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Dynamics of an infection age-space structured cholera model with Neumann boundary condition

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This paper investigates global dynamics of an infection age-space structured cholera model. The model describes the vibrio cholerae transmission in human population, where infection-age structure of vibrio cholerae and infectious individuals are incorporated to measure the infectivity during the different stage of disease transmission. The model is described by reaction–diffusion models involving the spatial dispersal of vibrios and the mobility of human populations in the same domain $\Omega \subset \mathbb{R}^n$. We first give the well-posedness of the model by converting the model to a reaction–diffusion model and two Volterra integral equations and obtain two constant equilibria. Our result suggest that the basic reproduction number determines the dichotomy of disease persistence and extinction, which is achieved by studying the local stability of equilibria, disease persistence and global attractivity of equilibria.

Key words: Cholera, infection age-space structured model, uniform persistence, basic reproduction number, global stability

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1 Introduction

Cholera belongs to severe waterborne diseases. It is estimated that cholera causes about 2.9 million cases and 95,000 deaths in 69 endemic countries per year [1]. Cholera are transmitted by pathogenic microorganisms vibrio cholerae contained in contaminated water, lakes and aquatic reservoirs. Like other waterborne diseases, such as giardiasis, diarrhoea, dysentery, and typhoid, cholera arises major public health concern and global burden of disease. In recent years, devastating outbreaks in Zimbabwe (2008–2009) [23], Haiti (2010–2012) [34], South Africa (2000–2001) [25] and Yemen (2017–2018) [14] have received worldwide attention. It is urgent that quantitative understanding of cholera transmission is needed to control cholera epidemics.

The complexity of cholera dynamics involves two different transmission routes, that is, vibrio cholerae transmission in human population takes place at human-to-human and environment-to-human transmission. Recently, modelling the transmission of cholera has attracted much attention by taking into account various aspects, such as multiple infection stages [32], age-structure [5, 39, 22], hyperinfectivity [21], patch/network structures [36, 33] and spatial heterogeneity [6, 40, 41, 49, 50, 53]. We list some literatures related to our work from the standpoint of mathematical modelling and analyis. A review of the literature of mathematical modelling of cholera outbreaks is given in [10].

- Cholera models based on ODEs deterministic model: since the earlier work [11] on cholera epidemics in the European Mediterranean region, subsequent contributions have been developed to investigate cholera epidemics. In the aspect of mathematical modelling, a water compartment (pathogen concentration) was introduced to classical Susceptible-Infected-Recovered (SIR) epidemic model [13]. By introducing a hyperinfectious state of vibrio cholerae (freshly shed vibrios), an extended model of [13] is proposed in [21]. The model in [31] incorporated both hyperinfectivity and temporary immunity. Due to the fact that differential infectivity, a staged progression model is formulated to model the multiple stages of infectious individuals [36]. Some models involving indirect pathway and/or direct pathway transmission of cholera can be founded in [4, 36, 26, 47].
- Cholera models based on patch/network structures: some models incorporating spatial effects have used patches, networks, and directed graphs [6, 19, 33]. Taking into account the cities, towns, and villages in the region as the nodes of the model, these discrete structure models are allowed to be studied from the standpoint of applications. For example, Mukandavire *et al.* [23] assessed the reproductive ratios for the 10 provinces in Zimbabwe and revealed that spatial heterogeneity remarkably affected the underlying transmission pattern for the 2008–2009 cholera outbreaks. After that, Tuite *et al.* [34] estimated the reproductive numbers using the data from 10 administrative departments and also revealed that spatial heterogeneity brings the difficulties in guiding practical control strategies for the 2010 cholera epidemic in Haiti.
- Cholera models based on infection-age structure: for the aspects of partial differential equations of cholera epidemics, Shuai *et al.* [5] proposed the model incorporating infection-age structure for infectious humans and vibrios. In particular, the infection age is interpreted as the the time since infection began, which can trace the history of infected individuals. The infection-age structure for vibrios reflects the differential infectivity as a continuous variable. In epidemic modelling, the age of infection was used to describe the period of latency (see [15, 35] and references therein). At time t, denote by $i(t, a)$ and $p(t, b)$ the densities of infectious humans of age a and the concentrations of pathogen in the contaminated water of age b, respectively. Then the dynamics of infectious humans and pathogens are described by:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) i(t,a)=-(\mu_i+\theta(a))i(t,a), \\ \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial b}\right) p(t,b)=-(\mu_p+\delta(b))p(t,b), \end{cases} \tag{1.1}$$

  for $a \geqslant 0$ and $b \geqslant 0$, where μ_i and μ_p, respectively, represent the death rates of infectious humans and pathogens. $\delta(b)$ and $\theta(a)$ represent the removal rate of the pathogen and the infectious humans, respectively. Generally, $i(t, 0)$ and $p(t, 0)$ are used to reflect the resources fluxing into compartment, since the infection occurs at age 0. In [5], the authors identified a sharp threshold called basic reproduction number (BRN) to determining whether or not cholera dies out. However, some necessary arguments are left in [5], including uniform persistence and relative compactness of orbit generated by (2.3), which two major issues to make use of the Lyapunov techniques and LaSalle's invariance principle. Hereafter, Yang *et al.* [51] and Wang *et al.* [39] give a supplement to [5] with different methods.

- Cholera models in nonhomogeneous environments: since the studies of cholera models with patch/network structures, it is evident that spatial heterogeneity is very important in the cholera transmission (see, e.g., [23, 34]). Recently, spatial heterogeneity (such as hygiene conditions, water resource availability and spatial position) has been considered as one of the main factors in understanding the spatial spread of infectious diseases. Reaction–diffusion models involving the mobility of human populations, the spatial dispersal of vibrios and environmental spatial effects have been formulated to get threshold dynamics of cholera epidemics and find practical control strategies (see, e.g., [6, 12, 27, 40, 41, 49, 50, 53, 54, 43, 44, 46]). Bertuzzo *et al*. [6] analysed a diffusive cholera system to investigate the effects of heterogeneity, where only saturating indirect transmission was adopted. Their results also revealed that the heterogeneities of the environment may be the reason of spatial patterns of the disease (e.g., secondary peaks). Wang *et al*. [45] further modified the model in [6] by incorporating direct transmission pathways to consider how human behaviours impact cholera transmission. They also obtained the threshold dynamics and the propagation of epidemic waves when convection of vibrios was theoretically set in a one-dimensional river. In [46], the habitat is assumed to be a bounded one-dimensional domain. The authors of [46] confirmed that spatial diffusion is not necessary to arise Turing instability and investigated the role of spatial diffusion in the disease spread. In a recent work [42], a spatiotemporally heterogeneous cholera epidemic model has been investigated.

In epidemic modelling, it is worth pointing out that threshold value called BRN determines the dichotomy of disease persistence and extinction. The core problem here is to define this threshold value such that above BRN, the disease persists; while below BRN, the disease vanishes. The BRN for ODEs follows the classic theory developed in [18, 38]. For diffusive epidemic model, the spectral radius of a resolvent-positive operator is usually used to define the BRN [37], which extends the application range and theoretical approach from finite dimensional to infinite dimensional. In a recent work [24], the connection between the BRN for reaction–diffusion epidemic models and the BRN for ODEs are established using a vector-host model. On the other hand, once the disease persists, it becomes important to investigate the long-time behaviour of solution, so that it can help decision-makers to conduct more effective control.

In this paper, inspired by the standard infection-age cholera model [5, 39, 46], we focus on a cholera model with infection age-space structure. However, the modelling process is not trivial due to the mobility of the individuals and cholera. Unlike in [45, 46] where the habitat is 1-D-bounded domain, we consider the situation that the vibrio cholerae and human population are living in the same domain $\Omega \subset \mathbb{R}^n$. This constitutes one motivation of the current paper. For $t \geqslant 0$ and $x \in \Omega$, if no infection occurs, denote by $d_S > 0$ the diffusion coefficient for susceptible individuals. Further, we use Λ and μ_S to denote the constant recruitment rate and the natural death rate for the susceptible individuals. Thus, the susceptible individuals $S(t,x)$ is governed by:

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_S \Delta S(t,x) + \Lambda - \mu_S S(t,x), & x \in \Omega,\ t > 0, \\[2ex] \dfrac{\partial S(t,x)}{\partial n} = 0, & x \in \partial\Omega,\ t > 0. \end{cases} \tag{1.2}$$

At time t and location x, we use $i(t,a,x)$ and $p(t,b,x)$ to denote the density of infected individuals with infection age a and the density of vibrios contained in contaminated water with infection age b, respectively. It is assumed that $\beta_1(a)$ and $\beta_2(b)$ measure the age-specific infectivity of infected individuals and vibrios, respectively. Considering the direct and indirect transmission, we assume that $i(t,a,x)$ in the domain is governed by:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial a}\right) i(t,a,x)=d_i\Delta i(t,a,x)-(\mu_i+\theta(a))i(t,a,x), & a\geqslant 0,\ x\in\Omega \\ i(t,0,x)=S(t,x)\displaystyle\int_0^\infty \beta_1(a)i(t,a,x)\mathrm{d}a+S(t,x)\int_0^\infty \beta_2(b)p(t,b,x)\mathrm{d}b, & \\ \dfrac{\partial i(t,a,x)}{\partial n}=0, & x\in\partial\Omega, \end{cases} \tag{1.3}$$

and $p(t,a,x)$ in the domain is governed by:

$$\begin{cases} \left(\dfrac{\partial}{\partial t}+\dfrac{\partial}{\partial b}\right) p(t,b,x)=d_p\Delta p(t,b,x)-(\mu_p+\delta(b))p(t,b,x), & b\geqslant 0,\ x\in\Omega, \\ p(t,0,x)=\displaystyle\int_0^\infty \xi(a)i(t,a,x)\mathrm{d}a, & \\ \dfrac{\partial p(t,b,x)}{\partial n}=0, & x\in\partial\Omega. \end{cases} \tag{1.4}$$

Here d_i and d_p represent, respectively, the diffusion coefficients for infective individuals and cholera. $\xi(a)$ is the age-specific shedding rate of infected individual.

Our second motivation comes from the recent studies on infection age-space structured models (see, e.g., [16, 17, 55, 9, 52]), which is spent on understanding the effects of the infection age and spatial heterogeneity on disease transmission. For the standard age-space structured SIR model, Chekroun and Kuniya [9] reformulated the model by a hybrid system of one diffusive equation and one Volterra integral equation and studied the threshold dynamics for the disease extinction and persistence in one-dimensional domain. Further, the global stability problem of constant equilibria was achieved. In another works, the travelling wave solutions of age-space structured SIR model with or without birth and death processes were studied in a spatially unbounded domain [16, 17]. For a age-space structured SIR model with seasonality, Zhang and Wang [55] established the threshold dynamics that BRN more that 1 or less than 1 determines whether or not disease extinction. Yang *et al.* [52] made an attempt to extend the methods and ideas in [9] to propose a model for the spatial spreading of brucellosis in a continuous bounded domain. Some basic mathematical arguments, including the existence and uniqueness of the solution and threshold dynamics, were successfully addressed. However, reaction–diffusion cholera model with infection-age structure seems to have received little attention.

Following this line and above settings, the main model of this paper is

$$\begin{cases} S_t(t,x)=d_1\Delta S(t,x)+\Lambda-\mu_S S(t,x)-i(t,0,x), \\ i_t(t,a,x)+i_a(t,a,x)=d_2\Delta i(t,a,x)-(\mu_i+\theta(a))i(t,a,x), \\ i(t,0,x)=S(t,x)\displaystyle\int_0^\infty \beta_1(a)i(t,a,x)\mathrm{d}a+S(t,x)\int_0^\infty \beta_2(b)p(t,b,x)\mathrm{d}b, \\ p_t(t,b,x)+p_b(t,b,x)=d_3\Delta p(t,b,x)-(\mu_p+\delta(b))p(t,b,x), \\ p(t,0,x)=\displaystyle\int_0^\infty \xi(a)i(t,a,x)\mathrm{d}a, \end{cases} \tag{1.5}$$

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associated with initial data:

$$(S(0,x), i(0,a,x), p(0,b,x)) = (\phi_1(x), \phi_2(a,x), \phi_3(b,x)),\ \ a,b \geqslant 0, x \in \Omega,$$

and boundary condition:

$$\frac{\partial S(t,x)}{\partial n} = \frac{\partial i(t,a,x)}{\partial n} = \frac{\partial p(t,b,x)}{\partial n} = 0,\ \ x \in \partial\Omega. \tag{1.6}$$

Our goal is to investigate the effect of infection-age structure and spatial diffusion on the threshold dynamics of diffusive cholera models. Let $\mathbb{F} = \beta_1, \theta, \xi, \beta_2, \delta$, respectively. We always assume that

(**A**1): $\mathbb{F}(\upsilon) \in L^\infty_+(0,+\infty)$ and $\mathbb{F}^+ := \text{ess.sup}\ \mathbb{F}(\upsilon) < +\infty$, where $\upsilon = a$ or b.
(**A**2): There exist $0 < \upsilon_1 < \upsilon_2 < +\infty$ such that $\mathbb{F}(\upsilon)$ is strictly positive for all $\upsilon \in (\upsilon_1, \upsilon_2)$.

In Section 2, we first reformulate the original model into a hybrid system. Then we investigate the basic properties of the solution of reformulated system, such as positivity, existence, uniqueness and boundedness. Section 3 is spent on defining BRN and giving the constant equilibria. We studied the local stability of equilibria by investigating the distribution of roots of characteristic equations in Section 4. Section 5 is devoted to exploring the disease persistence. By constructing suitable Lyapunov functions, we investigate the global stability of equilibria in Section 6. The paper ends with a brief conclusion.

2 Well-posedness of the model

2.1 Reformulating the model (1.5)

Denote Banach spaces $\mathbb{X} := C(\overline{\Omega}, \mathbb{R})$ and $\mathbb{Y} := L^1(\mathbb{R}_+, \mathbb{X})$ with norm $|\cdot|_{\mathbb{X}}$, $|\varphi|_{\mathbb{Y}} := \int_0^{+\infty} |\varphi(a)|_{\mathbb{X}}\, \mathrm{d}a$, respectively. The positive cones of $\mathbb{X}$ and $\mathbb{Y}$ are denoted by $\mathbb{X}^+$ and $\mathbb{Y}^+$, respectively. Letting

$$\Pi_i(a) = e^{-\int_0^a [\mu_i + \theta(\sigma)]d\sigma} \text{ and } \Pi_p(b) = e^{-\int_0^b [\mu_p + \delta(\sigma)]d\sigma},$$

and denoting Γ_2 and Γ_3 be the Green function associated with $d_2\Delta$ and $d_3\Delta$ subject to Neumann boundary condition. By directly solving the equations i and p by the method of characteristics yields

$$i(t,a,x) = \begin{cases} \Pi_i(a) \displaystyle\int_\Omega \Gamma_2(a,x,y) i(t-a,0,y)\mathrm{d}y, & t-a > 0,\ x \in \Omega, \\ \dfrac{\Pi_i(a)}{\Pi_i(a-t)} \displaystyle\int_\Omega \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y, & a-t \geqslant 0,\ x \in \Omega, \end{cases} \tag{2.1}$$

and

$$p(t,b,x) = \begin{cases} \Pi_p(b) \displaystyle\int_\Omega \Gamma_3(b,x,y) p(t-b,0,y)\mathrm{d}y, & t-b > 0,\ x \in \Omega, \\ \dfrac{\Pi_p(b)}{\Pi_p(b-t)} \displaystyle\int_\Omega \Gamma_3(t,x,y)\phi_3(b-t,y)\mathrm{d}y, & b-t \geqslant 0,\ x \in \Omega, \end{cases} \tag{2.2}$$

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where

Let $(\mathbf{u}_1(t,x), \mathbf{u}_2(t,x)) := (i(t,0,x), p(t,0,x))$. Substituting (2.1) and (2.2) into (1.5) yields

$$\begin{cases} \dfrac{\partial S(t,x)}{\partial t} = d_1 \Delta S(t,x) + \Lambda - \mu_S S(t,x) - \mathbf{u}_1(t,x), \\ \mathbf{u}_1(t,x) = S(t,x) \displaystyle\sum_{i=1}^{4} \mathscr{F}_i(t,x), \\ \mathbf{u}_2(t,x) = \mathscr{F}_5(t,x) + \mathscr{F}_6(t,x), \\ \dfrac{\partial S(t,x)}{\partial n} = 0, \;\; x \in \partial\Omega, \end{cases} \tag{2.3}$$

with

$$\begin{cases} \mathscr{F}_1(t,x) = \displaystyle\int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a, \\ \mathscr{F}_2(t,x) = \displaystyle\int_t^\infty \beta_1(a)\frac{\Pi_i(a)}{\Pi_i(a-t)} \int_\Omega \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y\mathrm{d}a, \\ \mathscr{F}_3(t,x) = \displaystyle\int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y)\mathbf{u}_2(t-b,y)\mathrm{d}y\mathrm{d}b, \\ \mathscr{F}_4(t,x) = \displaystyle\int_t^\infty \beta_2(b)\frac{\Pi_p(b)}{\Pi_p(b-t)} \int_\Omega \Gamma_3(t,x,y)\phi_3(b-t,y)\mathrm{d}y\mathrm{d}b, \\ \mathscr{F}_5(t,x) = \displaystyle\int_0^t \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a, \\ \mathscr{F}_6(t,x) = \displaystyle\int_t^\infty \xi(a)\frac{\Pi_i(a)}{\Pi_i(a-t)} \int_\Omega \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y\mathrm{d}a. \end{cases} \tag{2.4}$$

Corresponding to the initial data of original system (1.5), we note that

$$\begin{cases} \phi_2(a,x) = \displaystyle\int_\Omega \Gamma_2(0,x,y)\phi_2(a,y)\mathrm{d}y, \;\; a-t \geqslant 0, \\ \phi_3(b,x) = \displaystyle\int_\Omega \Gamma_3(0,x,y)\phi_3(b,y)\mathrm{d}y, \;\; b-t \geqslant 0, \end{cases} \tag{2.5}$$

we impose on the initial data of system (2.3) as:

$$S(0,x) = \phi_1(x), \;\; u_1(0,x) = \phi_1(x)[\int_0^\infty \beta_1(a)\phi_2(a,x)\mathrm{d}a + \int_0^\infty \beta_2(b)\phi_2(b,x)\mathrm{d}b] \tag{2.6}$$

and

$$u_2(0,x) = \int_0^\infty \xi(a)\phi_2(a,x)\mathrm{d}a. \tag{2.7}$$

In what follows, we focus on the system (2.3) with (2.6) and (2.7). If without specific requirements, we use $\phi \in \mathbb{W}^+$ instead of $(\phi_1, \phi_2, \phi_3) \in \mathbb{X}^+ \times \mathbb{Y}^+ \times \mathbb{Y}^+$. Our main result on the well-posedness of (2.3) reads as:

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Theorem 2.1 *For any $\phi \in \mathbb{W}^+$, system (2.3) with (2.6) and (2.7) admits a unique global nonnegative classical solution $(S, \boldsymbol{u}_1, \boldsymbol{u}_2)$ on $[0, +\infty) \times \overline{\Omega}$.*

We will show Theorem 2.1 by the following lemmas, which is achieved by the Banach–Picard fixed point theorem (i.e., contraction mapping theorem). In what follows, we omit (t, x) in the variable of equation (2.3) and (2.4) for simplicity.

Lemma 1 *For any $\phi \in \mathbb{W}^+$, system (2.3) with (2.6) and (2.7) admits a unique solution $(S, \boldsymbol{u}_1, \boldsymbol{u}_2)$ on $[0, t_{max}) \times \overline{\Omega}$, with $t_{max} \leq \infty$.*

Proof Let $\mathbb{Z} := C([0, t_{max}], \mathbb{X})$ with the norm $|\varphi|_{\mathbb{Z}} := \sup_{0 \leq t \leq t_{max}} |\varphi(t, \cdot)|_{\mathbb{X}}$. Solving the S-equation of (2.3) in $(t, x) \in [0, t_{max}) \times \overline{\Omega}$ obtains

$$\begin{cases} S = \mathscr{F}_0 + \displaystyle\int_0^t e^{-\mu_S(t-a)} \int_\Omega \Gamma_1(t-a, x, y)[\Lambda - \mathbf{u}_1(a, y)]\mathrm{d}y\mathrm{d}a, \\ \mathbf{u}_1 = S \displaystyle\sum_{i=1}^4 \mathscr{F}_i, \\ \mathbf{u}_2 = \mathscr{F}_5 + \mathscr{F}_6, \end{cases} \tag{2.8}$$

where $\mathscr{F}_0 := e^{-\mu_S t} \int_0^\pi \Gamma_1(t, x, y)\phi_1(y)\mathrm{d}y$, and Γ_1 is the Green function associated with $d_1\Delta$ subject to Neumann boundary condition. Substituting S and $\mathbf{u}_2$ into $\mathbf{u}_1$ allows us to define $\mathscr{F} : \mathbb{Z} \to \mathbb{Z}$ as:

$$\begin{aligned} \mathscr{F}[\mathbf{u}_1](t, x) := & \left[\mathscr{F}_0 + \int_0^t e^{-\mu_S(t-a)} \int_\Omega \Gamma_1(t-a, x, y)[\Lambda - \mathbf{u}_1(a, y)]\mathrm{d}y\mathrm{d}a\right] \\ & \times \left[\int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a, x, y)u_1(t-a, y)\mathrm{d}y\mathrm{d}a + \mathscr{F}_2\right. \\ & \left. + \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b, x, y)[\mathscr{F}_5(t-b, y) + \mathscr{F}_6(t-b, y)]\mathrm{d}y\mathrm{d}b + \mathscr{F}_4\right] \\ = & \left[\mathscr{F}_0 + \int_0^t e^{-\mu_S(t-a)} \int_\Omega \Gamma_1(t-a, x, y)[\Lambda - \mathbf{u}_1(a, y)]\mathrm{d}y\mathrm{d}a\right] \\ & \times \left[\int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a, x, y)u_1(t-a, y)\mathrm{d}y\mathrm{d}a + \mathscr{F}_2 + \mathscr{F}_4\right. \\ & + \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b, x, y) \int_0^{t-b} \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a, x, z)\mathbf{u}_1(t-b-a, z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b \\ & \left. + \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b, x, y)\mathscr{F}_6(t-b, y)\mathrm{d}y\mathrm{d}b\right]. \end{aligned} \tag{2.9}$$

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For ease of notations, we denote

$$\begin{cases} \overline{\mathscr{F}}_2 = \mathscr{F}_2 + \mathscr{F}_4, \\ G_1(\mathbf{u}_1) = \int_0^t e^{-\mu_S(t-a)} \int_\Omega \Gamma_1(t-a,x,y)[\Lambda - \mathbf{u}_1(a,y)]\mathrm{d}y\mathrm{d}a, \\ G_2(\mathbf{u}_1) = \int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a, \\ G_3(\mathbf{u}_1) = \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \int_0^{t-b} \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,z)\mathbf{u}_1(t-b-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b, \\ \overline{\mathscr{F}}_6(t,x)) = \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y)\mathscr{F}_6(t-b,y)\mathrm{d}y\mathrm{d}b. \end{cases} \tag{2.10}$$

It then follows that

$$\mathscr{F}[\mathbf{u}_1](t,x) = [\mathscr{F}_0 + G_1(\mathbf{u}_1)]\left[\overline{\mathscr{F}}_2 + G_2(\mathbf{u}_1) + G_3(\mathbf{u}_1) + \overline{\mathscr{F}}_6\right]. \tag{2.11}$$

By standard procedures, in order to obtain a strict contraction mapping $\mathscr{F}$ in $\mathbb{Z}$, we let $\mathbf{u}_1', \mathbf{u}_1'' \in \mathbb{Z}$ and set $\tilde{\mathbf{u}}_1 := \mathbf{u}_1' - \mathbf{u}_1''$. We then have

$$\begin{aligned} |\mathscr{F}\mathbf{u}_1' - \mathscr{F}\mathbf{u}_1''| &= \mathscr{F}_0[G_2(\tilde{\mathbf{u}}_1) + G_3(\tilde{\mathbf{u}}_1)] + [\overline{\mathscr{F}}_2 + \overline{\mathscr{F}}_6]\widehat{G}_1(\tilde{\mathbf{u}}_1) \\ &\quad + \widehat{G}_1(\tilde{\mathbf{u}}_1)\left[G_2\left(\mathbf{u}_1''\right) + G_3\left(\mathbf{u}_1''\right)\right] + G_1\left(\mathbf{u}_1'\right)[G_2(\tilde{\mathbf{u}}_1) + G_3(\tilde{\mathbf{u}}_1)] \\ &\le |(\mathscr{F}_0 + G_1)\left[\overline{G}_2 + \overline{G}_3\right] + \left[\overline{\mathscr{F}}_2 + \overline{\mathscr{F}}_6 + G_2 + G_3\right]\overline{G}_1| \cdot \sup_{0\le s\le t} |\tilde{\mathbf{u}}_1(s,\cdot)|_{\mathbb{X}}, \end{aligned} \tag{2.12}$$

where

$$\begin{cases} \widehat{G}_1(\mathbf{u}_1) = -\int_0^t e^{-\mu_S(t-a)} \int_\Omega \Gamma_1(t-a,x,y)\mathbf{u}_1(a,y)\mathrm{d}y\mathrm{d}a, \\ \overline{G}_1 = -\int_0^t e^{-\mu_S(t-a)} \int_\Omega \Gamma_1(t-a,x,y)\mathrm{d}y\mathrm{d}a, \\ \overline{G}_2 = \int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a, \\ \overline{G}_3 = \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \int_0^{t-b} \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b. \end{cases}$$

Set

$$c(t_{max}) := \sup_{0\le t\le t_{max}} \left|(\mathscr{F}_0 + G_1)\left(\overline{G}_2 + \overline{G}_3\right) + \left(\overline{\mathscr{F}}_2 + \overline{\mathscr{F}}_6 + G_2 + G_3\right)\overline{G}_1\right|_{\mathbb{X}}.$$

It follows that

$$\left|\mathscr{F}\mathbf{u}_1 - \mathscr{F}\mathbf{u}_1''\right|_{\mathbb{Z}} \le c(t_{max})\left|\mathbf{u}_1' - \mathbf{u}_1''\right|_{\mathbb{Z}}.$$

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Choose $0 < t_{max} \ll 1$ small enough that $c(t_{max}) < 1$ (and clearly $\lim_{\alpha \mapsto 0} c(\alpha) = 0$). Hence, $\mathscr{F}$ is a strict contraction in $\mathbb{Z}$. By contraction mapping theorem (see [28, Theorem 9.23]), we finish the proof of this lemma. □

Lemma 2 *For any $\phi \in \mathbb{W}^+$, solution $(S, \boldsymbol{u}_1, \boldsymbol{u}_2)$ of (2.3) with (2.6) and (2.7) satisfies*

$$S > 0, \ \boldsymbol{u}_1 \geqslant 0 \ \text{ and } \ \boldsymbol{u}_2 \geqslant 0 \ \text{ on } [0, t_{max}) \times \overline{\Omega}.$$

Proof For any $\varphi \in \mathbb{Y}$, denote

$$\begin{cases} \Phi_1(\varphi)(t,x) := \displaystyle\int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\varphi(t-a,y)\mathrm{d}y\mathrm{d}a, \\ \Phi_2(\varphi)(t,x) := \displaystyle\int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y)\varphi(t-b,y)\mathrm{d}y\mathrm{d}b, \\ \Phi_3(\varphi)(t,x) := \displaystyle\int_0^t \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\varphi(t-a,y)\mathrm{d}y\mathrm{d}a. \end{cases}$$

For $i = 1, 2, 3$, it follows from (**A**1) that $\Phi_i : \mathbb{Y} \to \mathbb{Y}$ is the positive linear operator in the sense that $\Phi_i(\mathbb{Y}^+) \subset \mathbb{Y}^+$. Noticing that $\mathscr{F}_1, \mathscr{F}_3$ and $\mathscr{F}_5$ defined in (2.4) can be expressed by $\Phi_1(\mathbf{u}_1)$, $\Phi_2(\mathbf{u}_2)$ and $\Phi_3(\mathbf{u}_1)$, respectively. Further, for $(t,x) \in [0, t_{max}) \times \overline{\Omega}$,

$$\begin{cases} \dfrac{\partial S}{\partial t} > d_1 \Delta S - S\left[\mu_S + \Phi_1(\mathbf{u}_1) + \mathscr{F}_2 + \Phi_2(\mathbf{u}_2) + \mathscr{F}_4\right], \\ \mathbf{u}_1 = S(t,x)\left[\Phi_1(\mathbf{u}_1) + \mathscr{F}_2 + \Phi_2(\mathbf{u}_2) + \mathscr{F}_4\right], \\ \mathbf{u}_2 = \Phi_3(\mathbf{u}_1) + \mathscr{F}_6, \\ \dfrac{\partial S}{\partial n} = 0, \ x \in \partial\Omega. \end{cases} \tag{2.13}$$

Due to the continuity and boundedness of $\mu_S + \Phi_1(\mathbf{u}_1) + \mathscr{F}_2 + \Phi_2(\mathbf{u}_2) + \mathscr{F}_4$, we directly have $S(t,x) > 0$, on $[0, t_{max}) \times \overline{\Omega}$.

Next, we prove the positivity of u_1. Assume for the contrary that there exist $0 < \varepsilon \ll 1$ and $(t_1, x_1) \in (0, t_{max}) \times \Omega$ such that

$$\begin{cases} \mathbf{u}_1(t,x) \geqslant 0, & t \in [0, t_1) \text{ and } x \in \Omega; \\ \mathbf{u}_1(t,x_1) = 0, & t = t_1 \text{ and } x_1 \in \Omega; \\ \mathbf{u}_1(t+\varepsilon,x_1) < 0, & t = t_1 \text{ and } x_1 \in \Omega. \end{cases}$$

However, due to the positivity of $\mathscr{F}_2$ and $\mathscr{F}_4 \geqslant 0$, and for sufficiently small ε,

$$\begin{aligned} &\mathbf{u}_1(t_1+\varepsilon, x_1) \\ &= S(t_1+\varepsilon, x_1)\left[\int_0^{t_1+\varepsilon} \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t_1+\varepsilon-a,y)\mathrm{d}y\mathrm{d}a + \mathscr{F}_2(t_1+\varepsilon,x_1)\right. \\ &\quad \left. + \mathscr{F}_4(t_1+\varepsilon,x_1)\right] \end{aligned}$$

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$$+\int_0^{t_1+\varepsilon}\beta_2(b)\Pi_p(b)\int_\Omega\Gamma_3(b,x,y)\int_0^{t_1+\varepsilon-b}\xi(a)\Pi_i(a)\int_\Omega\Gamma_2(a,x,z)\mathbf{u}_1(t_1+\varepsilon-b-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b$$
$$\left.+\int_0^{t_1+\varepsilon}\beta_2(b)\Pi_p(b)\int_\Omega\Gamma_3(b,x,y)\mathscr{F}_6(t_1+\varepsilon-b,y)\mathrm{d}y\mathrm{d}b\right]$$
$$\geqslant 0,$$

a contradiction. Similarly, due to the positivity of $\mathscr{F}_6(t,x)$, $\mathbf{u}_2(t,x)\geqslant 0$ directly follows on $(0,t_{max})\times\Omega$. This completes the proof. □

To extend the existence interval of solution from $[0,t_{max})\times\overline{\Omega}$ to $[0,+\infty)\times\overline{\Omega}$, we only need to prove that the solution does not blow up in $[0,t_{max})$.

Lemma 3 *For any $\phi\in\mathbb{W}^+$, solution $(S,\boldsymbol{u}_1,\boldsymbol{u}_2)$ of (2.3) with (2.6) and (2.7) is bounded in $[0,t_{max})$.*

Proof From Lemma 2, S is bounded above by Λ/μ_S since $\frac{\partial S}{\partial t}\leq d_1\Delta S+\Lambda-\mu_S S$ for $(0,\infty)\times\Omega$. If $\mathbf{u}_1$ is unbounded in the sense that there exist $t^*>0$ and $x^*\in\Omega$ such that $\lim_{t\to t^*-0}\mathbf{u}_1(t,x^*)=+\infty$. Then S-equation satisfies $\lim_{t\to t^*-0}\partial_t S(t,x^*)=-\infty$, that is, $S(t,x^*)$ is negative around of t^*, a contradiction. Hence, $\mathbf{u}_1(t,x)<+\infty$ for $(0,\infty)\times\Omega$. Further, the bounded of $\mathbf{u}_2(t,x)$ in finite time is implied by the boundedness of $\mathbf{u}_1$. □

Hence, Theorem 2.1 can be proved by the previous lemmas.

Proof of Theorem 2.1. The local existence and uniqueness of solution of (2.3) with (2.6) and (2.7) are demonstrated in Lemma 1. By Lemma 2, the solution of (2.3) with (2.6) and (2.7) is non-negative. By Lemma 3, the solution of (2.3) with (2.6) and (2.7) does not blow up in finite time. Hence, (2.3) with (2.6) and (2.7) admits a unique global non-negative classical solution in $C([0,+\infty),\mathbb{X})$.

3 BRN and equilibria

Obviously, (2.3) admits the disease-free equilibrium $E_0=(S_0,0,0)\in\mathbb{W}^+$ with $S_0=\frac{\Lambda}{\mu_S}$. Linearising the disease compartments $\mathbf{u}_1$ and $\mathbf{u}_2$ around E_0 yields

$$\begin{cases}\mathbf{u}_1(t,x)=S_0\int_0^t\beta_1(a)\Pi_i(a)\int_\Omega\Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a\\ \qquad\qquad+S_0\int_0^t\beta_2(b)\Pi_p(b)\int_\Omega\Gamma_3(b,x,y)\mathbf{u}_2(t-b,y)\mathrm{d}y\mathrm{d}b,\\ \mathbf{u}_2(t,x)=\int_0^t\xi(a)\Pi_i(a)\int_\Omega\Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a.\end{cases}\tag{3.1}$$

Inserting u_2-equation of (3.1) into u_2-equation gets

$$\mathbf{u}_1(t,x)=S_0\int_0^t\beta_1(a)\Pi_i(a)\int_\Omega\Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a$$
$$+S_0\int_0^t\beta_2(b)\Pi_p(b)\int_\Omega\Gamma_3(b,x,y)\int_0^{t-b}\xi(a)\Pi_i(a)\int_\Omega\Gamma_2(a,y,z)\mathbf{u}_1(t-b-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b.\tag{3.2}$$

Making the Laplace transformation to (3.2), we obtian

$$
\begin{aligned}
L[\mathbf{u}_1] :=& \int_0^\infty e^{-\lambda t}\mathbf{u}_1 \mathrm{d}t \\
=& S_0 \int_0^\infty e^{-\lambda t} \int_0^t \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a\mathrm{d}t \\
&+ S_0 \int_0^\infty e^{-\lambda t} \int_0^t \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \\
&\times \int_0^{t-b} \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,y,z)\mathbf{u}_1(t-b-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\mathrm{d}t.
\end{aligned}
$$

Consequently, after multiple interchanging the order of integration, we can obtain

$$
\begin{aligned}
L[\mathbf{u}_1] =& S_0 \int_0^\infty \beta_1(a)\Pi_i(a)e^{-\lambda a} \int_\Omega \Gamma_2(a,x,y) \int_0^\infty e^{-\lambda t}\mathbf{u}_1(t,y)\mathrm{d}t\mathrm{d}y\mathrm{d}a \\
&+ S_0 \int_0^\infty \beta_2(b)\Pi_p(b)e^{-\lambda b} \int_\Omega \Gamma_3(b,x,y) \int_0^\infty \xi(a)\Pi_i(a)e^{-\lambda a} \int_\Omega \Gamma_2(a,y,z) \\
&\times \int_0^\infty e^{-\lambda t}\mathbf{u}_1(t,y)\mathrm{d}t\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b.
\end{aligned} \tag{3.3}
$$

Setting $\lambda = 0$ results in

$$
\begin{aligned}
\int_0^\infty \mathbf{u}_1(t,x)\mathrm{d}t =& S_0 \int_0^\infty \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y) \int_0^\infty \mathbf{u}_1(t,y)\mathrm{d}t\mathrm{d}y\mathrm{d}a \\
&+ S_0 \int_0^\infty \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \int_0^\infty \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,y,z) \\
&\times \int_0^\infty \mathbf{u}_1(t,y)\mathrm{d}t\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b.
\end{aligned} \tag{3.4}
$$

Hence, the following operator $\mathscr{K} : \mathbb{X} \to \mathbb{X}$ is termed as the next-generation operator (NGO) (see, e.g., [18]),

$$
\begin{aligned}
\mathscr{K}[\varphi](x) :=& S_0 \int_0^\infty \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\varphi(y)\mathrm{d}y\mathrm{d}a \\
&+ S_0 \int_0^\infty \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y)\int_0^\infty \xi(a)\Pi_i(a)\int_\Omega \Gamma_2(a,y,z)\varphi(y)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b, \ \varphi \in \mathbb{X}.
\end{aligned} \tag{3.5}
$$

The following result is on the operator $\mathscr{K}$.

Lemma 4 *The NGO $\mathscr{K}$ is strictly positive and compact.*

Proof The positivity of the operator $\mathscr{K}$ is obvious. To prove the compactness of $\mathscr{K}$, we need the following two claims.

Claim 1 $\mathscr{K}$ is uniformly bounded. To this end,

- selecting a bounded sequence $\{\varphi_n\}_{n\in\mathbb{N}}$ in $\mathbb{X}$ with $|\varphi_n|_{\mathbb{X}} \le \mathbb{M}$ for some $\mathbb{M} > 0$.
- defining a sequence $\{\psi_n\}_{n\in\mathbb{N}}$ by $\psi_n := \mathscr{K}\varphi_n$.

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Hence, for all $n \in \mathbb{N}$ and $x \in \Omega$,

$$\begin{aligned}\psi_n(x) \le S_0 \int_0^\infty \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a\mathbb{M} \\ + S_0 \int_0^\infty \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \int_0^\infty \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,y,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\mathbb{M}.\end{aligned} \tag{3.6}$$

Hence, $\{\psi_n\}_{n\in\mathbb{N}}$ is uniformly bounded.

Claim 2 $\{\psi_n\}_{n\in\mathbb{N}}$ is equi-continuous. For $x, \tilde{x} \in \Omega$, directly calculation gives

$$\begin{aligned}|\psi_n(x) - \psi_n(\tilde{x})| &= |\mathscr{K}\varphi_n(x) - \mathscr{K}\varphi_n(\tilde{x})| \\ &\le S_0\Big(\int_0^\infty \beta_1(a)\Pi_i(a) \int_\Omega |\Gamma_2(a,x,y) - \Gamma_2(a,\tilde{x},y)|\varphi_n(y)\mathrm{d}y\mathrm{d}a \\ &\quad + \int_0^\infty \beta_2(b)\Pi_p(b) \int_\Omega |\Gamma_3(b,x,y) \\ &\quad - \Gamma_3(b,\tilde{x},y)| \int_0^\infty \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,y,z)\varphi_n(y)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\Big) \\ &\le S_0\beta_1^+ \int_0^\infty \Pi_i(a) \int_\Omega |\Gamma_2(a,x,y) - \Gamma_2(a,\tilde{x},y)|\mathrm{d}y\mathrm{d}a\mathbb{M} \\ &\quad + S_0\beta_2^+\xi^+ \int_0^\infty \Pi_p(b) \int_\Omega |\Gamma_3(b,x,y) \\ &\quad - \Gamma_3(b,\tilde{x},y)| \int_0^\infty \Pi_i(a) \int_\Omega \Gamma_2(a,y,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\mathbb{M},\end{aligned} \tag{3.7}$$

where $\beta_1^+, \beta_2^+, \xi^+$ are defined in (**A**1).

Due to the compactness of the operator Δ and the uniform continuity of $\Gamma_2(a,x,y)$ and $\Gamma_3(b,x,y)$, there exists $\delta > 0$ such that

$$|\Gamma_2(a,x,y) - \Gamma_2(a,\tilde{x},y)| \le \frac{\varepsilon_0}{2S_0\beta_1^+\mathbb{M}\mathbb{M}_1},$$

and

$$|\Gamma_3(b,x,y) - \Gamma_3(b,\tilde{x},y)| \le \frac{\varepsilon_0}{2S_0\beta_2^+\xi^+\mathbb{M}\mathbb{M}_2},$$

for any $\varepsilon_0 > 0$ and $|x - \tilde{x}| < \delta$, $y \in \Omega$, where $\mathbb{M}_1 = |\Omega| \int_0^\infty \Pi_i(a)\mathrm{d}a$ and $\mathbb{M}_2 = |\Omega|^2 \int_0^\infty \Pi_p(b) \int_0^\infty \Pi_i(a)\mathrm{d}a\mathrm{d}b$ and $|\Omega|$ is the volume of Ω. For this δ and ε_0, $|\psi_n(x) - \psi_n(\tilde{x})| < \varepsilon_0$, for all $|x - \bar{x}| < \delta$, that is, $\psi_n(x)_{n\in\mathbb{N}}$ is equi-continuous. □

As in [18], we define the BRN of (2.3) $\Re_0 = r(\mathscr{K})$, the spectral radius of $\mathscr{K}$. Lemma 4 together with Krein–Rutman theorem ([3, Theorem 3.2]) imply that BRN $\Re_0$ is the only positive eigenvalue of $\mathscr{K}$, corresponding to a positive eigenvector. Without loss of generality, substituting $\varphi(x) \equiv \mathbf{1} > 0$ into (3.5) and using $\int_\Omega \Gamma_i(\cdot,x,y)\mathrm{d}y = 1, i = 2, 3$, one gets

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$$\mathscr{K}[\mathbf{1}] = S_0 \int_0^\infty \beta_1(a)\Pi_i(a) \int_\Omega \Gamma_2(a,x,y)\mathrm{d}y\mathrm{d}a[\mathbf{1}]$$
$$+ \frac{\Lambda}{\mu_S} \int_0^\infty \beta_2(b)\Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \int_0^\infty \xi(a)\Pi_i(a) \int_\Omega \Gamma_2(a,y,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b[\mathbf{1}]$$
$$= S_0 \int_0^\infty \beta_1(a)\Pi_i(a)\mathrm{d}a[\mathbf{1}] + \frac{\Lambda}{\mu_S} \int_0^\infty \beta_2(b)\Pi_p(b)\mathrm{d}b \int_0^\infty \xi(a)\Pi_i(a)\mathrm{d}a[\mathbf{1}].$$

Hence, $\Re_0 = r(\mathscr{K})$ can be explicitly expressed by:

$$\Re_0 = S_0\mathbb{K} + S_0\mathbb{Q}\mathbb{L}, \tag{3.8}$$

where

$$\mathbb{K} = \int_0^\infty \beta_1(a)\Pi_i(a)\mathrm{d}a, \ \mathbb{L} = \int_0^\infty \xi(a)\Pi_i(a)\mathrm{d}a \text{ and } \mathbb{Q} = \int_0^\infty \beta_2(b)\Pi_p(b)\mathrm{d}b. \tag{3.9}$$

By simple calculation, we directly have the existence of positive space-independent endemic equilibrium.

Theorem 3.1 *Let $\Re_0$ be defined in (3.8). If $\Re_0 > 1$, then (1.5) possesses a space-independent endemic equilibrium $E^* = (S^*, i^*(a), p^*(b))$, where*

$$S^* = \frac{S_0}{\Re_0}, \ i^*(a) = \Lambda\left(1 - \frac{1}{\Re_0}\right)\Pi_i(a) \text{ and } p^*(b) = \Pi_p(b)\int_0^\infty \xi(a)i^*(a)da.$$

4 Local dynamics

This subsection is spent on proving that both E^0 and E^* are locally asymptotically stable (LAS).

Theorem 4.1 *Let $\mathscr{R}_0$ be defined in (3.8), then*

(i) E_0 is LAS if $\Re_0 < 1$,
(ii) E^* is LAS if $\Re_0 > 1$.

Proof We first prove (i). Let $\hat{S} = S - S_0$, $\hat{i}(t,a,x) = i(t,a,x)$ and $\hat{p}(t,b,x) = p(t,b,x)$. The linearised equation of (1.5) at E_0 reads as:

$$\begin{cases} \dfrac{\partial \hat{S}}{\partial t} = d_1\Delta\hat{S} - \mu_S\hat{S} - S_0\displaystyle\int_0^\infty \beta_1(a)\hat{i}\mathrm{d}a - S_0\int_0^\infty \beta_2(b)\hat{p}\mathrm{d}b, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\right)\hat{i} = d_2\Delta\hat{i}(t,a,x) - (\mu_i + \theta(a))\hat{i}, \\ \hat{i}(t,0,x) = S_0\displaystyle\int_0^\infty \beta_1(a)\hat{i}\mathrm{d}a + \frac{\Lambda}{\mu_S}\int_0^\infty \beta_2(b)\hat{p}\mathrm{d}b, \\ \left(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\right)\hat{p} = d_3\Delta\hat{p} - (\mu_p + \delta(b))\hat{p}, \\ \hat{p}(t,0,x) = \displaystyle\int_0^\infty \xi(a)\hat{i}\mathrm{d}a. \end{cases} \tag{4.1}$$

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Since the linear system contains Laplacian term, we introduce the related theory from [7]. Denote by $\chi_i(i = 1, 2, ...)$ the eigenvalues of operator $-\Delta$ on a bounded set Ω with boundary condition (1.6), that is, $\Delta\nu(x) = -\chi_i\nu(x)$. Hence,

$$0 = \chi_0 < \chi_1 < \chi_2 < \cdots,$$

corresponding to which, there is the space of eigenfunctions in $C^1(\Omega)$, denoted by $E(\chi_i)$. Denote by $\{\phi_{ij}|\ j = 1, 2, ..., \dim E(\chi_i)\}$ the orthogonal basis of $E(\chi_i)$. Further, let $\mathbb{X}_{ij} = \{c\phi_{ij}|\ c \in \mathbb{R}^3\}$ such that

$$\tilde{\mathbb{X}} = \bigoplus_{i=0}^{\infty} \mathbb{X}_i, \text{ where } \mathbb{X}_i = \bigoplus_{j=1}^{\dim E(\chi_i)} \mathbb{X}_{ij}.$$

Since the parabolic problem $\frac{\partial u}{\partial t} = \Delta u$ with $\frac{\partial u}{\partial n} = 0$ admits the exponential solution $u(t,x) = e^{\eta t}\nu(x)$, where $\nu(x) \in \mathbb{X}_i$. Substituting $(\hat{S}(t,x), \hat{i}(t,a,x), \hat{p}(t,b,x)) = e^{\eta t}(\gamma_1^0(x), \gamma_2^0(a,x), \gamma_3^0(b,x))$ into (4.1), one has that

$$\begin{cases} \eta\gamma_1^0(x) = -d_1\chi_i\gamma_1^0(x) - \mu_S\gamma_1^0(x) - S_0\left(\int_0^\infty \beta_1(a)\gamma_2^0(a,x)\mathrm{d}a + \int_0^\infty \beta_2(b)\gamma_3^0(b,x)\mathrm{d}b\right) \\ \eta\gamma_2^0(a,x) + \dfrac{\partial\gamma_2^0(a,x)}{\partial a} = -d_2\chi_i\gamma_2^0(a,x) - (\ \mu_i + \theta(a))\gamma_2^0(a,x), \\ \eta\gamma_3^0(b,x) + \dfrac{\partial\gamma_3^0(b,x)}{\partial b} = -d_3\chi_i\gamma_3^0(b,x) - (\mu_p + \delta(b))\gamma_3^0(b,x), \\ \gamma_2^0(0,x) = S_0\left(\int_0^\infty \beta_1(a)\gamma_2^0(a,x)\mathrm{d}a + \int_0^\infty \beta_2(b)\gamma_3^0(b,x)\mathrm{d}b\right), \\ \gamma_3^0(0,x) = \int_0^\infty \xi(a)\gamma_2^0(a,x)\mathrm{d}a. \end{cases} \tag{4.2}$$

Solving the last four equations yields

$$\begin{cases} \gamma_2^0(a,x) = \gamma_2^0(0,x)\tilde{\Pi}_i(a)e^{-\eta a}, \\ \gamma_3^0(b,x) = \gamma_3^0(0,x)\tilde{\Pi}_p(b)e^{-\eta b} = \gamma_2^0(0,x)\displaystyle\int_0^\infty \xi(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a\tilde{\Pi}_p(b)e^{-\eta b}, \end{cases} \tag{4.3}$$

where $\tilde{\Pi}_i(a) = \Pi_i(a)e^{-d_2\chi_i a}$ and $\tilde{\Pi}_p(b) = \Pi_p(b)e^{-d_3\chi_i b}$. Inserting (4.3) into the fourth equation of (4.1), we can obtain

$$1 = S_0\left(\int_0^\infty \beta_1(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a + \int_0^\infty \beta_2(b)\int_0^\infty \xi(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a\tilde{\Pi}_p(b)e^{-\eta b}\mathrm{d}b\right) := \mathscr{H}(\eta). \tag{4.4}$$

Obviously, $\mathscr{H}'(\eta) < 0$. If (4.4) has a unique real positive root η, one has that

$$1 = \frac{\Lambda}{\mu_S}\left(\int_0^\infty \beta_1(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a + \int_0^\infty \beta_2(b)\int_0^\infty \xi(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a\tilde{\Pi}_p(b)e^{-\eta b}\mathrm{d}b\right) < \Re_0.$$

which leads to a contradiction. Thus, all roots of (4.4) are negative.

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If (4.4) has complex roots in the form of $\eta = x_0 + iy_0$ with $x_0 > 0$, we can obtain

$$1 = S_0\bigg(\int_0^\infty \beta_1(a)\tilde{\Pi}_i(a)e^{-x_0 a}\cos(y_0 a)\mathrm{d}a + \int_0^\infty \beta_2(b)\int_0^\infty \xi(a)\tilde{\Pi}_i(a)e^{-x_0(a+b)}\cos(y_0(a+b))\mathrm{d}a\tilde{\Pi}_p(b)\mathrm{d}b\bigg) \le \Re_0,$$

a contradiction with $\Re_0 < 1$. Consequently, E^0 is LAS if $\Re_0 < 1$.

We next prove (ii). Denote by $\check{S}(t,x) = S(t,x) - S^*$, $\check{i}(t,a,x) = i(t,a,x) - i^*(a)$ and $\check{p}(t,b,x) = p(t,b,x) - p^*(b)$. The linearised equation of (1.5) at $E^* = (S^*, i^*(a), p^*(b))$ reads as:

$$\begin{cases} \dfrac{\partial \check{S}}{\partial t} = d_1\Delta\check{S} - \mu_S\Re_0\check{S} - S^*\bigg(\displaystyle\int_0^\infty \beta_1(a)\check{i}\mathrm{d}a + \int_0^\infty \beta_2(b)\check{p}\mathrm{d}b\bigg), \\ \bigg(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial a}\bigg)\check{i} = d_2\Delta\check{i} - (\mu_i + \theta(a))\check{i}, \\ \check{i}(t,0,x) = \mu_S(\Re_0 - 1)\check{S} + S^*\bigg(\displaystyle\int_0^\infty \beta_1(a)\check{i}\mathrm{d}a + \int_0^\infty \beta_2(b)\check{p}\mathrm{d}b\bigg), \\ \bigg(\dfrac{\partial}{\partial t} + \dfrac{\partial}{\partial b}\bigg)\check{p} = d_3\Delta\check{p} - (\mu_p + \delta(b))\check{p}, \\ \check{p}(t,0,x) = \displaystyle\int_0^\infty \xi(a)\check{i}\mathrm{d}a, \\ \dfrac{\partial \check{S}}{\partial n} = \dfrac{\partial \check{i}}{\partial n} = \dfrac{\partial \check{p}}{\partial n} = 0. \end{cases} \tag{4.5}$$

Here, we have used the fact that

$$S^* = \frac{S^0}{\Re_0} = \frac{\Lambda}{\mu_S\Re_0}, \quad \Lambda - \mu_S S^* = S^*\bigg(\int_0^\infty \beta_1(a)i^*(a)\mathrm{d}a + \int_0^\infty \beta_2(b)p^*(b)\mathrm{d}b\bigg). \tag{4.6}$$

Similarly, we substitute $(\check{S}(t,x), \check{i}(t,a,x), \check{p}(t,b,x)) = e^{\eta t}(\gamma_1(x), \gamma_2(a,x), \gamma_3(b,x))$ into (4.5) gets

$$\begin{cases} \eta\gamma_1(x) = -(d_1\chi_i + \mu_S\Re_0)\gamma_1(x) - S^*\bigg(\displaystyle\int_0^\infty \beta_1(a)\gamma_2(a,x)\mathrm{d}a + \int_0^\infty \beta_2(b)\gamma_3(b,x)\mathrm{d}b\bigg), \\ \eta\gamma_2(a,x) + \dfrac{\partial\gamma_2(a,x)}{\partial a} = -d_2\chi_i\gamma_2(a,x) - (\mu_i + \theta(a))\gamma_2(a,x), \\ \eta\gamma_3(b,x) + \dfrac{\partial\gamma_3(b,x)}{\partial a} = -d_3\chi_i\gamma_3(b,x) - (\mu_p + \delta(b))\gamma_3(b,x), \\ \gamma_2(0,x) = \mu_S(\Re_0 - 1)\gamma_1(x) + S^*\bigg(\displaystyle\int_0^\infty \beta_1(a)\gamma_2(a,x)\mathrm{d}a + \int_0^\infty \beta_2(b)\gamma_3(b,x)\mathrm{d}b\bigg), \\ \gamma_3(0,x) = \displaystyle\int_0^\infty \xi(a)\gamma_2(b,x)\mathrm{d}a. \end{cases} \tag{4.7}$$

Solving the last four equations of (4.7), we can obtain

$$\begin{cases} \gamma_2(a,x) = \gamma_2(0,x)\tilde{\Pi}_i(a)e^{-\eta a}, \\ \gamma_3(b,x) = \gamma_3(0,x)\tilde{\Pi}_p(b)e^{-\eta b} = \gamma_2(0,x)\displaystyle\int_0^\infty \xi(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a\tilde{\Pi}_p(b)e^{-\eta b}. \end{cases} \tag{4.8}$$

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Plugging (4.8) into the first and fourth equation of (4.7) yields the characteristic equation:

$$\begin{vmatrix} \mu_S(\Re_0 - 1) & \mathscr{H}_1(\eta) - 1 \\ d_1\chi_i + \mu_S\Re_0 + \eta & \mathscr{H}_1(\eta) \end{vmatrix} = 0, \tag{4.9}$$

where

$$\mathscr{H}_1(\eta) = S^* \left(\int_0^\infty \beta_1(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a + \int_0^\infty \beta_2(b) \int_0^\infty \xi(a)\tilde{\Pi}_i(a)e^{-\eta a}\mathrm{d}a\tilde{\Pi}_p(b)e^{-\eta b}\mathrm{d}b \right).$$

Thus,

$$(\eta + d_1\chi_i + \mu_S)\mathscr{H}_1(\eta) - (\eta + d_1\chi_i + \mu_S\Re_0) = 0 \quad . \tag{4.10}$$

If (4.10) has a real root $\eta > 0$. By $\Re_0 > 1$, we have

$$\mathscr{H}_1(\eta) = \frac{(\eta + d_1\chi_i + \mu_S\Re_0)}{(\eta + d_1\chi_i + \mu_S)} > 1. \tag{4.11}$$

Obviously, $\mathscr{H}_1'(\eta) < 0$. This together with $S^* = \frac{S_0}{\Re_0}$ indicate that

$$\mathscr{H}_1(\eta) < \mathscr{H}_1(0) < S^* \int_0^\infty \beta_1(a)\Pi_i(a)\mathrm{d}a + S^* \int_0^\infty \beta_2(b)\Pi_p(b)\mathrm{d}b \int_0^\infty \xi(a)\Pi_i(a)\mathrm{d}a = 1,$$

which leads to a contradiction with (4.11). Hence, all the real roots of (4.10) are negative.

If (4.10) has complex roots $\eta = x_1 + y_1 i$ with $x_1 \geqslant 0$, then

$$(x_1 + y_1 i + d_1\chi_i + \mu_S)\mathscr{H}_1(x_1 + y_1 i) - (x_1 + y_1 i + d_1\chi_i + \mu_S\Re_0) = 0.$$

It follows that

$$\mathrm{Re}\mathscr{H}_1(x_1 + y_1 i) = \frac{(x_1 + d_1\chi_i + \mu_S\Re_0)(x_1 + d_1\chi_i + \mu_S) + y_1^2}{(x_1 + d_1\chi_i + \mu_S)^2 + y_1^2} > 1. \tag{4.12}$$

On the other hand,

$$\mathrm{Re}\mathscr{H}_1(x_1 + y_1 i) \leq |H(x_1)| = H(x_1) \leq \mathscr{H}_1(0) < 1,$$

a contradiction with (4.12). Thus, E^* is LAS. □

5 Disease persistence

This section is to show the disease persistence when $\Re_0 > 1$. In order to using the methods in [30, Section 9.4], we first rewrite (2.1) and (2.2) as:

$$i(t,a,x) = \begin{cases} \dfrac{\Pi_i(a)}{\Pi_i(a-t)} \displaystyle\int_\Omega \Gamma_2(t,x,y)\phi_2(a-t,y)\mathrm{d}y, & a - t \geqslant 0, \ x \in \Omega, \\ \Pi_i(a) \displaystyle\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y, & t - a > 0, \ x \in \Omega, \end{cases} \tag{5.1}$$

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and

$$p(t,b,x)=\begin{cases}\dfrac{\Pi_p(b)}{\Pi_p(b-t)}\displaystyle\int_\Omega \Gamma_3(t,x,y)\phi_3(b-t,y)\mathrm{d}y, & b-t\geqslant 0,\ x\in\Omega,\\ \Pi_p(b)\displaystyle\int_\Omega \Gamma_3(b,x,y)\mathbf{u}_2(t-b,y)\mathrm{d}y, & t-b>0,\ x\in\Omega.\end{cases} \tag{5.2}$$

Now, we are in position to show the first lemma of this section.

Lemma 5 *For any $\phi\in\mathbb{W}^+$, system (2.3) defines a continuous semiflow:*

$$\Theta(t,\phi):=(S(t,\cdot,\phi_1),i(t,\cdot,\phi_2),p(t,\cdot,\phi_3))\in\mathbb{W}^+$$

for all $t\geqslant 0$.

Proof For any $\varsigma\geqslant 0$, $t\geqslant 0$, $a,b\geqslant 0$, and $x\in\Omega$, let

$$S_\varsigma(t,x)=S(\varsigma+t,x),\ \mathbf{u}_{1\varsigma}(t,x)=u_1(\varsigma+t,x),\ \mathbf{u}_{2\varsigma}(t,x)=u_2(r+t,x)$$

and

$$i_\varsigma(t,a,x)=i(\varsigma+t,a,x),\ p_\varsigma(t,b,x)=p(\varsigma+t,b,x).$$

Then,

$$\begin{cases}\dfrac{\partial S_\varsigma(t,x)}{\partial t}=d_1\Delta S_\varsigma(t,x)+\Lambda-\mu_S S_\varsigma(t,x)-\mathbf{u}_{1\varsigma}(t,x),\ \text{with } S_\varsigma(0,x)=S(r,x),\\ \mathbf{u}_{1\varsigma}(t,x)=S_\varsigma(t,x)\displaystyle\int_0^\infty \beta_1(a)i_\varsigma(t,a,x)\mathrm{d}a+S_\varsigma(t,x)\int_0^\infty \beta_2(b)p_\varsigma(t,b,x)\mathrm{d}b,\\ \mathbf{u}_{2\varsigma}(t,x)=\displaystyle\int_0^\infty \xi(a)i_\varsigma(t,a,x)\mathrm{d}a.\end{cases} \tag{5.3}$$

Hence (5.1) and (5.2) can be rewritten as:

$$i_\varsigma(t,a,x)=\begin{cases}\dfrac{\Pi_i(a)}{\Pi_i(a-\varsigma-t)}\displaystyle\int_\Omega \Gamma_2(\varsigma+t,x,y)\phi_2(a-\varsigma-t,y)\mathrm{d}y, & a\geqslant\varsigma+t,\\ \Pi_i(a)\displaystyle\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_{1\varsigma}(t-a,y)\mathrm{d}y, & a<\varsigma+t,\end{cases} \tag{5.4}$$

and

$$p_\varsigma(t,b,x)=\begin{cases}\dfrac{\Pi_p(b)}{\Pi_p(b-\varsigma-t)}\displaystyle\int_\Omega \Gamma_3(\varsigma+t,x,y)\phi_3(b-\varsigma-t,y)\mathrm{d}y, & b\geqslant\varsigma+t,\\ \Pi_p(b)\displaystyle\int_\Omega \Gamma_3(b,x,y)\mathbf{u}_{2\varsigma}(t-b,y)\mathrm{d}y, & b<\varsigma+t.\end{cases} \tag{5.5}$$

Additionally, for $r\geqslant 0$, $a>t\geqslant 0$ and $x\in\Omega$, we have

$$i_\varsigma(0,a-t,x)=\begin{cases}\dfrac{\Pi_i(a-t)}{\Pi_i(a-\varsigma-t)}\displaystyle\int_\Omega \Gamma_2(\varsigma,x,y)\phi_2(a-\varsigma-t,y)\mathrm{d}y, & a>\varsigma+t,\\ \Pi_i(a-t)\displaystyle\int_\Omega \Gamma_2(a-t,x,y)\mathbf{u}_{1\varsigma}(t-a,y)\mathrm{d}y, & a\in[t,\varsigma+t),\end{cases}$$

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and

$$p_\varsigma(0,b-t,x)=\begin{cases}\dfrac{\Pi_p(b-t)}{\Pi_p(b-\varsigma-t)}\displaystyle\int_\Omega \Gamma_3(\varsigma,x,y)\phi_3(b-\varsigma-t,y)\mathrm{d}y, & b>\varsigma+t,\\ \Pi_p(b-t)\displaystyle\int_\Omega \Gamma_3(b-t,x,y)\mathbf{u}_{2\varsigma}(t-b,y)\mathrm{d}y, & b\in[t,\varsigma+t).\end{cases}$$

Due to the properties of Γ_2 and Γ_3 (see [8]), we have

$$\begin{aligned}&\frac{\Pi_i(a)}{\Pi_i(a-t)}\int_\Omega \Gamma_2(t,x,y)i_\varsigma(0,a-t,y)\mathrm{d}y\\ &=\begin{cases}\dfrac{\Pi_i(a)}{\Pi_i(a-\varsigma-t)}\displaystyle\int_\Omega \Gamma_2(\varsigma+t,x,y)\phi_2(a-\varsigma-t,y)\mathrm{d}y, & a>\varsigma+t,\\ \Pi_i(a)\displaystyle\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_{1\varsigma}(t-a,y)\mathrm{d}y, & a\in[t,\varsigma+t),\end{cases}\end{aligned}\tag{5.6}$$

and

$$\begin{aligned}&\frac{\Pi_p(b)}{\Pi_p(b-t)}\int_\Omega \Gamma_3(t,x,y)p_\varsigma(0,b-t,y)\mathrm{d}y\\ &=\begin{cases}\dfrac{\Pi_p(b)}{\Pi_p(b-\varsigma-t)}\displaystyle\int_\Omega \Gamma_3(\varsigma+t,x,y)\phi_3(b-\varsigma-t,y)\mathrm{d}y, & b>\varsigma+t,\\ \Pi_p(b)\displaystyle\int_\Omega \Gamma_3(b,x,y)\mathbf{u}_{2\varsigma}(t-b,y)\mathrm{d}y, & b\in[t,\varsigma+t).\end{cases}\end{aligned}\tag{5.7}$$

Combined with (5.4) and (5.6), we can obtain

$$i_\varsigma(t,a,x)=\begin{cases}\dfrac{\Pi_i(a)}{\Pi_i(a-t)}\displaystyle\int_\Omega \Gamma_2(t,x,y)i_\varsigma(0,a-t,y)\mathrm{d}y, & a-t\geqslant 0,\\ \Pi_i(a)\displaystyle\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_{1\varsigma}(t-a,y)\mathrm{d}y, & t-a>0.\end{cases}\tag{5.8}$$

Combined with (5.5) and (5.7), we can obtain

$$p_\varsigma(t,b,x)=\begin{cases}\dfrac{\Pi_p(b)}{\Pi_p(b-t)}\displaystyle\int_\Omega \Gamma_3(t,x,y)i_\varsigma(0,b-t,y)\mathrm{d}y, & a-t\geqslant 0,\\ \Pi_p(b)\displaystyle\int_\Omega \Gamma_3(b,x,y)\mathbf{u}_{2\varsigma}(t-b,y)\mathrm{d}y, & t-b>0.\end{cases}\tag{5.9}$$

Consequently, from (5.3), (5.8) and (5.9), we have,

$$\Theta(t,S(\varsigma,\cdot),i(\varsigma,\cdot,\cdot),p(\varsigma,\cdot,\cdot))=(S_\varsigma(t),i_\varsigma(t,\cdot,\cdot),p_\varsigma(t,\cdot,\cdot))=\Theta(\varsigma+t,\phi_1,\phi_2,\phi_3),$$

for all $\varsigma\geqslant 0$ and $t\geqslant 0$. Hence, the time continuity of Θ follows from Theorem 2.1. □

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Following the procedures in [2, Lemma 6.1], let

$$\mathbb{D} := \left\{ \phi \in \mathbb{W}^+ \,\middle|\, \begin{array}{l} \phi_1(\cdot)\left[\int_0^\infty \beta_1(\cdot)\phi_2(a,\cdot)\mathrm{d}a + \int_0^\infty \beta_2(b)\phi_3(b,\cdot)\mathrm{d}b\right] > 0, \\ \phi_1(\cdot)\int_0^\infty \xi(a)\phi_2(a,\cdot)\mathrm{d}a > 0 \text{ for some } x \in \Omega \end{array} \right\},$$

we claim the following results.

Lemma 6 *If $\phi \in \mathbb{D}$ and $\Re_0 > 1$, then there exist $\rho_i > 0$ $(i = 1.2)$ such that*

$$\limsup_{t\to+\infty} |\boldsymbol{u}_i(t,\cdot)|_{\mathbb{X}} > \rho_i.$$

Proof By the expression of $\Re_0$ defined in (3.8), choose $\rho_1 > 0$ such that

$$\frac{\Lambda - \rho_1}{\mu_S}\left(\int_0^\infty \beta_1(a)\Pi_i(a)\mathrm{d}a + \int_0^\infty \beta_2(b)\Pi_p(b)\int_0^\infty \xi(a)\Pi_i(a)\mathrm{d}a\mathrm{d}b\right) > 1. \tag{5.10}$$

Suppose, by contradiction, assume that $u_1(t,x) \le \rho_1$ for all $x \in \Omega$ and $t \geqslant t_1 > 0$. Using inequality (5.10), there exist sufficiently large $t_2 > t_1$ and $\lambda > 0$ is small enough ensures that

$$\begin{aligned}\Re :=& \frac{\Lambda - \rho_1}{\mu_S}\left(1 - e^{-\mu h}\right)\left(\int_0^\infty \beta_1(a)\Pi_i(a)e^{-\lambda a}\mathrm{d}a\right.\\ &\left.+ \int_0^\infty \beta_2(b)\Pi_p(b)e^{-\lambda b}\int_0^\infty \xi(a)\Pi_i(a)e^{-\lambda a}\mathrm{d}a\mathrm{d}b\right) > 1,\end{aligned} \tag{5.11}$$

where $h = t_2 - t_1$. One has that

$$\frac{\partial S(t,x)}{\partial t} \geqslant d_1\Delta S(t,x) + \Lambda - \rho_1 - \mu_S S(t,x) \text{ on } [t_2,\infty)\times\Omega.$$

Solving the above equation and applying comparison principle yields

$$\begin{cases} S(t,x) \geqslant \dfrac{\Lambda - \rho_1}{\mu_S}(1 - e^{-\mu_S h}), \\ \mathbf{u}_2(t,x) \geqslant \displaystyle\int_0^t \xi(a)\Pi_i(a)\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a, \end{cases} \tag{5.12}$$

on $[t_2,\infty)\times\Omega$. Lemma 5 together with (5.12) allow us to take $S(t_2,x)$, $\mathbf{u}_1(t_2,x)$ and $\mathbf{u}_2(t_2,x)$ with $t_2 = 0$ as initial data. Hence,

$$\begin{aligned}\mathbf{u}_1(t,x) \geqslant& \frac{\Lambda - \rho_1}{\mu_S}\left(1 - e^{-\mu_S h}\right)\left(\int_0^t \beta_1(a)\Pi_i(a)\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a\right.\\ &\left.+ \int_0^t \beta_2(b)\Pi_p(b)\int_\Omega \Gamma_3(b,x,y)\int_0^{t-b}\xi(a)\Pi_i(a)\int_\Omega \Gamma_2(a,x,z)\mathbf{u}_1(t-b-a,z)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\right)\end{aligned} \tag{5.13}$$

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on $[0,\infty)\times\Omega$. Obviously, for all $x\in\Omega$, $L(\mathbf{u}_1)=\int_0^\infty e^{-\lambda t}\mathbf{u}_1(t,x)\mathrm{d}t<\infty$. Define $\tilde{x}\in\Omega$ such that $L(\mathbf{u}_1)(\tilde{x})=\min_{x\in\Omega}L(\mathbf{u}_1)$. By (5.13),

$$
\begin{aligned}
L(\mathbf{u}_1)(\tilde{x}) \geqslant \frac{\Lambda-\rho_1}{\mu_S}(1-e^{-\mu_S h})\Bigg(&\int_0^\infty e^{-\lambda t}\int_0^t \beta_1(a)\Pi_i(a)\int_\Omega \Gamma_2(a,x,y)\mathbf{u}_1(t-a,y)\mathrm{d}y\mathrm{d}a\mathrm{d}t\\
&+\int_0^\infty e^{-\lambda t}\int_0^t \beta_2(b)\Pi_p(b)\int_\Omega \Gamma_3(b,x,y)\int_0^{t-b}\xi(a)\Pi_i(a)\\
&\times\int_\Omega \Gamma_2(a,y,z)\mathbf{u}_1(t-b-a,y)\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\mathrm{d}t\Bigg).
\end{aligned}
$$

Consequently, after multiple interchanging the order of integration, we can obtain

$$
\begin{aligned}
L(\mathbf{u}_1)(\tilde{x}) &\geqslant \frac{\Lambda-\rho_1}{\mu_S}(1-e^{-\mu_S h})\Bigg(\int_0^\infty \beta_1(a)\Pi_i(a)e^{-\lambda a}\int_\Omega \Gamma_2(a,x,y)\int_0^\infty e^{-\lambda t}\mathbf{u}_1(t,y)\mathrm{d}t\mathrm{d}y\mathrm{d}a\\
&\quad+\int_0^\infty \beta_2(b)\Pi_p(b)e^{-\lambda b}\int_\Omega \Gamma_3(b,x,y)\int_0^\infty \xi(a)\Pi_i(a)e^{-\lambda a}\int_\Omega \Gamma_2(a,y,z)\\
&\quad\int_0^\infty e^{-\lambda t}\mathbf{u}_1(t,y)\mathrm{d}t\mathrm{d}z\mathrm{d}a\mathrm{d}y\mathrm{d}b\Bigg)\\
&\geqslant \Re L(\mathbf{u}_1)(\tilde{x}),
\end{aligned}
$$

a contradiction with (5.11). The second assertion directly follows from (5.12). □

With the help of Lemma 6, we next prove the strong $|\cdot|_{\mathbb{X}}$-persistence (see the definition in [20]).

Lemma 7 *For any $\phi\in\mathbb{D}$, if $\Re_0>1$, then there exist $\rho_i'>0$ $(i=1,2)$ such that*

$$\liminf_{t\to+\infty}|\boldsymbol{u}_i(t,\cdot)|_{\mathbb{X}}>\rho_i'.$$

Proof Assume that $\liminf_{t\to+\infty}|\mathbf{u}_1(t,\cdot)|_{\mathbb{X}}<\rho_1'$ for some $\rho_1'>0$. This together with Lemma 6 imply that there exist increasing sequences $\{t_{1k}\}_{k=1}^{+\infty}$, $\{t_{2k}\}_{k=1}^{+\infty}$, $\{t_{3k}\}_{k=1}^{+\infty}$ and decreasing sequence $\{t_{4k}\}_{k=1}^{+\infty}$ with $t_{1k}>t_{2k}>t_{3k}$, $\liminf_{k\to+\infty}t_{4k}=0$ and

$$
\begin{cases}
|\mathbf{u}_1(t_{3k},\cdot)|_{\mathbb{X}}>\rho_1, & t=t_{3k},\\
|\mathbf{u}_1(t_{2k},\cdot)|_{\mathbb{X}}=\rho_1, & t=t_{2k},\\
|\mathbf{u}_1(t_{1k},\cdot)|_{\mathbb{X}}<t_{4k}<\rho_1, & t=t_{1k},\\
|\mathbf{u}_1(t,\cdot)|_{\mathbb{X}}<\rho_1, & t\in(t_{2k},t_{1k}).
\end{cases}
\tag{5.14}
$$

Let $\{S_k\}_{k=1}^{+\infty}$, $\{\mathbf{u}_{1k}\}_{k=1}^{+\infty}$ and $\{\mathbf{u}_{2k}\}_{k=1}^{+\infty}$ such that $S_k:=S(t_{2k},\cdot)\in\mathbb{X}$, $\mathbf{u}_{1k}:=\mathbf{u}_1(t_{2k},\cdot)\in\mathbb{X}$ and $\mathbf{u}_{2k}:=\mathbf{u}_2(t_{2k},\cdot)\in\mathbb{X}$, respectively. From (2.8) and (2.9) and applying the Arzela–Ascoli theorem (see [29, Theorem 11.28]), there exists $(S^*,\mathbf{u}_1^*,\mathbf{u}_2^*)\in\mathbb{X}^+\times\mathbb{X}^+\times\mathbb{X}^+$ such that

$$\liminf_{k\to+\infty}S_k=S^*,\quad \liminf_{k\to+\infty}\mathbf{u}_{1k}=\mathbf{u}_1^*\quad\text{and}\quad \liminf_{k\to+\infty}\mathbf{u}_{2k}=\mathbf{u}_2^*.$$

Let $(\tilde{S}, \tilde{\mathbf{u}}_1, \tilde{\mathbf{u}}_2)$ be a solution of (2.3) with

$$\begin{cases} \phi_1(x) = S^*(x), \\ \phi_2(a,x) = \Pi_i(a) \displaystyle\int_\Omega \Gamma_2(a,x,y) u_1^*(y) \mathrm{d}y \mathrm{d}a + \Pi_p(b) \int_\Omega \Gamma_3(b,x,y) \Pi_i(a) \int_\Omega \Gamma_2(z,x,y) \mathbf{u}_1^*(y) \mathrm{d}z \mathrm{d}y, \\ \phi_3(x) = \Pi_i(a) \displaystyle\int_\Omega \Gamma_2(a,x,y) \mathbf{u}_1^*(y) \mathrm{d}y, \end{cases}$$

for all $a \geqslant 0$, $b \geqslant 0$ and $x \in \Omega$. Since ϕ_2 and ϕ_3 depend on (5.1) and (5.2). According to Lemma 6, there exist $\tau' > 0$, $\varpi > 0$ such that

$$\begin{cases} |\tilde{\mathbf{u}}_1(\tau', \cdot)|_{\mathbb{X}} > \rho_1, & t = \tau', \\ |\tilde{\mathbf{u}}_1(t, \cdot)|_{\mathbb{X}} > \varpi, & 0 < t < \tau'. \end{cases} \tag{5.15}$$

Let $\tilde{\mathbf{u}}_{1k}(t, \cdot) := \mathbf{u}_1(t_{2k} + t, \cdot)$ for each $k \in \mathbb{N}$, it follows from the semiflow property that

$$\begin{cases} |\tilde{\mathbf{u}}_{1k}(\tau', \cdot)|_{\mathbb{X}} > \rho_1, & t = \tau', \\ |\tilde{\mathbf{u}}_{1k}(t, \cdot)|_{\mathbb{X}} > \varpi > t_{4k}, & 0 < t < \tau', \end{cases} \tag{5.16}$$

for sufficiently large k. In contrast, for $\tilde{t}_k := t_{1k} - t_{2k}$, we have from (5.14) that

$$\begin{cases} |\tilde{\mathbf{u}}_{1k}(\tilde{t}_k, \cdot)|_{\mathbb{X}} < t_{4k} < \rho_1, & t = \tilde{t}_k, \\ |\tilde{\mathbf{u}}_{1k}(t, \cdot)|_{\mathbb{X}} < \rho_1, & 0 < t < \tau'. \end{cases} \tag{5.17}$$

It is easy to get a contradiction between (5.16) and (5.17). Here, we finish the proof of $\liminf_{t \to +\infty} |\mathbf{u}_1(t, \cdot)|_{\mathbb{X}} > \rho_1'$ for some constant $\rho_1' > 0$. Similarly, $\rho_2' > 0$ such that $\liminf_{t \to +\infty} |\mathbf{u}_2(t, \cdot)|_{\mathbb{X}} > \rho_2'$ for some $\rho_2' > 0$. □

6 Global stability of equilibria

Now, we spent on proving that both E_0 and E^* are globally asymptotically stable (GAS).

Theorem 6.1 *E^0 is GAS provided that $\Re_0 < 1$.*

Proof Define Lyapunov functional"

$$L_{E_0}(t) = \int_\Omega [V_S(t,x) + V_i(t,x) + V_p(t,x)] \mathrm{d}x,$$

where

$$V_S(t,x) = G[S, S_0], \quad V_i(t,x) = \int_0^\infty \Phi(a) i(t,a,x) \mathrm{d}a, \quad V_p(t,x) = \int_0^\infty \Psi(b) p(t,b,x) \mathrm{d}b,$$

and

$$G[\alpha, \beta](t,x) = \alpha - \beta - \beta \ln \frac{\alpha}{\beta} \geqslant 0 \quad \text{for } \alpha, \beta \in \mathbb{X}_+ \quad \text{with} \quad G[\alpha, \alpha](t,x) = 0. \tag{6.1}$$

We will determine the functions $\Phi(a)$ and $\Psi(b)$ later. The calculation of the derivative of V_S reads as:

$$\frac{\partial V_S(t,x)}{\partial t} = d_1 \frac{S-S_0}{S}\Delta S - \frac{\mu_S}{S}(S-S_0)^2 - \mathbf{u}_1(t,x) + S_0 \int_0^\infty \beta_1(a) i(t,a,x)\mathrm{d}a + S_0 \int_0^\infty \beta_2(b) p(t,b,x)\mathrm{d}b. \tag{6.2}$$

Note that

$$V_i(t,x) = \int_0^\infty \Phi(t+a)\frac{\Pi_i(t+a)}{\Pi_i(a)} \int_\Omega \Gamma_2(t,x,y)\phi_2(a,y)\mathrm{d}y\mathrm{d}a + \int_0^t \Phi(t-a)\Pi_i(t-a)\int_\Omega \Gamma_2(t-a,x,y)\mathbf{u}_1(a,y)\mathrm{d}y\mathrm{d}a.$$

Hence,

$$\begin{aligned}
\frac{\partial V_i(t,x)}{\partial t} &= \Phi(0)\int_\Omega \Gamma_2(0,x,y)\mathbf{u}_1(t,y)\mathrm{d}y + \int_0^t \frac{\mathrm{d}\Phi(t-a)}{\mathrm{d}t}\Pi_i(t-a)\int_\Omega \Gamma_2(t-a,x,y)\mathbf{u}_1(a,y)\mathrm{d}y\mathrm{d}a \\
&\quad + \int_0^t \Phi(t-a)\Pi_i(t-a)\int_\Omega \frac{\partial \Gamma_2(t-a,x,y)}{\partial t}\mathbf{u}_1(a,y)\mathrm{d}y\mathrm{d}a \\
&\quad - \int_0^t [\mu_i + \theta(t-a)]\Phi(t-a)\Pi_i(t-a)\int_\Omega \Gamma_2(t-a,x,y)\mathbf{u}_1(a,y)\mathrm{d}y\mathrm{d}a \\
&\quad + \int_0^\infty \frac{\mathrm{d}\Phi(t+a)}{\mathrm{d}t}\frac{\Pi_i(t+a)}{\Pi_i(a)}\int_\Omega \Gamma_2(t,x,y)\phi_2(a,y)\mathrm{d}y\mathrm{d}a \\
&\quad + \int_0^\infty \Phi(t+a)\frac{\Pi_i(t+a)}{\Pi_i(a)}\int_\Omega \frac{\partial \Gamma_2(t,x,y)}{\partial t}\phi_2(a,y)\mathrm{d}y\mathrm{d}a \\
&\quad - \int_0^\infty [\mu_i + \theta(t+a)]\Phi(t+a)\frac{\Pi_i(t+a)}{\Pi_i(a)}\int_\Omega \Gamma_2(t,x,y)\phi_2(a,y)\mathrm{d}y\mathrm{d}a \\
&= \Phi(0)\mathbf{u}_1(t,x) + \int_0^\infty \left[\frac{\mathrm{d}\Phi(a)}{\mathrm{d}a} - [\mu_i + \theta(a) - d_2\Delta]\Phi(a)\right] i(t,a,x)\mathrm{d}a.
\end{aligned} \tag{6.3}$$

Similarly,

$$V_p(t,x) = \int_0^\infty \Psi(t+b)\frac{\Pi_p(t+b)}{\Pi_p(b)}\int_\Omega \Gamma_3(t,x,y)\phi_3(b,y)\mathrm{d}y\mathrm{d}b + \int_0^t \Psi(t-b)\Pi_p(t-b)\int_\Omega \Gamma_3(t-b,x,y)\mathbf{u}_2(b,y)\mathrm{d}y\mathrm{d}b.$$

and

$$\frac{\partial V_p(t,x)}{\partial t} = \Psi(0)\int_0^\infty \xi(a) i(t,a,x)\mathrm{d}a + \int_0^\infty \left[\frac{\mathrm{d}\Psi(b)}{\mathrm{d}b} - [\mu_p + \delta(b) - d_3\Delta]\Psi(b)\right] p(t,b,x)\mathrm{d}b. \tag{6.4}$$

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Hence,

$$\frac{\partial L_{E_0}(t)}{\partial t} = -d_1\int_\Omega \frac{|\nabla S|^2}{S^2}\mathrm{d}x - \int_\Omega \frac{\mu_S}{S}(S-S^0)^2\mathrm{d}x - \int_\Omega (1-\Phi(0))u_1(t,x)\mathrm{d}x$$
$$+\int_\Omega\int_0^\infty \Big(S^0\beta_1(a)+\Phi'(a)+\Psi(0)\xi(a)-[\mu_i+\theta(a)-d_2\Delta]\Phi(a)\Big)i(t,a,x)\mathrm{d}a\mathrm{d}x$$
$$+\int_\Omega\int_0^\infty \Big(S^0\beta_2(b)+\Psi'(b)-[\mu_p+\delta(b)-d_3\Delta]\Psi(b)\Big)p(t,b,x)\mathrm{d}b\mathrm{d}x. \tag{6.5}$$

By (6.5), we define

$$\begin{cases} \Psi(b)=\dfrac{1}{\Pi_p(b)}\displaystyle\int_b^\infty S_0\beta_2(\varsigma)\Pi_p(\varsigma)\mathrm{d}\varsigma, \\ \Phi(a)=\dfrac{1}{\Pi_i(a)}\displaystyle\int_a^\infty [S_0\beta_1(\vartheta)+\Psi(0)\xi(\vartheta)]\Pi_i(\vartheta)\mathrm{d}\vartheta. \end{cases}$$

Obviously, $\Psi(b)$ and $\Phi(a)$ satisfy

$$\begin{cases} S_0\beta_2(b)+\dfrac{\mathrm{d}\Psi(b)}{\mathrm{d}b}-[\mu_p+\delta(b)]\Psi(b)=0, \\ \Psi(0)=S_0\mathbb{Q}, \\ S_0\beta_1(a)+\dfrac{\mathrm{d}\Phi(a)}{\mathrm{d}a}+\Psi(0)\xi(a)-[\mu_i+\theta(a)]\Phi(a)=0, \\ \Phi(0)=S_0\mathbb{K}+S_0\mathbb{QL}=\Re_0, \end{cases}$$

where $\mathbb{K}$, $\mathbb{Q}$ and $\mathbb{L}$ are defined in (3.9). Consequently, we have

$$\frac{\partial L_{E_0}(t)}{\partial t} = -d_1\int_\Omega \frac{|\nabla S|^2}{S^2}x - \int_\Omega \frac{\mu_S}{S}(S-S^0)^2\mathrm{d}x - \int_\Omega (1-\Re_0)\mathbf{u}_1(t,x)\mathrm{d}x \le 0 \ \text{ if } \Re_0\le 1.$$

Hence, $\{E_0\}$ is the largest invariant set such that $\frac{\partial L_{E_0}(t)}{\partial t}=0$ and it follows from invariance principle [48] that E_0 is globally attractive. □

Theorem 6.2 *E^* is GAS provided that $\Re_0>1$.*

Proof Let $\Theta(a)$ and $\Upsilon(b)$ be some functions to be determined later. Define

$$L_{E^*}(t)=\int_\Omega [\overline{V}_S(t,x)+\overline{V}_i(t,x)+\overline{V}_p(t,x)]\mathrm{d}x,$$

where

$$\overline{V}_S=G[S,S^*],\ \overline{V}_i=\int_0^\infty \Theta(a)G[i(t,a,x),i^*(a)]\mathrm{d}a,\ \overline{V}_p=\int_0^\infty \Upsilon(a)G[p(t,b,x),p^*(b)]\mathrm{d}b,$$

By direct calculation, one has that

$$\frac{\partial \overline{V}_S}{\partial t}=d_1\frac{S-S^*}{S}\Delta S-\frac{\mu_S}{S}(S-S^*)^2+i(t,0,x)\frac{S^*}{S}+i^*(0)-i(t,0,x)-i^*(0)\frac{S^*}{S}. \tag{6.6}$$

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Note that

$$\overline{V}_i = \int_0^t \Theta(t-a)G[m_1,n_1]\mathrm{d}y\mathrm{d}a + \int_0^\infty \Theta(t+a)G[m_2,n_2]\mathrm{d}y\mathrm{d}a.$$

where

$$m_1 = \Pi_i(t-a)\int_\Omega \Gamma_2(t-a,x,y)i(a,0,y)\mathrm{d}y,\ n_1 = i^*(t-a),$$

and

$$m_2 = \frac{\Pi_i(t+a)}{\Pi_i(a)}\int_\Omega \Gamma_2(t,x,y)\phi_2(a,y)\mathrm{d}y,\ n_2 = i^*(t+a).$$

Recalling that $i^*(a) = i^*(0)\Pi_i(a)$ and $\Pi_i(0) = 1$, we have

$$\begin{aligned}\frac{\partial \overline{V}_i(t,x)}{\partial t} &= \Theta(0)G\left[\int_\Omega \Gamma_2(0,x,y)i(a,0,y)\mathrm{d}y, i^*(0)\right] + \int_0^t \frac{\mathrm{d}\Theta(t-a)}{\mathrm{d}t}G[m_1,n_1]\mathrm{d}a\\
&+\int_0^\infty \frac{\mathrm{d}\Theta(t+a)}{\mathrm{d}t}G[m_2,n_2]\mathrm{d}a\\
&+\int_0^t \Theta(t-a)\left\{\left[\Pi(t-a)\int_\Omega \frac{\partial\Gamma_2(t-a,x,y)}{\partial t}i(a,0,y)\mathrm{d}y\right.\right.\\
&\left.\left.-[\mu_i+\theta(t-a)]m_1\right]\frac{\partial G[m_1,n_1]}{\partial m_1} - [\mu_i+\theta(t-a)]i^*(t-a)\frac{\partial G[m_1,n_1]}{\partial n_1}\right\}\mathrm{d}a\\
&+\int_0^\infty \Theta(t+a)\left\{\left[\frac{\Pi_i(t+a)}{\Pi_i(a)}\int_\Omega \frac{\partial\Gamma_2(t,x,y)}{\partial t}\phi_2(a,y)\mathrm{d}y\right.\right.\\
&\left.\left.-[\mu_i+\theta(t+a)]m_2\right]\frac{\partial G[m_2,n_2]}{\partial m_2} - [\mu_i+\theta(t+a)]i^*(t+a)\frac{\partial G[m_2,n_2]}{\partial n_2}\right\}\mathrm{d}a.\end{aligned}$$

It follows from $m\frac{\partial G[m,n]}{\partial m} + n\frac{\partial G[m,n]}{\partial n} = G[m,n]$ that

$$\begin{aligned}\frac{\partial \overline{V}_i(t,x)}{\partial t} =& \Theta(0)G\left[\int_\Omega \Gamma_2(0,x,y)i(a,0,y)\mathrm{d}y, i^*(0)\right]\\
&+\int_0^\infty \left[\Theta'(a) - [\mu_i+\theta(t-a)]\Theta(a)\right]G[i(t,a,x),i^*(a)]\mathrm{d}a\\
&+\int_0^t \Theta(t-a)\left[\Pi(t-a)\int_\Omega \frac{\partial\Gamma_2(t-a,x,y)}{\partial t}i(a,0,y)\mathrm{d}y\right]\frac{\partial G[m_1,n_1]}{\partial m_1}\mathrm{d}a\\
&+\int_0^\infty \Theta(t+a)\left[\frac{\Pi_i(t+a)}{\Pi_i(a)}\int_\Omega \frac{\partial\Gamma_2(t,x,y)}{\partial t}\phi(a,y)\mathrm{d}y\right]\frac{\partial G[m_2,n_2]}{\partial n_2}\mathrm{d}a.\end{aligned}$$

Define the semigroup $(T(0)[\phi])(x) = \int_\Omega \Gamma_2(0,x,y)\phi(y)\mathrm{d}y$ as the unit semigroup and note that $\frac{\partial\Gamma_2}{\partial t} = d_2\Delta\Gamma_2$, $\frac{\partial G[m,n]}{m} = 1 - \frac{n}{m}$. One has that

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$$\begin{aligned}\frac{\partial \overline{V}_i(t,x)}{\partial t} =&\Theta(0)G[i(t,0,x),i^*(0)]+\int_0^\infty \Theta(a)d_2\Delta i(t,a,x)\left[1-\frac{i^*(a)}{i(t,a,x)}\right]\mathrm{d}a\\&+\int_0^\infty\left[\frac{\mathrm{d}\Theta(a)}{\mathrm{d}a}-[\mu_i+\theta(a)]\Theta(a)\right]G[i(t,a,x),i^*(a)]\mathrm{d}a.\end{aligned}\tag{6.7}$$

Similarly,

$$\begin{aligned}\overline{V}_p=&\int_0^t \Upsilon(t-b)G\Big[\Pi_p(t-b)\int_\Omega \Gamma_3(t-b,x,y)p(a,0,y)\mathrm{d}y,p^*(t-b)\Big]\mathrm{d}b\\&+\int_0^\infty \Upsilon(t+b)G\Big[\frac{\Pi_p(t+b)}{\Pi_p(b)}\int_\Omega \Gamma_3(t,x,y)\phi_3(b,y)\mathrm{d}y,p^*(t+b)\Big]\mathrm{d}b.\end{aligned}$$

Hence,

$$\begin{aligned}\frac{\partial \overline{V}_p(t,x)}{\partial t} =&\Upsilon(0)G[p(t,0,x),p^*(0)]+\int_0^\infty \Upsilon(b)d_3\Delta p(t,b,x)\left[1-\frac{p^*(b)}{p(t,b,x)}\right]\mathrm{d}b\\&+\int_0^\infty\left[\frac{\mathrm{d}\Upsilon(b)}{\mathrm{d}b}-[\mu_p+\delta(b)-d_3\Delta]\Upsilon(b)\right]G[p(t,b,x),p^*(b)]\mathrm{d}b.\end{aligned}\tag{6.8}$$

Further, we let

$$\begin{cases}\Upsilon(b)=\dfrac{1}{\Pi_p(b)}\displaystyle\int_b^\infty S^*\beta_2(\varsigma)\Pi_p(\varsigma)\mathrm{d}\varsigma,\\ \Theta(a)=\dfrac{1}{\Pi_i(a)}\displaystyle\int_a^\infty [S^*\beta_1(\vartheta)+\Psi(0)\xi(\vartheta)]\Pi_i(\vartheta)\mathrm{d}\vartheta.\end{cases}\tag{6.9}$$

It then follows that

$$\begin{cases}\Upsilon(b)[\mu_p+\delta(b)]-\Upsilon'(b)=S^*\beta_2(b),\\ \Theta(a)[\mu_i+\theta(a)]-\Theta'(a)=S^*\beta_1(a)+\Upsilon(0)\xi(a),\\ \Upsilon(0)=\displaystyle\int_0^\infty S^*\beta_2(\varsigma)\Pi_p(\varsigma)\mathrm{d}\varsigma=S^*\mathbb{Q},\\ \Theta(0)=\displaystyle\int_0^\infty [S^*\beta_1(\vartheta)+\Upsilon(0)\xi(\vartheta)]\Pi_i(\vartheta)\mathrm{d}\vartheta=S^*\mathbb{K}+S^*\mathbb{Q}\mathbb{L}=1,\end{cases}\tag{6.10}$$

where $\mathbb{K}$, $\mathbb{Q}$ and $\mathbb{L}$ are defined in (3.9). Hence, we have

$$\begin{aligned}\frac{\partial \overline{V}_i(t,x)}{\partial t} =& G[i(t,0,x),i^*(0)]+\int_0^\infty \Theta(a)d_2\Delta i(t,a,x)\left[1-\frac{i^*(a)}{i(t,a,x)}\right]\mathrm{d}a\\&-\int_0^\infty [S^*\beta_1(a)+S^*\mathbb{Q}\xi(a)]G[i(t,a,x),i^*(a)]\mathrm{d}a,\end{aligned}\tag{6.11}$$

and

$$\begin{aligned}\frac{\partial \overline{V}_p(t,x)}{\partial t} =& S^*\mathbb{Q}G[p(t,0,x),p^*(0)]+\int_0^\infty \Upsilon(b)d_3\Delta p(t,b,x)\left[1-\frac{p^*(b)}{p(t,b,x)}\right]\mathrm{d}b\\&-\int_0^\infty S^*\beta_2(b)G[p(t,b,x),p^*(b)]\mathrm{d}b.\end{aligned}\tag{6.12}$$

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Let

$$\hat{W}(t,x) = \overline{V}_S + \overline{V}_i + \overline{V}_p.$$

Together with (6.6), (6.11) and (6.12), which implies that

$$\begin{aligned}\frac{\partial \hat{W}(t,x)}{\partial t} = W_0 - i^*(0)\frac{S^*}{S} + i(t,0,x)\frac{S^*}{S} + \int_0^\infty [S^*\beta_1(a) - S^*\mathbb{Q}\xi(a)]G[i(t,a,x), i^*(a)]\mathrm{d}a \\ + S^*\mathbb{Q}G[p(t,0,x), p^*(0)] - i^*(0)\ln\frac{i(t,0,x)}{i^*(0)} - \int_0^\infty S^*\beta_2(b)G[p(t,b,x), p^*(b)]\mathrm{d}b,\end{aligned} \tag{6.13}$$

where

$$\begin{aligned}W_0 = d_1\frac{S-S^*}{S}\Delta S - \frac{\mu_S}{S}(S-S^*)^2 + \int_0^\infty \Theta(a)d_2\Delta i(t,a,x)\left[1 - \frac{i^*(a)}{i(t,a,x)}\right]\mathrm{d}a \\ + \int_0^\infty \Upsilon(b)d_3\Delta p(t,b,x)\left[1 - \frac{p^*(b)}{p(t,b,x)}\right]\mathrm{d}b.\end{aligned}$$

With the help of the third and fifth equations of (1.5) and using the equilibrium condition, one has that

$$\begin{aligned}\frac{\partial \hat{W}(t,x)}{\partial t} =& W_0 + S^*\int_0^\infty \beta_1(a)i^*(a)\left[1 - \frac{S^*}{S} + \ln\frac{i(t,a,x)}{i^*(a)} - \ln\frac{i(t,0,x)}{i^*(0)}\right]\mathrm{d}a \\ &+ S^*\int_0^\infty \beta_2(b)p^*(b)\left[1 - \frac{S^*}{S} + \ln\frac{p(t,b,x)}{p^*(b)} - \ln\frac{i(t,0,x)}{i^*(0)}\right]\mathrm{d}b \\ &+ S^*\mathbb{Q}\int_0^\infty \xi(a)i^*(a)\left[\ln\frac{i(t,a,x)}{i^*(a)} - \ln\frac{p(t,0,x)}{p^*(0)}\right]\mathrm{d}a.\end{aligned} \tag{6.14}$$

By simple calculation, we can obtain the following zero tricks:

$$S^*\mathbb{Q}\int_0^\infty \xi(a)i^*(a)\left[1 - \frac{i(t,a,x)p^*(0)}{i^*(a)p(t,0,x)}\right]\mathrm{d}a = 0,$$

and

$$S^*\int_0^\infty \beta_1(a)i^*(a)\left[1 - \frac{Si^*(0)i(t,a,x)}{S^*i(t,0,x)i^*(a)}\right]\mathrm{d}a + S^*\int_0^\infty \beta_2(b)p^*(b)\left[1 - \frac{Si^*(0)p(t,b,x)}{i(t,0,x)p^*(b)}\right]\mathrm{d}b = 0,$$

Then, we can rewrite (6.14) as:

$$\begin{aligned}\frac{\partial \hat{W}(t,x)}{\partial t} =& W_0 + S^*\int_0^\infty \beta_1(a)i^*(a)\left[1 - \frac{S^*}{S} + \ln\frac{S^*}{S} + 1 - \frac{Si^*(0)i(t,a,x)}{S^*i(t,0,x)i^*(a)} + \ln\frac{Si^*(0)i(t,a,x)}{S^*i(t,0,x)i^*(a)}\right]\mathrm{d}a \\ &+ S^*\int_0^\infty \beta_2(b)p^*(b)\left[1 - \frac{S^*}{S} + \ln\frac{S^*}{S} + 1 - \frac{Si^*(0)p(t,b,x)}{S^*i(t,0,x)p^*(b)} + \ln\frac{Si^*(0)p(t,b,x)}{S^*i(t,0,x)p^*(b)}\right]\mathrm{d}b \\ &+ S^*\mathbb{Q}\int_0^\infty \xi(a)i^*(a)\left[1 - \frac{i(t,a,x)p^*(0)}{i^*(a)p(t,0,x)} + \ln\frac{i(t,a,x)p^*(0)}{i^*(a)p(t,0,x)}\right]\mathrm{d}a.\end{aligned} \tag{6.15}$$

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Consequently, we integrate (6.15) over Ω to get

$$\begin{aligned}\frac{\mathrm{d}L_{E^*}(t)}{\mathrm{d}t} = & -d_S S^* \int_\Omega \frac{|\nabla S(t,x)|^2}{S^2(t,x)}\mathrm{d}x - \int_\Omega \int_0^\infty \Theta(a) d_2 i^*(a) \frac{|\nabla i(t,a,x)|^2}{i^2(t,a,x)}\mathrm{d}a\mathrm{d}x \\ & - \int_\Omega \int_0^\infty \Upsilon(b) d_3 p^*(b) \frac{|\nabla p(t,b,x)|^2}{p^2(t,b,x)}\mathrm{d}b\mathrm{d}x \\ & - \int_\Omega \frac{\mu_S}{S}(S-S^*)^2 \mathrm{d}x + S^* \int_\Omega \int_0^\infty \beta_1(a) i^*(a)\left[g\left(\frac{S^*}{S}\right) + g\left(\frac{Si^*(0)i(t,a,x)}{S^* i(t,0,x) i^*(a)}\right)\right]\mathrm{d}a\mathrm{d}x \\ & + S^* \int_\Omega \int_0^\infty \beta_2(b) p^*(b)\left[g\left(\frac{Si^*(0)p(t,b,x)}{S^* i(t,0,x) p^*(b)}\right) + g\left(\frac{S^*}{S}\right)\right]\mathrm{d}b\mathrm{d}x \\ & + S^* \mathbb{Q} \int_\Omega \int_0^\infty \xi(a) i^*(a)\left[g\left(\frac{i(t,a,x)p^*(0)}{i^*(a)p(t,0,x)}\right)\right]\mathrm{d}a\mathrm{d}x \\ \leq & \ 0,\end{aligned}$$

where $g(s) = 1 - s + \ln s$, $s \in \mathbb{R}_+$ possesses the properties that $g(s) \leq 0$ when $s > 0$ and $g(s)$ reaches the global minimum 0 at $s = 1$. It can be verified that $\{E^*\}$ is the largest invariant set such that $L'_{E^*}(t) = 0$. From [48], we finish the proof. □

7 Conclusion

This paper focus on the dynamics of the cholera model with infection age-space structure. By using the reaction–diffusion model formulation, the mobility of human populations, the spatial dispersal of vibrios and the infection-age structure of vibrio cholerae and infectious individuals are incorporated into the model to describe the vibrio cholerae transmission in a general domain. Our main task is studying the dynamics for the model. Thanks to the Banach–Picard fixed point theorem, we are able to show the existence and uniqueness of solution for the model on $t \in [0, t_{max}) \times \overline{\Omega}$ with $t_{max} < \infty$. The positivity and boundedness for such solution are confirmed by way of contradiction. Hence, the model in the current paper has a unique global non-negative classical solution in $C([0, +\infty), \mathbb{X})$ (see Theorem 2.1). We introduce the BRN $\Re_0$ by the theory developed in [18], and we found that the spectral radius of NGO $\mathscr{K}$ is $\Re_0$. Further, with the help of Ascoli–Arzelá theorem, we have verified that $\mathscr{K}$ is strictly positive and compact (see Lemma 4), which allows us to use Krein–Rutman theorem to get the explicit expression of $\Re_0$ (see (3.8)). Our model (2.3) with (2.6) and (2.7) possesses two space-independent equilibria: disease-free equilibrium E_0 and endemic equilibrium E^*.

Our results imply that, despite the introduction of the spatial dispersal of vibrios and the mobility of human populations, $\Re_0$ still is the sharp threshold for cholera dynamics: (i) if $\Re_0 < 1$, then E_0 is LAS (see (i) of Theorem 4.1); (ii) If $\Re_0 > 1$, then E^* is LAS (see (ii) of Theorem 4.1); (iii) If $\Re_0 > 1$, strong $|\cdot|_{\mathbb{X}}$-persistence is confirmed; (iv) If $\Re_0 < 1$, then E_0 is GAS (see Theorem 6.1); (v) If $\Re_0 > 1$, then E^* is LAS (see Theorem 6.2). We proved the (i) and (ii) by checking the distribution of roots of characteristic equations. (iii) is verified based on the weak $|\cdot|_{\mathbb{X}}$-persistence of the disease and the way of contradiction. We proved (iv) and (v) by constructing Lyapunov functions. Biologically, (i) and (iv) imply that the disease dies out, and (ii), (iii) and (v) imply that disease will persist in the long time. In summary, $\Re_0$ plays the role in determining the dichotomy of disease persistence and extinction.

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Conflict of interest

None.

References

[1] ALI, M., NELSON, A. R., LOPEZ, A. L. & SACK, D. A. (2015) Updated global burden of cholera in endemic countries. *PLoS Negl. Trop. Dis.* **9**, e0003832.

[2] ADIMY, M., CHEKROUN, A. & KUNIYA, T. (2017) Delayed nonlocal reaction diffusion model for hematopoietic stem cell dynamics with Dirichlet boundary conditions. *Math. Model Nat. Phenom.* **12**, 1–22.

[3] AMANN, H. (1976) Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces. *SIAM Rev.* **18**, 620–709.

[4] ALAM, A., LAROCQUE, R. C., HARRIS, J. B., VANDERSPURT, C., RYAN, E. T., QADRI, F. & CALDERWOOD, S. B. (2005) Hyperinfectivity of human-passaged Vibrio cholerae can be modelled by growth in the infant mouse. *Infect. Immun.* **73**, 6674–6679.

[5] BRAUER, F., SHUAI, Z. & VAN DEN DRIESSCHE P. (2013) Dynamics of an age-of-infection cholera model. *Math. Biosci. Eng.* **10**, 1335–1349.

[6] BERTUZZO, E., MARITAN, A., GATTO, M., RODRIGUEZ-ITURBE, I. & RINALDO, A. (2007) River networks and ecological corridors: reactive transport on fractals, migration fronts, hydrochory. *Water Resour. Res.* **43**, W04419.

[7] CANTRELL, R. S. & COSNER, C. (2003) *Spatial Ecology via Reaction-Diffusion Equations*. Wiley Series in Mathematical and Computational Biology. John Wiley & Sons, Chichester.

[8] CHEKROUN, A. & KUNIYA, T. (2020) Global threshold dynamics of an infection age-structured SIR epidemic model with diffusion under the Dirichlet boundary condition. *J. Differential Equations* **269**, 117–148.

[9] CHEKROUN, A. & KUNIYA, T. (2020) An infection age-space structured SIR epidemic model with neumann boundary condition. *Appl. Anal.* **99**, 1972–1985.

[10] CHAO, D., LONGINI, I. M. & MORRIS, J. G. (2003) Modeling cholera outbreaks. In: *Cholera Outbreaks*. Current Topics in Microbiology and Immunology, 379. Springer, Berlin.

[11] CAPASSO, V. & PAVERI-FONTANA, S. L. (1979) A mathematical model for the 1973 cholera epidemic in the European Mediterranean region. *Rev. Epidemiol. Sante.* **27**, 121–132.

[12] CAPONE, F., DE CATALDIS, V. & DE LUCA, R. (2015) Influence of diffusion on the stability of equilibria in a reaction-diffusion system modeling cholera dynamic. *J. Math. Biol.* **27**, 1107–1131.

[13] CODEÇO, C. T. (2001) Endemic and epidemic dynamics of cholera: The role of the aquatic reservoir. *BMC Infect. Dis.* **1**, 1.

[14] CARFORA, M. F. & TORCICOLLO, I. (2020) Identification of epidemiological models: the case study of Yemen cholera outbreak. *Appl. Anal.* DOI: 10.1080/00036811.2020.1738402.

[15] D'AGATA, E. M. C., MAGAL, P., RUAN, S. & WEBB, G. F. (2006) Asymptotic behavior in nosocomial epidemic models with antibiotic resistance. *Differ. Int. Equ.* **19**, 573–600.

[16] DUCROT, A. & MAGAL, P. (2009) Travelling wave solutions for an infection-age structured model with diffusion. *Proc. R. Soc. Edinb.* **139**, 459–482.

[17] DUCROT, A. & MAGAL, P. (2011) Travelling wave solutions for an infection-age structured epidemic model with external supplies. *Nonlinearity* **24**, 2891–2911.

[18] DIEKMANN, O., HEESTERBEEK, J. A. P. & METZ, J. A. J. (1990) On the definition and the computation of the basic reproduction ratio R_0 in models for infectious diseases in heterogeneous populations. *J. Math. Biol.* **28**, 365–382.

[19] EISENBERG, M. C., SHUAI, Z., TIEN, J.H. & VAN DEN DRIESSCHE, P. (2013) A cholera model in a patchy environment with water and human movement. *Math. Biosci.* **246**, 105–112.

[20] FREEDMAN, H.I. & MOSON, P. (1990) Persistence definitions and their connections. *Proc. Am. Math. Soc.* **109**, 1025–1033.

[21] HARTLEY, D. M., MORRIS, J. G. & SMITH, D. L. (2006) Hyperinfectivity: A critical element in the ability of V. cholerae to cause epidemics? *PLoS Med.* **3**, 63–69.

[22] KOKOMO, E. & EMVUDU, Y. (2019) Mathematical analysis and numerical simulation of an age-structured model of cholera with vaccination and demographic movements. *Nonlinear Anal.-Real* **45**, 142–156.

[23] MUKANDAVIRE, Z., LIAO, S., WANG, J., GAFF, H., SMITH, D. L. & MORRIS, J. G. (2019) Estimating the reproductive numbers for the 2008–2009 cholera outbreaks in Zimbabwe. *Proc. Natl. Acad. Sci. USA* **108**, 8767–8772.

[24] MAGAL, P., WEBB, G. F. & WU, Y. (2019) On the basic reproduction number of reaction-diffusion epidemic models. *SIAM J. Appl. Math.* **79**, 284–304.

[25] MENDELSOHN, J. & DAWSON, T. (2008) Climate and cholera in KwaZulu-Natal, South Africa: The role of environmental factors and implications for epidemic preparedness. *Int. J. Hyg. Environ. Health* **211**, 156–162.

[26] NELSON, E. J., HARRIS, J. B., MORRIS, J. G., CALDERWOOD, S. B. & CAMILLI, A. (2009) Cholera transmission: The host, pathogen and bacteriophage dynamics. *Nat. Rev. Microbiol.* **7**, 693–702.

[27] RINALDO, A., BERTUZZO, E., MARI, L., RIGHETTO, L., BLOKESCH, M., GATTO, M., CASAGRANDI, R., MURRAY, M., VESENBECKH, S.M. & RODRIGUEZ-ITURBE, I. (2012) Reassessment of the 2010–2011 Haiti cholera outbreak and rainfall-driven multiseason projections. *Proc. Natl. Acad. Sci. USA* **109**, 6602–6607.

[28] RUDIN, W. (1976) *Principles of Mathematical Analysis* (3rd edn). International Series in Pure and Applied Mathematics. McGraw-Hill, New York.

[29] RUDIN, W. (1987) *Real and Complex Analysis* (3rd edn). McGraw-Hill, New York.

[30] SMITH, H. L. & THIEME, H. R. (2011) *Dynamical Systems and Population Persistence*. Graduate Studies in Mathematics, 118. Providence. American Mathematical Society.

[31] SHUAI, Z., TIEN, J. H. & VAN DEN DRIESSCHE, P. (2012) Cholera models with hyperinfectivity and temporary immunity. *Bull. Math. Biol.* **74**, 2423–2445.

[32] SHUAI, Z. & VAN DEN DRIESSCHE, P. (2011) Global dynamics of cholera models with differential infectivity. *Math. Biosci.* **234**, 118–126.

[33] SHUAI, Z. & VAN DEN DRIESSCHE, P. (2015) Modeling and control of cholera on networks with a common water source. *J. Biol. Dyn.* **9**, 90–103.

[34] TUITE, A. R., TIEN, J. H., EISENBERG, M., EARN, D. J. D., MA, J. & FISMAN, D. N. (2011) Cholera epidemic in Haiti, 2010: Using a transmission model to explain spatial spread of disease and identify optimal control interventions. *Ann. Internal. Med.* **154**, 593–601.

[35] THIEME, H. R. & CASTILLO-CHAVEZ, C. (1993) How may infection-age-dependent infectivity affect the dynamics of HIV/AIDS? *SIAM. J. Appl. Math.* **53**, 1447–1479.

[36] TIEN, J. H. & EARN, D. J. D. (2010) Multiple transmission pathways and disease dynamics in a waterborne pathogen model. *Bull. Math. Biol.* **72**, 1506–1533.

[37] THIEME, H. R. (2009) Spectral bound and reproduction number for infinite-dimensional population structure and time heterogeneity. *SIAM J. Appl. Math.* **70**, 188–211.

[38] VAN DEN DRIESSCHE, P. & WATMOUGH, J. (2002) Reproduction numbers and sub-threshold endemic equilibria for compartmental models of disease transmission. *Math. Biosci.* **180**, 29–48.

[39] WANG, J., ZHANG, R. & KUNIYA, T. (2016) A note on dynamics of an age-of-infection cholera model. *Math. Biosci. Eng.* **13**, 227–247.

[40] WANG, J., XIE, F. & KUNIYA, T. (2020) Analysis of a reaction-diffusion cholera epidemic model in a spatially heterogeneous environment. *Commun. Nonlinear Sci. Numer. Simulat.* **80**, 104951.

[41] WANG, J. & WANG, J. (2020) Analysis of a reaction-diffusion cholera model with distinct dispersal rates in the human population. *J. Dyn. Diff. Equat.* Doi: 10.1007/s10884-019-09820-8.

[42] WANG, X., ZHAO, X.-Q. & WANG, J. (2018) A cholera epidemic model in a spatiotemporally heterogeneous environment. *J. Math. Anal. Appl.* **468**, 893–912.

[43] WANG, X., POSNY, P. & WANG, J. (2016) A reaction-convection-diffusion model for Cholera spatial dynamics. *Discrete Contin. Dyn. Syst. Ser. B* **21**, 2785–2809.

[44] WANG, X. & WANG, J. (2015) Analysis of cholera epidemics with bacterial growth and spatial movement. *J. Math. Anal. Appl.* **9**, 233–261.

[45] WANG, X., GAO, D. & WANG, J. (2015) Influence of human behavior on cholera dynamics. *Math. Biosci.* **267**, 41–52.

[46] WANG, X. & WANG, J. (2015) Analysis of cholera epidemics with bacterial growth and spatial movement. *J. Biol. Dyn.* **9**, 233–261.

[47] WANG, J. & LIAO, S. (2012) A generalized cholera model and epidemic/endemic analysis. *J. Biol. Dyn.* **6**, 568–589.

[48] WALKER, J. A. (1980) *Dynamical Systems and Evolution Equations: Theory and Applications*. Mathematical Concepts and Methods in Science and Engineering, 20. Plenum Press, New York.

[49] YAMAZAKI, K. & WANG, X. (2016) Global well-posedness and asymptotic behavior of solutions to a reaction-convection-diffusion cholera epidemic model. *Discrete Contin. Dyn. Syst. Ser. B* **21**, 1297–1316.

[50] YAMAZAKI, K. & WANG, X. (2017) Global stability and uniform persistence of the reaction-convection-diffusion cholera epidemic model. *Math. Biosci. Eng.* **14**, 559–579.

[51] YANG, J., QIU, Z. & LI, X. (2014) Global stability of an age-structured cholera model. *Math. Biosci. Eng.* **11**, 641–665.

[52] YANG, J., XU, R. & LI, J. (2019) Threshold dynamics of an age-space structured brucellosis disease model with Neumann boundary condition. *Nonlinear Anal.-Real* **50**, 192–217.

[53] ZHANG, L., WANG, Z. & ZHANG, Y. (2016) Dynamics of a reaction-diffusion waterborne pathogen model with direct and indirect transmission. *Comput. Math. Appl.* **72**, 202–215.

[54] ZHANG, X. & ZHANG, Y. (2018) Spatial dynamics of a reaction-diffusion cholera model with spatial heterogeneity. *Discrete Contin. Dyn. Syst. Ser. B* **23**, 2625–2640.

[55] ZHANG, L. & WANG, Z. (2016) A time-periodic reaction-diffusion epidemic model with infection period. *Z. Angew. Math. Phys.* **67**, 117.