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Symmetry and invariants of kinematic chains and parallel manipulators

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SUMMARY
This paper presents applications of group theory tools to simplify the analysis of kinematic chains associated with mechanisms and parallel manipulators. For the purpose of this analysis, a kinematic chain is described by its properties, i.e. degrees-of-control, connectivity and redundancy matrices. In number synthesis, kinematic chains are represented by graphs, and thus the symmetry of a kinematic chain is the same as the symmetry of its graph. We present a formal definition of symmetry in kinematic chains based on the automorphism group of its associated graph. The symmetry group of the graph is associated with the graph symmetry. By using the group structure induced by the symmetry of the kinematic chain, we prove that degrees-of-control, connectivity and redundancy are invariants by the action of the automorphism group of the graph. Consequently, it is shown that it is possible to reduce the size of these matrices and thus reduce the complexity of the kinematic analysis of mechanisms and parallel manipulators in early stages of mechanisms design.

KEYWORDS: Kinematic chain; Parallel manipulators; Graph symmetry; Automorphism group; Actions; Orbits.

1. Introduction

Mathematical models are commonly difficult to handle in a general setting. Symmetry in mathematical models is useful to simplify the understanding of a model and to determine the patterns for which the model is appropriate. Thus, it is a common strategy to study cases of symmetry in order to learn more about a model. In nature, there are different types of mathematical models and also different types of symmetries, but thanks to the symmetry concept, many models are now reasonably well understood. In our setting, the mathematical model associates a kinematic chain with a graph. Graphs are extensively used in the literature of mechanisms and machine to describe kinematic chains.[3,35,50] Belfiore and Di Benedetto,[3] Liberati and Belfiore[27] and Martins and Carboni[30] discuss how the topological structure of a kinematic chain of a parallel manipulator can be described quite extensively by degrees-of-control, connectivity and redundancy matrices. These matrices are square symmetric with dimension $n \times n$, where n is the number of links of the kinematic chain. One aim of this paper is to develop a method to reduce the size of the degrees-of-control, connectivity and redundancy matrices of a kinematic chain associated with kinematic chains of mechanisms and parallel manipulators. In this context, the graph symmetry plays an important role because it provides a group structure that fits our purposes.

The graph of a kinematic chain is a graph on which the vertices represent the links and the edges represent the joints of the kinematic chain.[35] Hence, in early stages of mechanisms design, such as number synthesis,[35,50] the analysis of a kinematic chain is reduced to the analysis of its graph, and thus in this paper the term graph will be synonymous with kinematic chain.

In order to achieve our aims, we investigate symmetries and invariants by the action of the automorphism group of the graph representing kinematic chains of mechanisms and parallel manipulators. Symmetries of graphs are related to automorphisms.[10,37] By exploring these symmetries it is possible to reduce the matricial representation of important properties to the kinematic analysis of kinematic chains.

The main result of this study was to prove that the degrees-of-freedom (DoF), connectivity and redundancy matrices are all invariants by the action of the automorphism group of the graph. This invariance is the main tool used to reduce the size of the matrices. It is shown that the matrix size is reduced from $n \times n$ to $o \times n$, where n is the number of links and o is the number of orbits by the action of the automorphism group of the graph. Higher graph symmetry means a smaller number of orbits o, as will be clearly shown through examples.

The group theory has been used by some authors in the context of analyzing kinematic chains. Tsai[50] uses the symmetry group of a kinematic chain to identify when two kinematic chains are identical (isomorphism problem). Tuttle[51] uses group theory to identify all distinct bases of a kinematic chain enumeration process. Simoni *et al.*[44,45] have applied the group theory tools in the enumeration of kinematic chains, mechanisms and parallel manipulators.

The group theory has several applications in mechanisms and robotics such as to characterize all equivalent ways a modular robots can be constructed or assembled from its components,[8,9] assembly planning[18,29] and for positioning robots or robotic end effectors.[36,52]

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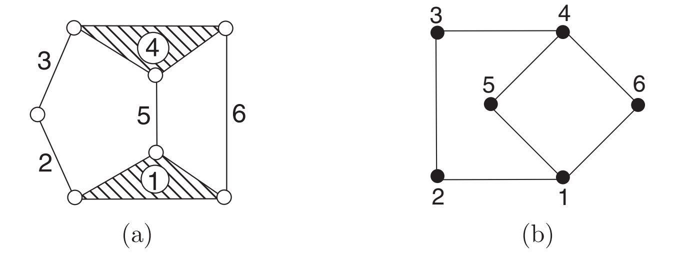

Fig. 1. Biunivocal correspondence between graphs and kinematic chains used extensively in early stages of mechanism design. (a) Stephenson kinematic chain; (b) Stephenson graph.

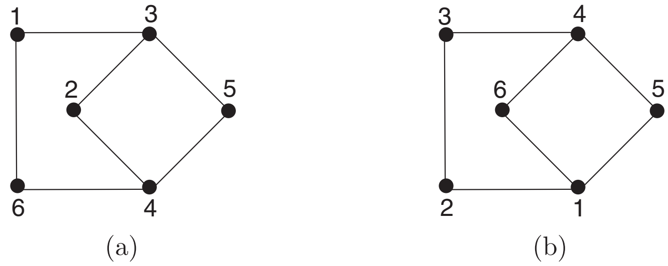

Fig. 2. Action of σ_1 and σ_2 in the Stephenson graph X. (a) $\sigma_1(X)$; (b) $\sigma_2(X)$.

The remainder of this paper is organized as follows: Section 2 describes the analogy of kinematic chains and graphs, and the application of group theory to graphs/kinematic chains. Concepts that are important in terms of the content of this paper, such as actions and orbits of the automorphism group, are discussed and examples of their application to graphs are presented. Details of group theory can be found in Appendix 6. Section 2 also gives a precise definition of symmetry of a kinematic chain. Section 3 presents the definitions found in the literature to degrees-of-control, connectivity and redundancy in terms of the graph associated with a kinematic chain. Section 4 details the applications of group theory to the kinematic analysis of kinematic chains and a method to reduce the size of these matrices. Section 6 presents the conclusions and suggestions for further work.

2. Graphs and Symmetry

A graph $X = (V, E)$ consists of a finite set $V(X)$ of vertices and a family $E(X)$ of subsets of $V(X)$ of size two called edges. Usually, the pair {x,y} denotes an edge, and the number of edges incident to a vertex v is the degree of the vertex v ($deg(v)$). It is important to remember that in early stages of design a kinematic chain can be uniquely represented by a graph whose vertices correspond to the links of the chain and whose edges correspond to the joints of the chain.[3,27,35,50] Figure 1 shows this correspondence: Fig. 1(a) shows the classical Stephenson kinematic chain with labeled links and Fig. 1(b) shows the corresponding graph.

A subgraph of a graph X is a graph Y such that $V(Y) \subseteq V(X)$, $E(Y) \subseteq E(X)$. A path between two vertices x and y is a sequence $x_0, x_1, x_2, \ldots, x_k$ of vertices such that $x_0 = x$, $x_k = y$ and for all $i \in [1, k]$, $(x_{i-1}, x_i) \in E$. The length of a path is its number of edges. The distance between two vertices x and y, denoted by $\delta(x, y)$, is the length of the shortest path between x and y.

Our aim is to develop a technique to reduce the size of the degrees-of-control, connectivity and redundancy matrices of a kinematic chain. The analysis of these matrices is developed in early stages of mechanisms design where the representation of kinematic chains by graphs is classical.[3,27,35,50] Thus, for our purpose, a kinematic chain is represented univocally by a graph. From now on, the term graph will be used to mean a kinematic chain and vice versa, unless otherwise stated.

2.1. Actions in kinematic chains

Given a graph X, a bijective map $\sigma : V(X) \to V(X)$ defines a permutation of the elements of $V(X)$. Assuming $V(X)$ has n elements, the set of permutations endowed with the operation of composition is the group $\mathcal{S}_n$ and we can apply the definitions present in Appendix 6.

Example 1 (Actions). *Figure 1(a) shows the Stephenson kinematic chain and Fig. 1(b) its graph X. Figures 2(a) and (b) show the action of $\sigma_1(X)$ and $\sigma_2(X)$, respectively, on the labels of the Stephenson graph, where*

$$\sigma_1(X) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 3 & 5 & 4 & 1 & 6 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 3 & 4 \\ 3 & 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 5 & 6 \\ 5 & 6 & 2 \end{pmatrix} = (134)(256)$$

and

$$\sigma_2(X) = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 3 & 2 & 1 & 6 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 \\ 4 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 3 & 2 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 6 & 5 \end{pmatrix} = (14)(23)(56).$$

2.2. Automorphisms in kinematic chains

Definition 1 (Isomorphisms and automorphisms of graphs). *Two graphs X and Y are isomorphic if there is a bijection $\sigma : V(X) \to V(Y)$ such that*

$$\{xy\} \in E(X) \Leftrightarrow \{\sigma(x)\sigma(y)\} \in E(Y).$$

If isomorphism exists between two graphs, then the graphs are called isomorphic and we write $X \simeq H$.
The automorphism of a graph is the graph's isomorphism with itself. The automorphism group of a graph X is denoted by $Aut(X)$.

Most isomorphism tests are based on graph invariants, which preserve the properties or parameters of graphs under isomorphism, such as degree sequence, distance matrix, vertex ordering, etc.[21]

Herein, we use some results for invariants of isomorphism and automorphism groups of graphs found in the literature.[4,12,16,17,25,33,46] These results are important to

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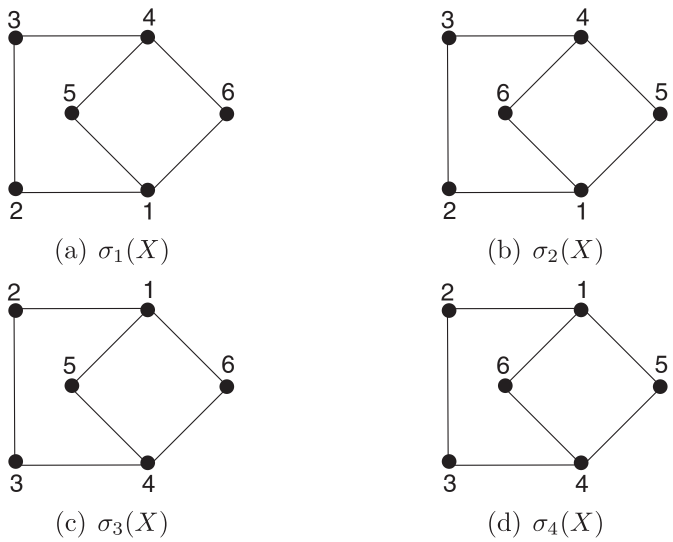

Fig. 3. Action of the automorphism group in the Stephenson graph.

prove Theorems 9 and 10 in Section 4 and are summarized below:

Remark 1. *Let X be a graph, Y a subgraph of X and σ an element of Aut(X).*

(*1*) *Degree invariance:* $deg(\sigma(x)) = deg(x)$, *for all* $x \in V(X)$;
(2) *Distance invariance:* $\delta(\sigma(x), \sigma(y)) = \delta(x, y)$, *for all* $x, y \in V(X)$;
(*3*) *Subgraph invariance:* $\sigma(Y) \simeq Y$, i.e. *they are isomorphic.*

Proofs of these invariant remarks are found in refs. [12, 46].

2.3. Orbits in kinematic chains

The orbit of a graph vertex corresponds to the set of vertices for which the vertex is moved by the action of the automorphism group of the graph. Let us consider an example.

Example 2. *Let X be the Stephenson graph shown in Fig. 1(b). In this case,*

$$Aut(X) = \begin{Bmatrix} \sigma_1 = (1)(2)(3)(4)(5)(6), & \sigma_2 = (1)(2)(3)(4)(56), \\ \sigma_3 = (14)(23)(5)(6), & \sigma_4 = (14)(23)(56) \end{Bmatrix}.$$

The generator set is $Aut(X) =< \sigma_2, \sigma_3 >$. *The action of the automorphism group in the Stephenson graph is shown in Figs. 3(a)–(d).*

The orbits are:

$$\mathcal{O} = \{\{1, 4\}, \{2, 3\}, \{5, 6\}\} = \{\mathcal{O}_1, \mathcal{O}_2, \mathcal{O}_3\},$$

where

- $\mathcal{O}_1 = \{1, 4\}$;
- $\mathcal{O}_2 = \{2, 3\}$ *and*
- $\mathcal{O}_5 = \{5, 6\}$.

The Stephenson graph has three different orbits resulting in classical Stephenson I ($\mathcal{O}_5$), *II* ($\mathcal{O}_2$) *and III* ($\mathcal{O}_3$) *mechanisms fixing a representative link of each orbit* (for more details, consult[44,45]).

2.4. Symmetry in kinematic chains

Since our aim is to apply symmetry to simplify the kinematic chain analysis, and kinematic chains are represented in biunivocal correspondence by graphs, it is necessary to define what is meant by graph symmetry. This is carried out using the concept of a group defined in the Appendix 6 and used in previous sections. Rao[39] discusses symmetries in kinematic chains but does not present a formal definition or a technique to obtain the symmetries of a kinematic chain.

The symmetry of a graph corresponds to an element of the automorphism group of the graph. According to Erdős and Rényi[10] and Petitjean,[37] a graph is considered to be symmetric when it has more than one automorphism, i.e. the automorphism group has a degree greater than 1. In the definition below, we extend the concept of graph symmetry to kinematic chains.

Definition 2 (Symmetry of a kinematic chain). *The symmetry of a kinematic chain is the symmetry of its corresponding graph. A kinematic chain is symmetric when it has more than one automorphism.*

This definition can be applied to kinematic chains with geometrical information attached and their correspondent valued graphs. Despite this fact, this paper considers only "topological" kinematic chains and their correspondent non-valued graphs. In all contexts, symmetric links can be identified by the orbits of the automorphism group of the graph.

3. Fundamental Properties of Kinematic Chains

In this section, some fundamental properties of kinematic chains are introduced. These are essential for topological analysis and number synthesis of mechanisms and parallel manipulators and are extensively used in the literature of mechanism and machine.[3,20,27,30,35,50]

Definition 3 (Mobility). *The number of degrees-of-freedom, or* mobility (*M*), *of a kinematic chain is the number of independent parameters required to completely specify the configuration of the kinematic chain in space with respect to one link chosen as the reference.*

The mobility of a kinematic chain, with n links and g single degree-of-freedom joints, may be calculated by the general mobility criterion[20,35] applied to a set of n links and g single degree-of-freedom joints,

$$M = \lambda(n - g - 1) + g, \qquad (1)$$

where λ is the order of the screw system to which all the joint screws belong. Using the graph representation of a kinematic chain (see Fig. 1), the general mobility criterion is given by

$$M = \lambda\,(|V| - |E| - 1) + |E|, \qquad (2)$$

where $|V|$ is the number of graph vertices (i.e. links) and $|E|$ is the number of graph edges (i.e. joints).[35,50]

A recent review of mobility calculation was presented by Gogu,[15] who presents a critical review on the mobility of

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mechanisms and discusses all the faults of mobility equations found in the literature. However, in the early stages of mechanism design (classical number synthesis[35]), where the kinematic structure is not known, it is necessary to have a quick calculation of mobility to design new kinematic chains of mechanisms and parallel manipulators.

The *connectivity* C_{ij} between two links i and j of a kinematic chain is the relative mobility between links i and j.[20] Different algorithms for connectivity calculations have been proposed by Shoham and Roth,[43] Belfiore and Di Benedetto,[3] Liberati and Belfiore[27] and Martins and Carboni.[30] Below we present the definitions found in the literature for connectivity, degrees-of-control and redundancy in terms of graphs.

Definition 4 (Connectivity[30]). *In a kinematic chain represented by a graph X, the connectivity between two links i and j is*

$$C_{ij}(X) = \min : \{D[i, j], M'_{\min}, \lambda\}, \tag{3}$$

where $D[i, j]$ *is the distance between vertices i and j of X,* $M'_{\min}$ *is the minimum mobility of any closed-loop biconnected subchain of X containing vertices i and j, and* λ *is the order of the screw system.*

Definition 5 (Degrees-of-control[30]). *In a kinematic chain represented by a graph X, the degrees-of-control between two links i and j is*

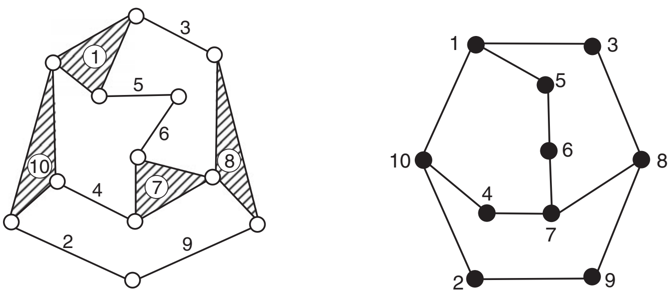
$$K_{ij}(X) = \min : \{D[i, j], M'_{\min}\}. \tag{4}$$

Definition 6 (Redundancy[3]). *In a kinematic chain represented by a graph X, the redundancy between two links i and j is the difference between* $K_{ij}(X)$ *and* $C_{ij}(X)$,

$$R_{ij}(X) = K_{ij}(X) - C_{ij}(X). \tag{5}$$

The importance of the connectivity and redundancy is emphasized by several authors.[3,20,27,47–49]

Example 3. *Consider the planar kinematic chain X shown in Fig. 4 where the mobility is* $M = 3$.



(a) Kinematic chain (b) Graph

Fig. 4. Kinematic chain X: (a) structural, and (b) graph representations.

The adjacency matrix $A(X)$ *is given by*

$$A(X) = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array} \\ \begin{bmatrix} 0&0&1&0&1&0&0&0&0&1\\ 0&0&0&0&0&0&0&0&1&1\\ 1&0&0&0&0&0&0&1&0&0\\ 0&0&0&0&0&0&1&0&0&1\\ 1&0&0&0&0&1&0&0&0&0\\ 0&0&0&0&1&0&1&0&0&0\\ 0&0&0&1&0&1&0&1&0&0\\ 0&0&1&0&0&0&1&0&1&0\\ 0&1&0&0&0&0&0&1&0&0\\ 1&1&0&1&0&0&0&0&0&0 \end{bmatrix} \end{array}. \tag{6}$$

The connectivity $C(X)$ *is given by*

$$C(X) = \begin{array}{c} \\ 1\\2\\3\\4\\5\\6\\7\\8\\9\\10 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1&2&3&4&5&6&7&8&9&10 \end{array} \\ \begin{bmatrix} 0&2&1&2&1&2&3&2&3&1\\ 2&0&3&2&3&3&3&2&1&1\\ 1&3&0&3&2&3&2&1&2&2\\ 2&2&3&0&3&2&1&2&3&1\\ 1&3&2&3&0&1&2&3&3&2\\ 2&3&3&2&1&0&1&2&3&3\\ 3&3&2&1&2&1&0&1&2&2\\ 2&2&1&2&3&2&1&0&1&3\\ 3&1&2&3&3&3&2&1&0&2\\ 1&1&2&1&2&3&2&3&2&0 \end{bmatrix} \end{array}. \tag{7}$$

In this case, the degrees-of-control matrix $K(X)$ *is equal to the connectivity matrix* $C(X)$, $K(X) = C(X)$, *and the redundancy matrix is null,* $R(X) = 0$.

This example shows that a kinematic chain can be described by its properties, i.e. *adjacency, connectivity, degrees-of-control and redundancy matrices.*

3.1. The relevance of the connectivity, the degrees-of-control and the redundancy

The connectivity, though is not a new concept in the mechanism and machine literature, is as important as mobility for kinematic chains, see e.g. Phillips,[38] Hunt,[20] Tischler *et al.*,[47,48] Belfiore and Di Benedetto,[3] Liberati and Belfiore[27] and Martins and Carboni.[30]

The connectivity computation is very important for the structural analysis and synthesis of mechanisms and parallel manipulators. The structural analysis and synthesis of mechanisms are fundamental to the invention and innovation of mechanisms.[7] In the structural synthesis, several kinematic chains are enumerated, so it is necessary to analyze those kinematic chains that fit in the customer's specifications. The connectivity is an important criterion for selecting kinematic chains. For a better understanding of the importance of the connectivity, we consider the kinematic chain as shown in Fig. 5. In Fig. 5 it is represented as a closed-loop kinematic chain with mobility $M = 3$ and the connectivity between any two links not exceeding 2. From this simple example, as outlined in previous works,[3,6,27,43] it is evident that the connectivity, not the mobility, determines the ability of an output link to perform a task relative to a frame.

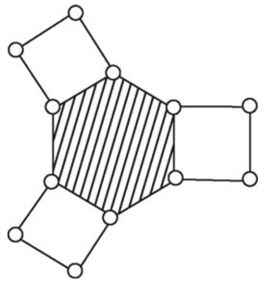


Fig. 5. Planar kinematic chain with maximum connectivity between links of 2, i.e. $C_{ij} \leq 2 \; \forall \; i, j$.[6] This kinematic chain will be eliminated for connectivity.

As shown in the above example, the connectivity is very important to select kinematic chains in the structural analysis. Since the degrees-of-control and the redundancy are related to connectivity, they are also important parameters for selecting kinematic chains as emphasized by Belfiore and Di Benedetto,[3] Liberati and Belfiore,[27] Martins and Carboni[30] and Shoham and Roth.[43]

More discussions and applications of connectivity can be found in Shoham and Roth,[43] Agrawal and Rao,[1] Liu and Yu.[28] Similar applications for connectivity are described in Gogu[13] and Tsai.[50]

4. Invariants of Kinematic Chains

This section considers the applications of the group and graph theory presented above. First, we prove theorems about the invariance of degrees-of-control, connectivity and redundancy of kinematic chains by the action of the automorphism group of the associated graph. Then, with the definition of the symmetry of kinematic chains (Definition 2) and the result of these theorems, we develop a technique to reduce the matricial representation of the degrees-of-control, connectivity and redundancy matrices simplifying the kinematic analysis of kinematic chains in early stages of mechanism design/analysis.

Lemma 7 (Mobility invariance). *The mobility M of a graph (kinematic chain) is invariant by the action of the automorphism group of the graph.*

Proof. The proof follows from Definition 1. An automorphism of a graph is an isomorphism with itself and thus the graph structure is preserved. As we can see in Example 2, the automorphism group of the graph results in the relabeling of the graph vertices and consequently the number of vertices $|V|$, the number of edges $|E|$ and the order of the screw system λ remain the same. Consequently, the mobility, Eq. (2), is invariant. □

Lemma 8 (Subgraph mobility invariance). *The mobility M of a subgraph (subchain) is invariant by the action of the automorphism group of the graph.*

Proof. The proof follows from Remark 1 and Lemma 7. Remark 1 proves that a subgraph is invariant by the action of its automorphism group and thus the structure of the subgraph ($|V|$, $|E|$, λ) remains the same. Lemma 7 proves that the mobility is invariant. Consequently, the subgraph mobility is invariant. □

Theorem 9 (Degrees-of-control invariance). *Let X be a graph (kinematic chain) and $Aut(X)$ its automorphism group. The degrees-of-control matrix $K(X)$ of the kinematic chain is invariant by the action of the automorphism group of the graph.*

Proof. The degrees-of-control is given by $K_{ij} = \min : \{D[i, j], M'_{\min}\}$, see Eq. (4). To prove this theorem, it is necessary to show that D matrix and $M'_{\min}$ are invariant by the action of the automorphism group. According to Remark 1, the distance of any pair of vertices is invariant by the action of the automorphism group of the graph, i.e. $D[i, j] = D[\sigma(i), \sigma(j)]$. Therefore, the D matrix is invariant by the action of the automorphism group of the graph. According to Remark 1, any subgraph is invariant by the action of the automorphism group of the graph, therefore $M'_{\min}$ is also invariant. □

Theorem 10 (Connectivity invariance). *Let X be a graph (kinematic chain) and $Aut(X)$ its automorphism group. The connectivity matrix $C(X)$ of the kinematic chain is invariant by the action of the automorphism group of the graph.*

Proof. The proof follows from Theorem 9. The connectivity is given by $C_{ij} = \min : \{K_{ij}, \lambda\}$, see Eq. (3). K_{ij} is invariant according to Theorem 9 and λ is a property of the kinematic chain (it is not dependent on the graph) and therefore it is constant. □

Corollary 11 (Redundancy invariance). *Let X be a graph (kinematic chain) and $Aut(X)$ its automorphism group. The redundancy matrix $R(X)$ of the kinematic chain is invariant by the action of the automorphism group of the graph.*

Proof. The proof follows straightforwardly from Theorems 9 and 10. The redundancy is given by $R_{ij}(X) = K_{ij}(X) - C_{ij}(X)$ (see Eq. (4)). $K_{ij}(X)$ and $C_{ij}(X)$ are invariants according to Theorems 9 and 10; consequently, $R_{ij}(X)$ is invariant. □

Theorems 9 and 10 and Corollary 11 state that the degrees-of-control, connectivity and redundancy are symmetric properties of a kinematic chain, i.e. elements that are symmetric by the action of the automorphism group of the graph have the same properties. Considering that symmetric links are identified by the orbits of the automorphism group of the graph, it is possible to reduce the matricial representation considering one representative element of each orbit. Figure 6 presents the technique to reduce the matricial representation of the properties of a kinematic chain.

5. Applications

To show the potential of the proposal reduction in the matricial representation, we have selected examples of mechanisms and parallel manipulators found in the literature where the connectivity, degrees-of-control and redundancy matrices are presented. We use the notation $A_r(X)$, $K_r(X)$, $C_r(X)$ and $R_r(X)$ to represent the reduced adjacency matrix, the reduced degrees-of-control matrix, the reduced connectivity matrix and the reduced redundancy matrix, respectively.

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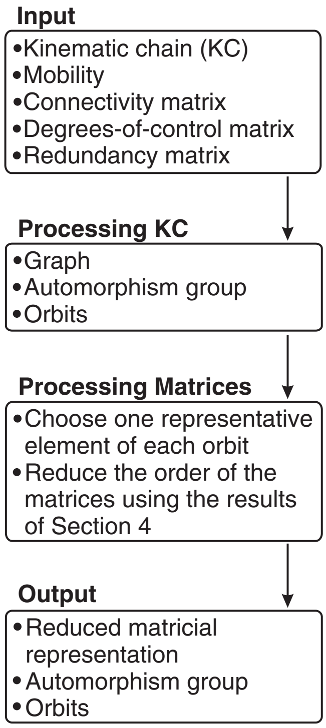

Fig. 6. Technique to reduce the size of the matrices associated to kinematic chains.

5.1. *Example 1: Planar parallel mechanisms*

Let X be the kinematic chain of the planar parallel mechanism shown in Fig. 4. In this case, using, for example, Nauty program[33] on its graph, we find the automorphism group

$$Aut(X) = \left\{ \begin{array}{l} \sigma_1 = (1)(2)(3)(4)(5)(6)(7)(8)(9)(10) \\ \sigma_2 = (1\ 7)(2\ 9)(3\ 4)(5\ 6)(8\ 10) \\ \sigma_3 = (1\ 8)(2\ 6)(5\ 9)(7\ 10) \\ \sigma_4 = (1\ 10)(2\ 5)(3\ 4)(6\ 9)(7\ 8) \end{array} \right\}. \quad (8)$$

The generator set is $Aut(X) = <\sigma_2, \quad \sigma_3>$. The automorphism group of the graph can be obtained with the Nauty program[31,32,34] or sage program.[41]

The orbits are:

$$\mathcal{O} = \{\{1\ 7\ 8\ 10\}; \{2\ 5\ 6\ 9\}; \{3\ 4\}\},$$

where

- $\mathcal{O}_1 = \{1\ 7\ 8\ 10\}$,
- $\mathcal{O}_2 = \{2\ 5\ 6\ 9\}$ and
- $\mathcal{O}_3 = \{3\ 4\}$.

The adjacency matrix presented in Eq. (6) is reduced to

$$A_r(X) = \begin{array}{c} \\ \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccccccccc} 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \end{array} \right] \end{array} \quad (9)$$

and the connectivity matrix presented in Eq. (7) is reduced to

$$C_r(X) = \begin{array}{c} \\ \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \end{array} \begin{array}{c} \begin{array}{cccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{array} \\ \left[\begin{array}{cccccccccc} 0 & 2 & 1 & 2 & 1 & 2 & 3 & 2 & 3 & 1 \\ 2 & 0 & 3 & 2 & 3 & 3 & 3 & 2 & 1 & 1 \\ 1 & 3 & 0 & 3 & 2 & 3 & 2 & 1 & 2 & 2 \end{array} \right] \end{array}, \quad (10)$$

where we chose as representative elements of each orbit the elements (links) 1, 2 and 3, rows with a single element of each orbit.

Note that the matrices are reduced from 10×10 to 3×10. Appendix 6 shows how to rebuilt the original matrices using the automorphism group and the reduced matrices.

The more symmetric the kinematic chain, the smaller is its representation. As most parallel mechanisms found in the literature are symmetric, this representation is particularly advantageous.

5.2. *Example 2: Hybrid 6-DoF mechanisms*

Let X be the kinematic chain of the hybrid 6-DoF manipulator presented in Fig. 11 of Belfiore Di Benedetto[3] and shown in Fig. 7(a). In this case, $Aut(X)$ in terms of the generator set is given by

$$Aut(X) = \left\langle \begin{array}{l} \sigma_1 = (7\ 11)(8\ 12)(9\ 13)(10\ 14) \\ \sigma_2 = (20\ 24)(21\ 25)(22\ 26)(23\ 27) \\ \sigma_3 = (15\ 20)(16\ 21)(17\ 22)(18\ 23) \\ \sigma_4 = (2\ 7)(3\ 8)(4\ 9)(5\ 10) \\ \sigma_5 = (1\ 19)(2\ 18)(3\ 17)(4\ 16)(5\ 15)(7\ 23)(8\ 22) \\ \qquad (9\ 21)(10\ 20)(11\ 27)(12\ 26)(13\ 25)(14\ 24) \end{array} \right\rangle$$

The orbits are

$$\mathcal{O} = \{\{1\ 19\}; \{2\ 7\ 11\ 18\ 23\ 27\}; \{3\ 8\ 12\ 17\ 22\ 26\}; \{4\ 9\ 13\ 16\ 21\ 25\}; \{5\ 10\ 14\ 15\ 20\ 24\}; \{6\}\}.$$

Following the same procedure applied in the example above, it is possible to reduce the matricial representation. The connectivity matrix $C(X)$ presented in Appendix B of Belfiore and Di Benedetto,[3] which is 27×27, is reduced to

$$C_r(X) = \begin{array}{c} \\ \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \\ \mathcal{O}_4 \\ \mathcal{O}_5 \\ \mathcal{O}_6 \end{array} \begin{array}{c} \begin{array}{ccccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 \end{array} \\ \left[\begin{array}{ccccccccccccccccccccccccccc} 0 & 1 & 2 & 3 & 3 & 3 & 1 & 2 & 3 & 3 & 1 & 2 & 3 & 3 & 4 & 5 & 6 & 6 & 6 & 4 & 5 & 6 & 6 & 4 & 5 & 6 & 6 \\ 1 & 0 & 1 & 2 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 4 & 5 & 6 & 6 & 6 & 4 & 5 & 6 & 6 & 4 & 5 & 6 & 6 \\ 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 5 & 6 & 6 & 6 & 4 & 5 & 6 & 6 & 4 & 5 & 6 & 6 \\ 3 & 2 & 1 & 0 & 1 & 2 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 3 & 4 & 5 & 5 & 5 & 3 & 4 & 5 & 5 & 3 & 4 & 5 & 5 \\ 3 & 3 & 2 & 1 & 0 & 1 & 3 & 3 & 3 & 2 & 3 & 3 & 3 & 2 & 2 & 3 & 4 & 4 & 4 & 2 & 3 & 4 & 4 & 2 & 3 & 4 & 4 \\ 3 & 3 & 3 & 2 & 1 & 0 & 3 & 3 & 2 & 1 & 3 & 3 & 2 & 1 & 1 & 2 & 3 & 3 & 3 & 1 & 2 & 3 & 3 & 1 & 2 & 3 & 3 \end{array} \right] \end{array},$$

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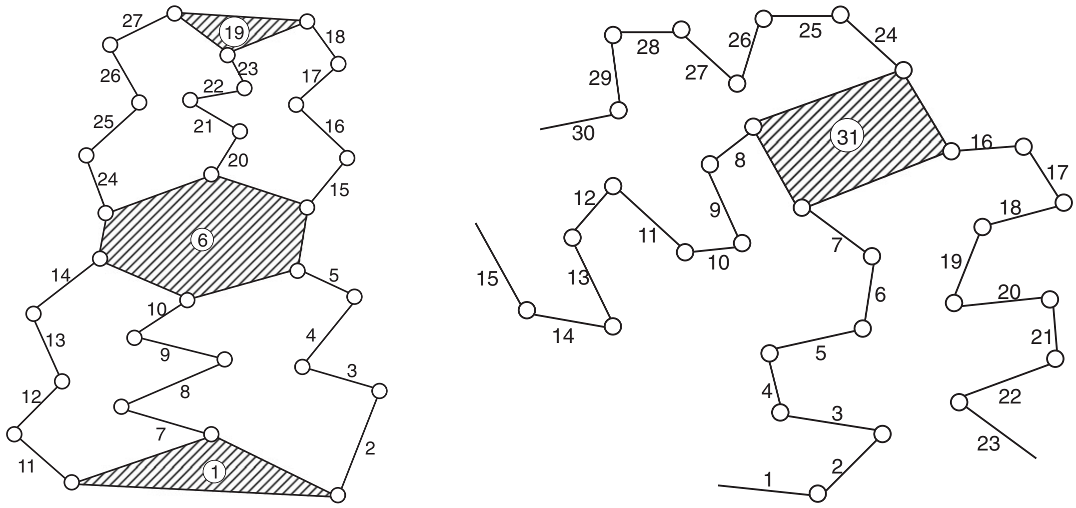

(a) (b)

Fig. 7. Examples of mechanisms found in the literature.[3,27] (a) Hybrid 6-DoF mechanisms, (b) redundant mechanism.

where we chose as representative elements of each orbit the elements (links) 1, 2, 3, 4, 5 and 6.

In this case, the connectivity matrix is reduced from 27×27 to 6×27. Other properties represented by matrices, such as degrees-of-control, redundancy and adjacency, are also reduced from 27×27 to 6×27.

5.3. *Example 3: Redundant mechanism employed in space missions*

Let X be the kinematic chain of a multiple-arm robot employed in space missions presented by Belfiore and Di Benedetto[3] and shown schematically in Fig. 7(b). In this case, $Aut(X)$ in terms of the generator set is given by

$$Aut(X) = \left\langle \begin{array}{l} \sigma_1 = (1\ 30)(2\ 29)(3\ 28)(4\ 27)(5\ 26)(6\ 25)(7\ 24) \\ \sigma_2 = (8\ 16)(9\ 17)(10\ 18)(11\ 19)(12\ 20)(13\ 21)(14\ 22)(15\ 23) \end{array} \right\rangle.$$

The orbits are

$$\mathcal{O} = \{\{1\ 30\}; \{2\ 29\}; \{3\ 28\}; \{4\ 27\}; \{5\ 26\}; \{6\ 25\}; \{7\ 24\}; \{8\ 16\}; \{9\ 17\}, \{10\ 18\}; \{11\ 19\}; \{12\ 20\}; \{13\ 21\}; \{14\ 22\}; \{15\ 23\}; \{31\}\}.$$

The redundancy matrix $R(X)$ presented in Appendix B of Belfiore and Di Benedetto[3] is reduced to

where we chose as representative elements of each orbit the elements (links) 1, 2, 3, . . ., 15 and 31.

In this case the redundancy matrix is reduced from 31×31 to 16×31.

6. Conclusions

The main contribution of this paper is to prove the invariance of connectivity, degrees-of-control and redundancy by the action of the automorphism group of the graph. The connectivity, degrees-of-control and redundancy are symmetric properties of a kinematic chain, i.e. links which are symmetric by the action of the automorphism group of the graph have the same properties. Considering that symmetric links are identified by the orbits of the automorphism group of the graph, we reduce the matricial representation considering one representative element of each orbit. Thus, the order of the matrices are reduced from $n \times n$ to $o \times n$, where n is the number of links of the kinematic chain and o is the number of orbits of the automorphism group of the graph.

$$R_r(X) = \begin{array}{c} \\ \mathcal{O}_1 \\ \mathcal{O}_2 \\ \mathcal{O}_3 \\ \mathcal{O}_4 \\ \mathcal{O}_5 \\ \mathcal{O}_6 \\ \mathcal{O}_7 \\ \mathcal{O}_8 \\ \mathcal{O}_9 \\ \mathcal{O}_{10} \\ \mathcal{O}_{11} \\ \mathcal{O}_{12} \\ \mathcal{O}_{13} \\ \mathcal{O}_{14} \\ \mathcal{O}_{15} \\ \mathcal{O}_{31} \end{array} \begin{array}{c} \begin{array}{ccccccccccccccccccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 & 16 & 17 & 18 & 19 & 20 & 21 & 22 & 23 & 24 & 25 & 26 & 27 & 28 & 29 & 30 & 31 \end{array} \\ \left[\begin{array}{ccccccccccccccccccccccccccccccc}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 \\
4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 0 \\
5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 0 \\
6 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 0 \\
7 & 6 & 5 & 4 & 3 & 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 0 \\
8 & 7 & 6 & 5 & 4 & 3 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 1 \\
9 & 8 & 7 & 6 & 5 & 4 & 3 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 2 \\
2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 3 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0
\end{array} \right] \end{array},$$

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Another contribution is a precise definition of the symmetry of a kinematic chain in terms of the automorphism group of the graph (see Section 2.4). The reduced representation presented is a minimal representation of the properties of kinematic chains in terms of symmetry. This definition is applied in early stages of mechanisms design where a kinematic chain can be represented by a graph.

Considering that the majority of parallel manipulators in the literature have symmetric kinematic chains,[11,13,14,19,22–24,26] the reduced representation offers considerable advantages. As shown in the examples, if a kinematic chain has symmetry, it is possible to obtain a gain in terms of the storage of matrices, and in the simplicity of the kinematic analysis. These techniques can also be applied to kinematic chains of serial and hybrid manipulators. The only cases for which the theory presented herein is not advantageous is when the graph is fully asymmetric, i.e. in rare practical cases.

The authors believe that the results presented in this paper can be applied to network analysis and optimization.

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Appendix A. Group theory

The group theory concepts used in this paper are self-contained in this appendix. The appendix was written with the aim of making group theory more accessible to those not acquainted with the main techniques. Thus, a brief review of definitions, theorems and examples are discussed. More details on group theory can be found in ref. [2, 5, 40, 42, 43].

A group is a set G endowed with a binary operation $\cdot : G \times G \to G$ satisfying certain axioms, detailed below. Thus, whenever a set has a group structure, the whole group can be described in terms of a set of generators. This follows from the fact that the equation $a \cdot x = b$ always admits a unique solution $x = a^{-1} \cdot b$ in G.

Definition 12 (Group). *Let G be a set and $\cdot : G \times G \to G$. The pair $(G, \cdot)$ is a group if the following conditions are satisfied:*

(1) *Associativity: for all a, b and c in G, $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.*
(2) *Identity element: there exists an element $e \in G$ such that for all $a \in G$, $e \cdot a = a \cdot e = a$.*
(3) *Inverse element: for every $a \in G$, there exists an element $a^{-1} \in G$ such that $a \cdot a^{-1} = a^{-1} \cdot a = e$.*

Definition 13 (Subgroup). *A subset $H \subset G$ is a subgroup of a group G if the operation induced by the operation on $(G, \cdot)$ satisfies the three conditions in Definition 12. This is equivalent to a requirement that $x = h^{-1} \cdot g \in H$, for all $h, g \in H$.*

Definition 14 (Group generators). *A set $\beta = \{g_1, \ldots, g_n\} \subset G$ is a set of generators for a group G if any element $g \in G$ can be written as the product of elements in β. In this case, we denote $G =< g_1, \ldots, g_n >$.*

Example 4 (Symmetric group). *Let $X_n = \{x_1, x_2, \ldots, x_n\}$ and $\mathcal{S}_n = \{\sigma : X_n \to X_n \mid \sigma \text{ is bijective}\}$ (permutations). Consider $\cdot : \mathcal{S}_n \times \mathcal{S}_n \to \mathcal{S}_n$ the operation given by the composition law $\sigma \cdot \tau = \sigma \circ \tau : X_n \to X_n$. Thus, $(\mathcal{S}_n, \cdot)$ is the nth-symmetric group. In order to describe the elements of $\mathcal{S}_n$ in a convenient way, let us consider a bijection $\sigma : X_n \to X_n$:*

$$\sigma = \begin{pmatrix} 1 & 2 & \cdots & n \\ \sigma(1) & \sigma(2) & \cdots & \sigma(n) \end{pmatrix}.$$

For $n = 2$, we have $2! = 2$ elements,

$$\mathcal{S}_2 = \left\{ \begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \right\}.$$

For $n = 3$, we have $3! = 6$ elements,

$$\mathcal{S}_3 = \left\{ \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{pmatrix} \right\}.$$

The group $\mathcal{S}_n$ has $n!$ elements. Also, the symmetric group is a matrix group as shown by the following example:

$$\sigma = \begin{pmatrix} a & b & c \\ b & a & c \end{pmatrix} \to \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

since

$$\sigma = \begin{bmatrix} b \\ a \\ c \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} a \\ b \\ c \end{bmatrix}.$$

A group structure may appear in different sets. Sometimes two distinct sets, when endowed with an operation, represent the same group structure. From an algebraic point of view they are the same groups.

Definition 15 (Isomorphism and automorphism groups). *Consider the groups $(G_1, \cdot_1)$ and $(G_2, \cdot_2)$.*

(1) *A map $\phi : G_1 \to G_2$ is a homomorphism if $\phi(x \cdot_1 y) = \phi(x) \cdot_2 \phi(y)$, for all $x, y \in G_1$.*
(2) *A homomorphism $\phi : G_1 \to G_2$ is an isomorphism if ϕ is bijective.*
(3) *Whenever $G_1 = G_2$, the isomorphism $\phi : G \to G$ is named an automorphism.*

The group structure is present in a model in the form of the group action, also called group representation. For the sake

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Table I. Actions of the elements of the automorphism group of the graph on the rows of the reduced adjacency matrix $A_r(X)$ for reconstruction of the original adjacency matrix $A(X)$.

Rebuilt	Applied element of $Aut(X)$	Row of $A_r(X)$	Row of $A(X)$
4	(1 7)(2 9)**(3 4)**(5 6)(8 10)	1 0 0 0 0 0 0 1 0 0	0 0 0 0 0 0 1 0 0 1
5	(1 10)**(2 5)**(3 4)(6 9)(7 8)	0 0 0 0 0 0 0 0 1 1	1 0 0 0 0 1 0 0 0 0
6	(1 8)**(2 6)**(5 9)(7 10)	0 0 0 0 0 0 0 0 1 1	0 0 0 0 1 0 1 0 0 0
7	**(1 7)**(2 9)(3 4)(5 6)(8 10)	0 0 1 0 1 0 0 0 0 1	0 0 0 1 0 1 0 1 0 0
8	**(1 8)**(2 6)(5 9)(7 10)	0 0 1 0 1 0 0 0 0 1	0 0 1 0 0 0 1 0 1 0
9	(1 7)**(2 9)**(3 4)(5 6)(8 10)	0 0 0 0 0 0 0 0 1 1	0 1 0 0 0 0 0 1 0 0
10	**(1 10)**(2 5)(3 4)(6 9)(7 8)	0 0 1 0 1 0 0 0 0 1	1 1 0 1 0 0 0 0 0 0

of simplicity, from now on let us denote the product of two group elements $g, h \in G$ by gh.

Definition 16 (Left group action). *A left group action of a group G on a set X is a map $\alpha : G \times X \to X$, usually denoted by $\alpha(g, x) = g \cdot x$, satisfying the following conditions*:

(*1*) *For all $g, h \in G$ and $x \in X$, $g \cdot (h \cdot x) = (gh) \cdot x$.*
(2) *For all $x \in X$, $e \cdot x = x$.*

Analogously, a right group action can be defined. From now on, we use the term action for left action. A space X endowed with a G-action is named a G-space.

Definition 17 (Orbits). *Let X be a G-space. The orbit of a point $x \in X$, by the action of G, is the space*

$$\mathcal{O}_x = \{g \cdot x \mid g \in G\}.$$

A partition of a G-space X is obtained by considering the space of G-orbits. This can be seen by defining the following equivalence relation: $x \sim y$ if and only if there exists an element $g \in G$ such that $y = g \cdot x$. The equivalence classes are exactly the orbits under the G-action. Therefore, if $x \sim y$, then $\mathcal{O}_x = \mathcal{O}_y$. It is well known that the equivalent classes define a partition.

Appendix B. From reduced to original matrices

This appendix presents an example of the reconstruction of the original matrices from the reduced matrices and the orbits.

With the reduced adjacency and connectivity matrices shown in Eqs. (9) and (10) and the automorphism group shown in Eq. (8), it is possible to rebuild the original matrices shown in Eqs. (6) and (7), respectively, just considering the action of the automorphism group elements on the rows of the reduced matrices.

Note that it is necessary to rebuild rows 4, 5, …, 10. Tables I and II show the actions that should be applied to rebuild the original matrices $A(X)$ and $C(X)$, respectively, where the first column shows the row to be rebuilt.

Observe the action of each element of $Aut(X)$. To rebuild row 4 we need to choose an element of the automorphism group whose action changes a determined label x to 4. For example, the action of (1 7)(2 9)**(3 4)**(5 6)(8 10) change the label $x = 3$ to 4 and thus it can be used to rebuild row 4 from row 3. Note that, while the results are the same, the way to rebuild the matrices is not unique, i.e. to rebuild row 10 we can use the elements **(1 10)**(2 5)(3 4)(6 9)(7 8), (1 8)(2 6)(5 9)**(7 10)** and (1 7)(2 9)(3 4)(5 6)**(8 10)**.

Table II. Actions of the elements of automorphism group of the graph on the rows of the reduced connectivity matrix $C_r(X)$ for reconstruction of original connectivity matrix $C(X)$.

Rebuilt	Applied element of $Aut(X)$	Row of $C_r(X)$	Row of $C(X)$
4	(1 7)(2 9)**(3 4)**(5 6)(8 10)	1 3 0 3 2 3 2 1 2 2	2 2 3 0 3 2 1 2 3 1
5	(1 10)**(2 5)**(3 4)(6 9)(7 8)	2 0 3 2 3 3 3 2 1 1	1 3 2 3 0 1 2 3 3 2
6	(1 8)**(2 6)**(5 9)(7 10)	2 0 3 2 3 3 3 2 1 1	2 3 3 2 1 0 1 2 3 3
7	**(1 7)**(2 9)(3 4)(5 6)(8 10)	0 2 1 2 1 2 3 2 3 1	3 3 2 1 2 1 0 1 2 2
8	**(1 8)**(2 6)(5 9)(7 10)	0 2 1 2 1 2 3 2 3 1	2 2 1 2 3 2 1 0 1 3
9	(1 7)**(2 9)**(3 4)(5 6)(8 10)	2 0 3 2 3 3 3 2 1 1	3 1 2 3 3 3 2 1 0 2
10	**(1 10)**(2 5)(3 4)(6 9)(7 8)	0 2 1 2 1 2 3 2 3 1	1 1 2 1 2 3 2 3 2 0

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