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BOUNDARY CROSSING PROBABILITIES FOR HIGH-DIMENSIONAL BROWNIAN MOTION

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Abstract

The two-sided nonlinear boundary crossing probabilities for one-dimensional Brownian motion and related processes have been studied in Fu and Wu (2010) based on the finite Markov chain imbedding technique. It provides an efficient numerical method to computing the boundary crossing probabilities. In this paper we extend the above results for high-dimensional Brownian motion. In particular, we obtain the rate of convergence for high-dimensional boundary crossing probabilities. Numerical results are also provided to illustrate our results.

Keywords: Boundary crossing probability; finite Markov chain imbedding; Brownian motion; Y-type time tunnel; irregular boundary; two-dimensional; high-dimensional; rate of convergence

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1. Introduction

Several fundamental theorems for the boundary crossing probabilities (BCPs) or the first passage time distributions have been established in, for example, [1], [6], and [12]. Erdős and Kac [6] derived the invariant theorem for constant boundaries. Robbins and Siegmund [12] and Anderson [1] obtained the BCPs for one-sided and two-sided linear boundaries, respectively. For piecewise linear boundaries, BCPs have been studied by Robbins and Siegmund [12] and Scheike [14]. Approximations for nonlinear BCPs can be found in [3], [4], [5], [9], [11], and [13].

In contrast to the above mentioned results for one-dimensional BCPs, only limited results are known for high-dimensional BCPs; see, for example, [2], [15], and [17]. For the two-dimensional case, Iyengar [8] and Metzler [10] derived the first passage time distribution as an infinite series associated with the modified Bessel function of the first kind.

Recently, Fu and Wu [7] developed a computationally efficient method for BCPs based on the finite Markov chain imbedding (FMCI) technique. In this paper we extend the approach in [7] to high-dimensional Brownian motion. The reason for adopting the method are four fold: conceptually simple, flexible toward the boundaries, easy to program, and efficient in computation. The paper is organized in the following way. In Section 2, notation and results in [7] are reviewed, including the approximate BCPs for one-dimensional Brownian motion and their convergence rates. In Section 3, two-dimensional and higher-dimensional Brownian

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motions are considered and their BCPs are studied. The rate of convergence is derived in Section 4. Numerical results, examples, and discussions are given in Section 5.

2. Preliminary results: one-dimensional Brownian motion

Throughout this paper, we denote the lower and upper boundaries by $a(t)$ and $b(t)$, respectively. We assume that $a(t)$ and $b(t)$ satisfy the following conditions:

(A) $a(t)$ and $b(t)$ are continuous for $t \in [0, 1]$ except finite k ($k = 0, 1, \ldots$) discontinuous points of the first kind $0 = t_0^* < t_1^* < \cdots < t_k^* < t_{k+1}^* = 1$,

(B) $a(0) < 0 < b(0)$, and

(C) there exists a constant K such that $|a(t + \varepsilon) - a(t)| < K\varepsilon$ and $|b(t + \varepsilon) - b(t)| < K\varepsilon$ for $t, t + \varepsilon \in (t_i^*, t_{i+1}^*)$, $i = 0, 1, \ldots, k$, except finite $t_1^* < \cdots < t_k^*$ points (the Lipschitz condition).

Without loss of generality, we also assume that the initial probability of the process $\mathbb{P}(W(0) = 0) = 1$. Let $h = \max(\sup_{0\le t\le 1} |a(t)|, \ \sup_{0\le t\le 1} |b(t)|)$.

Given large positive integers m and n, define $\Delta x = h/m$ and $\Delta t = 1/n$ in such a way that $\Delta t = \Delta x^2$, i.e. $n = m^2/h^2$, or $n = \lfloor m^2/h^2 \rfloor$ if n is not an integer. Let us define a family of discrete distributions $\mathcal{F} = \{f_\gamma\}$ on $\{0, \pm 1\Delta x, \pm 2\Delta x, \ldots\}$ as

$$f_\gamma(k\Delta x) = \begin{cases} \dfrac{C^{-1}}{k^\gamma\sqrt{2\pi}} \exp\Big(-\dfrac{k^2}{2}\Big) & \text{if } k \neq 0, \\ \dfrac{C^{-1}}{\sqrt{2\pi}} \displaystyle\sum_{\ell\neq 0} \Big(\dfrac{1}{\ell^{\gamma-2}} - \dfrac{1}{\ell^\gamma}\Big) \exp\Big(-\dfrac{\ell^2}{2}\Big) & \text{if } k = 0, \end{cases} \tag{1}$$

where $C = (1/\sqrt{2\pi})\sum_{\ell\neq 0}(1/\ell^{\gamma-2})\exp(-\ell^2/2)$ and γ is an even nonnegative integer. For any given t, n, and γ, we define

$$\hat{W}_n(t: \gamma) = \sum_{i=1}^{\lfloor nt \rfloor} \hat{X}_i(\gamma),$$

where $\hat{X}_i(\gamma)$ are independent and identically distributed (i.i.d.) random variables having common distribution $f_\gamma \in \mathcal{F}$. For given γ, $\{\hat{W}_n(t: \gamma)\}$ is a homogeneous Markov chain. For example, if $\gamma = 0$, it has transition probabilities

$$\begin{aligned} p(k \mid j) &= \mathbb{P}(\hat{W}_n(t + \Delta t: 0) = k\Delta x \mid \hat{W}_n(t: 0) = j\Delta x) \\ &= \begin{cases} \dfrac{C^{-1}}{\sqrt{2\pi}} \exp\Big(-\dfrac{(k-j)^2}{2}\Big) & \text{if } k - j \neq 0, \\ \dfrac{C^{-1}}{\sqrt{2\pi}} \displaystyle\sum_{\ell\neq 0}(\ell^2 - 1)\exp\Big(-\dfrac{\ell^2}{2}\Big) & \text{if } k - j = 0. \end{cases} \end{aligned}$$

The following theorem is cited directly from [7] with the minor modification of condition (A) by allowing finite discontinuous points of the first kind. We do not repeat the proof here.

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Theorem 1. *Given* $f_\gamma \in \mathcal{F}$, $\gamma = 0, 2, 4, \dots,$

(i) *we have* $\hat{W}_n(t\colon \gamma) \xrightarrow{\text{D}} W(t)$ *as* $n \to \infty$, *where* '$\xrightarrow{\text{D}}$' *stands for convergence in distribution,*

(ii) *the BCP can be approximated by*

$$\mathbb{P}(W(t) \le a(t) \text{ or } W(t) \ge b(t) \text{ for some } t \in [0, 1]) = 1 - \lim_{m\to\infty} \boldsymbol{\xi}_0 \bigg(\prod_{i=1}^{\lfloor m^2/h^2 \rfloor} \boldsymbol{N}_i(\gamma) \bigg) \mathbf{1}^\top,$$

where $\mathbf{1}^\top$ *is the transpose of the row vector* $\mathbf{1} = (1, \dots, 1)$, *and* $\boldsymbol{N}_i(\gamma)$, $i = 1, \dots, \lfloor m^2/h^2 \rfloor$ *are essential transition matrices of the imbedded Markov chain associated with discrete boundaries induced by* $a(t)$ *and* $b(t)$ *and can be constructed using (1), and*

(iii) *if the boundaries* $a(t)$ *and* $b(t)$ *satisfy the conditions (A), (B), and (C), then the error bounds are*

$$\bigg| \mathbb{P}\bigg(a^*\bigg(\frac{k}{n}\bigg) < \hat{W}_n(t_k\colon \gamma) < b^*\bigg(\frac{k}{n}\bigg),\ k = 1, \dots, n \bigg) - \mathbb{P}(a(t) < \hat{W}_n(t_k\colon \gamma) < b(t),\ t \in [0, 1]) \bigg| = \mathcal{O}\bigg(\frac{1}{m}\bigg),$$

and

$$\bigg| \mathbb{P}\bigg(a^*\bigg(\frac{k}{n}\bigg) < \hat{W}_n(t_k\colon \gamma) < b^*\bigg(\frac{k}{n}\bigg),\ k = 1, \dots, n \bigg) - \mathbb{P}(a(t) < W(t) < b(t), t \in [0, 1]) \bigg| = \mathcal{O}\bigg(\frac{1}{m}\bigg) \quad \text{as } n \to \infty,$$

where $n = \lfloor m^2/h^2 \rfloor$, $a^*(k/n)$, *and* $b^*(k/n)$ *are the boundaries for* $\hat{W}_n(t_k\colon \gamma)$.

Note that for $\gamma = 0$, f_γ is a discrete normal distribution and for $\gamma = \infty$, (1) reduces to a simple random walk moving one step in either the right or left with probability $\frac{1}{2}$. It is known that for $\gamma = 0$ the imbedded Markov chain $\hat{W}_n(t\colon 0)$ has the smallest error bound among all $\hat{W}_n(t\colon \gamma)$, $\gamma = 0, 2, 4, \dots$ (see [7, Table 1]). If there is no special specification, throughout this paper, we will use the notation $\hat{W}_n(t)$ for $\hat{W}_n(t\colon 0)$.

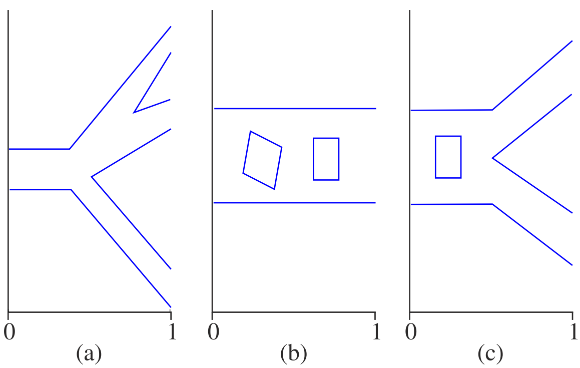

Figure 1: Y-type time tunnels.

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Basically, the above theorem shows that the BCPs of Brownian motion $W(t)$ are casted as limiting probabilities of the imbedded Markov chain $\hat{W}_n(t)$ staying in the absorbing state. One can see the simplicity and flexibility of the method toward the shape of the boundaries. Hence, we expect that the above results could be extended to Y-type time tunnels illustrated in Figure 1. Further using a one-to-one transformation, we also expect that the above results are able to extend Y-type time tunnels for certain diffusion processes satisfying $\mathrm{d}X(t) = b(t, X(t))\,\mathrm{d}t + \sigma(t, X(t))\,\mathrm{d}W(t)$, for example the Ornstein–Uhlenbeck (OU) processes and the Brownian bridge; see [7]. Numerical examples of BCPs of Y-type time tunnels for Brownian motion and the OU process will be provided to illustrate the theoretical results in Section 5.

3. High-dimensional Brownian motion

In this section we extend our results to the high-dimensional Brownian motion. If the high-dimensional Brownian motion is not a standard but correlated one, then it can be transformed into a standard one. We consider the two-dimensional Brownian motion first, and the higher-dimensional one would follow in a similar fashion.

Let $\{\boldsymbol{X}(t) = (X_1(t), X_2(t)),\ t \geq 0\}$ be a two-dimensional correlated Brownian motion with mean $t\boldsymbol{\mu}$ and covariance matrix $t\boldsymbol{\Sigma}$. It is well known that

(i) $\boldsymbol{W}(t) = (\boldsymbol{X}(t) - t\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1/2}$ is a standard two-dimensional Brownian motion, and

(ii) if $B(t)$ is a compact convex set then $\tilde{B}(t) = \{(\boldsymbol{b} - t\boldsymbol{\mu})\boldsymbol{\Sigma}^{-1/2} \colon \boldsymbol{b} \in B(t)\}$ remains a compact convex set; see, for example, [16].

Hence, it suffices to study the boundary crossing probabilities for the standard Brownian motion for compact convex sets.

3.1. Two-dimensional standard Brownian motion

Let $\{\boldsymbol{W}(t) = (W_1(t), W_2(t)),\ t \geq 0\}$ be a standard two-dimensional Brownian motion with drift $\mathbf{0}$, where $\mathbf{0} = (0, \ldots, 0)$, and covariance matrix $t\boldsymbol{\Sigma} = t\boldsymbol{I}$. Throughout this paper we let $\mathbb{P}(\boldsymbol{W}(0) = (0, 0)) = 1$ and assume the compact convex set $B(t)$ satisfies the following conditions:

(i) for every $t \in [0, 1]$, $B(t)$ is a compact convex set in $\mathbb{R}^2$,

(ii) the boundary of $B(t)$ is a continuous function in t for $t \in [0, 1]$ and satisfies the Lipschitz conditions, and

(iii) $(0, 0) \in B(0)$.

For example, $B(t) \triangleq \{(\omega_1, \omega_2, t) \colon \omega_1^2 + \omega_2^2 \leq 1 + t, t \in [0, 1] \subset \mathbb{R}^2 \times [0, 1]\}$, where '$\triangleq$' stands for 'defined as'. To simplify the notation further, we define $B = \operatorname{int}\{B(t) \colon t \in [0, 1]\}$ and its complement is denoted by B^c. Hence, the BCP is given by

$$\mathbb{P}(\boldsymbol{W}(t) \in B^c \text{ for some } t \in [0, 1]) = 1 - \mathbb{P}(\boldsymbol{W}(t) \in B \text{ for all } t \in [0, 1]).$$

Given $B(t)$, let $h = \sup_{0 \leq t \leq 1} \sup\{\|\boldsymbol{b}\| \colon \boldsymbol{b} \in B(t)\}$, where $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. Choose a large integer m, define $\Delta x = h/m$ and discretize $\mathbb{R}^2$ as $\mathbb{R}^2_m = \{(k_1\Delta x, k_2\Delta x),\ k_1, k_2 = 0, \pm 1, \pm 2, \ldots\}$. The time interval $[0, 1]$ is correspondingly partitioned into n equal sub-intervals, preserving the scale relationship $\Delta x^2 = \Delta t$, i.e. $n = m^2/h^2$, or $n = \lfloor m^2/h^2 \rfloor$ if n is not an integer.

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Let $\{0 = t_0 < t_1 < \cdots < t_n = 1\}$ be an equal-spaced partition of $[0, 1]$ with $t_i = i\Delta t$. The partial sums

$$\hat{W}_{jn}(t) = \sum_{i=1}^{\lfloor nt \rfloor} \hat{X}_{ji}, \qquad j = 1, 2,$$

where $\hat{X}_{1i}$ and $\hat{X}_{2i}$ are i.i.d. random variables having common distribution $f_0 \in \mathcal{F}$, converge in distribution to independent one-dimensional Brownian motions. Since the components are independent, it follows from the construction in Section 2 that the transition probabilities of the two-dimensional Markov chain $\hat{\boldsymbol{W}}_n(t) = (\hat{W}_{1n}(t), \hat{W}_{2n}(t))$ are given by

$$\begin{aligned} p((k_1, k_2) \mid (j_1, j_2)) &= \mathbb{P}(\hat{\boldsymbol{W}}_n(t + \Delta t) = (k_1\Delta x, k_2\Delta x) \mid \hat{\boldsymbol{W}}_n(t) = (j_1\Delta x, j_2\Delta x)) \\ &= p(k_1 \mid j_1)p(k_2 \mid j_2), \end{aligned} \tag{2}$$

where, for $i = 1, 2$,

$$p(k_i \mid j_i) = \begin{cases} C^{-1} \exp\biggl(-\dfrac{(k_i - j_i)^2}{2}\biggr) & \text{if } k_i - j_i \neq 0, \\ C^{-1} \displaystyle\sum_{\ell \neq 0} (\ell^2 - 1) \exp\biggl(-\frac{\ell^2}{2}\biggr) & \text{if } k_i - j_i = 0, \end{cases}$$

and $C = \sum_{\ell \neq 0} \ell^2 \exp(-\ell^2/2)$. Note that any distribution in the family $\mathcal{F}$ can be used to construct the partial sums and transition probabilities.

Theorem 2. *Given $t \in [0, 1]$ and $\Delta t = \Delta x^2$ $(n = m^2/h^2)$, we have*

$$\hat{\boldsymbol{W}}_n(t) \xrightarrow{\text{D}} \boldsymbol{W}(t) \quad \textit{as } n \to \infty.$$

Proof. We show that the characteristic function of $\hat{\boldsymbol{W}}_n(t)$ converges to that of $\boldsymbol{W}(t)$ for all t. From the proof of Theorem 1, we know that

$$\mathbb{E}[\mathrm{e}^{\mathrm{i}(s_1,s_2)(\hat{X}_{11},\hat{X}_{21})^\top}] = \mathbb{E}[\mathrm{e}^{\mathrm{i}s_1\hat{X}_{11}}]\mathbb{E}[\mathrm{e}^{\mathrm{i}s_2\hat{X}_{21}}] = 1 - \frac{s_1^2h^2 + s_2^2h^2}{2m^2} + \mathcal{O}\biggl(\frac{1}{m^4}\biggr).$$

Thus, as $m \to \infty$,

$$\varphi_{\hat{\boldsymbol{W}}_n(t)}(\boldsymbol{s}) = \biggl(1 - \frac{(s_1^2 + s_2^2)h^2}{2m^2} + \mathcal{O}\biggl(\frac{1}{m^4}\biggr)\biggr)^{m^2t/h^2} \to \exp\biggl(-\frac{t\boldsymbol{s}\boldsymbol{s}^\top}{2}\biggr),$$

where $\varphi_{\boldsymbol{W}(t)}(s) = \exp(-t\boldsymbol{s}\boldsymbol{s}^\top/2)$ is the characteristic function of the bivariate normal distribution with mean $\boldsymbol{0}$ and covariance matrix $t\boldsymbol{\Sigma} = t\boldsymbol{I}$. □

Imbedding procedure. Given an open set $A \subseteq \mathbb{R}^2$, we introduce an oriented distance function $g(x, A) = d(x, A) - d(x, A^c)$, where $d(x, A) = \inf\{\|x - y\| \colon y \in A\}$ and $\|\cdot\|$ is the Euclidean norm in $\mathbb{R}^2$. The function g is continuous since d is continuous. It is easy to see that if $x \in A$ then $g(x, A) < 0$, and if $x \in \bar{A}$ then $g(x, A) \geq 0$. In the sequel, we construct an imbedded Markov chain with absorbing states induced by B^c. For each t_i, we define the *inner* and *outer* approximations of $B(t_i)$ as follows. Let Q be the collection of all squares whose lengths are Δx and whose centers are in $\mathbb{R}_m^2$. Then, the inner and outer approximations of $B(t_i)$ are, respectively, given by

$$\underline{B}_i(\Delta x) = \cup\{q \in Q \colon q \subset B(t_i)\} \quad \text{and} \quad \bar{B}_i(\Delta x) = \cup\{q \in Q \colon q \cap B(t_i) \neq \varnothing\},$$

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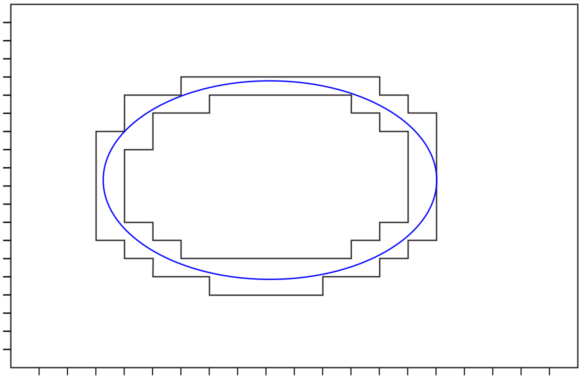

FIGURE 2: Inner and outer approximations at t_i.

and $\underline{B}_i(\Delta x) \subset B(t_i) \subset \bar{B}_i(\Delta x)$, $\underline{B}_i(\Delta x) \uparrow B(t_i)$, and $\bar{B}_i(\Delta x) \downarrow B(t_i)$; see Figure 2. We can use either the inner or outer approximations of $B(t_i)$. Here we choose the inner approximation and define the set $\hat{B}_i(t)$ to be the collection of centers of squares in $\underline{B}_i(\Delta x)$. Thus for inner approximation, we can define a finite Markov chain $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ on the state spaces; $i = 1, \dots, n$, $\Omega_i = \hat{B}_i(t) \cup \{\alpha_i\}$, where α_i stands for all values outside $\hat{B}_i(t)$.

Then $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ forms a nonhomogeneous Markov chain having transition probabilities

$$\mathbb{P}(\boldsymbol{Y}_n(i) = (k_1, k_2) \mid \boldsymbol{Y}_n(i-1)) = (j_1, j_2))$$
$$= \begin{cases} p((k_1, k_2) \mid (j_1, j_2)) & \text{if } (j_1, j_2) \in \Omega_{i-1} \setminus \alpha_{i-1},\ (k_1, k_2) \in \Omega_i \setminus \alpha_i, \\ p(\alpha_i \mid (j_1, j_2)) & \text{if } (j_1, j_2) \in \Omega_{i-1} \setminus \alpha_{i-1},\ (k_1, k_2) = \alpha_i, \\ 1 & \text{if } (j_1, j_2) = \alpha_{i-1},\ (k_1, k_2) = \alpha_i, \\ 0 & \text{if } (j_1, j_2) = \alpha_{i-1},\ (k_1, k_2) \in \Omega_i \setminus \alpha_i, \end{cases} \tag{3}$$

where $p((k_1, k_2) \mid (j_1, j_2))$ is given by (2), and $\Omega_0 = \{(0, 0)\}$ and $\mathbb{P}(\boldsymbol{Y}_n(0) = (0, 0)) \equiv 1$. For given $(j_1, j_2) \in \Omega_{i-1} \setminus \alpha_{i-1}$, the absorption probabilities are given by

$$p(\alpha_i \mid (j_1, j_2)) = 1 - \sum_{(k_1, k_2) \in \hat{B}_i(t)} p((k_1, k_2) \mid (j_1, j_2)).$$

Again, all the transition probability matrices of the imbedded Markov chain $\{\boldsymbol{Y}_n(i)\}_{i=0}^n$ have the form

$$\boldsymbol{M}_i = \left[\begin{array}{c|c} p((k_1, k_2) \mid (j_1, j_2)) & p(\alpha_i \mid (j_1, j_2)) \\ \hline \boldsymbol{0} & 1 \end{array}\right] = \left[\begin{array}{c|c} \boldsymbol{N}_i & \boldsymbol{C}_i \\ \hline \boldsymbol{0} & 1 \end{array}\right], \qquad i = 1, 2, \dots, n.$$

Together with Theorem 2, we are now in a position to prove our main theorem.

Theorem 3. *Let $\boldsymbol{W}(t)$ be a standard two-dimensional Brownian motion. Given a compact convex set $B(t)$, then*

$$\mathbb{P}(\boldsymbol{W}(t) \in B \text{ for all } t \in [0, 1]) = \lim_{m \to \infty} \boldsymbol{\xi}_0 \bigg(\prod_{i=1}^{m^2/h^2} \boldsymbol{N}_i \bigg) \boldsymbol{1}^\top,$$

where the transition probabilities are given in (3).

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Proof. Let $h(x) = \sup_{0 \le t \le 1} x(t)$. From the definition of function g, it is not difficult to see that the following two sets are equal:

$$\Big\{\max_{1 \le i \le n} g(\hat{\boldsymbol{W}}_n(t_i), \operatorname{int}(\underline{B}_i(\Delta x))) < 0\Big\} \quad \Longleftrightarrow \quad \Big\{\max_{1 \le i \le n} g(\hat{\boldsymbol{W}}_n(t_i), \hat{B}_i(t)) < 0\Big\}.$$

Since $\boldsymbol{W}(t)$ is continuous and $\boldsymbol{W}(0) \in B(0)$, $\sup_{0 \le t \le 1} g(\boldsymbol{W}(t), B) < 0$ represents that $\boldsymbol{W}(t)$ stays in B for all $t \in [0, 1]$. Thus, due to the continuity of the probability measure and of the functions h and g, the boundary crossing probabilities can be obtained by

$$\begin{aligned}\mathbb{P}(\boldsymbol{W}(t) \in B^c \text{ for some } t \in [0, 1]) &= 1 - \lim_{m \to \infty} \mathbb{P}(\boldsymbol{Y}_n(1) \neq \alpha_1, \dots, \boldsymbol{Y}_n(n) \neq \alpha_n) \\ &= 1 - \lim_{m \to \infty} \boldsymbol{\xi}_0 \bigg(\prod_{i=1}^{m^2/h^2} \boldsymbol{N}_i\bigg) \mathbf{1}^\top.\end{aligned}$$

For the second last equality, $g(\hat{\boldsymbol{W}}_n(t_i), \hat{B}_i(t)) < 0$ represents $\hat{\boldsymbol{W}}_n(t_i)$ stays inside $\hat{B}_i(t)$. In other words, it is equivalent to saying that $Y_i \neq \alpha_i$. The last equality follows from the FMCI technique. This completes the proof. □

The above result is proved for a two-dimensional Brownian motion; however, it is straightforward to extend to a higher-dimensional Brownian motion. It is important to point out that for four and up dimensional Brownian motions their corresponding state spaces of imbedded Markov chains are very large and a high-speed computer is required in order to obtain a numerical solution.

4. Convergence rate

Note that the convergence rate of the approximation depends on the geometry of the boundary $B(t)$. First we introduce a product topology $\mathcal{T}$ on the $\mathbb{R}^2 \times [0, 1]$ space. Let us define a family of open sets on $\mathbb{R}^2 \times [0, 1]$ as

$$\mathfrak{O} = \{(\omega_1, \omega_2, t) \colon a(t) < \omega_1 < b(t), \text{ and } c(t) < \omega_2 < d(t),\ t \in [0, 1]\}, \tag{4}$$

where $a(t)$, $b(t)$, $c(t)$, and $d(t)$ are continuous functions defined on $\mathbb{R} \times [0, 1]$ and satisfy the Lipschitz condition. Let

$$\mathfrak{F} = \{\mathfrak{O} \colon \text{ all open sets defined on } \mathbb{R}^2 \times [0, 1] \text{ by (4)}\},$$

and $\mathcal{T}$ be the topology induced by the open sets in $\mathfrak{F}$. For any small $\delta > 0$, it follows from convexity of set B, finite covering theorem, and continuity of the probability that there exist outer and inner approximations $\bar{B} = \bigcup_{i=1}^{k_1} \mathfrak{O}_i$ and $\underline{B} = \bigcup_{j=1}^{k_2} \mathfrak{O}_j$ for B and such that $\bar{B} \supseteq B \supseteq \underline{B}$ and

$$|\mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \text{ for all } t \in [0, 1]) - \mathbb{P}(\boldsymbol{W}(t) \in \underline{B} \text{ for all } t \in [0, 1])| < \delta.$$

To show the convergence rate remains $\mathcal{O}(1/m)$ for $d = 2$, we need the following lemma.

Lemma 1. *For any $\mathfrak{O} \in \mathcal{T}$, we have*

$$|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \mathfrak{O} \text{ for all } t \in [0, 1]) - \mathbb{P}(\boldsymbol{W}(t) \in \mathfrak{O} \text{ for all } t \in [0, 1])| = \mathcal{O}\Big(\frac{1}{m}\Big) \quad \text{as } n \to \infty.$$

Proof. From the independence between $\hat{W}_{1n}(t)$ and $\hat{W}_{2n}(t)$, $W_1(t)$ and $W_2(t)$, the triangle inequality, and Theorem 1(iii), we have

$$\begin{aligned}
&|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \mathfrak{O} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \mathfrak{O} \text{ for all } t \in [0,1])| \\
&\quad = |\mathbb{P}(a(t) < \hat{W}_{1n}(t) < b(t) \text{ for all } t \in [0,1])\mathbb{P}(c(t) < \hat{W}_{2n}(t) < d(t) \text{ for all } t \in [0,1]) \\
&\qquad - \mathbb{P}(a(t) < W_1(t) < b(t) \text{ for all } t \in [0,1])\mathbb{P}(c(t) < W_2(t) < d(t) \text{ for all } t \in [0,1])| \\
&\quad \le |\mathbb{P}(a(t) < \hat{W}_{1n}(t) < b(t) \text{ for all } t \in [0,1]) \\
&\qquad - \mathbb{P}(a(t) < W_1(t) < b(t) \text{ for all } t \in [0,1])| \\
&\qquad + |\mathbb{P}(c(t) < \hat{W}_{2n}(t) < d(t) \text{ for all } t \in [0,1]) \\
&\qquad\quad - \mathbb{P}(c(t) < W_2(t) < d(t) \text{ for all } t \in [0,1])| \\
&\quad = \mathcal{O}\left(\frac{1}{m}\right).
\end{aligned}$$

□

Lemma 2. *As $n \to \infty$, we have*

(i) $|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \bar{B} \textit{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \textit{ for all } t \in [0,1])| = \mathcal{O}(1/m)$, *and*

(ii) $|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \underline{B} \textit{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \underline{B} \textit{ for all } t \in [0,1])| = \mathcal{O}(1/m)$.

Proof. Note that

$$\begin{aligned}
&|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \bar{B} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \text{ for all } t \in [0,1])| \\
&\quad = \left|\mathbb{P}\left(\hat{\boldsymbol{W}}_n(t) \in \bigcup_{i=1}^{k} \mathfrak{O}_i \text{ for all } t \in [0,1]\right) - \mathbb{P}\left(\boldsymbol{W}(t) \in \bigcup_{i=1}^{k} \mathfrak{O}_i \text{ for all } t \in [0,1]\right)\right|.
\end{aligned}$$

Since k is finite, Lemma 2(i) is an immediate consequence of the inclusive–exclusive identity, the triangle inequality, and Lemma 1. The proof of Lemma 2(ii) is the same as given above.

□

Theorem 4. *As $n \to \infty$, we have*

$$|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in B \textit{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in B \textit{ for all } t \in [0,1])| = \mathcal{O}\left(\frac{1}{m}\right).$$

Proof. From the triangle inequality, it follows that

$$\begin{aligned}
&|\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in B \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in B \text{ for all } t \in [0,1])| \\
&\quad \le |\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in B \text{ for all } t \in [0,1]) - \mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \bar{B} \text{ for all } t \in [0,1])| \\
&\qquad + |\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \bar{B} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \text{ for all } t \in [0,1])| \\
&\qquad + |\mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in B \text{ for all } t \in [0,1])| \\
&\quad = A_n + B_n + C. \qquad (5)
\end{aligned}$$

Further, since $\bar{B} \supseteq B \supseteq \underline{B}$, it follows that the following inequality holds:

$$\begin{aligned}
A_n &\le |\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \bar{B} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \text{ for all } t \in [0,1])| \\
&\quad + |\mathbb{P}(\hat{\boldsymbol{W}}_n(t) \in \underline{B} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \underline{B} \text{ for all } t \in [0,1])| \\
&\quad + |\mathbb{P}(\boldsymbol{W}(t) \in \bar{B} \text{ for all } t \in [0,1]) - \mathbb{P}(\boldsymbol{W}(t) \in \underline{B} \text{ for all } t \in [0,1])| \\
&= B_n + E_n + F. \qquad (6)
\end{aligned}$$

It follows from Lemma 2(i) and (ii) that $B_n = \mathcal{O}(1/m)$ and $E_n = \mathcal{O}(1/m)$ as $n \to \infty$. Since $C < F < \delta$ for arbitrarily small δ (independent of n), hence the theorem follows from (5) and (6). □

By the same token, the results hold for $d \geq 3$. The convergence rate remains $\mathcal{O}(1/m)$.

5. Numerical results and discussions

To illustrate the procedure, we provide two numerical examples: BCP of Y-type time tunnel for Brownian motion and the OU process, and BCP for two-dimensional Brownian motion.

Example 1. (*Y-type time tunnel.*) We consider a Y-type time tunnel $\mathcal{Y}(t)$ in Figure 3 with $a(t) = -1 - t$, $b(t) = 1 + t$, $c(t) = 1 - 2t$, and $d(t) = 2t - 1$. The BCP of a Brownian motion to $\mathcal{Y}(t)$ is

$$1 - \mathbb{P}(W(t) \in \operatorname{int}(\mathcal{Y}(t)) \text{ for all } t \in [0, 1]) \approx 0.9046.$$

It is well known (see, for example, [7]) that the OU process can also be written as a time-changed Brownian motion $X(t) = \mathrm{e}^{-\mu t}\tilde{W}(\tau_t)$, where $\tilde{W}(\tau_t)$ is a Brownian motion with $\tau_t = \sigma^2(\mathrm{e}^{2\mu t} - 1)/2\mu$. Using time-change and the scaling property of a Brownian motion, we convert the time interval back to [0, 1], and the transformed boundaries $b'(t)$ and $d'(t)$ are $(1 + \log(1 + t(\mathrm{e} - 1)))(1 + t(\mathrm{e} - 1))^{0.5}(2(\mathrm{e} - 1))^{-0.5}$ and $(2\log(1 + t(\mathrm{e} - 1)) - 1)(1 + t(\mathrm{e} - 1))^{0.5}(2(\mathrm{e} - 1))^{-0.5}$, respectively. Hence, using Theorem 1 the approximate crossing probability for $\mathcal{Y}'(t)$ equals

$$1 - \mathbb{P}(W(t) \in \operatorname{int}(\mathcal{Y}'(t)) \text{ for all } t \in [0, 1]) \approx 0.9854.$$

Example 2. (*Cone.*) Let $\boldsymbol{W}(t)$ be a standard two-dimensional Brownian motion starting at the origin and $B(t) = \{(x_1, x_2): x_1^2 + x_2^2 \leq 1 + t\}$. Since the set $B(t)$ is a function of t (see Figure 4), the imbedded Markov chain is not homogeneous, and the boundary crossing probability is obtained by multiplying the essential matrices which might not be square or might not be of the same sizes. Take $m = 50$, and for each t_i, collect the nodes inside the boundary as the states associated with the essential transition matrices $\boldsymbol{N}_i$ which can be constructed using (3).

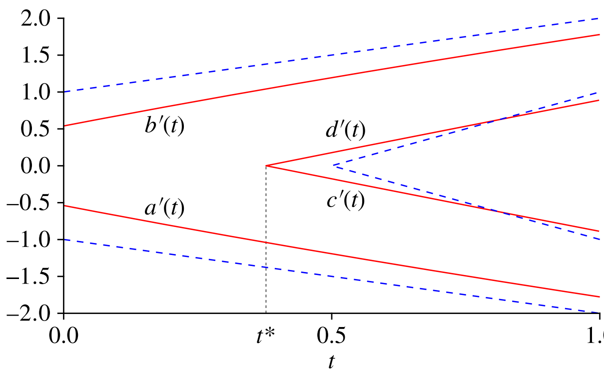

FIGURE 3: The boundaries $a'(t)$, $b'(t)$, $c'(t)$, and $d'(t)$ (solid lines) are transformed boundaries. The dashed lines are original boundaries for the OU process.

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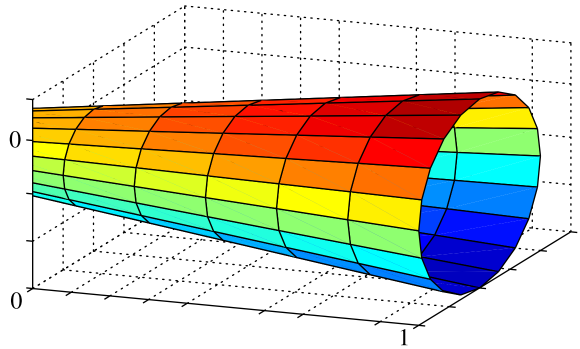

FIGURE 4: Cone boundary.

Then, the boundary crossing probability can be approximated by

$$\mathbb{P}(\boldsymbol{W}(t) \in B^c \text{ for some } t \in [0, 1]) \approx 1 - \boldsymbol{\xi}_0 \prod_{i=1}^{n} \boldsymbol{N}_i \mathbf{1}^\top = 0.7019.$$

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References

[1] ANDERSON, T. W. (1960). A modification of the sequential probability ratio test to reduce the sample size. *Ann. Math. Statist.* **31,** 165–197.

[2] BUCKHOLTZ, P. G. AND WASAN, M. T. (1979). First passage probabilities of a two-dimensional Brownian motion in an anisotropic medium. *Sankhyā A* **41,** 198–206.

[3] DI NARDO, E., NOBILE, A. G., PIROZZI, E. AND RICCIARDI, L. M. (2001). A computational approach to first-passage-time problems for Gauss–Markov processes. *Adv. Appl. Prob.* **33,** 453–482.

[4] DURBIN, J. (1985). The first-passage density of a continuous Gaussian process to a general boundary. *J. Appl. Prob.* **22,** 99–122. (Correction: **25** (1988), 804.)

[5] DURBIN, J. (1992). The first-passage density of the Brownian motion process to a curved boundary. *J. Appl. Prob.* **29,** 291–304.

[6] ERDŐS, P. AND KAC, M. (1946). On certain limit theorems of the theory of probability. *Bull. Amer. Math. Soc.* **52,** 292–302.

[7] FU, J. C. AND WU, T.-L. (2010). Linear and nonlinear boundary crossing probabilities for Brownian motion and related processes. *J. Appl. Prob.* **47,** 1058–1071.

[8] IYENGAR, S. (1985). Hitting lines with two-dimensional Brownian motion. *SIAM J. Appl. Math.* **45,** 983–989.

[9] LEHMANN, A. (2002). Smoothness of first passage time distributions and a new integral equation for the first passage time density of continuous Markov processes. *Adv. Appl. Prob.* **34,** 869–887.

[10] METZLER, A. (2010). On the first passage problem for correlated Brownian motion. *Statist. Prob. Lett.* **80,** 277–284.

[11] NOVIKOV, A., FRISHLING, V. AND KORDZAKHIA, N. (1999). Approximations of boundary crossing probabilities for a Brownian motion. *J. Appl. Prob.* **36,** 1019–1030.

[12] ROBBINS, H. AND SIEGMUND, D. (1970). Boundary crossing probabilities for the Wiener process and sample sums. *Ann. Math. Statist.* **41,** 1410–1429.

[13] Sacerdote, L. and Tomassetti, F. (1996). On evaluations and asymptotic approximations of first-passage-time probabilities. *Adv. Appl. Prob.* **28,** 270–284.
[14] Scheike, T. H. (1992). A boundary-crossing result for Brownian motion. *J. Appl. Prob.* **29,** 448–453.
[15] Spitzer, F. (1958). Some theorems concerning 2-dimensional Brownian motion. *Trans. Amer. Math. Soc.* **87,** 187–197.
[16] Walsh, J. L. (1921). On the transformation of convex point sets. *Ann. Math. (2)* **22,** 262–266.
[17] Wendel, J. G. (1980). Hitting spheres with Brownian motion. *Ann. Prob.* **8,** 164–169.