The Journal of Symbolic Logic
Volume 82, Number 1, March 2017

LEFT MAXIMAL AND STRONGLY RIGHT MAXIMAL IDEMPOTENTS IN G^*

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Abstract. Let G be a countably infinite discrete group, let βG be the Stone–Čech compactification of G, and let $G^* = \beta G \setminus G$. An idempotent $p \in G^*$ is left (right) maximal if for every idempotent $q \in G^*$, $pq = p$ $(qp = p)$ implies $qp = q$ $(pq = q)$. An idempotent $p \in G^*$ is strongly right maximal if the equation $xp = p$ has the unique solution $x = p$ in G^*. We show that there is an idempotent $p \in G^*$ which is both left maximal and strongly right maximal.

§1. Introduction. Throughout the paper, G will be an arbitrary countably infinite discrete group.

The operation of G extends to the Stone–Čech compactification βG of G so that for each $a \in G$, the left translation $\beta G \ni x \mapsto ax \in \beta G$ is continuous, and for each $q \in \beta G$, the right translation $\beta G \ni x \mapsto xq \in \beta G$ is continuous.

We take the points of βG to be the ultrafilters on G, the principal ultrafilters being identified with the points of G, and $G^* = \beta G \setminus G$. The topology of βG is generated by taking as a base the subsets $\overline{A} = \{p \in \beta G : A \in p\}$, where $A \subseteq G$. For $p, q \in \beta G$, the ultrafilter pq has a base consisting of subsets $\bigcup_{x \in A} xB_x$, where $A \in p$ and $B_x \in q$.

The semigroup βG is interesting both for its own sake and for its applications to Ramsey theory and to topological dynamics. An elementary introduction to βG can be found in [3].

As any compact Hausdorff right topological semigroup, G^* has idempotents [1, Corollary 2.10]. The inclusion relation on principal left and right ideals of βG induces the left and right preorderings on idempotents of G^*:

$$p \leq_L q \Leftrightarrow (\beta G)p \subseteq (\beta G)q \Leftrightarrow pq = p,$$
$$p \leq_R q \Leftrightarrow p(\beta G) \subseteq q(\beta G) \Leftrightarrow qp = p.$$

The maximal idempotents with respect to $\leq_L$ $(\leq_R)$ are called *left* (*right*) *maximal*. Thus, an idempotent $p \in G^*$ is left (right) maximal if and only if for every idempotent $q \in G^*$, $pq = p$ $(qp = p)$ implies $qp = q$ $(pq = q)$.

Received November 1, 2015.
2010 *Mathematics Subject Classification.* Primary 22A15, 54H20, Secondary 54D80, 54H11.
Key words and phrases. Stone–Čech compactification, ultrafilter, principal left ideal, orbit closure, left maximal idempotent, strongly right maximal idempotent.

0022-4812/17/8201-0002
DOI:10.1017/jsl.2016.62

As any compact Hausdorff right topological semigroup, G^* has right maximal idempotents [5, Theorem I.2.7]. For every right maximal idempotent $p \in G^*$, $\{x \in G^* : xp = p\}$ is a finite right zero semigroup [3, Theorem 9.4]. An idempotent $p \in G^*$ is *strongly right* (*left*) *maximal* if the equation $xp = p$ ($px = p$) has the unique solution $x = p$ in G^*. There are strongly right maximal idempotents in G^* [4]. Assuming Martin's Axiom (MA), there is an idempotent $p \in \mathbb{Z}^*$ such that, for any $q, r \in \mathbb{Z}^*$, $q + r = p$ implies $q, r \in \mathbb{Z} + p$ [2], and consequently, p is both strongly right maximal and strongly left maximal.

Recently, it was shown in ZFC, the system of usual axioms of set theory, that there are left maximal idempotents in G^* [8]. More specifically, it was shown that there are idempotents in G^* which are both minimal and left maximal. Since the minimal idempotents are not right maximal [3, Exercise 9.1.4], the idempotents constructed in [8] are left maximal but not right maximal.

In this paper we prove (in ZFC) the following result.

THEOREM 1.1. *Let X be a G_δ subset of G^* containing an idempotent. Then there is an idempotent $p \in X$ such that p is strongly right maximal in G^* and $(\beta G)p$ is a maximal principal left ideal of βG.*

As an immediate consequence, we obtain from Theorem 1.1 that

COROLLARY 1.2. *Every G_δ subset of G^* containing an idempotent contains an idempotent which is both left maximal and strongly right maximal.*

The proof of Theorem 1.1 is based on a special construction of regular left invariant topologies on G and closed left ideals of βG and on deep subsets of ω^*.

For every closed subset $Y \subseteq \omega^*$, the *character* of Y in ω^*, denoted $\chi(Y)$, is the minimum cardinality of a family $\mathcal{F}$ of subsets of ω such that $\bigcap_{A\in\mathcal{F}} \overline{A} = Y$. A nonempty closed subset $Z \subseteq \omega^*$ is *deep* if for every closed subset $Y \subseteq \omega^*$ with $\chi(Y) < \mathfrak{c}$, $Y \cap Z$ is either empty or infinite.

THEOREM 1.3 ([8, Theorem 3.1]). *There is a deep subset $Z \subseteq \omega^*$.*

As in [8], we use Theorem 1.3 as a replacement of MA.

In Section 2 we characterize idempotents from Theorem 1.1. In Section 3 we give that special construction. And in Section 4 we prove Theorem 1.1 itself.

§2. Left invariant topologies and closed left ideals. A topology on a group is *left invariant* if left translations are continuous. All topologies are assumed to satisfy the T_1 separation axiom unless otherwise specified. For every left invariant topology $\mathcal{T}$ on G, $\mathrm{Ult}(\mathcal{T}) = \{p \in G^* : p \text{ converges to } 1 \text{ in } \mathcal{T}\}$. That is, $\mathrm{Ult}(\mathcal{T})$ consists of all nonprincipal ultrafilters on G containing the neighborhood filter of 1 in $\mathcal{T}$. This is a closed subsemigroup of G^* (see [6, Lemma 7.1]).

Every idempotent $p \in G^*$ determines naturally two left invariant topologies on G. The first one is the topology $\mathcal{T}(p)$ with $\mathrm{Ult}(\mathcal{T}(p)) = \{p\}$. That is, the neighborhood filter of 1 in $\mathcal{T}(p)$ consists of subsets $A \cup \{1\}$, where $A \in p$. It is Hausdorff and *maximal* (= maximal among all dense in itself topologies). The second one is the topology $\mathcal{T}_\rho(p)$ with $\mathrm{Ult}(\mathcal{T}_\rho(p)) = \{x \in G^* : xp = p\}$. It is the largest regular left invariant topology on G in which p converges to 1. It is induced by the mapping $G \ni x \mapsto xp \in G^*$. See [3, Section 9.2] or [6, Proposition 6.30 and Theorem 7.17] for the proofs. These results imply the following characterizations of strongly right maximal idempotents.

LEMMA 2.1. *Let $p \in G^*$ be an idempotent. Then the following statements are equivalent*:

(1) *p is strongly right maximal,*
(2) *$\mathcal{T}_p(p) = \mathcal{T}(p)$,*
(3) *$\mathcal{T}(p)$ is regular,*
(4) *$Gp \subseteq G^*$ is a maximal space.*

Given a left invariant topology on G, an ultrafilter q on G is *fundamental* if for every neighborhood U of 1, there is $x \in G$ such that $xU \in q$.

LEMMA 2.2. *An ultrafilter $q \in \beta G$ is fundamental if and only if there is $r \in \beta G$ such that the ultrafilter rq converges to* 1.

PROOF. Suppose that q is fundamental. For every neighborhood U of 1, pick $x_U \in G$ such that $x_U U \in q$, so $U \in x_U^{-1} q$. Let r be an ultrafilter on G extending the family of subsets $\{x_V^{-1} : V$ is a neighborhood of 1 contained in $U\}$, where U runs over neighborhoods of 1. Then rq converges to 1.

Conversely, suppose that rq converges to 1. Then for every neighborhood U of 1, there is $x_U \in G$ such that $x_U q \in U$, so $x_U^{-1} U \in q$. Consequently, q is fundamental. ⊣

A left invariant topology $\mathcal{T}$ on G is *complete* if every fundamental ultrafilter is convergent. We say that $\mathcal{T}$ is *weakly complete* if every fundamental ultrafilter containing a discrete subset is convergent.

LEMMA 2.3. *For every idempotent $p \in G^*$, $(G, \mathcal{T}(p))$ contains no nonclosed discrete subset.*

PROOF. It suffices to show that p contains no discrete subset. Let $A \in p$ and let $B = \{x \in G : A \in xp\}$. Since $pp = p$, one has $B \in p$. Then every $x \in A \cap B$ is a limit point of A, so A is not discrete. ⊣

Since by Lemma 2.3, no nonprincipal ultrafilter on $(G, \mathcal{T}(p))$ containing a discrete subset is convergent, we obtain that

COROLLARY 2.4. *For every idempotent $p \in G^*$, $\mathcal{T}(p)$ is weakly complete if and only if no nonprincipal ultrafilter containing a discrete subset is fundamental.*

The next lemma gives us a characterization of maximal principal left ideals generated by idempotents.

LEMMA 2.5. *Let $p \in G^*$ be an idempotent. Then*

(1) *$(\beta G)p = \bigcap\{\overline{A} : G \setminus A$ is discrete in $\mathcal{T}(p)\}$,*
(2) *$(\beta G)p$ is maximal if and only if $\mathcal{T}(p)$ is weakly complete.*

PROOF. (1) Let $A \subseteq G$. Suppose that $G \setminus A$ is discrete in $\mathcal{T}(p)$. Then, by Lemma 2.3, it is also closed. For every $x \in G$, pick $A_x \in p$ such that $(xA_x) \cap (G \setminus A) = \emptyset$, and let $B = \bigcup_{x \in G} xA_x$. Then $(\beta G)p \subseteq \overline{B}$ and $B \subseteq A$.

Now suppose that $(\beta G)p \subseteq \overline{A}$. For every $x \in G$, there is $A_x \in p$ such that $xA_x \subseteq A$. Then for every $x \in G \setminus A$, $U = A_x \cup \{x\}$ is a neighborhood of x and $U \cap (G \setminus A) = \{x\}$. It follows that $G \setminus A$ is discrete in $\mathcal{T}(p)$.

(2) Suppose that $(\beta G)p$ is maximal. Let D be a discrete subset of $(G, \mathcal{T}(p))$ and let $D \in q \in \beta G$. By (1), $\overline{D} \cap ((\beta G)p) = \emptyset$, so $q \notin (\beta G)p$. Since $(\beta G)p$ is maximal, there is no $r \in \beta G$ such that $rq = p$. Consequently by Lemma 2.2, q is not fundamental. Hence by Corollary 2.4, $\mathcal{T}(p)$ is weakly complete.

Suppose that $(\beta G)p$ is not maximal. Then there is $q \in G^* \setminus ((\beta G)p)$ such that $(\beta G)p \subseteq (\beta G)q$, and so there is $r \in \beta G$ such that $rq = p$. Consequently, q is fundamental, and by (1), q contains a discrete subset. Hence, $\mathcal{T}(p)$ is not weakly complete. ⊣

Notice that for every $p \in G^*$, $(\beta G)p = \overline{Gp}$ (here, $\overline{Gp}$ denotes $\text{cl}_{G^*}(Gp)$), so the principal left ideal generated by p is the same as the orbit closure of p (under the action $G \times G^* \ni (a, p) \mapsto ap \in G^*$).

Combining Lemmas 2.1 and 2.5, we obtain the following characterizations of idempotents from Theorem 1.1.

PROPOSITION 2.6. *Let $p \in G^*$ be an idempotent. Then the following statements are equivalent*:

(1) *p is strongly right maximal and $(\beta G)p$ is a maximal principal left ideal,*
(2) *$Gp \subseteq G^*$ is a maximal space and $\overline{Gp}$ is a maximal orbit closure,*
(3) *$\mathcal{T}(p)$ is regular and weakly complete.*

We conclude this section with two more lemmas needed in the proof of Theorem 1.1.

LEMMA 2.7. *Let $\mathcal{T}_0$ be a Hausdorff (regular) left invariant topology on G and let $(U_n)_{n<\omega}$ be any sequence of neighborhoods of 1 in $\mathcal{T}_0$. Then $\mathcal{T}_0$ can be weakened to a first countable Hausdorff (regular) left invariant topology $\mathcal{T}$ on G in which each U_n remains a neighborhood of* 1.

PROOF. We consider the Hausdorff case, the regular one is [6, Lemma 9.28].

Without loss of generality one may suppose that $U_0 = G$. Enumerate $G \setminus \{1\}$ as $\{x_n : 1 \leq n < \omega\}$. Construct inductively a sequence $(V_n)_{n<\omega}$ of open neighborhoods of 1 in $\mathcal{T}_0$ with $V_0 = G$ such that for every $n \geq 1$ the following conditions are satisfied:

(i) $V_n \subseteq V_{n-1}$,
(ii) $x_n V_n \subseteq V_k$, where $k = \max\{i \leq n-1 : x_n \in V_i\}$,
(iii) $(x_n V_n) \cap V_n = \emptyset$, and
(iv) $V_n \subseteq U_n$.

It follows from (i)–(ii) that there is a left invariant topology $\mathcal{T}$ on G, not necessarily Hausdorff, in which $\{V_n : n < \omega\}$ is a neighborhood base at 1 (see [6, Corollary 4.4]). Condition (iii) implies that $\mathcal{T}$ is Hausdorff, and (iv) that each U_n remains a neighborhood of 1 in $\mathcal{T}$. ⊣

LEMMA 2.8. *Let $I \subseteq G^*$ be a closed left ideal of βG. Then there is a left invariant topology $\mathcal{T}$ on G with* $\text{Ult}(\mathcal{T}) = I$.

PROOF. Let $\mathcal{F}$ be the intersection of all ultrafilters from I. Then $\mathcal{F}$ is a filter on G such that the set of all ultrafilters on G containing $\mathcal{F}$ is I [6, Lemma 2.28]. The filter $\mathcal{F}$ has also the property that for every $A \in \mathcal{F}$ and $x \in G$, there is $B_x \in \mathcal{F}$ such that $xB_x \subseteq A$. Indeed, for every $p \in I$, one has $xp \in I$, so there is $B_{x,p} \in p$ such that $xB_{x,p} \subseteq A$. Put $B_x = \bigcup_{p \in I} B_{x,p}$.

Now let $\mathcal{N} = \{A \cup \{1\} : A \in \mathcal{F}\}$. Then

(i) $\bigcap \mathcal{N} = \{1\}$, and
(ii) for every $U \in \mathcal{N}$ and $x \in U$, there is $V \in \mathcal{N}$ and $xV \subseteq U$.

It follows that there is a left invariant topology $\mathcal{T}$ on G for which $\mathcal{N}$ is the neighborhood filter of 1, and so $\mathrm{Ult}(\mathcal{T}) = I$. ⊣

§3. Special construction. By [7, Lemma 6], there is a surjective finite-to-one function $f : G \to \omega$ such that

(1) $f(1) = 0$,
(2) for every $x \in G$, $f(x) = f(x^{-1})$, and
(3) for every $x, y \in G$, $f(xy) \le \max\{f(x), f(y)\}+1$, and if $|f(x)-f(y)| \ge 2$, then $f(xy) \ge \max\{f(x), f(y)\} - 1$.

The function $f : G \to \omega$ extends continuously to $\beta G \to \beta\omega$. We use the same letter f to denote this extension. Notice that for any $p \in \beta G$ and $q \in G^*$, $f(pq) = f(q) + i$ for some $i \in \{-1, 0, 1\}$.

A left ideal $I \subseteq G^*$ of βG is *locally maximal* if $G^* \setminus I$ is also a left ideal.

THEOREM 3.1 ([8, Theorem 2.6]). *Let $(A_n)_{n<\omega}$ be a decreasing sequence of subsets of G such that*

(a) *for every $n < \omega$ and $x \in A_n$, $f(x) \ge n$, and*
(b) *both the sets $f(A_0) + i$, $i \in \{-1, 0, 1\}$, and the sets $f(A_n \setminus A_{n+1})$, $n < \omega$, are pairwise disjoint.*

For every $n < \omega$, let $W_n = \bigcup_{x\in G} xA_{n+f(x)}$, and let $I = \bigcap_{n<\omega} \overline{W_n}$. Then I is a locally maximal closed left ideal of βG.

For every filter $\mathcal{F}$ on G with $\bigcap \mathcal{F} = \emptyset$, there is a largest left invariant topology $\mathcal{T}[\mathcal{F}]$ on G in which $\mathcal{F}$ converges to 1. The topology $\mathcal{T}[\mathcal{F}]$ has a neighborhood base at 1 consisting of subsets

$$[M] = \{x_0x_1 \cdots x_n : n < \omega, x_0 = 1 \text{ and } x_{i+1} \in M(x_0 \cdots x_i) \text{ for each } i < n\},$$

where $M : G \to \mathcal{F}$ [6, Theorem 4.8].

A filter $\mathcal{F}$ on G is *strongly discrete* if $\bigcap \mathcal{F} = \emptyset$ and there is $M : G \to \mathcal{F}$ such that the subsets $xM(x) \subseteq G$, $x \in G$, are pairwise disjoint.

THEOREM 3.2 ([6, Theorem 4.18]). *For every strongly discrete filter $\mathcal{F}$ on G, the topology $\mathcal{T}[\mathcal{F}]$ is zero-dimensional and Hausdorff, and consequently, regular.*

THEOREM 3.3. *Let $\mathcal{T}$ be a Hausdorff left invariant topology on G and let $(\mathcal{F}_n)_{n<\omega}$ be a sequence of filters on G converging to 1 in $\mathcal{T}$. Suppose that*

(i) *there is a neighborhood U of 1 in $\mathcal{T}$ such that the subsets $f(U \setminus \{1\}) + i \subseteq \omega$, $i \in \{-1, 0, 1\}$, are pairwise disjoint,*
(ii) *for every $n < \omega$, there is $A_n \in \mathcal{F}_n$ such that the subsets $f(A_n) \subseteq \omega$, $n < \omega$, are pairwise disjoint.*

Let $\mathcal{F}$ be the filter on G with a base of subsets $\bigcup_{n\le i<\omega} B_i$, where $n < \omega$ and $B_i \in \mathcal{F}_i$. Then $\mathcal{F}$ is strongly discrete.

PROOF. For every $n < \omega$, choose a neighborhood U_n of 1 in $\mathcal{T}$ such that

(a) the subsets xU_n, where $x \in G$ with $f(x) \le n$, are pairwise disjoint, and choose $C_n \in \mathcal{F}_n$ such that
(b) $C_n \subseteq U_n$,
(c) for every $x \in C_n$, $f(x) \ge n + 2$, and
(d) $C_n \subseteq U \cap A_n$.

We claim that the subsets

$$x \bigcup_{n \geq f(x)} C_n,$$

where $x \in G$, are pairwise disjoint.

Let $x, y \in G$, $x \neq y$. Since

$$x \bigcup_{n \geq f(x)} C_n = \bigcup_{n \geq f(x)} xC_n,$$

$$y \bigcup_{m \geq f(y)} C_m = \bigcup_{m \geq f(y)} yC_m,$$

it suffices to check that the subsets xC_n and yC_m are disjoint for any $n \geq f(x)$, $m \geq f(y)$. If $n = m$, they are disjoint by (a) and (b). Now let $n \neq m$. Then by (c),

$$f(xC_n) \subseteq \bigcup_{i=-1}^{1} (f(C_n) + i),$$

$$f(yC_m) \subseteq \bigcup_{j=-1}^{1} (f(C_m) + j),$$

so by (d),

$$f(xC_n) \subseteq \bigcup_{i=-1}^{1} (f(U \cap A_n) + i),$$

$$f(yC_m) \subseteq \bigcup_{j=-1}^{1} (f(U \cap A_m) + j).$$

But by (i) and (ii),

$$\bigcup_{i=-1}^{1} (f(U \cap A_n) + i) \text{ and } \bigcup_{j=-1}^{1} (f(U \cap A_m) + j)$$

are disjoint. Hence, $f(xC_n)$ and $f(yC_m)$ are disjoint, and so are xC_n and yC_m. ⊣

§4. Proof of Theorem 1.1. Let $f : G \to \omega$ be as at the beginning of Section 3. Pick an idempotent $p_0 \in X$. Since the ultrafilters $f(p_0) + i \in \omega^*$, $i \in \{-1, 0, 1\}$, are distinct (see [3, Lemma 6.28]), there is $E \in f(p_0)$ such that the subsets $E + i \subseteq \omega$, $i \in \{-1, 0, 1\}$, are pairwise disjoint. Put $P = f^{-1}(E)$. Then $P \in p_0$ and the subsets $f(P) + i \subseteq \omega$, $i \in \{-1, 0, 1\}$, are pairwise disjoint. Let $\mathcal{T}_0' = \mathcal{T}(p_0)$ (that is, $\mathrm{Ult}(\mathcal{T}_0') = \{p_0\}$). By Lemma 2.7, $\mathcal{T}_0'$ can be weakened to a first countable Hausdorff left invariant topology $\mathcal{T}_0$ on G such that

$$T_0 = \mathrm{Ult}(\mathcal{T}_0) \subseteq X \cap \overline{P}.$$

Since $T_0 \subseteq \overline{P}$, we have that for any $p, q \in T_0$, $f(pq) = f(q)$. Since the character of $T_0 \subseteq G^*$ is countable, there is an infinite $D \subseteq \omega$ such that $D^* \subseteq f(T_0)$. By Theorem 1.3, there is a deep subset $Z \subseteq D^*$. Let

$$J = f^{-1}(Z) \cap T_0.$$

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Then

(1) J is a closed left ideal of T_0,
(2) $f(J) \subseteq \omega^*$ is deep, and
(3) $J = f^{-1}(f(J)) \cap T_0$.

Next, enumerate the subsets of G as $\{C_\alpha : \alpha < \mathfrak{c}\}$ with $C_0 = G$, and inductively, for every $\alpha > 0$, construct a first countable regular left invariant topology $\mathcal{T}_\alpha$ on G and a locally maximal closed left ideal $I_\alpha \subseteq G^*$ of countable character such that

(4) $T_\alpha = \mathrm{Ult}(\mathcal{T}_\alpha) \subseteq T_0$,
(5) $T_\alpha \subseteq I_\alpha$,
(6) $T_\alpha \subseteq \overline{C_\alpha}$ or $T_\alpha \subseteq \overline{G \setminus C_\alpha}$,
(7) if there is an idempotent $p \in \bigcap_{\gamma<\alpha} T_\gamma \cap J$ such that $(\beta G)p \subseteq \overline{C_\alpha}$, then $I_\alpha \subseteq \overline{C_\alpha}$, and
(8) $\bigcap_{\gamma\leq\alpha} T_\gamma \cap J \neq \emptyset$.

Fix $\alpha > 0$ and suppose that we have already constructed I_γ and $\mathcal{T}_\gamma$ for all $\gamma < \alpha$ as required. Let

$$K_\alpha = \bigcap_{\gamma<\alpha} T_\gamma \cap J.$$

By (1) and (8), K_α is a closed subsemigroup of T_0.

Suppose that there is an idempotent $p_\alpha \in K_\alpha$ such that $(\beta G)p_\alpha \subseteq \overline{C_\alpha}$. Pick $D_\alpha \in p_\alpha$ such that

(i) $D_\alpha \subseteq C_\alpha$.

Then for every $n < \omega$, pick $P_\alpha^n \in p_\alpha$ such that

(ii) for each $x \in G$ with $f(x) \leq n$, $xP_\alpha^n \subseteq C_\alpha$,
(iii) $P_\alpha^n \subseteq P$, and
(iv) for every $x \in P_\alpha^n$, $f(x) \geq n$.

Let $\mathcal{T}'_\alpha = \mathcal{T}(p_\alpha)$. By Lemma 2.7, $\mathcal{T}'_\alpha$ can be weakened to a first countable Hausdorff left invariant topology $\mathcal{T}''_\alpha$ such that

$$T''_\alpha = \mathrm{Ult}(\mathcal{T}''_\alpha) \subseteq T_0 \cap \overline{D_\alpha} \cap \bigcap_{n<\omega} \overline{P_\alpha^n}.$$

Let

$$Y_\alpha = \bigcap_{\gamma<\alpha} T_\gamma \cap T''_\alpha.$$

Since $p_\alpha \in Y_\alpha \cap J$ and $\chi(Y_\alpha) \leq |\alpha| + \omega < \mathfrak{c}$, it follows from (2) that $f(Y_\alpha) \cap f(J)$ is infinite. For every $n < \omega$, choose

$$u_\alpha^n \in f(Y_\alpha) \cap f(J)$$

and $E_\alpha^n \in u_\alpha^n$ such that the subsets $E_\alpha^n \subseteq \omega$, $n < \omega$, are pairwise disjoint.

This can be done by induction on n as follows. Pick $u_\alpha^n \in (f(Y_\alpha) \cap f(J)) \setminus \overline{F_\alpha^{n-1}}$ and $E_\alpha^n \in u_\alpha^n$, where $F_\alpha^{n-1} = \bigcup_{j\leq n-1} E_\alpha^j$, such that E_α^n is disjoint from F_α^{n-1} and $(f(Y_\alpha) \cap f(J)) \setminus \overline{F_\alpha^n} \neq \emptyset$.

For every $n < \omega$, pick $q_\alpha^n \in Y_\alpha$ such that $f(q_\alpha^n) = u_\alpha^n$. By (3), $q_\alpha^n \in J$, so

$$q_\alpha^n \in Y_\alpha \cap J.$$

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Then for every $n < \omega$, choose $Q_\alpha^n \in q_\alpha^n$ such that

(v) $Q_\alpha^n \subseteq P_\alpha^n$ and
(vi) $f(Q_\alpha^n) \subseteq E_\alpha^n$.

Let

$$A_\alpha^n = \bigcup_{n \leq i < \omega} Q_\alpha^i, \; W_\alpha^n = \bigcup_{x \in G} x A_\alpha^{n+f(x)}, \text{ and } I_\alpha = \bigcap_{n<\omega} \overline{W_\alpha^n}.$$

Then by (iv) and (v), for every $n < \omega$ and $x \in A_\alpha^n$, $f(x) \geq n$, and by (iii) and (vi), both the sets $f(A_\alpha^0) + i$, $i \in \{-1, 0, 1\}$, and the sets $f(A_\alpha^n \setminus A_\alpha^{n+1})$, $n < \omega$, are pairwise disjoint. Consequently by Theorem 3.1, I_α is a locally maximal closed left ideal. By (ii) and (v), $I_\alpha \subseteq C_\alpha$.

If there is no idempotent $p \in K_\alpha$ such that $(\beta G)p \subseteq \overline{C_\alpha}$, then pick any idempotent of K_α as p_α and take care of (i)' $D_\alpha \subseteq C_\alpha$ or $D_\alpha \subseteq G \setminus C_\alpha$, (vi), (iii)' $Q_\alpha^n \subseteq P$, and (iv)' for every $x \in Q_\alpha^n$, $f(x) \geq n$.

Let $\mathcal{F}_\alpha$ be the filter on G with a base consisting of subsets $\bigcup_{n \leq i < \omega} R_\alpha^i$, where $n < \omega$ and $R_\alpha^i \in q_\alpha^i$, and let $\mathcal{T}_\alpha''' = \mathcal{T}[\mathcal{F}_\alpha]$. By Theorem 3.3, $\mathcal{F}_\alpha$ is strongly discrete, so $\mathcal{T}_\alpha'''$ is regular. Notice that $\{q \in \beta G : \mathcal{F}_\alpha \subseteq q\} \subseteq I_\alpha$. Consequently by Lemma 2.8, $\mathrm{Ult}(\mathcal{T}_\alpha''') \subseteq I_\alpha$. Using Lemma 2.7, weaken $\mathcal{T}_\alpha'''$ to a first countable regular left invariant topology $\mathcal{T}_\alpha$ such that $\mathrm{Ult}(\mathcal{T}_\alpha) \subseteq T_\alpha'' \cap I_\alpha$.

Clearly, (4), (5), (6), and (7) are satisfied. To see (8), let q be any limit point of $\{q_\alpha^n : n < \omega\}$. Then $\mathcal{F}_\alpha \subseteq q$ and $q \in \bigcap_{\gamma<\alpha} T_\gamma \cap J$, so $q \in \bigcap_{\gamma \leq \alpha} T_\gamma \cap J$.

Now let $\mathcal{T}$ be the least upper bound of topologies $\mathcal{T}_\alpha$, $1 \leq \alpha < \mathfrak{c}$. That is, $\mathcal{T}$ is the left invariant topology on G with a neighborhood base at 1 consisting of subsets $\bigcap_{i=1}^n U_{\alpha_i}$, where $1 \leq n < \omega$, $1 \leq \alpha_1 < \cdots < \alpha_n < \mathfrak{c}$, and U_{α_i} is a neighborhood of 1 in $\mathcal{T}_{\alpha_i}$ for each i. Then

$$T = \mathrm{Ult}(\mathcal{T}) = \bigcap_{1 \leq \alpha < \mathfrak{c}} T_\alpha.$$

If each U_{α_i} is closed in $\mathcal{T}_{\alpha_i}$, $\bigcap_{i \leq n} U_{\alpha_i}$ is closed in $\mathcal{T}$. Consequently, $\mathcal{T}$ is regular. By (4), $T \subseteq T_0$, and since $T_0 \subseteq X$, one has $T \subseteq X$. By (6) and (8), T is a one-element semigroup, so $T = \{p\}$ for some idempotent $p \in X$, that is, $\mathcal{T} = \mathcal{T}(p)$. Hence by Lemma 2.1, p is strongly right maximal.

Let

$$I = \bigcap_{1 \leq \alpha < \mathfrak{c}} I_\alpha.$$

Then $p \in I$ by (5), and I is a locally maximal closed left ideal. We claim that $I = (\beta G)p$.

To see this, assume the contrary. Pick $C \subseteq G$ such that $(\beta G)p \subseteq \overline{C}$ and $I \setminus \overline{C} \neq \emptyset$. There is $\alpha < \mathfrak{c}$ such that $C = C_\alpha$. Then by (7), $I_\alpha \subseteq \overline{C_\alpha}$, and so $I \subseteq \overline{C_\alpha}$, a contradiction.

We conclude this paper with the following question.

Question 4.1. *Can it be shown in ZFC that there is a strongly left maximal idempotent in* $\mathbb{Z}^*$?

§5. Acknowledgements. The work was supported by NRF grant IFR2011033100072. The author thanks the referee for a careful reading of the paper and useful comments.

https://doi.org/10.1017/jsl.2016.62 Published online by Cambridge University Press

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