A Contribution to the Theory of Ferromagnetism. By G. S. MAHAJANI, B.A., St John's College.

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The following gives a brief summary of a paper, which it is hoped to publish *in extenso* later. It is a contribution to the theory of ferromagnetic crystals, and contains in particular a theoretical explanation of Webster's* experimental results.

Introduction. The fundamental fact of interest with regard to experiments on magnetic substances is that the direction (ψ) of magnetization (I) does not in general coincide with the direction (ϕ) of the external, applied field (H). Experiments on iron crystals have now revealed the fact that this "deviation-effect" $(\phi - \psi)$ exists even in crystals of cubic structure and a symmetrical boundary. In other words, cubical symmetry of structure does not make a body magnetically isotropic, as was erroneously assumed by Faraday, Tyndall and Plücker. To account for this "deviation-effect", and in general to relate the two directions (ϕ, ψ) with each other, must certainly be the first object of any theory bearing on this subject. Accordingly Weiss has developed a "macroscopic" theory by introducing the conception of the "molecular field". The idea served many useful purposes, especially in connection with experiments on Pyrhotite. But, as is shown in the present paper, it is quite inadequate to explain the phenomena in the case of crystals with cubic structure.

It seemed, therefore, better to give up the idea of the "molecular field", and start right from the beginning with atomic actions. This alternative was indeed conceived but not adopted by both Weber and Maxwell. It is quite possible that, apart from the external field (H), the internal field which governs the magnetization of the body arises partly from the mutual actions of the "molecular magnets" and partly from some other force, of which we do not know the origin. The rational way of proceeding, however, is to try to see whether we cannot do with only the mutual actions unaided by a force of an obscure origin. This is the point of view adopted in this investigation.

General method. Two types of ferromagnetic crystals are considered:

1. Iron, which has a cubic structure, space-centred cubic, i.e. consisting of two simple cubic lattices, interlocked in such a way that the lattice-points of either occupy the centres of cells of the other.

* "Magnetic Properties of Iron Crystals", by W. L. Webster, *Proc. Roy. Soc.* A, **107** (1925).

2. Pyrhotite, which has a non-cubic structure. Two ortho-rhombic simple lattices are interlocked in a "space-centred" manner, and give on the whole a hexagonal symmetry.

The ultimate particles that form the lattice-points of crystals are supposed to derive their magnetic properties from the presence of electronic orbits. A parameter (a) is introduced to denote the radius of an orbit. The magnetic energy due to mutual actions of the particles is calculated as a series of terms involving third, fifth, seventh, ..., etc. powers of $(1/s)$, where s is the lattice-constant. The parameter (a), the radius of an electronic orbit, is also naturally involved. The total magnetic energy of the system in an external field (H) is determined per unit volume; and it is made stationary to search for the steady states of magnetization.

In either of the two types considered the structure consists of two simple lattices. In calculating the internal magnetic energy, it is assumed that the "magnetic elements" (or, the axes of the electronic orbits in our case) arrange themselves, in any steady state, into two groups with two different directions. The two groups severally belong to the two component simple lattices. In saturation states, the two directions coincide; while in the non-magnetized state of the substance they are opposite to each other.

The two distinctive features of the theory in this paper are the association of two distinct directions with magnetic elements in the two component simple lattices, and the replacement of the usual "doublet-conception" by the one of electronic orbits.

Notation:

$O\text{-}xyz$	Coincide with three edges of a cell of either lattice.
(l, m, n)	Direction of I, magnetization.
(l', m', n')	Direction of H, applied field.
ω^3	Volume of a cell.
L_1, L_2	The two component lattices.
$1/\omega^3$	Density of magnetic elements in either lattice; and
$2/\omega^3$	Density in the crystal as a whole.

The work is divided into three parts, and may now be summarized as follows:

PART I.

This deals with cubic structures with saturation states and is devoted mainly to explaining Webster's results. A space-centred cubic structure is specified by

$$\left. \begin{array}{l} x = ps \\ y = qs \\ z = rs \end{array} \right\} p = q = r \ (\mathrm{mod}\ 2),$$

where $2s$ is the side of an elementary cubic cell.

The mutual potential energy of two magnetic elements, if identified with two parallel electronic equal orbits, radius a, is $2w$, where

$$w = -\frac{\mu^2}{r^3} P_2 + 3\frac{\mu^2 a^2}{r^5} P_4*$$

(neglecting higher terms) and

$r =$ distance between the centres,

$\mu = \pi a^2 i$,

$P_2, P_4 = P_2(\cos\theta), P_4(\cos\theta), \ldots,$

and $\qquad \theta =$ angle between r and the axis of an orbit.

The contribution to the internal energy due to any one magnetic element, say the one at the origin, in the presence of all others at the other lattice-points is

$$W_2 = \underset{p,q,r}{\Sigma'} w_2 = -\underset{p,q,r}{\Sigma'}\left(\frac{\mu^2}{r^3}P_2 - \frac{3\mu^2 a^2}{r^5}P_4\right),$$
$$p = q = r \pmod 2,$$

and the accent indicates that $p = q = r = 0$ is a combination to be omitted.

It is easily seen that

$$r^2 = (p^2 + q^2 + r^2)s^2, \qquad \cos\theta = \frac{lp + mq + nr}{(p^2 + q^2 + r^2)^{\frac{1}{2}}}.$$

It is shown in Part III that for cubic structures the third order terms which lead to divergent series involve no variable part of the energy and we are left with

$$W_2' = \text{effective part of } W_2 = +\underset{p,q,r}{\Sigma'}\frac{3\mu^2 a^2}{r^5}P_4(\cos\theta)$$
$$= +\underset{p,q,r}{\Sigma'}\frac{3\mu^2 a^2}{(p^2+q^2+r^2)^{\frac{5}{2}}\cdot s^5}P_4(\cos\theta), \quad p = q = r \pmod 2.$$

This reduces, omitting the non-variable part (i.e. not depending upon lmn), to

$$\frac{105}{4}\frac{\mu^2 a^2}{s^5}(3Q - P)(l^2 m^2 + m^2 n^2 + n^2 l^2),$$

where
$$\left.\begin{array}{l} P = \underset{p,q,r}{\Sigma'}\dfrac{p^4}{(p^2+q^2+r^2)^{\frac{9}{2}}} \\[4mm] Q = \underset{p,q,r}{\Sigma}\dfrac{p^2 q^2}{(p^2+q^2+r^2)^{\frac{9}{2}}} \end{array}\right\} p = q = r \pmod 2.$$

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* The mutual potential energy of two linear magnets is, however,

$$-\frac{2\mu^2}{r^3}P_2 - \frac{2\mu^2 a^2}{r^5}P_4, \text{ etc.}$$

The second term differs in sign. Actually experimental results require $(+)$ sign. Hence the orbit-conception.

Actual calculation shows that

$$0\cdot 15 \ < P < 0\cdot 18,$$
$$0\cdot 068 < Q < 0\cdot 076,$$
$$0\cdot 032 < 3Q - P < 0\cdot 064.$$

Next, the energy due to an external field (H) is

$$- \mu H \left(ll' + mm' + nn' \right).$$

Therefore the total energy density per unit volume is

$$E = \frac{2\mu^2}{\omega^3} \left[\alpha \left(l^2 m^2 + m^2 n^2 + n^2 l^2 \right) - \frac{H}{\mu} \left(ll' + mm' + nn' \right) \right],$$

where $$\alpha = \frac{105}{4} \frac{a^2}{s^5} \left(3Q - P \right).$$

(This formula does not take account of temperature effects.)

Let now $$\lambda x + \mu y + \nu z = 0$$

be a symmetry plane of the crystal structure, and consider a circular plate of the crystal cut parallel to this plane. If H be applied in the plane of the disk, we have

$$\lambda l' + \mu m' + \nu n' = 0.$$

By symmetry of the structure the vector of magnetization will also lie in the plane of the disk and therefore

$$\lambda l + \mu m + \nu n = 0.$$

Making now E stationary subject to

$$\Sigma l^2 = 1, \qquad \Sigma \lambda l = 0,$$

we get $$\begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} \div \begin{vmatrix} l' & m' & n' \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = - \frac{H}{2\mu\alpha}.$$

This is a quite general result and includes all the cases considered.

Obviously (H) and (I) will coincide in direction, i.e. the deviation-effect will vanish, when $(l', m', n') = (l, m, n)$, or

$$\begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ \lambda & \mu & \nu \end{vmatrix} = 0.$$

This result coupled with

$$\Sigma l^2 = 1, \qquad \Sigma \lambda l = 0,$$

gives the directions in the symmetry plane $\lambda x + \mu y + \nu z = 0$, along which the deviation-effect vanishes.

Particular cases.

(1) *Circular plate cut parallel to the face of an octahedron:*

$$\lambda = \mu = \nu = 1/\sqrt{3}.$$

In this case $\Sigma l = 0$ makes

$$\begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ 1 & 1 & 1 \end{vmatrix} = 0$$

an identity, and thus the following experimental fact is explained:

"The amplitude of the waves of σ_n, i.e. the component of magnetization perpendicular to the direction of H, vanishes in stronger fields almost completely*."

(2) *Circular plate cut parallel to the face of a rhombic dodecahedron.*

Here $\lambda = 1/\sqrt{2}, \qquad \mu = 1/\sqrt{2}, \qquad \nu = 0.$

Solving
$$\left. \begin{array}{c} \begin{vmatrix} l^3 & m^3 & n^3 \\ l & m & n \\ 1 & 1 & 0 \end{vmatrix} = 0 \\ \Sigma l^2 = 1, \qquad l + m = 0 \end{array} \right\},$$

we get exactly the same result as required by experimental data:

"...The normal components σ_n show four zeros within $180°$ in the direction of the axes at $0, 55°, 90°, 125°,$ and $180°$†."

Strictly speaking, theory yields not $55°$ and $125°$, but $\cos^{-1}(1/\sqrt{3})$ and $\pi - \cos^{-1}(1/\sqrt{3})$, that is $54° 44'$ and $125° 16'$.

(3) *Circular plate cut parallel to the face of the cube.* This is the case of Webster's experiments. We have $\lambda = \mu = 0$, $\nu = 1$, and therefore $n = n' = 0$. The result reduces to

$$\frac{\begin{vmatrix} l^3 & m^3 \\ l & m \end{vmatrix}}{\begin{vmatrix} l' & m' \\ l & m \end{vmatrix}} = -\frac{H}{2\mu\alpha}.$$

Putting
$$\left. \begin{array}{c} (l, m) = (\cos\psi, \sin\psi) \\ (l', m') = (\cos\phi, \sin\phi) \end{array} \right\},$$

we get
$$\frac{\sin(\phi - \psi)}{\sin\psi \cos\psi \cos 2\psi} = \frac{2\alpha\mu}{H}.$$

* Kunz, *Bull. Nat. Res. Council*, vol. 3, Part 3, p. 180 (1922).
† *Ibid.*

Further, if we write $\Im$ instead of I, to denote that we are dealing with saturation states, we see that

$$\frac{\Im_n}{\sin\psi\cos\psi\cos 2\psi} = \frac{2a\mu\Im}{H},$$

i.e. $\qquad (\Im_n)_{\max}. H = \tfrac{1}{2}a\mu\Im = \text{constant} \qquad \ldots\ldots(1).$

The explanation of all the experimental curves of Webster is contained in the two formulae

$$\left.\begin{array}{c} \dfrac{H\sin(\phi-\psi)}{\Im\sin\psi\cos\psi\cos 2\psi} = \dfrac{2a\mu}{\Im} = a\cdot\omega^3 \\ (\Im_n)_{\max}. H = \tfrac{1}{2}a\mu\Im \end{array}\right\}.$$

Quantitatively, the second of the results above is made to fit Webster's Graph 4, by adjusting a. All the quantities in equation (1) are known, or can be calculated except the parameter a, the radius of an electronic orbit. Thus the proper adjustment of a to fit the experimental curves yields the magnitude of a.

The results are:

Disk $A \qquad 0\cdot 212 > a \times 10^8 > 0\cdot 147 \rbrace$
Disk $B \qquad 0\cdot 331 > a \times 10^8 > 0\cdot 232 \rbrace$.

The conclusion is reached that

$$a = O\,[10^{-9}].$$

The molecular field. Our work shows that if we start from the bulk-conception of the molecular field, we must adopt a "third-power" law and not a "fourth-power" one, as is empirically suggested by Webster, in the case of cubic structures. Incidentally a better generalization of Weiss's law of simple proportionality in the case of non-cubic structures is suggested, which includes both the cubic and the non-cubic cases. Thus in place of Weiss's

$$(H_m)_\mu = N_\mu I_\mu,$$

we write $\qquad (H_m)_\mu = N_\mu\,[a_1 I_\mu + a_3 I_\mu{}^3 + a_5 I_\mu{}^5 + \ldots].$

Adopting this form, we obtain in the case of a circular plate the following result:

$$\frac{H\begin{vmatrix} l & m \\ l' & m' \end{vmatrix}}{Ilm\,[a_1(N_x - N_y) + a_3(l^2 N_x - m^2 N_y)\,I^2 + a_5(l^4 N_x - m^4 N_y)\,I^4 + \ldots]} = 1.$$

In the non-cubic case $\qquad N_x \neq N_y.$

Retaining only the first non-vanishing term in the denominator gives

$$\frac{H \begin{vmatrix} l & m \\ l' & m' \end{vmatrix}}{I\,lm} = a_1\,(N_x - N_y).$$

Take $a_1 = 1$ and we obtain exactly Weiss's result

$$\frac{H \sin (\phi - \psi)}{I \sin \psi \cos \psi} = (N_x - N_y).$$

On the other hand, in the cubic case

$$N_x = N_y = N, \text{ say.}$$

Therefore retaining only the first non-vanishing term gives

$$\frac{H \begin{vmatrix} l & m \\ l' & m' \end{vmatrix}}{I^3 lm\,(l^2 - m^2)} = a_3 N,$$

i.e. in the saturation case

$$\frac{H \sin (\phi - \psi)}{\Im \sin \psi \cos \psi \cos 2\psi} = a_3 N \Im^2,$$

which agrees with the result we obtained by the consideration of "atomic actions".

Quantitatively, the agreement is fairly satisfactory. We obtain the following result:

	Present theory	Webster's experiments
M_B	576	620
M_A	368	470

where M_B, M_A denote the magnitude in gauss of the molecular field in the case of saturation when H acts along one of the structural axes, for disks B and A respectively.

PART II.

This deals with non-cubic structures. Two methods are pointed out for dealing with the "third-order" terms in the energy function. It is shown that, in the cubic structures, these involve no variable part of the energy; while in the non-cubic cases, these involve the most effective part of the variable energy. Hence the simple doublet conception suffices in this case. The same results are arrived at as Weiss's.

PART III.

In this, an attempt is made to extend the theory to non-saturation states. The two general results that are obtained may be stated here. Cubic case:

$$\frac{H \cos (\phi - \psi)}{I \left[2q - \frac{a^2}{8} \{4 \, (h - 3j) \cos 4\psi . f_1 + 2h' \, (1 + \cos^2 2\psi) + 3j' \, (1 - \cos 4\psi)\} \right]} = - \tfrac{1}{2}\omega^3,$$

and

$$\frac{I - H \sin (\phi - \psi)}{\sin 4\psi \, [(hf_2 + h'f_1) - 3 \, (jf_2 + j'f_1)]} = - \frac{a^2}{16} \, \Im^2\omega^3,$$

where q, h, h', j, j' are constants defined by the structure thus:

$$\left. \begin{aligned} h &= \sum_{p \, q, r} \frac{\partial^4}{\partial x^4} \left(\frac{1}{r} \right) \\ j &= \sum_{p, q, r} \frac{\partial^4}{\partial x^2 \partial y^2} \left(\frac{1}{r} \right) \end{aligned} \right\} \; (p, q, r) \; all \; even,$$

$$\left. \begin{aligned} h' &= \sum_{p, q, r} \frac{\partial^4}{\partial x^4} \left(\frac{1}{r} \right) \\ j' &= \sum_{p, q, r} \frac{\partial^4}{\partial x^2 \partial y^2} \left(\frac{1}{r} \right) \end{aligned} \right\} \; (p, q, r) \; all \; odd,$$

$$r^2 = (p^2 + q^2 + r^2) \, s^2,$$

$$f_n = f_n \, (I/\Im) = \cos n \, (\psi_1 - \psi_2),$$

ψ_1 and ψ_2 being the directions of the magnetic elements in the two component simple lattices. The constant q is rather too complicated to define in this summary.

Formulae are obtained for the non-cubic structures also, in non-saturated states.

Concluding remarks. The theory requires to be completed in three directions:

1. To consider which of the steady states are stable. This will probably bear on the hysteresis phenomena.

2. To consider the effect of temperature.

3. To put the formulae for the third order terms in Part II in a form suitable for numerical computation, and thus to test the theory in the case of non-cubic structures quantitatively.

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