Reinhard Kahle and Michael Rathjen, editors, ***Gentzen's Centenary: The Quest for Consistency***, Springer, 2015.

Gerhard Gentzen, generally considered the first modern proof theorist, was born in 1909. *Gentzen's Centenary* is a volume of papers in his honor, conceived at the *Leeds Symposium on Proof Theory and Constructivism* in 2009. One of the contributors, Grigori Mints, passed away shortly before the volume was published, and the volume is also dedicated to his memory.

Gentzen is primarily associated with several proofs of one spectacular result, the consistency of first-order arithmetic using transfinite induction up to the ordinal ε_0. The nineteen papers in the volume are concerned, in one form or another, with this proof and its modern consequences for the consistency of formal theories of arithmetic.

For most readers, the more interesting papers will be the less technical ones. One particularly interesting feature of the volume is that it represents a rare chance to reflect on the development and history of Gentzen's work, not just its modern consequences. We often focus on the most recent, advanced, and streamlined arguments, but the authors took the occasion of the centenary to look back in detail on the original form of Gentzen's proof.

Gentzen originally worked out his proof of the consistency of arithmetic in late 1934 and submitted it to the *Mathematische Annalen* the following year. Other logicians, most importantly Gödel, questioned whether the techniques (in particular, the use of induction on well-founded trees) were properly finitistic. Gentzen's response to this criticism was to elaborate the transfinite recursion using ordinal numbers, leading him to identify the critical role of the ordinal ε_0; this lead to Gentzen's first consistency proof, published in 1936. Only two years later, in 1938, Gentzen produced a modified and somewhat simpler proof; it is this later proof which is usually referred to as Gentzen's proof.

Von Plato's contribution, in particular, is a rare example of an investigation of the history of an individual proof. Von Plato recounts the order in which Gentzen seems to have encountered and then surmounted the most important obstacles, providing some insight into how he might have come to his ideas. The reader needs to understand the proof itself, and the issues around it, quite well, but anyone with those prerequisites and interested in where a groundbreaking proof like Gentzen's could come from would appreciate the article.

Several other papers wrestle with features of Gentzen's original proofs, asking what more we might glean, or what might have been gleaned earlier, from Gentzen's arguments rather than later refinements. Buchholz gives a modern account of the most difficult part of the first consistency proof, which may make that little-read paper more accessible. Tait discusses in more detail the use of induction in Gentzen's original paper, elaborating on the common claim that Gentzen's 1934 proof is repaired by the use of Brouwer's Bar Theorem (a form of induction on well-founded trees). Rathjen's paper concerns itself, not with Gentzen's result, but with Goodstein's: Goodstein related the ordinals below ε_0 to the termination of *Goodstein sequences*. Today Goodstein's theorem—every Goodstein sequence terminates—is known to be unprovable in Peano arithmetic, but this was not actually proven until Kirby and Paris' work in the 1980s. Rathjen argues that Goodstein's letters suggest that he was quite close to proving some form of independence result from Peano arithmetic.

After Gentzen's work, the natural question was to ask for an analogous proof of the consistency of second-order arithmetic by transfinite induction up to some larger ordinal. For many decades, the pursuit of such a result—the branch of proof theory known as *ordinal analysis*—was central to the field. By the turn of the millennium that situation had changed, in part because of the seeming intractability of further advances, at least in the original direction of pushing directly towards analyses of stronger and stronger systems. As Mints observed, "This used to be the main problem of proof theory, but it was too difficult, so now it is not." Instead attention shifted to the other branches of proof theory that had been developed in the previous few decades.

The make-up of the volume reflects this change. Even in a volume devoted to Gentzen, there is only one paper that represents a direct contribution to the ordinal analysis of strong subsystems of analysis (Arai's paper, part of an ongoing series working towards an analysis of Π^1_2-comprehension). Two other papers (one by Jäger and Probst and one by Mints) reflect a closely related line of research, giving refined analyses of relatively weak systems, in part

https://doi.org/10.1017/bsl.2016.31 Published online by Cambridge University Press

motivated by the hope that a better understanding and new techniques for dealing with weaker systems might make a return to the analysis of stronger systems more viable.

The other technical papers in the volume are still based on Gentzen's techniques but apply them to other areas of proof theory and logic. In the realm of pure proof theory, Meskens and Weiermann investigate fast-growing functions in subsystems of Peano arithmetic, Oliva and Powell give a game-theoretic version of Gödel's Dialectica interpretation, Ferreira discusses Spector's consistency proof for analysis using higher order functionals, and Jervall discusses tree representations of ordinals. Further from proof theory, Rathjen and Vizcaíno look at the connection to reverse mathematics, giving an equivalence between a reverse mathematics system and the statement that a certain operation inspired by proof-theoretic ordinals maps well-orderings to well-orderings, and Pohlers discusses so-called semiformal calculi (proof systems where rules may have infinitely many premises), with a focus on applications outside of proof theory.

The remaining papers in the volume consider the broader impact of Gentzen's ideas. Kahle's contribution gives a survey of consistency results and how they have been viewed, both historically since Gentzen's work and across several subfields of modern logic. Detlefsen discusses Gentzen's own philosophical views, and Setzer considers the revision of Hilbert's original program to prove the consistency of mathematics in the context of modern proof theory.

Because Gentzen's cut-elimination results for Peano arithmetic are so central to the volume, even the less technical papers will be difficult to fully appreciate for readers who are not comfortable with that proof. However it is difficult to imagine a more thorough investigation of what that result means eighty years after its publication, both in the broad scope of its consequences, the field it generated, and the way it transformed our understanding of what consistency means in mathematics.

HENRY TOWSNER

Department of Mathematics, University of Pennsylvania, Philadelphia, PA 19143, USA, htowsner@math.upenn.edu.

Five papers on reverse mathematics and Ramsey-theoretic principles

C. T. CHONG, THEODORE A. SLAMAN, and YUE YANG, *The metamathematics of Stable Ramsey's Theorem for Pairs*. ***Journal of the American Mathematical Society***, vol. 27 (2014), no. 3, pp. 863–892.

MANUEL LERMAN, REED SOLOMON, and HENRY TOWSNER, *Separating principles below Ramsey's Theorem for Pairs*. ***Journal of Mathematical Logic***, vol. 13 (2013), no. 2, 1350007, 44 pp.

JIAYI LIU, *RT^2_2 does not imply WKL_0*. ***Journal of Symbolic Logic***, vol. 77 (2012), no. 2, pp. 609–620.

LU LIU, *Cone avoiding closed sets*. ***Transactions of the American Mathematical Society***, vol. 367 (2015), no. 3, pp. 1609–1630.

WEI WANG, *Some logically weak Ramseyan theorems*. ***Advances in Mathematics***, vol. 261 (2014), pp. 1–25.

These five papers are all major advances in the reverse-mathematical and computability-theoretic analysis of combinatorial principles related to Ramsey's Theorem. Reverse mathematics seeks to calibrate the strength of theorems provable in the theory Z_2 of second-order arithmetic. Typically, given such a theorem T, one endeavors to find a subsystem S of Z_2 that is equivalent to T, in the sense that T is provable in S, but also each axiom of S is provable from T over a weak base system. This base system is usually RCA_0, which roughly corresponds to the practice of computable mathematics (and hence lends the area a distinctively computability-theoretic flavor). A celebrated phenomenon is that there are a few such subsystems that suffice to classify many theorems across mathematics, most famously the "big five" systems RCA_0, WKL_0, ACA_0, ATR_0, and Π^1_1-CA_0. Furthermore, these systems are linearly ordered by strength.

https://doi.org/10.1017/bsl.2016.31 Published online by Cambridge University Press