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INDESTRUCTIBILITY WHEN THE FIRST TWO MEASURABLE CARDINALS ARE STRONGLY COMPACT

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Abstract. We prove two theorems concerning indestructibility properties of the first two strongly compact cardinals when these cardinals are in addition the first two measurable cardinals. Starting from two supercompact cardinals $\kappa_1 < \kappa_2$, we force and construct a model in which κ_1 and κ_2 are both the first two strongly compact and first two measurable cardinals, κ_1's strong compactness is fully indestructible (i.e., κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing), and κ_2's strong compactness is indestructible under $\mathrm{Add}(\kappa_2, \delta)$ for any ordinal δ. This provides an answer to a strengthened version of a question of Sargsyan found in [17, Question 5]. We also investigate indestructibility properties that may occur when the first two strongly compact cardinals are not only the first two measurable cardinals, but also exhibit nontrivial degrees of supercompactness.

§1. Introduction and preliminaries. The study of indestructibility properties that non-supercompact strongly compact cardinals may possess has been carried out in several papers, including [3, 5, 17]. In particular, [17, Question 5] asks whether it is possible for the first two strongly compact cardinals κ_1 and κ_2 to be the first two measurable cardinals, with the second strongly compact cardinal (κ_2) having its strong compactness indestructible under $\mathrm{Add}(\kappa_2, \kappa_2^{++})$ (where for $\kappa \geq \aleph_0$ a regular cardinal and λ an ordinal, $\mathrm{Add}(\kappa, \lambda)$ is the standard partial ordering for adding λ many Cohen subsets of κ).

The purpose of this paper is to answer [17, Question 5] in the affirmative, and also prove a theorem showing that the first two strongly compact cardinals κ_1 and κ_2 can be the first two measurable cardinals, where in addition, each κ_i (for $i = 1, 2$) is κ_i^+ supercompact and also exhibits certain indestructibility properties. Specifically, we will prove the following two theorems, where we take as terminology for the rest of this paper that a supercompact cardinal κ has its *supercompactness indestructible under forcing with a class of partial orderings* $\mathcal{C}$ if κ remains supercompact after forcing with members of $\mathcal{C}$.

Theorem 1.1. *Suppose $V \models$ "ZFC + $\kappa_1 < \kappa_2$ are supercompact." There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC +κ_1 and κ_2 are both the first two strongly compact and first two measurable cardinals." In $V^{\mathbb{P}}$, κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing, and κ_2's strong compactness is indestructible under forcing with* $\mathrm{Add}(\kappa_2, \delta)$ *for any ordinal* δ.

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THEOREM 1.2. *Suppose $V \models$ "ZFC $+\kappa_1 < \kappa_2$ are supercompact." There is then a partial ordering $\mathbb{P} \subseteq V$ such that $V^{\mathbb{P}} \models$ "ZFC $+\kappa_1$ and κ_2 are both the first two strongly compact and first two measurable cardinals." Further, in $V^{\mathbb{P}}$, the following properties hold.*

1. *Each κ_i (for $i = 1, 2$) is κ_i^+ supercompact and satisfies $2^{\kappa_i} = 2^{\kappa_i^+} = \kappa_i^{++}$.*
2. *$2^\delta = \delta^+$ for every $\delta \geq \kappa_2^{++}$.*
3. *κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing.*
4. *κ_2's strong compactness is indestructible under κ_2-directed closed, (κ_2^+, ∞)-distributive forcing.*
5. *Let $\lambda = (\kappa_2^+)^{V^{\mathbb{P}}}$. Then the λ supercompactness of κ_2 is indestructible under κ_2-directed closed forcing having size at most λ.*

We take this opportunity to make a few remarks concerning Theorems 1.1 and 1.2. We note that Theorem 1.1 provides a model witnessing a strengthened version of [17, Question 5], in that the first strongly compact and measurable cardinal κ_1 is fully indestructible, and the second strongly compact cardinal κ_2 has its strong compactness indestructible under $\mathrm{Add}(\kappa_2, \delta)$ for *arbitrary* δ, rather than being indestructible only under $\mathrm{Add}(\kappa_2, \kappa_2^{++})$. In addition, Theorem 1.2 provides a generalization of both [3, Theorem 1] and [5, Theorem 2]. Also, in Theorem 1.2, it is impossible for $i = 1, 2$ to have that κ_i is $2^{\kappa_i} = \kappa_i^{++}$ supercompact. This is since κ_1 and κ_2 are the first two measurable cardinals, and it is a well-known fact (see [13, Lemma 20.16]) that if κ is 2^κ supercompact, then κ is a limit of measurable cardinals. Further, because for $i = 1, 2$, any partial ordering which is (κ_i^+, ∞)-distributive adds no new subsets of κ_i^+ and hence also adds no new subsets of $P_{\kappa_i}(\kappa_i^+)$, each κ_i automatically has its κ_i^+ supercompactness indestructible under *every* (κ_i^+, ∞)-distributive forcing notion. However, unlike the situation with κ_2, the current state of forcing technology doesn't appear to provide a way for one to force the κ_1^+ supercompactness of κ_1 to be indestructible under κ_1-directed closed forcing having size at most κ_1^+. We will discuss this issue in greater detail towards the end of the paper.

Before continuing, we mention the overall structure of this paper. Section 1 contains our introductory comments. Section 2 contains the proofs of Theorems 1.1 and 1.2. Section 3 contains our concluding remarks.

We now briefly mention some preliminary information and terminology. Essentially, our notation and terminology are standard, and when this is not the case, this will be clearly noted. When forcing, $q \geq p$ will mean that *q is stronger than p*. If G is V-generic over $\mathbb{P}$, we will abuse notation slightly and use both $V[G]$ and $V^{\mathbb{P}}$ to indicate the universe obtained by forcing with $\mathbb{P}$. If $x \in V[G]$, then $\dot{x}$ will be a term in V for x. We may, from time to time, confuse terms with the sets they denote and write x when we actually mean $\dot{x}$ or $\check{x}$, especially when x is some variant of the generic set G, or x is in the ground model V. The abuse of notation mentioned above will be compounded by writing $x \in V^{\mathbb{P}}$ instead of $\dot{x} \in V^{\mathbb{P}}$. Any term for trivial forcing will always be taken as a term for the partial ordering $\{\emptyset\}$. If φ is a formula in the forcing language with respect to $\mathbb{P}$ and $p \in \mathbb{P}$, then $p \parallel \varphi$ means that *p decides φ*.

If $\mathbb{P}$ is an arbitrary partial ordering and κ is a regular cardinal, $\mathbb{P}$ is (κ, ∞)-*distributive* if for every sequence $\langle D_\alpha \mid \alpha < \kappa \rangle$ of dense open subsets of $\mathbb{P}$, $\bigcap_{\alpha<\kappa} D_\alpha$ is dense open. $\mathbb{P}$ is κ-*directed closed* if for every cardinal $\delta < \kappa$ and every directed set $\langle p_\alpha \mid \alpha < \delta \rangle$ of elements of $\mathbb{P}$ (where $\langle p_\alpha \mid \alpha < \delta \rangle$ is *directed* if every two elements p_ρ and p_ν have a common upper bound of the form p_σ), there is an upper bound $p \in \mathbb{P}$. $\mathbb{P}$ is κ-*strategically closed* if in the two person game of length $\kappa + 1$ in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha \le \kappa \rangle$, where Player I plays odd stages and Player II plays even stages (choosing the trivial condition at stage 0), Player II has a strategy which ensures the game can always be continued. $\mathbb{P}$ is $\prec\kappa$-*strategically closed* if in the two person game in which the players construct an increasing sequence $\langle p_\alpha \mid \alpha < \kappa \rangle$, where Player I plays odd stages and Player II plays even and limit stages (again choosing the trivial condition at stage 0), then Player II has a strategy which ensures the game can always be continued. Note that if $\mathbb{P}$ is κ^+-directed closed, then $\mathbb{P}$ is $\prec\kappa^+$-strategically closed.

An example of a partial ordering which is $\prec\kappa$-strategically closed and which will be used in the proof of Theorem 1.2 is the partial ordering $\mathbb{P}$ for adding a nonreflecting stationary set of ordinals of cofinality λ to κ, where $\lambda < \kappa$ is a regular cardinal. Specifically, $\mathbb{P} = \{p \mid \text{For some } \alpha < \kappa,\ p : \alpha \to \{0, 1\}$ is a characteristic function of S_p, a subset of α not stationary at its supremum nor having any initial segment which is stationary at its supremum, such that $\beta \in S_p$ implies $\beta > \lambda$ and $\mathrm{cof}(\beta) = \lambda\}$, ordered by $q \ge p$ iff $q \supseteq p$ and $S_p = S_q \cap \sup(S_p)$, i.e., S_q is an end extension of S_p. It is virtually immediate that $\mathbb{P}$ is λ-directed closed. For additional details, readers are urged to consult [6, second paragraph of Section 1, p. 106].

We recall for the benefit of readers the definition given by Hamkins in [11, Section 3] of the lottery sum of a collection of partial orderings. If $\mathfrak{A}$ is a collection of partial orderings, then the *lottery sum* is the partial ordering $\oplus\mathfrak{A} = \{\langle \mathbb{P}, p \rangle \mid \mathbb{P} \in \mathfrak{A}$ and $p \in \mathbb{P}\} \cup \{0\}$, ordered with 0 below everything and $\langle \mathbb{P}, p \rangle \le \langle \mathbb{P}', p' \rangle$ iff $\mathbb{P} = \mathbb{P}'$ and $p \le p'$. Intuitively, if G is V-generic over $\oplus\mathfrak{A}$, then G first selects an element of $\mathfrak{A}$ (or as Hamkins says in [11], "holds a lottery among the posets in $\mathfrak{A}$") and then forces with it.[1]

A corollary of Hamkins' work on gap forcing found in [10, 12] will be employed in the proof of Theorems 1.1 and 1.2. We therefore state as a separate theorem what is relevant for this paper, along with some associated terminology, quoting from [10, 12] when appropriate. Suppose $\mathbb{P}$ is a partial ordering which can be written as $\mathbb{Q} * \dot{\mathbb{R}}$, where $|\mathbb{Q}| < \delta$, $\mathbb{Q}$ is nontrivial, and $\Vdash_{\mathbb{Q}}$ "$\dot{\mathbb{R}}$ is δ^+-directed closed." In Hamkins' terminology of [10, 12], $\mathbb{P}$ *admits a gap at* δ. Also, as in the terminology of [10, 12] and elsewhere, an embedding $j : V \to M$ is *amenable to* V when $j \restriction A \in V$ for any $A \in V$. The specific corollary of Hamkins' work from [10, 12] we will be using is then the following.

Theorem 1.3 (Hamkins). *Suppose that $V[G]$ is a generic extension obtained by forcing with $\mathbb{P}$ that admits a gap at some regular $\delta < \kappa$. Suppose further that $j : V[G] \to M[j(G)]$ is an elementary embedding with critical point κ for*

[1]The terminology "lottery sum" is due to Hamkins, although the concept of the lottery sum of partial orderings has been around for quite some time and has been referred to at different junctures via the names "disjoint sum of partial orderings," "side-by-side forcing," and "choosing which partial ordering to force with generically."

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which $M[j(G)] \subseteq V[G]$ and $M[j(G)]^\delta \subseteq M[j(G)]$ in $V[G]$. Then $M \subseteq V$; indeed, $M = V \cap M[j(G)]$. If the full embedding j is amenable to $V[G]$, then the restricted embedding $j \restriction V : V \to M$ is amenable to V. If j is definable from parameters (such as a measure or extender) in $V[G]$, then the restricted embedding $j \restriction V$ is definable from the names of those parameters in V.

Theorem 1.3 immediately implies that if κ is measurable in a generic extension by a partial ordering admitting a gap at $\delta < \kappa$, then κ had to have been measurable in the ground model V.

Finally, we mention that we are assuming familiarity with the large cardinal notions of measurability, strong compactness, and supercompactness. Interested readers may consult [13] for further details.

§2. The proofs of Theorems 1.1 and 1.2.

We turn now to the proof of Theorem 1.1.

Proof. Suppose $V \vDash$ "ZFC $+\kappa_1 < \kappa_2$ are supercompact." Without loss of generality, by first doing a preliminary forcing and truncating the universe if necessary, we assume in addition that $V \vDash$ "GCH $+$ No cardinal $\lambda > \kappa_2$ is measurable."

The proof of Theorem 1.1 may now be divided into four modules, as follows:

Module 1: Let $\mathbb{P}^0 \in V$ be the partial ordering used in the proof of [3, Theorem 1] defined with respect to κ_1. Set $V_1 = V^{\mathbb{P}^0}$. It is then the case that $V_1 \vDash$ "κ_1 is both the least strongly compact and least measurable cardinal $+$ κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing." Since $\mathbb{P}^0$ may be defined so that $|\mathbb{P}^0| = \kappa_1$, standard arguments in conjunction with the Lévy–Solovay results [15] yield that $V_1 \vDash$ "GCH holds at and above κ_1 $+$ κ_2 is supercompact $+$ No cardinal $\lambda > \kappa_2$ is measurable."

Module 2: Let $\mathbb{P}^1 \in V_1$ be the Easton support iteration of length κ_2 which adds, to every measurable cardinal $\delta \in (\kappa_1, \kappa_2)$, a nonreflecting stationary set of ordinals of cofinality κ_1. Set $V_2 = V_1^{\mathbb{P}^1}$. Since $V_1 \vDash$ "No cardinal $\lambda > \kappa_2$ is measurable," by an argument due to Magidor (unpublished by him, but given in the proof of [1, Theorem 2]), $V_2 \vDash$ "κ_2 is both the least measurable and least strongly compact cardinal greater than κ_1." Because $V_1 \vDash$ "κ_1 is both the least strongly compact and least measurable cardinal $+$ κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing $+$ $\mathbb{P}^1$ is κ_1-directed closed," it consequently follows that $V_2 \vDash$ "κ_1 and κ_2 are both the first two strongly compact and first two measurable cardinals $+$ κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing."

Module 3: Working in V_2, let κ_1' be the least inaccessible cardinal greater than κ_1. Let $\mathbb{P}^2 \in V_2$ be $\mathrm{Add}(\kappa_1, 1) * \dot{\mathbb{F}}_{\kappa_1', \kappa_2}$, where $\mathbb{F}_{\kappa_1', \kappa_2}$ is Hamkins' presentation from [11, Section 1] of Woodin's notion of fast function forcing defined using functions whose domain lies in the half-open interval $[\kappa_1', \kappa_2)$. ($\mathbb{F}_{\kappa_1', \kappa_2} = \{p \mid p : \kappa_2 \to \kappa_2$ is a function such that $|\mathrm{dom}(p)| < \kappa_2$, $\mathrm{dom}(p) \subseteq [\kappa_1', \kappa_2)$, $\delta \in \mathrm{dom}(p)$ implies that δ is inaccessible, and if $\delta \in \mathrm{dom}(p)$, then $p''\delta \subseteq \delta$ and $|p \restriction \delta| < \delta\}$, ordered by inclusion.) Set $V_3 = V_2^{\mathbb{P}^2}$. By [11, Section 1, paragraph 3], $\Vdash_{\mathrm{Add}(\kappa_1, 1)}$ "$\dot{\mathbb{F}}_{\kappa_1', \kappa_2}$ is κ_1'-directed closed." Thus, $\mathbb{P}^2$ is κ_1-directed closed, from which it immediately follows

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that $V_3 \models$ "κ_1 is both the least strongly compact and least measurable cardinal + κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing." By [11, Theorem 1.7], $V_3 \models$ "κ_2 is strongly compact (and hence measurable)." In addition, since $|\text{Add}(\kappa_1, 1)| = \kappa_1 < \kappa_1^+ < \kappa_1'$ and $\Vdash_{\text{Add}(\kappa_1,1)}$ "$\dot{\mathbb{F}}_{\kappa_1', \kappa_2}$ is κ_1'-directed closed," $\mathbb{P}^2$ admits a gap at κ_1^+. Consequently, by our remarks immediately following the statement of Theorem 1.3, any cardinal greater than κ_1^+ which is measurable in V_3 had to have been measurable in V_2. Because $V_2 \models$ "κ_2 is the second measurable cardinal," it therefore follows that $V_3 \models$ "κ_2 is the second measurable cardinal" as well.

Module 4: We will now use methods from a theorem due to Usuba (specifically, [2, Theorem 3.1] and the techniques in its proof) to force over V_3 in order to create a model V_4 witnessing the conclusions of Theorem 1.1. Since Usuba's theorem and proof have only recently appeared in print, we will provide a detailed exposition of his arguments, feeling free to quote verbatim when appropriate from [2].

By our work from Module 3, we can assume that there is a fast function $f : \kappa_2 \to \kappa_2$ in V_3. Define $\mathbb{P}^3 = \langle\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha\rangle \mid \alpha < \kappa_2\rangle$, an Easton support iteration of length κ_2, as follows. Let $\mathbb{P}_0 = \text{Add}(\kappa_1, 1)$. $\dot{\mathbb{Q}}_\alpha$ is then a name for the trivial forcing notion, unless $\alpha \in \text{dom}(f)$. In this case, $\dot{\mathbb{Q}}_\alpha$ is a name for the lottery sum $\oplus_{\beta < f(\alpha)} \text{Add}(\alpha, \beta)$, as defined in $V_3^{\mathbb{P}_\alpha}$, where by convention, we take $\text{Add}(\alpha, 0)$ to be trivial forcing.

Let $G \subseteq \mathbb{P}^3$ be V_3-generic. Set $V_4 = V_3[G]$. The arguments of [11, Theorem 3.7] show that κ_2 remains strongly compact in V_4. We wish to show that in V_4, the strong compactness of κ_2 is indestructible under $\text{Add}(\kappa_2, \delta)$ for all δ. Fix δ, and let $g \subseteq \text{Add}(\kappa_2, \delta)$ be V_4-generic. If we let $Q = \bigcup g$, then $Q : \kappa_2 \times \delta \to 2$ is a function.

Let $\lambda > \max(\kappa_2, \delta)$ be a regular cardinal, and fix a cardinal $\theta \geq 2^{\lambda^{<\kappa_2}}$. By [11, Theorem 1.12], let $j : V_3 \to M$ be an ultrapower embedding by a κ_2-complete, fine ultrafilter $\mathcal{U} \in V_3$ over $P_{\kappa_2}(\theta)$ with $\text{crit}(j) = \kappa_2$ such that $|[\text{id}]_\mathcal{U}|^M < j(f)(\kappa_2)$. Since there is no source of confusion, we will drop the subscript from elements of M and denote them as $[h]$. As usual, $j''\theta \subseteq [\text{id}]$, so $|\theta|^M \leq |[\text{id}]|^M$.

Claim 2.1. *There is in M a function $\pi : [\text{id}] \to \theta$ such that for all $\alpha < \theta$, $\pi(j(\alpha)) = \alpha$.*

Proof. For each $\alpha < \theta$, let $g_\alpha : P_{\kappa_2}(\theta) \to V_3$ be a function such that $[g_\alpha] = \alpha$. Without loss of generality, we can assume that $g_\alpha(p)$ is defined for every $p \in P_{\kappa_2}(\theta)$. Let $h : P_{\kappa_2}(\theta) \to V_3$ be so that for each $p \in P_{\kappa_2}(\theta)$, $h(p)$ is the function having domain p such that for every $\alpha \in p$, $h(\alpha) = g_\alpha(p)$. It follows that $[h]$ is a function with domain $[\text{id}]$, and by the fineness of $\mathcal{U}$, for each $\alpha < \theta$, on a measure 1 subset of $\{p \mid \alpha \in p\}$, $[h](j(\alpha)) = [g_\alpha] = \alpha$. This completes the proof, since we can easily use $[h]$ to define a function π with the desired properties by setting $\pi = [h]$. ⊣

We now proceed by lifting j through $\mathbb{P}^3 * \dot{\text{Add}}(\kappa_2, \delta)$. As usual, $j(\mathbb{P}^3)$ can be factorized as $\mathbb{P}^3 * \dot{\mathbb{Q}} * \dot{\mathbb{P}}_{\text{tail}}$, where $\dot{\mathbb{Q}}$ is a name for the lottery sum $\oplus_{\beta < j(f)(\kappa_2)} \text{Add}(\kappa_2, \beta)$, and $\dot{\mathbb{P}}_{\text{tail}}$ is a name for the remaining stages through $j(\kappa_2)$. Using G as an M-generic filter for $\mathbb{P}^3$, we can form $M[G]$. Also, since $\delta < \theta \leq |[\text{id}]|^M < j(f)(\kappa_2)$, we can choose to force above a condition in $\mathbb{Q} = \text{int}_G(\dot{\mathbb{Q}})$ that opts for $\text{Add}(\kappa_2, \delta)$. Thus, we can use g as an $M[G]$-generic filter for $\mathbb{Q}$. Furthermore, note that since $j(f)(\kappa_2) > |[\text{id}]|^M$, $\mathbb{P}_{\text{tail}} = \text{int}_{G*g}(\dot{\mathbb{P}}_{\text{tail}})$ is at least $(|[\text{id}]|^+)^M$-directed closed in $M[G][g]$.

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Force over $V_4[g]$ to add a generic filter G_{tail} for $\mathbb{P}_{\text{tail}}$. Using G_{tail} as an $M[G][g]$-generic filter for $\mathbb{P}_{\text{tail}}$, since $j''G \subseteq G$, we can lift j in $V_4[g][G_{\text{tail}}]$ to $j : V_4 \to M[j(G)]$, where $j(G) = G * g * G_{\text{tail}}$. In order to further lift j through $\text{Add}(\kappa_2, \delta)$, we will use a master condition argument. Consider the function π given by Claim 2.1, and note that $|[\text{id}] \cap j(\delta)|^M \leq |[\text{id}]|^M < j(\kappa_2)$. Define in $M[j(G)]$ a function $q : \kappa_2 \times ([\text{id}] \cap j(\delta)) \to 2$ given by $q(\langle \beta, \gamma \rangle) = Q(\langle \beta, \pi(\gamma) \rangle)$ if $\pi(\gamma) < \delta$, and 0 otherwise. Clearly, q is a condition in $j(\text{Add}(\kappa_2, \delta))$.

Claim 2.2. *$q \geq j(p)$ for all $p \in g$.*

Proof. By elementarity and the fact that $\text{crit}(j) = \kappa_2$, for each $p \in g$, $j(p)$ is a function with domain $j''\text{dom}(p) = \{\langle \beta, j(\gamma) \rangle \mid \langle \beta, \gamma \rangle \in \text{dom}(p)\}$. Hence, $\text{dom}(j(p)) \subseteq \text{dom}(q)$. For $\langle \beta, j(\gamma) \rangle \in \text{dom}(j(p))$, we have $j(p)(\langle \beta, j(\gamma) \rangle) = p(\langle \beta, \gamma \rangle) = Q(\langle \beta, \gamma \rangle) = Q(\langle \beta, \pi(j(\gamma)) \rangle) = q(\langle \beta, j(\gamma) \rangle)$. ⊣

Force over $V_4[g][G_{\text{tail}}]$ to add a generic filter $h^* \subseteq j(\text{Add}(\kappa_2, \delta))$ containing q. By Claim 2.2, we can lift j in $V_4[g][G_{\text{tail}}][h^*]$ to $j : V_4[g] \to M[j(G)][h^*]$.

Let $\vec{X} = \langle X_\xi \mid \xi < 2^{[\lambda]^{<\kappa_2}} \rangle \in V_4[g]$ be an enumeration of $\wp(P_{\kappa_2}(\lambda))^{V_4[g]}$. In $M[j(G)][h^*]$, consider the set $B = \{\xi \in [\text{id}] \mid [\text{id}] \cap j(\lambda) \in j(\vec{X})_\xi\}$. Since $\mathbb{P}_{\text{tail}} * j(\dot{\text{A}}\text{dd}(\kappa_2, \delta))$ is at least $(|[\text{id}]|^+)^M$-directed closed in $M[G][g]$, $B \in M[G][g] \subseteq V_4[g]$. Hence, $\mathcal{W} = \{X_\xi \in \wp(P_{\kappa_2}(\lambda))^{V_4[g]} \mid j(\xi) \in B\} \in V_4[g]$, and since $\theta \geq 2^{[\lambda]^{<\kappa_2}}$, $\mathcal{W}$ is easily seen to be a κ_2-complete, fine ultrafilter over $P_{\kappa_2}(\lambda)$. Thus, κ_2 is λ strongly compact in $V_4[g]$. Since λ can be chosen arbitrarily large, we have shown that κ_2 remains strongly compact in $V_4[g]$, from which we immediately infer that $V_4 \vDash$ "κ_2's strong compactness is indestructible under forcing with $\text{Add}(\kappa_2, \delta)$ for any ordinal δ."

We finish the proof of Theorem 1.1 by combining the work of Modules 1–4. Because $V_3 \vDash$ "$\mathbb{P}^3$ is κ_1-directed closed," $V_4 \vDash$ "κ_1 is both the least strongly compact and least measurable cardinal + κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing." By the arguments of Module 4, since we have taken $\text{Add}(\kappa_2, 0)$ to be trivial forcing, $V_4 \vDash$ "κ_2 is strongly compact and hence measurable." In addition, by its definition, $\mathbb{P}^3 = \text{Add}(\kappa_1, 1) * \dot{\mathbb{Q}}$, where $|\text{Add}(\kappa_1, 1)| = \kappa_1$ and $\Vdash_{\text{Add}(\kappa_1,1)}$ "$\dot{\mathbb{Q}}$ is κ_1'-directed closed." This means that as in Module 3, $V_4 \vDash$ "κ_2 is the second measurable cardinal." It consequently follows that V_4 is the desired model witnessing the conclusions of Theorem 1.1. By writing $\mathbb{P} = \mathbb{P}^0 * \dot{\mathbb{P}}^1 * \dot{\mathbb{P}}^2 * \dot{\mathbb{P}}^3$, we have completed the proof of Theorem 1.1. ⊣

We turn now to the proof of Theorem 1.2.

Proof. Suppose $V \vDash$ "ZFC $+\kappa_1 < \kappa_2$ are supercompact." Without loss of generality, by truncating the universe if necessary, we assume in addition that $V \vDash$ "No cardinal $\lambda > \kappa_2$ is inaccessible." We explicitly mention it will of course be the case that in any of our generic extensions of V, there will be no inaccessible cardinals greater than κ_2.

Lemma 2.3. *There is a partial ordering $\mathbb{P}^* \subseteq V$ such that $V^{\mathbb{P}^*} \vDash$ "For $i = 1, 2$, $2^{\kappa_i} = 2^{\kappa_i^+} = \kappa_i^{++}$ and κ_i is supercompact + There are supercompact ultrafilters $\mathcal{U}_i$ over $P_{\kappa_i}(\kappa_i^+)$ such that κ_i isn't measurable in the ultrapower by $\mathcal{U}_i$ + $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa_2^{++}$."*

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Proof. To prove Lemma 2.3, begin by forcing GCH using the partial ordering $\mathbb{Q}^0$. Next, force indestructibility for κ_1's supercompactness as in [14], using a partial ordering $\mathbb{Q}^1$ having cardinality κ_1. Since $|\mathbb{Q}^1| = \kappa_1$, $2^\delta = \delta^+$ for every $\delta \geq \kappa_1$, and by the results of [15], κ_2 remains supercompact after forcing with $\mathbb{Q}^1$. Now, force with the partial ordering $\mathbb{Q}^2$ used in the proof of [1, Lemma 2] defined with respect to κ_2 and acting nontrivially only on inaccessible cardinals in the open interval (κ_1, κ_2). Since this definition ensures that $\mathbb{Q}^2$ is κ_1-directed closed, κ_1 remains supercompact after forcing with $\mathbb{Q}^2$. In addition, after forcing with $\mathbb{Q}^2$, as in [1], κ_2 remains supercompact, $2^{\kappa_2} = 2^{\kappa_2^+} = \kappa_2^{++}$, $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa_2^{++}$, and there is a supercompact ultrafilter $\mathcal{U}_2$ over $P_{\kappa_2}(\kappa_2^+)$ such that κ_2 isn't measurable in the ultrapower by $\mathcal{U}_2$. Finally, force with the partial ordering $\mathbb{Q}^3$ used in the proof of [1, Lemma 2] defined with respect to κ_1 on inaccessible cardinals $\delta < \kappa_1$ such that $2^\delta = \delta^+$. Let $\mathbb{P}^* = \mathbb{Q}^0 * \dot{\mathbb{Q}}^1 * \dot{\mathbb{Q}}^2 * \dot{\mathbb{Q}}^3$. Because $\mathbb{Q}^3$ can be defined so as to have cardinality κ_1, our previous work, [1, Lemma 2] applied to $\mathbb{Q}^3$, another application of the results of [15], and standard arguments for calculating the size of power sets in generic extensions yield that $V^{\mathbb{P}^*} \vDash$ "For $i = 1, 2$, $2^{\kappa_i} = 2^{\kappa_i^+} = \kappa_i^{++}$ and κ_i is supercompact + There are supercompact ultrafilters $\mathcal{U}_i$ over $P_{\kappa_i}(\kappa_i^+)$ such that κ_i isn't measurable in the ultrapower by $\mathcal{U}_i$ + $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa_2^{++}$." This completes the proof of Lemma 2.3. ⊣

With a slight abuse of notation, we relabel $V^{\mathbb{P}^*}$ as V. Now, as in the proof of Theorem 1.1, let $\mathbb{P}^0 \in V$ be the partial ordering used in the proof of [3, Theorem 1] defined with respect to κ_1. Set $V_1 = V^{\mathbb{P}^0}$. Since $\mathbb{P}^0$ is an Easton support iteration of length κ_1 such that $|\mathbb{P}^0| = \kappa_1$, standard arguments in conjunction with the results of [15] and Lemma 2.3 yield that $V_1 \vDash$ "For $i = 1, 2$, $2^{\kappa_i} = 2^{\kappa_i^+} = \kappa_i^{++}$ + $2^\delta = \delta^+$ for every cardinal $\delta \geq \kappa_2^{++}$ + κ_2 is supercompact + There is a supercompact ultrafilter $\mathcal{U}_2^*$ over $P_{\kappa_2}(\kappa_2^+)$ such that κ_2 isn't measurable in the ultrapower by $\mathcal{U}_2^*$." Because [9, Lemmas 1.2 and 1.4] require no GCH assumptions, and because the proof of [3, Lemma 2] only requires that $2^\delta = \delta^+$ holds for sufficiently large δ, it is then the case that $V_1 \vDash$ "κ_1 is both the least strongly compact and least measurable cardinal + κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing."

Lemma 2.4. *$V_1 \vDash$ "κ_1 is κ_1^+ supercompact."*

Proof. We will follow to a certain extent the proofs of [9, Lemma 1.5] and [16, Theorem 2.5]. Let G be V-generic over $\mathbb{P}^0$. By Lemma 2.3, let $j : V \to M$ be an elementary embedding (which we take to be generated by the ultrafilter $\mathcal{U}_1$ over $P_{\kappa_1}(\kappa_1^+)$ witnessing the κ_1^+ supercompactness of κ_1) such that $M \vDash$ "κ_1 isn't measurable." Because we may assume that $\mathbb{P}^0$ does nontrivial forcing only at V-measurable cardinals, $j(\mathbb{P}^0) = \mathbb{P}^0 * \dot{\mathbb{Q}}$, where the first nontrivial stage in $\dot{\mathbb{Q}}$ is forced to be well above κ_1^+. In addition, because $2^{\kappa_1^+} = \kappa_1^{++}$, we may let $\langle \dot{A}_\alpha \mid \alpha < \kappa_1^{++} \rangle \in V$ be an enumeration of all of the canonical $\mathbb{P}^0$-names for subsets of $(P_{\kappa_1}(\kappa_1^+))^{V[G]}$. Also, as $\mathbb{P}^0$ is an Easton support iteration of length κ_1 and is thus κ_1-c.c., $M[G]$ remains κ_1^+-closed with respect to $V[G]$. We now use terminology and results from [9], to which we refer readers for further details and explanations. Specifically, since $\mathbb{P}^0$ is an Easton support iteration of partial orderings satisfying the Prikry property and $\mathbb{Q} = j(\mathbb{P}^0)/G$ is κ_1^{++}-weakly closed, we may define in $V[G]$ an increasing sequence of Easton extensions $\langle p_\alpha \mid \alpha < \kappa_1^{++} \rangle$ of members of $\mathbb{Q}$ such that in $M[G]$, for every $\alpha < \kappa_1^{++}$, $p_{\alpha+1} \parallel_{\mathbb{Q}}$ "$\langle j(\beta) \mid \beta < \kappa_1^+ \rangle \in j(\dot{A}_\alpha)$."[2] Note that because $M[G]^{\kappa_1^+} \subseteq M[G]$, $\langle p_\alpha \mid \alpha < \kappa_1^{++} \rangle$ is well defined, and every initial segment of this sequence is a member of $M[G]$. In analogy to the proof of [9, Lemma 1.5], this now allows us to define $\mathcal{U}_1^* \in V_1$ by $A \in \mathcal{U}_1^*$ iff for some $\alpha < \kappa_1^{++}$ and some $\mathbb{P}^0$-name $\dot{A}$ for A, in $M[G]$, $p_\alpha \Vdash_{\mathbb{Q}}$ "$\langle j(\beta) \mid \beta < \kappa_1^+ \rangle \in j(\dot{A})$." Because $j''G = G$, as in [9, Lemma 1.5], the definition of $\mathcal{U}_1^*$ doesn't depend on a particular name for A, and so $\mathcal{U}_1^*$ is well defined. The arguments given on [3, proof of Lemma 2, p. 1408] (suitably modified) then allow us to infer that $V_1 = V[G] \vDash$ "κ_1 is κ_1^+ strongly compact."

To show that $V_1 \vDash$ "κ_1 is κ_1^+ supercompact" (i.e., that $V_1 \vDash$ "$\mathcal{U}_1^*$ is normal"), we modify Magidor's argument found in the proof of [16, Theorem 2.5, pp. 47–48]. Specifically, suppose $\Vdash_{\mathbb{P}^0}$ "$\dot{f} : \dot{P}_{\kappa_1}(\kappa_1^+) \to \kappa_1^+$ is such that $\dot{f}(p) \in p$ for every $p \in \dot{P}_{\kappa_1}(\kappa_1^+)$." It is then the case that $\Vdash_{j(\mathbb{P}^0)}$ "$j(\dot{f}) : \dot{P}_{j(\kappa_1)}(j(\kappa_1^+)) \to j(\kappa_1)$ is such that $j(\dot{f})(p) \in p$ for every $p \in \dot{P}_{j(\kappa_1)}(j(\kappa_1^+))$." Let $\Phi = \langle \varphi_\alpha \mid \alpha < \kappa_1^+ \rangle \in V[G]$ be the sequence where each φ_α is the statement "$j(\dot{f})(\langle j(\beta) \mid \beta < \kappa_1^+ \rangle) = j(\alpha)$." Since $M[G]^{\kappa_1^+} \subseteq M[G]$, $\Phi \in M[G]$. And, working in V, for each $\alpha < \kappa_1^+$, let γ_α be the least ordinal such that $\dot{A}_{\gamma_\alpha}$ is a term for $\{p \in P_{\kappa_1}(\kappa_1^+) \mid f(p) = \alpha\}$. By the regularity of κ_1^{++} in V, $\gamma = \sup_{\alpha < \kappa_1^+}(\gamma_\alpha + 1) < \kappa_1^{++}$. Working for the rest of the proof of Lemma 2.4 in $M[G]$, we may define the increasing sequence of Easton extensions $\langle q_\alpha \mid \alpha < \kappa_1^+ \rangle$ of members of $\mathbb{Q}$ such that $q_0 = p_\gamma$ and for every $\alpha < \kappa_1^+$, $q_{\alpha+1} \parallel_{\mathbb{Q}} \varphi_\alpha$. Let q be an upper bound to $\langle q_\alpha \mid \alpha < \kappa_1^+ \rangle$. If $q' \geq q$ and $\delta < \kappa_1^+$ are such that $q' \Vdash_{\mathbb{Q}}$ "$j(\dot{f})(\langle j(\beta) \mid \beta < \kappa_1^+ \rangle) = j(\delta)$," then since by construction $q \geq q_{\delta+1}$, $q' \geq q_{\delta+1}$. Because $q_{\delta+1} \parallel_{\mathbb{Q}} \varphi_\delta$, it consequently follows that $q_{\delta+1} \Vdash_{\mathbb{Q}}$ "$j(\dot{f})(\langle j(\beta) \mid \beta < \kappa_1^+ \rangle) = j(\delta)$," i.e., that $q_{\delta+1} \Vdash_{\mathbb{Q}}$ "$\langle j(\beta) \mid \beta < \kappa_1^+ \rangle \in j(\dot{A}_{\gamma_\delta})$." As again by construction, $q_{\delta+1} \geq p_\gamma \geq p_{\gamma_\delta+1}$ and $p_{\gamma_\delta+1} \parallel_{\mathbb{Q}}$ "$\langle j(\beta) \mid \beta < \kappa_1^+ \rangle \in j(\dot{A}_{\gamma_\delta})$," $p_{\gamma_\delta+1} \Vdash_{\mathbb{Q}}$ "$\langle j(\beta) \mid \beta < \kappa_1^+ \rangle \in j(\dot{A}_{\gamma_\delta})$." Thus, $A_{\gamma_\delta} = \{p \in P_{\kappa_1}(\kappa_1^+) \mid f(p) = \delta\} \in \mathcal{U}_1^*$. This completes the proof of Lemma 2.4. ⊣

Working now in V_1, let $\langle \delta_\alpha \mid \alpha < \kappa_2 \rangle$ enumerate the inaccessible cardinals in the open interval (κ_1, κ_2). We define an Easton support iteration of length κ_2, $\mathbb{P}^1 = \langle \langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\alpha \rangle \mid \alpha < \kappa_2 \rangle$, as follows:

1. $\mathbb{P}_0 = \mathrm{Add}(\kappa_1^{+3}, 1)$.
2. If $V_1 \vDash$ "δ_α is inaccessible but not measurable," then $\dot{\mathbb{Q}}_\alpha$ is a term for the lottery sum of all δ_α-directed closed partial orderings in $V^{\mathbb{P}_\alpha}$ (including trivial forcing) having size at most δ_α^+.
3. If $V_1 \vDash$ "δ_α is measurable," then $\mathbb{P}_{\alpha+1} = \mathbb{P}_\alpha * \dot{\mathbb{Q}}' * \dot{\mathbb{R}}$, where $\dot{\mathbb{Q}}'$ is a term for the lottery sum of all δ_α-directed closed partial orderings in $V^{\mathbb{P}_\alpha}$ (again including trivial forcing) having rank below $\delta_{\alpha+1}$, and $\dot{\mathbb{R}}$ is a term for the partial ordering which adds a nonreflecting stationary set of ordinals of cofinality κ_1 to δ_α.

Let $V_2 = V_1^{\mathbb{P}^1}$. Because $V_1 \vDash$ "$\mathbb{P}^1$ is a κ_1^{+3}-directed closed, κ_2-c.c . partial ordering having size κ_2," $V_2 \vDash$ "κ_1 is both the least strongly compact and least measurable cardinal + κ_1 is κ_1^+ supercompact + For $i = 1, 2$, $2^{\kappa_i} = 2^{\kappa_i^+} = \kappa_i^{++}$ + Cardinals and

[2] Here and throughout the rest of the proof of Lemma 2.4, we slightly abuse notation and assume that a statement which is actually in the forcing language with respect to $j(\mathbb{P}^0) = \mathbb{P}^0 * \dot{\mathbb{Q}}$ has been rewritten in the forcing language with respect to $\mathbb{Q}$.

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cofinalities at and above κ_2 are the same as in $V_1 + 2^\delta = \delta^+$ for every $\delta \geq \kappa_2^{++} + \kappa_1$'s strong compactness is indestructible under arbitrary κ_1-directed closed forcing."

LEMMA 2.5. *$V_2 \models$ "No cardinal $\delta \in (\kappa_1, \kappa_2)$ is measurable."*

PROOF. Since $\mathbb{P}^1 = \mathrm{Add}(\kappa_1^{+3}, 1) * \dot{\mathbb{Q}}$, $|\mathrm{Add}(\kappa_1^{+3}, 1)| = \kappa_1^{+3} < \kappa_1^{+4} < \delta_0$, and $\Vdash_{\mathrm{Add}(\kappa_1^{+3},1)}$ "$\dot{\mathbb{Q}}$ is δ_0-directed closed," $\mathbb{P}^1$ admits a gap at κ_1^{+4}. As in Module 3, by our remarks immediately following the statement of Theorem 1.3, any $\delta \in (\kappa_1, \kappa_2)$ which is measurable in V_2 had to have been measurable in V_1. However, by the definition of $\mathbb{P}^1$, $V_2 \models$ "Any $\delta \in (\kappa_1, \kappa_2)$ which is measurable in V_1 contains a nonreflecting stationary set of ordinals of cofinality κ_1 and hence is not measurable (or even weakly compact)." This completes the proof of Lemma 2.5. ⊣

LEMMA 2.6. *For $\lambda = (\kappa_2^+)^{V_2} = (\kappa_2^+)^{V_1}$, $V_2 \models$ "κ_2 is λ supercompact and has its λ supercompactness indestructible under κ_2-directed closed forcing having size at most λ."*

PROOF. Let $\mathbb{Q} \in V_2$ be any partial ordering (including trivial forcing) such that $V_2 \models$ "$\mathbb{Q}$ is a κ_2-directed closed partial ordering such that $|\mathbb{Q}| \leq \lambda$," with $\dot{\mathbb{Q}}$ a canonical term for $\mathbb{Q}$. By our remarks in the paragraph immediately following the proof of Lemma 2.3, let $j : V_1 \to M$ be an elementary embedding (which we take to be generated by the supercompact ultrafilter $\mathcal{U}_2^*$ over $P_{\kappa_2}(\kappa_2^+)$) such that $M \models$ "κ_2 isn't measurable." By forcing in M above a condition opting for $\mathbb{Q}$ in the stage κ_2 lottery held in M in the definition of $j(\mathbb{P}^1)$, we may assume that $j(\mathbb{P}^1 * \dot{\mathbb{Q}})$ is forcing equivalent in M to $\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}} * j(\dot{\mathbb{Q}})$, where $\dot{\mathbb{R}}$ is a term for the portion of $j(\mathbb{P}^1)$ acting on ordinals in the open interval $(\kappa_2, j(\kappa_2))$.

Let G_0 be V_1-generic over $\mathbb{P}^1$ and G_1 be $V_1[G_0]$-generic over $\mathbb{Q}$. Because $M[G_0][G_1] \models$ "$|\mathbb{R}| = j(\kappa_2)$," κ_2 is inaccessible, and $2^{\kappa_2} = 2^{\kappa_2^+} = 2^\lambda = \kappa_2^{++} = \lambda^+$, $V_1 \models$ "$|j(\kappa_2^{++})| = |j(\lambda^+)| = |j(2^{\kappa_2})| = |\{f \mid f : P_{\kappa_2}(\lambda) \to \kappa_2^{++}\}| = |\{f \mid f : \lambda \to \kappa_2^{++}\}| = |\{f \mid f : \lambda \to \lambda^+\}| = |[\lambda^+]^\lambda| = 2^\lambda = \lambda^+$." Consequently, $V_1[G_0][G_1] \models$ "There are (at most) $\lambda^+ = 2^\lambda = |j(\kappa_2^{++})| = |j(2^{\kappa_2})|$ many dense open subsets of $\mathbb{R}$ present in $M[G_0][G_1]$." It is therefore possible to let $\langle D_\alpha \mid \alpha < \lambda^+ \rangle$ enumerate the dense open subsets of $\mathbb{R}$ which are members of $M[G_0][G_1]$. Since standard arguments show that $M[G_0][G_1]$ remains λ-closed with respect to $V_1[G_0][G_1]$ and $\mathbb{R}$ is $\prec\lambda^+$-strategically closed in both $M[G_0][G_1]$ and $V_1[G_0][G_1]$, working in $V_1[G_0][G_1]$, we may then meet all of these sets in order to build an $M[G_0][G_1]$-generic object G_2 over $\mathbb{R}$ such that $j''G_0 \subseteq G_0 * G_1 * G_2$. Still working in $V_1[G_0][G_1]$, j lifts to $j : V_1[G_0] \to M[G_0][G_1][G_2]$. Since $M[G_0][G_1][G_2]$ remains λ-closed with respect to $V_1[G_0][G_1][G_2] = V_1[G_0][G_1]$, $V_1[G_0] \models$ "$|\mathbb{Q}| \leq \lambda$," $j(\kappa_2) > \lambda$, and $M[G_0][G_1][G_2] \models$ "$j(\mathbb{Q})$ is $j(\kappa_2)$-directed closed," there is a master condition $q \in V_1[G_0][G_1]$ for $\{j(p) \mid p \in G_1\}$. Because $V_1 \models$ "$|j(\lambda^+)| = |j(2^\lambda)| = \lambda^+$" and $M[G_0][G_1][G_2] \models$ "$|j(\mathbb{Q})| \leq j(\lambda)$," there are (at most) λ^+ many dense open subsets of $j(\mathbb{Q})$ present in $V_1[G_0][G_1]$. As $j(\mathbb{Q})$ is λ^+-directed closed and hence $\prec\lambda^+$-strategically closed in both $M[G_0][G_1][G_2]$ and $V_1[G_0][G_1]$, we may thus as was done for G_2 build in $V_1[G_0][G_1]$ an $M[G_0][G_1][G_2]$-generic object G_3 for $j(\mathbb{Q})$ containing q. It is then the case that $j''(G_0 * G_1) \subseteq G_0 * G_1 * G_2 * G_3$, so we may fully lift j in $V_1[G_0][G_1]$ to a λ supercompactness embedding $j : V_1[G_0][G_1] \to M[G_0][G_1][G_2][G_3]$. This completes the proof of Lemma 2.6. ⊣

Lemma 2.7. *$V_2 \vDash$ "κ_2 is a strongly compact cardinal whose strong compactness is indestructible under κ_2-directed closed forcing which is also (κ_2^+, ∞)-distributive."*

Proof. We mimic to a certain extent the proof of [5, Lemma 3], once again feeling free to quote verbatim as appropriate. We begin as in the proof of Lemma 2.6. Let $\mathbb{Q} \in V_2$ be any partial ordering (including trivial forcing) such that $V_2 \vDash$ "$\mathbb{Q}$ is both κ_2-directed closed and (κ_2^+, ∞)-distributive," with $\dot{\mathbb{Q}}$ a canonical term for $\mathbb{Q}$. Let $\gamma = (\max(|\mathrm{TC}(\dot{\mathbb{Q}})|, \kappa_2^{++}))^{+\omega}$, and let $\lambda > \gamma$ be a regular cardinal. Take $j : V_1 \to M$ to be an elementary embedding witnessing the λ supercompactness of κ_2 generated by a supercompact ultrafilter over $P_{\kappa_2}(\lambda)$. By forcing in M above a condition opting for $\mathbb{Q}$ in the stage κ_2 lottery held in M in the definition of $j(\mathbb{P}^1)$, we may assume that $j(\mathbb{P}^1 * \dot{\mathbb{Q}})$ is forcing equivalent in M to $\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}' * \dot{\mathbb{S}} * j(\dot{\mathbb{Q}})$, where $\dot{\mathbb{Q}} * \dot{\mathbb{R}}'$ is a term for the forcing taking place at stage κ_2 in M, $\dot{\mathbb{R}}'$ is a term for the partial ordering which adds a nonreflecting stationary set of ordinals of cofinality κ_1 to κ_2, and $\dot{\mathbb{S}}$ is a term for the portion of $j(\mathbb{P}^1)$ acting on ordinals in the open interval $(\kappa_2, j(\kappa_2))$.

Because λ has been chosen large enough (so that in particular, $\lambda > 2^{[\kappa_2^+]^{<\kappa_2}} = 2^{\kappa_2^+} = \kappa_2^{++}$), $\mathcal{U}_2^* \in M$. Let $k : M \to N$ be the elementary embedding generated by the ultrapower via $\mathcal{U}_2^*$. It is then true that $N \vDash$ "κ_2 isn't measurable." It is the case that if $i : V_1 \to N$ is an elementary embedding having critical point κ_2 and for any $x \subseteq N$ with $|x| \leq \lambda$, there is some $y \in N$ such that $x \subseteq y$ and $N \vDash$ "$|y| < i(\kappa_2)$," then i witnesses the λ strong compactness of κ_2. Using this fact, it is easily verifiable that $i = k \circ j$ is an elementary embedding witnessing the λ strong compactness of κ_2. We show that i lifts in $V_1^{\mathbb{P}^1 * \dot{\mathbb{Q}}}$ to $i : V_1^{\mathbb{P}^1 * \dot{\mathbb{Q}}} \to N^{i(\mathbb{P}^1 * \dot{\mathbb{Q}})}$. Since this lifted embedding witnesses the λ strong compactness of κ_2 in $V_1^{\mathbb{P}^1 * \dot{\mathbb{Q}}}$ and λ was arbitrarily chosen, this completes the proof of Lemma 2.7.

Let G_0 be V_1-generic over $\mathbb{P}^1$, and let H be $V_1[G_0]$-generic over $\mathbb{Q}$. By forcing in N above a condition opting for trivial forcing in the stage κ_2 lottery held in N in the definition of $i(\mathbb{P}^1)$, we may assume that $i(\mathbb{P}^1)$ is forcing equivalent in N to $\mathbb{P}^1 * \dot{\mathbb{Q}}^1 * \dot{\mathbb{Q}}^2 * \dot{\mathbb{Q}}^3$, where $\dot{\mathbb{Q}}^1$ is a term for the portion of the forcing acting on ordinals in the open interval $(\kappa_2, k(\kappa_2))$, $\dot{\mathbb{Q}}^2$ is a term for the forcing done at stage $k(\kappa_2)$, and $\dot{\mathbb{Q}}^3$ is a term for the remainder of the forcing, i.e., the portion acting on ordinals in the half-open interval $(k(\kappa_2), k(j(\kappa_2))]$ (inclusive of the term $i(\dot{\mathbb{Q}})$ for the forcing done at stage $k(j(\kappa_2)) = i(\kappa_2)$). We will build in $V_1[G_0][H]$ generic objects for the different portions of $i(\mathbb{P}^1)$.

To do this, we use a modification of an argument initially due to Magidor, unpublished by him but presented in, among other places, [1, Theorem 2]. The modification is due to Sargsyan and is found in [5, Lemma 3]. In particular, we begin by constructing an $N[G_0]$-generic object G_1 for $\mathbb{Q}^1$. The argument used will be carried out in $M[G_0] \subseteq V_1[G_0] \subseteq V_1[G_0][H]$. Specifically, since we are assuming that $\dot{\mathbb{Q}}^1$ is forced to act nontrivially only on ordinals in the open interval $(\kappa_2, k(\kappa_2))$, we may therefore build in $M[G_0]$ an $N[G_0]$-generic object G_1 for $\mathbb{Q}^1$ in the same manner as the construction of the generic object G_2 given in the proof of Lemma 2.6.

We next analyze the exact nature of $\dot{\mathbb{Q}}^2$. As we have already observed, we may assume that $j(\mathbb{P}^1 * \dot{\mathbb{Q}})$ is forcing equivalent in M to $\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}' * \dot{\mathbb{S}} * j(\dot{\mathbb{Q}})$, where

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$\dot{\mathbb{Q}} * \dot{\mathbb{R}}'$ is a term for the forcing taking place at stage κ_2 in M, $\dot{\mathbb{R}}'$ is a term for the partial ordering which adds a nonreflecting stationary set of ordinals of cofinality κ_1 to κ_2, and $\dot{\mathbb{S}}$ is a term for the portion of $j(\mathbb{P}^1)$ acting on ordinals in the open interval $(\kappa_2, j(\kappa_2))$. By elementarity, since $\dot{\mathbb{Q}}^2$ is a term for the forcing which takes place at stage $k(\kappa_2)$ in N, we may write $\dot{\mathbb{Q}}^2 = k(\dot{\mathbb{Q}}) * k(\dot{\mathbb{R}}')$. We will construct in $M[G_0][H]$ generic objects for $k(\mathbb{Q})$ and $k(\mathbb{R}')$.

For $k(\mathbb{Q})$, we use an argument containing ideas due to Woodin. First, note that since N is given by an ultrapower, $N = \{k(h)(\langle k(\beta) \mid \beta < \kappa_2^+\rangle) \mid h : P_{\kappa_2}(\kappa_2^+) \to M$ is a function in $M\}$. Further, since by the definition of G_1, $k''G_0 \subseteq G_0 * G_1$, k lifts in both $M[G_0]$ and $M[G_0][H]$ to $k : M[G_0] \to N[G_0][G_1]$. From these facts, we may now show that $k''H \subseteq k(\mathbb{Q})$ generates an $N[G_0][G_1]$-generic object G_2 over $k(\mathbb{Q})$. Specifically, given a dense open subset $D \subseteq k(\mathbb{Q})$, $D \in N[G_0][G_1]$, $D = \mathrm{int}_{G_0 * G_1}(\dot{D})$ for some N-name $\dot{D} = k(\vec{D})(\langle k(\beta) \mid \beta < \kappa_2^+\rangle)$, where $\vec{D} = \langle D_p \mid p \in P_{\kappa_2}(\kappa_2^+)\rangle$ is a function in M. We may assume that every D_p is a dense open subset of $\mathbb{Q}$. Since $\mathbb{Q}$ is (κ_2^+, ∞)-distributive and $|P_{\kappa_2}(\kappa_2^+)| = \kappa_2^+$, it follows that $D' = \bigcap_{p \in P_{\kappa_2}(\kappa_2^+)} D_p$ is also a dense open subset of $\mathbb{Q}$. As $k(D') \subseteq D$ and $H \cap D' \neq \emptyset$, $k''H \cap D \neq \emptyset$. Thus, $G_2 = \{p \in k(\mathbb{Q}) \mid \exists q \in k''H[q \geq p]\}$, which is definable in $M[G_0][H]$, is our desired $N[G_0][G_1]$-generic object over $k(\mathbb{Q})$. Then, since $k(\mathbb{R}')$ is in $N[G_0][G_1][G_2]$ the partial ordering which adds a nonreflecting stationary set of ordinals of cofinality $k(\kappa_1)$ to $k(\kappa_2)$, we know that $N[G_0][G_1][G_2] \vDash$ "$|k(\mathbb{R}')| = k(\kappa_2)$ and $|\wp(k(\mathbb{R}'))| = 2^{k(\kappa_2)} = k(\kappa_2^{++})$." Hence, since $N[G_0][G_1][G_2]$ remains κ_2^+-closed with respect to $M[G_0][H]$, which means $k(\mathbb{R}')$ is $\prec\kappa_2^{++}$-strategically closed in $N[G_0][G_1][G_2]$ and $M[G_0][H]$, the same argument used in the construction of G_1 allows us to build in $M[G_0][H]$ an $N[G_0][G_1][G_2]$-generic object G_3 for $k(\mathbb{R}')$.

We construct now (in $V_1[G_0][H]$) an $N[G_0][G_1][G_2][G_3]$-generic object for $\mathbb{Q}^3$. As in the proof of [5, Lemma 3], we do this by combining the term forcing argument found in [1, Theorem 2] with the argument for the creation of a "master condition" found in [3, Lemma 2]. Specifically, we begin by showing the existence of a term $\tau \in M$ for a "master condition" for $j(\dot{\mathbb{Q}})$, i.e., we show the existence of a term $\tau \in M$ in the language of forcing with respect to $j(\mathbb{P}^1)$ such that in M, $\Vdash_{j(\mathbb{P}^1)}$ "$\tau \in j(\dot{\mathbb{Q}})$ extends every $j(\dot{q})$ for $\dot{q} \in \dot{H}$." We first note that since $\mathbb{P}^1$ is κ_2-c.c. in both V_1 and M, as $\Vdash_{\mathbb{P}^1}$ "$\dot{\mathbb{Q}}$ is κ_2-directed closed and $|\dot{\mathbb{Q}}| < \lambda$," the usual arguments show $M[G_0][H]$ remains λ-closed with respect to $V_1[G_0][H]$. This means $T = \{j(\dot{q}) \mid \exists r \in G_0[\langle r, \mathrm{int}_{G_0 * H}(\dot{q})\rangle \in G_0 * H]\} \in M[G_0][H]$, so T has a name $\dot{T} \in M$ such that in M, $\Vdash_{j(\mathbb{P}^1)}$ "$|\dot{T}| < \lambda < j(\kappa_2)$, any two elements of $\dot{T}$ are compatible, and $\dot{T}$ is a subset of a partial ordering (namely $j(\dot{\mathbb{Q}})$) which is $j(\kappa_2)$-directed closed." Thus, in M, since $j(\kappa_2) > \lambda$, and $M^\lambda \subseteq M$, $\Vdash_{j(\mathbb{P}^1)}$ "There is a condition in $j(\dot{\mathbb{Q}})$ extending each element of $\dot{T}$." A term τ for this common extension is as desired.

We work for the time being in M. Consider the "term forcing" partial ordering $\mathbb{S}^*$ (see [8] for the first published account of term forcing or [7, Section 1.2.5, p. 8]—the notion is originally due to Laver) associated with $\dot{\mathbb{S}} * j(\dot{\mathbb{Q}})$, i.e., $\sigma \in \mathbb{S}^*$ iff σ is a term in the forcing language with respect to $\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'$ and $\Vdash_{\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'}$ "$\sigma \in \dot{\mathbb{S}} * j(\dot{\mathbb{Q}})$," ordered by $\sigma_1 \geq \sigma_0$ iff $\Vdash_{\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'}$ "$\sigma_1 \geq \sigma_0$." Note that τ' defined as the term in the language of forcing with respect to $\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'$ composed of the tuple all of whose members are forced to be the trivial condition, with the exception of the last member, which is τ, is an element of $\mathbb{S}^*$.

Clearly, $\mathbb{S}^* \in M$. In addition, since $V_1 \vDash$ "No cardinal above κ_2 is inaccessible," because $M^\lambda \subseteq M$, $M \vDash$ "The first stage at which $\dot{\mathbb{S}} * j(\dot{\mathbb{Q}})$ is forced to do nontrivial forcing is above λ." Thus, $\Vdash_{\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'}$ "$\dot{\mathbb{S}} * j(\dot{\mathbb{Q}})$ is $\prec\lambda^+$-strategically closed," which, since $M^\lambda \subseteq M$, immediately implies that $\mathbb{S}^*$ itself is $\prec\lambda^+$-strategically closed in both V_1 and M. Further, since $\Vdash_{\mathbb{P}^1}$ "$|\dot{\mathbb{Q}}| < \gamma < \lambda$," in M, $\Vdash_{\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'}$ "$|\dot{\mathbb{S}} * j(\dot{\mathbb{Q}})| < j(\gamma) < j(\lambda)$." We also have that $2^\delta = \delta^+$ for $\delta \geq \kappa_2^+$ in V_1 and $\delta \geq j(\kappa_2^+)$ in M. Consequently, as j is given via an ultrapower embedding by a normal measure over $P_{\kappa_2}(\lambda)$, $|j(\lambda^+)| = |\{f \mid f : P_{\kappa_2}(\lambda) \to \lambda^+| = |[\lambda^+]^\lambda| = \lambda^+$ and $\Vdash_{\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}'}$ "$|\wp(\dot{\mathbb{S}} * j(\dot{\mathbb{Q}}))| < 2^{j(\gamma)} = j(\gamma^+) \leq j(\lambda) < 2^{j(\lambda)} = j(\lambda^+)$." Therefore, since as in the footnote given in the proof of [4, Lemma 8], we may assume that $\mathbb{S}^*$ has cardinality below $j(\gamma)$ in M, we may once again build in V_1 an M-generic object G_4^* for $\mathbb{S}^*$ containing τ' as in the construction of the generic object G_1 of this lemma.

Note now that since N is given by an ultrapower of M via a normal measure over $P_{\kappa_2}(\kappa_2^+)$, [7, Section 1.2.2, Fact 2] tells us that $k''G_4^*$ generates an N-generic object G_4^{**} over $k(\mathbb{S}^*)$ containing $k(\tau')$. By elementarity, $k(\mathbb{S}^*)$ is the term forcing in N defined with respect to $k(j(\mathbb{P}^1)_{\kappa_2+1})$, which is forcing equivalent to $\mathbb{P}^1 * \dot{\mathbb{Q}}^1 * \dot{\mathbb{Q}}^2$. Therefore, since $i(\mathbb{P}^1 * \dot{\mathbb{Q}}) = k(j(\mathbb{P}^1 * \dot{\mathbb{Q}}))$ is forcing equivalent to $\mathbb{P}^1 * \dot{\mathbb{Q}}^1 * \dot{\mathbb{Q}}^2 * \dot{\mathbb{Q}}^3$, G_4^{**} is N-generic over $k(\mathbb{S}^*)$, and $G_0 * G_1 * G_2 * G_3$ is $k(\mathbb{P}^1 * \dot{\mathbb{Q}} * \dot{\mathbb{R}}')$-generic over N, [7, Section 1.2.5, Fact 1] (see also [8]) tells us that for $G_4 = \{\text{int}_{G_0 * G_1 * G_2 * G_3}(\sigma) \mid \sigma \in G_4^{**}\}$, G_4 is $N[G_0][G_1][G_2][G_3]$-generic over $\mathbb{Q}^3$. In addition, since the definition of τ tells us that in M, the statement "$\langle p, \dot{q}\rangle \in j(\mathbb{P}^1 * \dot{\mathbb{Q}})$ implies that $\langle p, \dot{q}\rangle \Vdash_{j(\mathbb{P}^1 * \dot{\mathbb{Q}})}$ 'τ extends $\dot{q}$' " is true, by elementarity, in N, the statement "$\langle p, \dot{q}\rangle \in k(j(\mathbb{P}^1 * \dot{\mathbb{Q}}))$ implies that $\langle p, \dot{q}\rangle \Vdash_{k(j(\mathbb{P}^1 * \dot{\mathbb{Q}}))}$ '$k(\tau)$ extends $\dot{q}$' " is true. In other words, since $k \circ j = i$, in N, the statement "$\langle p, \dot{q}\rangle \in i(\mathbb{P}^1 * \dot{\mathbb{Q}})$ implies that $\langle p, \dot{q}\rangle \Vdash_{i(\mathbb{P}^1 * \dot{\mathbb{Q}})}$ '$k(\tau)$ extends $\dot{q}$' " is true. Thus, in N, $k(\tau)$ functions as a term for a "master condition" for $i(\dot{\mathbb{Q}})$, so since G_4^{**} contains $k(\tau')$, the construction of all of the above generic objects immediately yields that $i''(G_0 * H) \subseteq G_0 * G_1 * G_2 * G_3 * G_4$. This means that i lifts in $V_1^{\mathbb{P}^1 * \dot{\mathbb{Q}}}$ to $i : V_1^{\mathbb{P}^1 * \dot{\mathbb{Q}}} \to N^{i(\mathbb{P}^1 * \dot{\mathbb{Q}})}$. This completes the proof of Lemma 2.7. ⊣

Let $\mathbb{P} = \mathbb{P}^* * \dot{\mathbb{P}}^0 * \dot{\mathbb{P}}^1$. Lemmas 2.3–2.7 and the intervening remarks then complete the proof of Theorem 1.2. ⊣

§3. Concluding remarks. We conclude this paper with several remarks, noting that in what we are about to say, we always assume that we begin with a model V of ZFC in which $\kappa_1 < \kappa_2$ are both supercompact. First, we note that it is possible to obtain a version of Theorem 1.1 where $2^{\kappa_2} = \kappa_2^{++}$, κ_2 is strongly compact, κ_2 is κ_2^+ supercompact, and the indestructibility property for κ_2 is that κ_2's strong compactness is indestructible under forcing with $\text{Add}(\kappa_2, \delta)$ for any ordinal δ and for $\lambda = (\kappa_2^+)^{V^{\mathbb{P}}}$, the λ supercompactness of κ_2 is indestructible under κ_2-directed closed forcing having size at most λ. We present a brief sketch of how this is done, and leave it to interested readers to provide any missing details. Specifically, we may assume from the proof of Theorem 1.2 that we are forcing over a ground model V in which κ_1 is both the first strongly compact and first measurable cardinal and has its strong compactness indestructible under arbitrary κ_1-directed closed forcing, κ_2 is supercompact, $2^{\kappa_2} = 2^{\kappa_2^+} = \kappa_2^{++}$, $2^\delta = \delta^+$ for every $\delta \geq \kappa_2^{++}$, and κ_2

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carries a supercompact ultrafilter $\mathcal{U}_2$ such that κ_2 isn't measurable in the ultrapower by $\mathcal{U}_2$. We then force as in Modules 2 and 3 of the proof of Theorem 1.1. By the proofs of [1, Lemma 5], [11, Theorems 1.7 and 1.8], and Theorem 1.1, we are now in a new ground model V in which κ_1 is both the first strongly compact and first measurable cardinal and has its strong compactness indestructible under arbitrary κ_1-directed closed forcing, κ_2 is both the second strongly compact and second measurable cardinal, κ_2 is κ_2^+ supercompact, and there is a fast function $f : \kappa_2 \to \kappa_2$ for κ_2. We then force with the lottery preparation defined as on [11, p. 127] using f, where the partial orderings allowed in the lottery sum at a nontrivial stage of forcing α all must be α-directed closed. Since the proof found in Module 4 of Theorem 1.1 requires no GCH assumptions about κ_2, the arguments of Module 4 and Lemma 2.6 now show that we have constructed our desired model.

We next observe that with slight modifications to our proofs, it is possible to change the assumption in the proof of Theorem 1.2 that $V \vDash$ "No cardinal $\lambda > \kappa_2$ is inaccessible" to, e.g., $V \vDash$ "No cardinal $\lambda > \kappa_2$ is Mahlo." This is accomplished by letting $\langle \delta_\alpha \mid \alpha < \kappa_2 \rangle$ enumerate the Mahlo cardinals in the open interval (κ_1, κ_2) and replacing "inaccessible" with "Mahlo" in clause (2) in the definition of $\mathbb{P}^1$. It is in addition possible to have in Theorem 1.2 that, e.g., $2^{\kappa_1} = \kappa_1^{+38}$, κ_1 is κ_1^{+37} supercompact, $2^{\kappa_2} = \kappa_2^{+75}$, κ_2 is κ_2^{+74} supercompact, $2^\delta = \delta^+$ for every $\delta \ge \kappa_2^{+75}$, κ_1's strong compactness is indestructible under arbitrary κ_1-directed closed forcing, κ_2's strong compactness is indestructible under κ_2-directed closed, (κ_2^{+74}, ∞)-distributive forcing, and for $\lambda = (\kappa_2^{+74})^{V^{\mathbb{P}}}$, the λ supercompactness of κ_2 is indestructible under κ_2-directed closed forcing having size at most λ. Other variations for Theorems 1.1 and 1.2 along these same lines (including versions of Theorem 1.1 where κ_1 exhibits nontrivial degrees of supercompactness) are possible as well. We leave it to readers to fill in the details for themselves.

We also return to the issue (first raised in Section 1) of why in Theorem 1.2, the current state of forcing technology doesn't appear to provide a way for one to force the κ_1^+ supercompactness of κ_1 to be indestructible under κ_1-directed closed forcing having size at most $(\kappa_1^+)^{V^{\mathbb{P}}}$. This is since $\mathbb{P}^0$ is not an iteration of the type considered in [10, 12], and so no analogue of Theorem 1.3 is presently known which allows us to establish the appropriate version of Lemma 2.5.

We finally ask the very broad and general question of what other sorts of indestructibility theorems are possible when considering the class of strongly compact cardinals. As examples, can the first i strongly compact cardinals κ_i (for $i > 1$ an ordinal) be the first i measurable cardinals, and also have their strong compactness indestructible under arbitrary κ_i-directed closed forcing? Since the proof of Theorem 1.1 doesn't seem to have a generalization beyond $i = 2$, is it even possible for the first i strongly compact cardinals κ_i (for $i > 2$ a natural number) to have their strong compactness indestructible under $\mathrm{Add}(\kappa_i, \delta)$ for any ordinal δ? These are the questions with which we end this paper.

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