_k}\log r \leqslant \text{height}(\infty_k) + \frac{1}{n_k}\log|v_{k,i}| \leqslant \text{height}(\infty_k) + \frac{\sigma}{n_k}\log r,$$
$$r^{1-\sigma} < |v_{k,i}(z)| < r^{\sigma}.$$

So z is in the domain of point (5). The images of these domains (for varying i) are contained in disjoint vertical cylinders by point (4). Hence the intersection of the surface with the slab under consideration has n_k disjoint components, each converging to a catenoid. So it is embedded. Consider the horizontal slab

$$\text{height}(\infty_k) - \frac{\sigma}{n_k}|\log r| \leqslant x_3 \leqslant \text{height}(\infty_k) + \frac{\sigma}{n_k}|\log r|.$$

Let $z \in \Sigma$ such that $\psi(z)$ is in this slab. Then $z \in \overline{\mathbb{C}}_k$ and satisfies $|v_{k,i}| \geqslant r^{\sigma}$, $|w_{k,i}| \geqslant r^{\sigma}$ for all i. Hence $|g(z)| \neq 1$ so the Gauss map is never horizontal on this domain. On the boundary, the surface is a graph since it converges to a catenoid. In a neighbourhood of infinity, the surface is also a graph since we have an embedded planar end. This implies that the intersection of the surface with the slab under consideration is a graph above the horizontal plane, hence embedded. Since these slabs cover all of $\mathbb{R}^3$, this proves that the surface is embedded. Proposition 7.1 implies that the surface, scaled by $2\sqrt{r}$, satisfy the Hypotheses 1.1 and 1.2. In particular point (4) says that $p_{k,i}$ is the asymptotic position of the neck. The uniqueness statement in Theorem 1.4 comes from the uniqueness in the implicit function theorem, and the fact that the Weierstrass data of a family of minimal surface satisfying our hypotheses may be written as in § 3. We will see this in the next section. This concludes the proof of Theorem 1.4.

8. Proof of Theorem 1.3

Our strategy is to prove that if M_t satisfies the hypotheses of the introduction, we may write its Weierstrass representation as in § 3, and then use Proposition 6.1.

Without loss of generality we may assume that $\Omega_{k,t}$, $U_{k,i,t}$ are closed domains with disjoint interiors. Let g_t be the Gauss map of M_t. We may assume that $g_t(\infty_k)$ is equal to 0 if k is even and ∞ otherwise. First assume that k is odd and consider the planar domain $\Omega_{k,t}$. By Hypothesis 1.2 (a), g_t converges to ∞ on this domain. Consider the domain $U_{k,i,t}$ and the circle $(\partial U_{k,i,t}) \cap \Omega_{k,t}$. The Gauss map sends this circle to a small circle near ∞ in $\mathbb{C} \cup \infty$. Let $D_{k,i,t}$ be the disk bounded by this circle, containing 0. Glue this disk to $\Omega_{k,t}$ by identifying the point $p \in \partial U_{k,i,t}$ with $g_t(p) \in \partial D_{k,i,t}$. Do the same for the circles $(\partial U_{k-1,i,t}) \cap \Omega_{k,t}$. Let $\tilde{\Omega}_{k,t}$ be the resulting genus zero compact Riemann surface. Let

λ_t be the scaling factor of Hypothesis 1.2 (c), i.e such that the necksizes of $\lambda_t M_t$ have non-zero limits. We define a meromorphic function $g_{k,t}$ on $\tilde{\Omega}_{k,t}$ by $g_{k,t} = 2\lambda_t/g_t$ in $\Omega_{k,t}$ and $g_{k,t} = 2\lambda_t/z$ in each disk. So $g_{k,t}$ has one simple pole in each disk. Now we would like to write $g_{k,t}$ as in § 3.1 but for this we need to identify $\tilde{\Omega}_{k,t}$ with $\mathbb{C} \cup \infty$. In other words, we need to define a global coordinate $z : \tilde{\Omega}_{k,t} \to \mathbb{C} \cup \infty$ (not to be confused with the z coordinate on the disks above).

We choose z as follows. Let $\pi_t : \Omega_{k,t} \to \mathbb{R}^2$ be the projection to the horizontal plane. Since $g_t \simeq \infty$ on $\Omega_{k,t}$, π_t is close to be an orientation preserving isometry. We choose z such that $\pi_t \simeq -z$ on $\Omega_{k,t}$. (The minus sign is here so that the notation agrees with the rest of the paper.) To prove that this is possible, we use quasiconformal mappings as follows. It is well known that we may prescribe the value of z at three points. Let ζ_1, ζ_2 be two points in $\pi_t(\Omega_{k,t})$. We define $z = -z'$, where z' is uniquely defined by

$$\begin{cases} z'(\pi_t^{-1}(\zeta_i)) = \zeta_i, \qquad i = 1, 2, \\ z'(\infty_k) = \infty. \end{cases}$$

By the analytic definition of quasiconformal mappings (see [**4**, p. 168]) π_t is K_t-quasiconformal on $\Omega_{k,t}$ with $K_t \to 1$. Hence $z' \circ \pi_t^{-1}$ is also K_t-quasiconformal. Let $\zeta \in \pi_t(\Omega_{k,t})$. Consider the quadrilateral $Q = (\zeta_1, \zeta_2, \zeta, \infty)$. By the geometric definition of quasiconformal mappings (see [**4**, p. 16]), the conformal modulus of $z' \circ \pi_t^{-1}(Q)$ converges to the modulus of Q. Since these quadrilaterals already agree at three points, this means that $z' \circ \pi_t^{-1}(\zeta) \to \zeta$. This proves that $\pi_t \simeq -z$ on $\Omega_{k,t}$. We may write

$$g_{k,t} = \sum_{i=1}^{n_k} \frac{\alpha_{k,i,t}}{z - a_{k,i,t}} - \sum_{i=1}^{n_{k-1}} \frac{\beta_{k,i,t}}{z - b_{k,i,t}}.$$

Here $z = a_{k,i,t}$ is the pole in the disk $D_{k,i,t}$. By Hypothesis 1.2 (b), $\pi_t(\partial D_{k,i,t})$ is contained in a disk whose radius goes to 0 and whose centre converges to $p_{k,i}$. Hence $a_{k,i,t} \to -p_{k,i}$ and in a similar way, $b_{k,i,t} \to -p_{k-1,i}$. To see that $\alpha_{k,i,t}$ and $\beta_{k,i,t}$ have non-zero limits, let η_t be the height differential of $\lambda_t M_t$. The necksize of $\lambda_t U_{k,i,t}$ is the imaginary part of $\int_{A_{k,i}} \eta_t$. Since the real part is zero, we may write

$$\int_{A_{k,i}} \eta_t = 2\pi \mathrm{i} \gamma_{k,i,t},$$

where $\gamma_{k,i,t}$ is real and has a non-zero limit $\gamma_{k,i}$ by Hypothesis 1.2 (c).

Since $g_t \simeq \infty$ on $\Omega_{k,t}$ we have $\mathrm{d}\pi_t \simeq -\frac{1}{2} g_t \eta_t \simeq -\mathrm{d}z$,

$$\eta_t = 2 g_t^{-1}(1 + \varepsilon_t(z))\,\mathrm{d}z = \frac{1}{\lambda_t} g_{k,t}(1 + \varepsilon_t(z))\,\mathrm{d}z,$$

where $\varepsilon_t(z)$ converges uniformly to 0. Integrating on a representative of $A_{k,i}$ contained in $\Omega_{k,t}$, we find

$$2\pi \mathrm{i} \gamma_{k,i,t} = 2\pi \mathrm{i} \alpha_{k,i,t} + \int_{A_{k,i}} \varepsilon_t(z) \left(\sum_{j \neq i} \frac{\alpha_{k,j,t}}{z - a_{k,j,t}} - \sum_{j} \frac{\beta_{k,j,t}}{z - b_{k,j,t}} \right) \mathrm{d}z.$$

It is easy to see from this formula that $\alpha_{k,i,t} \to \gamma_{k,i}$ and similarly, $\beta_{k,i,t} \to \gamma_{k-1,i}$. It remains to see that $\alpha_{k,i,t}$, $\beta_{k,i,t}$ and $\gamma_{k,i,t}$ satisfy the normalization (3.4). Since the flux is homology invariant, $\sum_{i=1}^{n_k} \gamma_{k,i,t}$ does not depend on k. Hence we may assume it is equal to 1 by choosing suitably λ_t. Since g_t has at least a double pole at ∞_k, we also have $\sum \alpha_{k,i,t} = \sum \beta_{k,i,t}$. Since $\alpha_{k,i,t} \to \gamma_{k,i}$, this sum converges to 1. Note that $\alpha_{k,i,t}$ is the residue at $a_{k,i,t}$ of $g_{k,t}\,\mathrm{d}z$, so depends on the choice of the coordinate z. By multiplying z by a suitable constant (converging to 1), we can assume that $\sum \alpha_{k,i,t} = 1$. When k is even, the above definitions have to be changed as follows. $g_{k,t} = (g_t/2\lambda_t)$ and $\pi_t \simeq \bar{z}$ (π_t reverses orientation when $g \simeq 0$). The construction of § 3 with $\sqrt{r} = (1/2\lambda_t)$ gives back the Weierstrass data of $\lambda_t M_t$ (the notation is the same up to the indices t). Note that $\lambda_t \to \infty$ implies that $r \to 0$. Since the period problem is solved for M_t, Proposition 6.1 implies that $\gamma_{k,i} = 1/n_k$ and $p_{k,i}$ is a balanced configuration. □

9. The case of multiple zeros

In this section we remove the restriction that $g_k\,\mathrm{d}z$ has simple zeros (see the beginning of § 6). We only have to change the definition of the map $\mathcal{F}_1$. The problem is that $g_k\,\mathrm{d}z$ might have a multiple zero for some value of the parameter X, and simple zeros for nearby values of X, so we have to define $\mathcal{F}_1$ without knowing a priori the multiplicity of the zeros. The following lemma is useful:

Lemma 9.1. *Let P be a polynomial of degree n in $\mathbb{C}$. Let Ω be a bounded domain in $\mathbb{C}$ containing all the zeros of P. Let f be a holomorphic function on Ω. Let*

$$F_k = \int_{\partial\Omega} \frac{P^{(k)} f}{P}, \quad k = 1, \dots, n.$$

Then $F_k = 0$, $k = 1, \dots, n$, if and only if P divides f in the ring of holomorphic functions on Ω, i.e f/P is holomorphic in Ω.

Proof. It is well known (see [**3**, p. 11]) that we may write $f = Ph + Q$, where h is holomorphic on Ω and Q is a polynomial with $\deg(Q) < \deg(P)$. In fact, h and Q are given by contour integration

$$h(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{f(w)\,\mathrm{d}w}{P(w)(w-z)},$$
$$Q(z) = \frac{1}{2\pi \mathrm{i}} \int_{\partial\Omega} \frac{f(w)(P(w)-P(z))\,\mathrm{d}w}{P(w)(w-z)}.$$

We want to prove that $Q = 0$. Using the Cauchy Theorem we get

$$F_k = \int_{\partial\Omega} \frac{P^{(k)} Q}{P}.$$

By the residue theorem on the complementary of Ω,

$$F_k = -2\pi \mathrm{i}\,\mathrm{Res}_\infty \frac{P^{(k)} Q}{P}\,\mathrm{d}z.$$

Let $w = 1/z$. The fraction $P^{(k)}/P$ has a zero of multiplicity k at ∞ so we may write

$$\frac{P^{(k)}}{P} = \sum_{\nu=k}^{\infty} a_{k,\nu} w^{\nu} \quad \text{with } a_{k,k} \neq 0,$$

$$Q = \sum_{\mu=1}^{n} b_{\mu} z^{\mu-1},$$

$$F_k = 2\pi \mathrm{i} \operatorname{Res}_{w=0} \sum_{\nu=k}^{\infty} \sum_{\mu=1}^{n} a_{k,\nu} b_{\mu} w^{\nu-\mu-1} = 2\pi \mathrm{i} \sum_{\mu=k}^{n} a_{k,\mu} b_{\mu}.$$

The system of n equation $F_k = 0$ in the unknowns b_μ is triangular with non-zero coefficients $a_{k,k}$ on the diagonal, so $b_\mu = 0$ and $Q = 0$. □

First assume that g_k only has a double zero at infinity, for X in a neighbourhood of X_0. We define $\mathcal{F}_{1,k}$ as follows. Let Ω be a bounded domain in $\mathbb{C}$ which contains the zeros of $g_k \, \mathrm{d}z$ and none of its poles. Let

$$P(z) = g_k(z) \times \prod_{i=1}^{n_k} (z - a_{k,i}) \times \prod_{i=1}^{n_{k-1}} (z - b_{k,i}).$$

P is clearly a polynomial and since g_k has a double zero at infinity, P has degree $n_k + n_{k-1} - 2$. Also P and $g_k \, \mathrm{d}z$ have the same zeros. Define $\mathcal{F}_{1,k}$ by

$$\mathcal{F}_{1,k,i} = \int_{\partial\Omega} \frac{P^{(i)}\eta}{P}, \quad i = 1, \ldots, n_k + n_{k-1} - 2.$$

By Lemma 9.1, $\mathcal{F}_{1,k} = 0$ if and only if η/g_k is holomorphic in Ω which is what we want. When g_k has more than a double zero at infinity, we first do an inversion so that the zeros of $g_k \, \mathrm{d}z$ are in a bounded domain and we define $\mathcal{F}_{1,k}$ in a similar way.

Proof of Proposition 6.3 in the general case. It remains to prove Proposition 6.3 with this new definition of $\mathcal{F}_{1,k}$. We write D for the partial differential with respect to the variables (α_k, β_k). Let $(\dot\alpha_k, \dot\beta_k) \in E$ such that $D\mathcal{F}_{1,k}(\dot\alpha_k, \dot\beta_k) = 0$. We compute

$$D\mathcal{F}_{1,k,i}(\dot\alpha_k, \dot\beta_k) = \int_{\partial\Omega} \frac{DP^{(i)}(\dot\alpha_k, \dot\beta_k)\eta}{P} - \int_{\partial\Omega} \frac{P^{(i)} DP(\dot\alpha_k, \dot\beta_k)\eta}{P^2}.$$

Since we compute the differential at a point where $\mathcal{F}_1 = 0$, η/P is holomorphic so the first integral vanishes by the Cauchy Theorem. By Lemma 9.1, the vanishing of the second integral for all i implies that P divides $\eta DP(\dot\alpha_k, \dot\beta_k)/P$ as holomorphic functions in Ω. Since η/P has no zero in Ω, this means that P divides $DP(\dot\alpha_k, \dot\beta_k)$ as polynomials, and since they have the same degree, we may write $DP(\dot\alpha_k, \dot\beta_k) = \lambda P$. Since $(\alpha_k, \beta_k) \mapsto P$ is linear, this implies that $\dot\alpha_{k,i} = \lambda\alpha_{k,i}$ and $\dot\beta_{k,i} = \lambda\beta_{k,i}$. From $\sum \dot\alpha_{k,i} = 0$ and $\sum \alpha_{k,i} = 1$, we get $\lambda = 0$. Hence $\dot\alpha_k = \dot\beta_k = 0$. This proves Proposition 6.3. □

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The idea to interpret a set of algebraic equations as a balancing condition in term of forces is remindful of the paper by Douady and Douady [**1**].

Wei never published his result but his 'Riemann example with a handle' was an important motivation for this paper. I also thank Michael Wolf and Matthias Weber for many discussions about this work, and the referee for pointing out some mistakes in the first version of this paper.

References

1. A. Douady and R. Douady, Changements de cadres à partir des surfaces minimales, *Didactique des Mathématiques, Cahier IREM 23-1, Université Paris 7* (1994)
2. J. D. Fay, *Theta functions on Riemann surfaces*, Lecture Notes in Mathematics, no. 352 (Springer, 1973).
3. P. Griffiths and J. Harris, *Principles of algebraic geometry* (Wiley Interscience, 1978).
4. O. Lehto and K. I. Virtanen, *Quasiconformal maps in the plane*, 2nd edn (Springer, 1973).
5. W. H. Meeks, J. Perez and A. Ros, Uniqueness of the Riemann minimal example, *Invent. Math.* **133** (1998), 107–132.
6. W. H. Meeks and H. Rosenberg, The geometry of periodic minimal surfaces, *Comment. Math. Helvetici* **68** (1993), 538–578.
7. M. Traizet, Weierstrass representation of some simply-periodic minimal surfaces, *Ann. Global Analysis Geom.* **20** (2001), 77–101.