

On topological rank of factors of Cantor minimal systems

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(Received 20 September 2020 and accepted in revised form 7 May 2021)

Abstract. A Cantor minimal system is of finite topological rank if it has a Bratteli–Vershik representation whose number of vertices per level is uniformly bounded. We prove that if the topological rank of a minimal dynamical system on a Cantor set is finite, then all its minimal Cantor factors have finite topological rank as well. This gives an affirmative answer to a question posed by Donoso, Durand, Maass, and Petite in full generality. As a consequence, we obtain the dichotomy of Downarowicz and Maass for Cantor factors of finite-rank Cantor minimal systems: they are either odometers or subshifts.

Key words: Cantor minimal system, topological rank, topological factor, ordered Bratteli diagram, ordered premorphism

2020 Mathematics Subject Classification: 54H20 (Primary); 37B05, 37B10 (Secondary)

1. Introduction

A Cantor minimal system is a pair (X, T) , where T is a minimal homeomorphism of the Cantor space X . Inspired by the definition of Rokhlin towers for approximating an ergodic system by a finite union of towers of measurable sets [21], the definition of Kakutani–Rokhlin towers for Cantor minimal systems was established [20, 22]. As a topological analogue, a Kakutani–Rokhlin partition for a Cantor minimal system is a finite union of towers of closed and open (clopen) sets in which every level of a tower is mapped onto its upper level up to the top level. The difference in this analogy is that a Kakutani–Rokhlin partition topologically covers the space X and the dynamical system is generated pointwise by a nested sequence of Kakutani–Rokhlin partitions so that the intersection of bases of their towers converges to a point. When the number of towers of each partition in the sequence is uniformly bounded (equivalently, when

the Bratteli–Vershik diagram associated to the sequence of Kakutani–Rokhlin partitions has uniformly bounded number of vertices per level), the associated system has *finite topological rank* or $\text{rank}_{\text{top}}(X, T) < \infty$.

A system of finite topological rank has zero entropy and bounded number of invariant ergodic measures [4]. Moreover, the rank of the additive group of the *continuous spectrum* (that is, the eigenvalues of the Koopman operator $U_T(f) = f \circ T$ defined on $C(X)$) as an abelian group is not more than the topological rank of the system [5, 15].

Examples of finite topological rank Cantor minimal systems include symbolic systems generated by the coding of interval exchange transformations, substitution subshifts, and linearly recurrent systems on Cantor sets [2, 8, 9, 18]. It is folklore that *odometers* (minimal isometries on Cantor sets) are the only rank-one minimal Cantor systems.

The Kakutani–Rokhlin partition associated to the minimal Cantor system (X, T) ‘approximates’ T by a *shift* map up to the top levels of the towers. However, as the system is eventually generated by a sequence of partitions, it can be far from being a *subshift* [17]. A remarkable theorem of Downarowicz and Maass [7] states that when a Cantor minimal system is of finite topological rank, then it is either an odometer or a subshift. The latter holds if the rank is bigger than one. Although odometers are the only topological factors of odometers, a topological factor of a subshift is not necessarily a subshift. For instance, odometers are topological factors of any dynamical system with non-trivial continuous rational spectrum. Moreover, there are minimal dynamical systems on Cantor sets that are neither subshifts nor conjugate to any odometer [17]. Even though a minimal factor of a finite topological rank system is a subshift, it is not clear whether it is of finite topological rank [6]. As a corollary of our main theorem, one can guarantee this for all Cantor factors of (essentially) minimal systems. We recall that a dynamical system is essentially minimal if it has a unique minimal subsystem and thus every minimal system is essentially minimal.

THEOREM 1.1. *Let (X, T) be an essentially minimal Cantor system of finite topological rank and (Y, S) be a minimal system on a Cantor set such that for some continuous map $\alpha : X \rightarrow Y$, $\alpha \circ T = S \circ \alpha$. Then*

$$\text{rank}_{\text{top}}(Y, S) < \infty.$$

Indeed, we prove that $\text{rank}_{\text{top}}(Y, S) \leq 3 \text{rank}_{\text{top}}(X, T)$.

After submitting this article, for the case that (X, T) is minimal and (Y, S) is a *subshift system* (which means that S is a shift map on a Cantor space, such a system being called a *symbolic system* in some literature), Espinoza proved that $\text{rank}_{\text{top}}(Y, S) \leq \text{rank}_{\text{top}}(X, T)$ in [13]. He had previously proved that for the case of subshifts $\text{rank}_{\text{top}}(Y, S) < \infty$ [12]. Here, though, we do not have subshift assumptions for (Y, S) and it can be any minimal factor of (X, T) on the Cantor set. Moreover, a weaker condition for (X, T) is considered, which is essential minimality. In Remark 4.7, we discuss some special cases in which the inequality $\text{rank}_{\text{top}}(Y, S) \leq \text{rank}_{\text{top}}(X, T)$ holds.

As a corollary of Theorem 1.1, one can get the dichotomy of the main result of [7] for the Cantor factors of finite topological rank Vershik systems.

COROLLARY 1.2. *Let (X, T) be an essentially minimal system on a compact totally disconnected metrizable space. Then every minimal Cantor factor of (X, T) is an odometer or a subshift.*

From the algebraic point of view, given the assumptions of Theorem 1.1, $\text{rank}_{\text{alg}}(Y, S)$, the algebraic rank (that is, the dimension of the \mathbb{Q} -vector space on the abelian group) of the *dimension group* $K^0(Y, S)$ associated to (Y, S) as an (essentially) minimal Cantor system, does not exceed $\text{rank}_{\text{alg}}K^0(X, T)$. In fact, every factor map $\alpha : X \rightarrow Y$ between two Cantor minimal systems induces an order embedding $\alpha_* : K^0(Y, S) \rightarrow K^0(X, T)$ [19] and this implies that $\text{rank}_{\text{alg}}(Y, S) \leq \text{rank}_{\text{alg}}(X, T)$. Therefore, it seems reasonable to ask whether $\text{rank}_{\text{top}}(Y, S) \leq \text{rank}_{\text{top}}(X, T)$ (although the topological rank of a system is an upper bound for its algebraic rank). We will consider this question in Remark 4.7 after the proof of Theorem 1.1. This theorem answers Question 8.4 of [6] affirmatively and in full generality.

Recently, S -adic subshifts, which are Cantor systems generated by sequences of morphisms between finite alphabets, have also been studied by those who are interested in ‘approximating’ finite-rank minimal subshifts with primitive substitutions [10, 6]. In [6, Theorem 4.1], the authors proved that every minimal S -adic subshift with *bounded alphabet rank* is a topological factor of a finite topological rank minimal Cantor system. Combining this result with Theorem 1.1, one obtains the following statement.

COROLLARY 1.3. *Every minimal aperiodic S -adic subshift with bounded alphabet rank is conjugate to a subshift of finite topological rank.*

To prove Theorem 1.1, we use Bratteli–Vershik representations of (X, T) and (Y, S) . Then we apply the representation of the factor map α in terms of an *ordered premorphism* between the associated ordered Bratteli diagrams. This notion has been recently defined in [1]. Here, we will show how the existence of an ordered premorphism from an ordered Bratteli diagram B_1 to B_2 may reduce the number of vertices of levels of B_1 needed to construct an ordered Bratteli diagram equivalent to B_1 whose rank is dominated by $3 \text{rank}(B_2)$.

The paper starts by recalling basic definitions and tools in §2. In §3, the notion of an ordered premorphism via a sequence of morphisms is defined and the main result of [1] that is also a main tool of this paper is presented. Then, in §4, Theorem 1.1 will be proved. In §5, a combinatorial condition will be given for the verification of conjugacy.

2. Preliminaries

2.1. *Topological dynamical systems on Cantor sets.* A *topological dynamical system* is a pair (X, T) , where X is a compact metric space and T is a homeomorphism on X . If Z is an invariant closed subset of X , then (Z, T) is called a *subsystem*. The orbit of a point $x \in X$, denoted by $\mathcal{O}(x)$, is the sequence $(T^n x)_{n \in \mathbb{Z}}$. If X is a Cantor space (that is, a non-empty compact metrizable totally disconnected space with no isolated points), then the system is called a *Cantor system*. Two topological dynamical systems (X, T) and (Y, S) are *semi-conjugate* if there exists a surjective continuous map $\alpha : X \rightarrow Y$ such that

$\alpha \circ T = S \circ \alpha$. In this case (Y, S) is called a *factor* of (X, T) , (X, T) is called an *extension* of (Y, S) , and α is called a *factor map*.

When $X = A^{\mathbb{Z}}$, where A is a finite alphabet of cardinality $n \geq 2$ and X is equipped with the compact product topology that makes X homeomorphic to the Cantor set, together with the shift map T acting on the bi-infinite sequences of X , then (X, T) is called a *shift system*. Every subsystem of a shift is called a *subshift system*.

For a topological dynamical system (X, T) , if the orbits of all points are dense in X , then the system is called *minimal*. This is equivalent to the absence of non-trivial invariant closed subsets. When (X, T) has a unique minimal subsystem, the system is called *essentially minimal*. Every essentially minimal system on a Cantor set has realizations by sequences $\{\mathcal{T}_n\}_{n \geq 1}$ of *Kakutani–Rokhlin* (briefly called *K-R partitions*) [20]. Each K-R partition $\mathcal{T}_n = \bigcup_{i=1}^k \bigcup_{j=1}^h B_{ij}$ is a finite union of towers, $\bigcup_{j=1}^h B_{ij}$, of clopen sets, B_{ij} , so that $T(B_{ij}) = B_{ij+1}$ for $j < h$. The *base* of the tower is $\bigcup_{i=1}^k B_{i1}$. The towers construction is based on the first return time of the points of bases to them.

2.2. Ordered Bratteli diagrams. A *Bratteli diagram* is an infinite directed graph $B = ((V_i)_{i \geq 0}, (E_i)_{i \geq 1})$, where $V = \bigcup_{i \geq 0} V_i$ is the set of vertices with $V_0 = \{v_0\}$ and, for each $i \geq 1$, E_i is the set of edges between V_{i-1} and V_i . Each V_i and each E_i is a finite non-empty set. There are two maps $r, s : E \rightarrow V$, called the range and the source maps, respectively, with $r(E_i) \subseteq V_i$ and $s(E_i) \subseteq V_{i-1}$. A vertex $v \in V_i$ is connected to a vertex $w \in V_{i-1}$ if there exists an edge $e \in E_i$ such that $r(e) = v$ and $s(e) = w$. In this way, for each $n \geq 1$ there is a $|V_n| \times |V_{n-1}|$ incidence matrix A_n whose entry a_{ij} counts the number of edges between $v_i \in V_n$ and $w_j \in V_{n-1}$. We assume that every row and every column of each A_n is non-zero.

For $m, n \geq 0$ with $m < n$, let $E_{m,n}$ be the set of finite paths from V_m to V_n , that is, $E_{m,n}$ consists of the tuples (e_{m+1}, \dots, e_n) , where $e_i \in E_i$ for $i = m + 1, \dots, n$ and $r(e_i) = s(e_{i+1})$ for $i = m + 1, \dots, n - 1$. In particular, $E_{m,m} = \{(v, v) \mid v \in V_m\}$ is an edge set from V_m to itself.

For a strictly increasing sequence of integers $n = \{n_k\}_{k \geq 0}$ with $n_0 = 0$, one can define the *telescoping* of the diagram along n by defining a new Bratteli diagram $B' = ((V'_i)_{i \geq 0}, (E'_i)_{i \geq 1})$ in which for every $i \geq 1$, $V'_i = V_{n_i}$, $E'_i = E_{n_i, n_{i+1}}$, and $V'_0 = V_0$. So, the incidence matrices of B' are $A'_i = A_{n_i} \times A_{n_i-1} \times \dots \times A_{n_{i-1}+1}$. A Bratteli diagram is called *simple* if there exists a telescoping of that along a sequence such that all the incidence matrices have just positive entries.

An *ordered Bratteli diagram*, $B = ((V_i)_{i \geq 0}, (E_i)_{i \geq 1}, \geq)$, is a Bratteli diagram $((V_i)_{i \geq 0}, (E_i)_{i \geq 1})$ together with a partial ordering on the set of its edges in which two edges e and e' are comparable if and only if $r(e) = r(e')$. In fact, for every $n \geq 1$ and every $v \in V_n \setminus V_0$, $r^{-1}(v)$ is linearly ordered. For each v , the edge with the largest (smallest) number in the ordering of $r^{-1}(v)$ is called the *max edge* (*min edge*). For every telescoping of B , there is an induced ordering on the edge set. In fact, $(e_{k+1}, e_{k+2}, \dots, e_l) > (f_{k+1}, f_{k+2}, \dots, f_l)$ as two finite paths in $E_{k,\ell}$ if $r(e_\ell) = r(f_\ell)$ and there exists some i with $k + 1 \leq i \leq l$ such that for all j with $i < j \leq l$, $e_j = f_j$ and $e_i > f_i$.

Let $B = ((V_i)_{i \geq 0}, (E_i)_{i \geq 1}, \geq)$ be an ordered Bratteli diagram. The set of *infinite paths* is

$$X_B = \{(e_1, e_2, \dots) \mid e_i \in E_i, r(e_i) = s(e_{i+1}), i = 1, 2, \dots\}.$$

Two paths are *cofinal* if all but finitely many of their edges agree. The set X_B is equipped with the usual compact product topology so that its basis consists of cylinder sets of the form

$$U(e_1, e_2, \dots, e_k) = \{(f_1, f_2, \dots) \in X_B : f_i = e_i, 1 \leq i \leq k\}.$$

The set X_B is a compact Hausdorff space with a countable basis consisting of clopen sets and is homeomorphic to the Cantor set if it is infinite and B is simple. Let X_B^{\max} denote the set of all those elements (e_1, e_2, \dots) in X_B such that each e_n is a max edge and define X_B^{\min} analogously. An ordered Bratteli diagram is called *properly ordered* if it is simple and X_B^{\max} and X_B^{\min} each contains only one element; when this occurs, the max and min paths are denoted by x_{\max} and x_{\min} , respectively. For any Bratteli diagram, there exists an ordering which makes it properly ordered [20].

Let $B = ((V_i)_{i \geq 0}, (E_i)_{i \geq 1}, \geq)$ be a properly ordered Bratteli diagram. The *Vershik (or adic) map* is the homeomorphism $\varphi_B : X_B \rightarrow X_B$ wherein $\varphi_B(x_{\max}) = x_{\min}$ and, for any other point $(e_1, e_2, \dots) \neq x_{\max}$, the map sends the path to its successor [20]; in particular, let k be the smallest number that e_k is not a max edge and let f_k be the immediate successor of e_k . Then $\varphi_B(e_1, e_2, \dots) = (f_1, \dots, f_{k-1}, f_k, e_{k+1}, e_{k+2}, \dots)$, where (f_1, \dots, f_{k-1}) is the min path in $E_{0,k-1}$ having the range $s(f_k)$.

A Bratteli diagram is of bounded rank if the number of vertices at every level is uniformly bounded, that is, $\max_n \#V_n = d < \infty$. When a finite-rank diagram B is properly ordered, the Vershik system (X_B, T_B) on that is said to be of *finite topological rank* and the minimum such d (as there are equivalent ordered Bratteli diagrams) is called the rank of (X_B, T_B) [6].

Using Kakutani–Rokhlin partitions for Cantor minimal systems, Herman, Putnam, and Skau proved the following result.

THEOREM 2.1. [20] *Let (X, T) be a Cantor minimal system. Then T is topologically conjugate to a Vershik map on a Bratteli compactum X_B of a simple properly ordered Bratteli diagram B . Furthermore, given $x_0 \in X$, we may choose the conjugating map $\beta : X \rightarrow X_B$ so that $\beta(x_0)$ is the unique infinite max path in B .*

We use the notation $\text{rank}_{\text{top}}(X, T) = d$ when the conjugate Vershik system to (X, T) is of rank d .

2.3. Dimension groups. Here we recall some preliminaries about dimension groups in as much detail as we need in the following (mainly in Example 3.4). For more detailed information, we refer the reader to [11, 20].

Every Bratteli diagram $B = ((V_n)_{n \geq 0}, (E_n)_{n \geq 1})$ can be represented by a sequence of ordered groups $\mathbb{Z}^{|V_n|}$ mapped into each other by a sequence of positive group homomorphisms $\{A_n\}_n$:

$$\mathbb{Z}^{|V_0|} \xrightarrow{A_0} \mathbb{Z}^{|V_1|} \xrightarrow{A_1} \mathbb{Z}^{|V_2|} \xrightarrow{A_2} \dots$$

One can associate a dimension group, denoted by $K^0(V, E)$, which is the *direct limit* of the above system of ordered groups and homomorphisms [11]. There will be a partial ordering induced on $K^0(V, E)$ inducing a positive cone $K^0_+(V, E)$. Each non-zero element of $K^0_+(V, E)$ can be considered as an *order unit*. The order unit corresponding to the class of $1 \in \mathbb{Z}^{|V_0|}$ is called the *distinguished order unit*. By definition, the dimension group of a Bratteli diagram is independent of the choice of ordering on the diagram. This gives an equivalence relation between two Vershik systems called *strong orbit equivalence* in terms of the ordered isomorphism of dimension groups with order unit. Let us recall that two Cantor minimal systems are *orbit equivalent* if there exists a homeomorphism between them preserving the orbits.

The dimension group can be defined abstractly as an ordered group [11] or for the Vershik system (X, T) on a properly ordered Bratteli diagram $B = (V, E, \geq)$ [20]. There is also an equivalent ‘dynamical definition’ for $K^0(X, T)$. That is,

$$K^0(X, T) = C(X, \mathbb{Z}) / (1 - T)(C(X, \mathbb{Z})), K^0_+(X, T) = \{[f] \mid f \geq 0\}, \mathbf{1} = [1_X],$$

where $(1 - T)(C(X, \mathbb{Z}))$ denotes the subgroup of $C(X, \mathbb{Z})$ containing all the integer-valued co-boundaries:

$$(1 - T)(C(X, \mathbb{Z})) = \{f \in C(X, \mathbb{Z}) \mid \text{there exists } g \in C(X, \mathbb{Z}) \text{ } f = g - g \circ T\}.$$

Let $\mathcal{M}_T(X)$ denote the set of all invariant measures of (X, T) . Then the *infinitesimal subgroup* of $K^0(X, T)$ is defined by

$$\text{Inf}(K^0(X, T)) = \left\{ [f] \mid \text{for all } \mu \in \mathcal{M}_T(X) \int f d\mu = 0 \right\}.$$

The equivalence of these definitions was established in the following remarkable results.

THEOREM 2.2. [20] *Let (G, G^+, u) be a dimension group with distinguished order unit. Then the following are equivalent.*

- (1) *There exists an essentially minimal Cantor system (X, T) such that $K^0(X, T) \cong G$ as ordered groups with order units.*
- (2) *There is a properly ordered Bratteli diagram $B = (V, E, \geq)$ such that $K^0(V, E) \cong G$ as ordered groups with order units.*

THEOREM 2.3. [16] *Two Cantor minimal systems (X, T) and (Y, S) are:*

- (1) *strongly orbit equivalent if and only if $K^0(X, T)$ and $K^0(Y, S)$ are isomorphic as ordered groups with order units;*
- (2) *orbit equivalent if and only if $K^0(X, T) / \text{Inf}(K^0(X, T))$ and $K^0(Y, S) / \text{Inf}(K^0(Y, S))$ are isomorphic as ordered groups with order units.*

The *rank* of a dimension group (as an abelian group) is defined as the dimension of its vector space over \mathbb{Q} . For a Cantor minimal system (X, T) , the rank of $K^0(X, T)$ is called the *algebraic rank* of (X, T) .

2.4. *S-adic representation of minimal subshifts.* One of the models to represent a minimal system of finite topological rank is the *S-adic* representation [10, 6]. We use the definitions of [6].

Let $\{A_n\}_{n \geq 0}$ be a sequence of finite alphabets and suppose that $\tau = (\tau_n : A_{n+1} \rightarrow A_n^*)_{n \geq 0}$ is a *directive sequence* of *morphisms* such that for every $a \in A_{n+1}$, $\tau_n(a)$ is not the empty word. Then there is a sequence of $|A_{n+1}| \times |A_n|$ matrices M_n (also denoted by M_{τ_n}) so that for every $n \geq 0$ each entry $(M_n)_{ij}$ counts the number of occurrences of the j th letter of A_n in $\tau_n(a_i)$, $a_i \in A_{n+1}$. When all the matrices are positive, the sequence of morphisms is called *positive*. The sequence τ is *proper* if every τ_n is proper and the latter means that for every n , there exist letters a, b in A_n such that for all $c \in A_{n+1}$, $\tau_n(c)$ starts with a and ends with b . Moreover, τ is called *primitive* if for every $n \geq 1$ there exists some $N \geq n$ such that $M_{\tau_{[n,N]}} > 0$, where $\tau_{[n,N]} = \tau_n \circ \tau_{n+1} \circ \dots \circ \tau_{N-1}$. For every $n \geq 0$, let

$$\mathcal{L}^{(n)}(\tau) = \{w \in A_n^* \mid \text{there exists } N > n \text{ there exists } a \in A_N \text{ } w \text{ occurs in } \tau_{[n,N]}(a)\}.$$

Suppose that $X_\tau^{(n)}$ is the set of points $x \in A_n^{\mathbb{Z}}$ so that all the factors of x belong to $\mathcal{L}^{(n)}(\tau)$. Then $(X_\tau^{(n)}, \sigma)$ is a subshift and, if τ is primitive, it will be a minimal subshift. The minimal subshift $(X, S) := (X_\tau^{(0)}, S)$ is called the *S-adic subshift* generated by the directive sequence τ .

Let $B = ((V_k)_{k \geq 0}, (E_k)_{k \geq 1}, \geq)$ be an ordered Bratteli diagram. There exists a sequence of morphisms $\sigma^B = (\sigma_i^B : V_i \rightarrow V_{i-1})_{i \geq 1}$ defined by, for $i \geq 2$,

$$\sigma_i^B(v) = s(e_1(v))s(e_2(v)) \cdots s(e_k(v)),$$

where $\{e_j(v) \mid j = 1, \dots, k(v)\}$ is the ordered set of edges in E_i with range v and, for $i = 1$, $\sigma_1^B : V_1^* \rightarrow E_1^*$, $\sigma_1^B(v) = e_1(v) \cdots e_\ell(v)$, where $e_1(v), \dots, e_\ell(v)$ are all the edges with range v .

Each σ_i^B extends to V_i^* by concatenation. For every $i, j \in \mathbb{N}$ with $i < j$, we define $\sigma_{[i,j]}^B : V_j^* \rightarrow V_i^*$ by $\sigma_{[i,j]}^B = \sigma_{i+1}^B \circ \sigma_{i+2}^B \circ \dots \circ \sigma_j^B$. Also, let $\sigma_{[i,i]}^B : V_i^* \rightarrow V_i^*$ be the identity map.

PROPOSITION 2.4. [6, Proposition 4.6] *Let (X, T) be a minimal Cantor system given by a Bratteli–Vershik representation B . If (X, T) is a subshift, then, after an appropriate telescoping, the S-adic subshift generated by the sequence of morphisms $\sigma^B = (\sigma_n^B : V_i \rightarrow V_{i-1})_{i \geq 1}$ read on B is conjugate to (X, T) .*

2.5. *Fine–Wilf theorem.* One of the well-known theorems of symbolic dynamics that we need in the following is a form of the so-called Fine–Wilf theorem [14] that follows easily from an induction argument and the original form of that theorem, and we state it here for the sake of completeness. Let us first recall that if A is an alphabet and if $w = w_1 \cdots w_n$ with $w_1, \dots, w_n \in A$, then w is called *periodic* with period $p \leq n$ whenever $w_i = w_{i+p}$ for every $1 \leq i \leq n - p$.

LEMMA 2.5. Let A be a finite alphabet and let $k \in \mathbb{N}$. If $w \in A^*$ has periods p_1, p_2, \dots, p_k such that $|w| \geq p_1 + p_2 + \dots + p_k - \gcd(p_1, p_2, \dots, p_k)$, then w is periodic with period $\gcd(p_1, p_2, \dots, p_k)$.

3. Factoring and Bratteli diagrams

Let (X, T) and (Y, S) be two Cantor minimal systems such that for some continuous map $\alpha : X \rightarrow Y$, $\alpha \circ T = S \circ \alpha$. It is natural to realize this relation between the two systems in terms of a relation between their associated Bratteli diagrams. For the special case that α is almost one-to-one (that is, α is one-to-one for a generic point in X), some characterizations have been proved in [23]. For the general case, in [1] the authors defined the notion of an *ordered premorphism* between two ordered Bratteli diagrams and proved the equivalence of the existence of an ordered premorphism between two properly ordered Bratteli diagrams with the existence of a factor map between their associated Bratteli–Vershik systems. Here for the sake of completeness and to be more precise we first recall the definition of an ordered premorphism and Proposition 4.6 of [1] and then, to be prepared for the proof of Theorem 1.1, we will show how such an ordered premorphism induces a sequence of morphisms between the two diagrams. There will be an example at the end of this section.

Definition 3.1. [1, Definition 3.1] Let $B_1 = (V, E, \geq)$ and $B_2 = (W, S, \geq')$ be ordered Bratteli diagrams. By an *ordered premorphism* (or just a *premorphisms* if there is no confusion) $f : B_1 \rightarrow B_2$, we mean a triple $(F, (f_n)_{n=0}^\infty, \geq)$, where $(f_n)_{n=0}^\infty$ is a cofinal (that is, unbounded) sequence of positive integers with $f_0 = 0 \leq f_1 \leq f_2 \leq \dots$, F consists of a disjoint union of sets of edges, say $F_0 \cup F_1 \cup F_2 \cup \dots$ together with a pair of range and source maps $r : F \rightarrow W, s : F \rightarrow V$, and \geq is a partial ordering on F such that:

- (1) each F_n is a non-empty finite set, $s(F_n) \subseteq V_n, r(F_n) \subseteq W_{f_n}, F_0$ is a singleton, $s^{-1}\{v\}$ is non-empty for all v in V , and $r^{-1}\{w\}$ is non-empty for all w in W ;
- (2) $e, e' \in F$ are comparable if and only if $r(e) = r(e')$, and \geq is a linear order on $r^{-1}\{w\}$ for all $w \in W$;
- (3) the following diagram of $f : B_1 \rightarrow B_2$

$$\begin{array}{ccccccc}
 V_0 & \xrightarrow{E_1} & V_1 & \xrightarrow{E_2} & V_2 & \xrightarrow{E_3} & \dots \\
 F_0 \downarrow & & F_1 \downarrow & & F_2 \downarrow & & \\
 W_{f_0} & \xrightarrow{S_{f_0, f_1}} & W_{f_1} & \xrightarrow{S_{f_1, f_2}} & W_{f_2} & \xrightarrow{S_{f_2, f_3}} & \dots
 \end{array}$$

commutes. The ordered commutativity of the diagram of f means that for each $n \geq 0, E_{n+1} \circ F_{n+1} \cong F_n \circ S_{f_n, f_{n+1}}$, that is, there is a (necessarily unique) bijective map from $E_{n+1} \circ F_{n+1}$ to $F_n \circ S_{f_n, f_{n+1}}$ preserving the order and intertwining the respective source and range maps.

To see how the ordered premorphism $f : B_1 \rightarrow B_2$ induces a factoring $\alpha : X_{B_2} \rightarrow X_{B_1}$ between the two Vershik systems, let $x = (s_1, s_2, \dots)$ be an infinite path in X_{B_2} . Define

the path $\alpha(x) = (e_1, e_2, \dots)$ in X_{B_1} as follows. Fix $n \geq 1$. By Definition 3.1, the diagram

$$\begin{array}{ccc} V_0 & \xrightarrow{E_{0,n}} & V_n \\ F_0 \downarrow & & \downarrow F_n \\ W_0 & \xrightarrow{S_{0,f_n}} & W_{f_n} \end{array}$$

commutes, that is, $F_0 \circ S_{0,f_n} \cong E_{0,n} \circ F_n$. Thus, there is a unique path $(e_1, e_2, \dots, e_n, d_n)$ in $E_{0,n} \circ F_n$ (in fact, $(e_1, e_2, \dots, e_n) \in E_{0,n}$ and $d_n \in F_n$) corresponding to the path $(d_0, s_1, \dots, s_{f_n})$ in $F_0 \circ S_{0,f_n}$, where d_0 is the unique element of F_0 . So, the path $\alpha(x) = (e_1, e_2, \dots)$ in X_B is associated to the path $x = (s_1, s_2, \dots)$ in X_{B_2} .

The correspondence of factor maps between two Vershik systems and ordered premorphisms between the associated properly ordered Bratteli diagrams was established in the proof of the following proposition.

PROPOSITION 3.2. [1, Proposition 4.6] *Let (X, T) and (Y, S) be Cantor minimal systems, and let $x \in X$ and $y \in Y$. Suppose that B_1 and B_2 are Bratteli–Vershik models for (Y, S, y) and (X, T, x) , respectively. The following statements are equivalent:*

- (1) *there is a factor map $\alpha : (X, T) \rightarrow (Y, S)$ with $\alpha(x) = y$;*
- (2) *there is an ordered premorphism f from B_1 to B_2 .*

More precisely, there is a one-to-one correspondence between the set of factor maps α as in (1) and the set of equivalence classes of ordered premorphisms f from B_1 to B_2 .

Definition 3.3. Let $B_1 = (V, E, \geq)$ and $B_2 = (W, S, \geq')$ be two ordered Bratteli diagrams with an ordered premorphism $f : B_1 \rightarrow B_2$ between them. One can describe an induced sequence of morphisms

$$\tau = (\tau_n : W_n \rightarrow V_n^*)_{n \geq 0}$$

between the two diagrams as follows. Let $f = (F, (f_n)_{n=0}^\infty, \geq)$. Assume that $f_n = n$ for every $n \geq 0$. (We can always make this assumption using a telescoping of B_2 along a strictly increasing subsequence of $(f_n)_{n=0}^\infty$.) Let $V = \bigcup_{n=0}^\infty V_n$ and $W = \bigcup_{n=0}^\infty W_n$ be the sets of vertices of B_1 and B_2 , respectively. Then, for every $n \geq 0$, the morphism τ_n is a map $\tau_n : W_n \rightarrow V_n^*$ such that for every vertex $w \in W_n$, $\tau(w)$ represents the ordered set of vertices of V_n connected to w via f . More precisely, let $F = \bigcup_{n=0}^\infty F_n$ be the decomposition of the set of edges of f and let $\{g_1, g_2, \dots, g_m\}$ be the ordered set of edges in F_n with range w . Then

$$\tau_n(w) = s(g_1)s(g_2) \cdots s(g_m).$$

We extend τ_n to W_n^* by concatenation. Observe that the essential property of the ordered premorphism f , which is the ordered commutativity, reads as

$$\tau_n \circ \sigma_{n+1}^{B_2} = \sigma_{n+1}^{B_1} \circ \tau_{n+1} \tag{3.1}$$

for all $n \geq 1$. Conversely, every sequence of morphisms satisfying (3.1) induces an ordered premorphism between the two diagrams; however, this direction is not needed in this note.

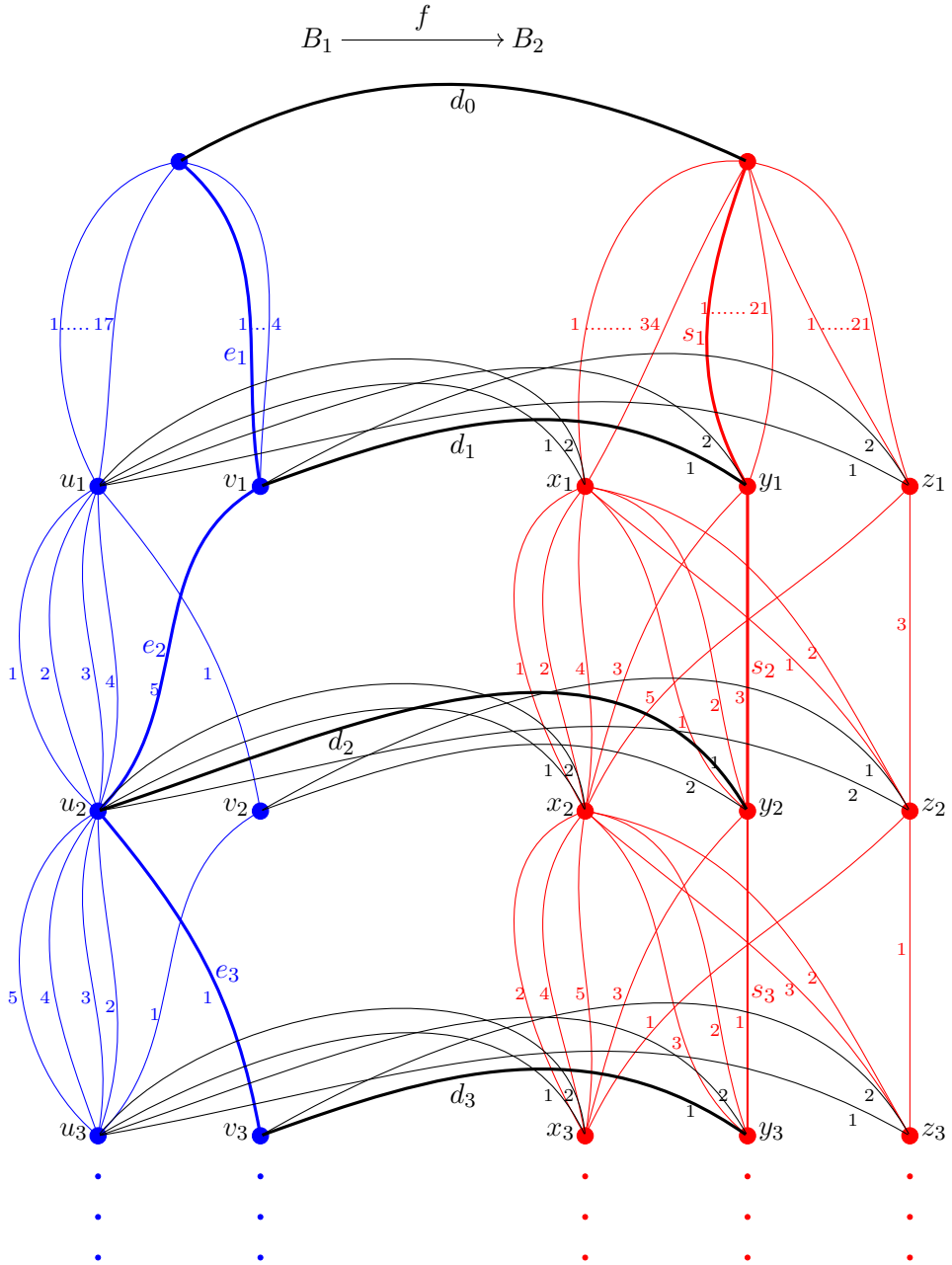


FIGURE 1. An ordered premorphism $f : B_1 \rightarrow B_2$ inducing a factor map $\alpha : X_{B_2} \rightarrow X_{B_1}$. The thick path (s_1, s_2, s_3, \dots) in the right-hand diagram is mapped by α to the thick path (e_1, e_2, e_3, \dots) in the left-hand diagram.

Example 3.4. Figure 1 shows an example of an ordered premorphism $f : B_1 \rightarrow B_2$ between two ordered Bratteli diagrams which induces a factor map $\alpha : X_{B_2} \rightarrow X_{B_1}$. An

infinite path (s_1, s_2, s_3, \dots) on B_2 and its image (e_1, e_2, e_3, \dots) on B_1 under α are drawn in Figure 1 to illustrate the formula of α as explained in the remarks following Definition 3.1. Note that the orderings on the two diagrams as well as the orderings of the premorphisms are repeated at every other level. In fact, the two diagrams are stationary. The left-hand diagram is associated to the Sturmian system with rotation number $\theta = 1 + \sqrt{5}$ and the right-hand diagram, which is a substitution, is in fact orbit equivalent to the Sturmian system with rotation number θ . To see the orbit equivalence (for the sake of completeness) by Theorem 2.3(3), it is sufficient to compute the dimension group $K^0(B_2)$ and to show that $K^0(B_2)/\text{Inf}(K^0(B_2)) \cong \mathbb{Z} + \theta\mathbb{Z}$. The direct limit system associated to $K^0(B_2)$ is

$$\mathbb{Z} \xrightarrow{A_0} \mathbb{Z}^3 \xrightarrow{A_1} \mathbb{Z}^3 \xrightarrow{A_2} \dots \longrightarrow K^0(V, E)$$

with

$$A_0 = \begin{bmatrix} 34 \\ 21 \\ 21 \end{bmatrix}, \quad A_n = A = \begin{bmatrix} 3 & 1 & 1 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \quad n \geq 1.$$

The Perron eigenvalue of A is $\theta = 2 + \sqrt{5}$ associated to the (normalized) eigenvector $\gamma = (2/(5 + \sqrt{5}))(1 + \sqrt{5})/2, 1, 1)$. As for all stationary systems, by the well-known Perron–Frobenius theorem, the positive cone is

$$K^0_+(B_2) = \{v \in \mathbb{Z}^3 \mid \text{there exists } k \in \mathbb{N}, A^k v \geq 0\} = \{v \in \mathbb{Z}^3 \mid \gamma \cdot v \geq 0\}$$

and the unique invariant measure (which is associated to the unique state of $K^0(B_2)$) is determined by

$$\tau([v]) = (\gamma \cdot v)/\lambda^n,$$

where v is in the n th group of the direct system and $[v]$ is its corresponding equivalence class in $K^0(B_2)$. Then it is not hard to see that

$$\text{for all } v = (0, n, -n) \text{ with } n \geq 1, \quad \tau([v]) = 0.$$

So, the infinitesimal subgroup is non-trivial and isomorphic to \mathbb{Z} ; hence, $K^0(B_2)/\text{Inf}(K^0(B_2)) \cong \mathbb{Z} + \theta\mathbb{Z}$.

Now let us show how to determine the sequence of morphisms induced by the ordered premorphism f between the two diagrams. To do that, we first label the vertices of the two diagrams by $V_i = \{u_i, v_i\}$ and $W_i = \{x_i, y_i, z_i\}$ for every $i \geq 1$. Then the morphisms are

$$\begin{aligned} \tau_i(x_i) &= u_i u_i, & \tau_i(y_i) &= v_i u_i, & \tau_i(z_i) &= u_i v_i, & \text{if } i \text{ is odd,} \\ \tau_i(x_i) &= u_i u_i, & \tau_i(y_i) &= u_i v_i, & \tau_i(z_i) &= v_i u_i, & \text{if } i \text{ is even.} \end{aligned}$$

Let us examine the order commutativity in terms of the morphisms on a fixed vertex of the right-hand diagram. Consider for example $y_2 \in W_2$. Then

$$\tau_1 \circ \sigma_2^{B_2}(y_2) = \tau_1(x_1 x_1 y_1) = \tau_1(x_1) \tau_1(x_1) \tau_1(y_1) = u_1 u_1 u_1 u_1 v_1 u_1$$

and

$$\sigma_2^{B_1} \circ \tau_2(y_2) = \sigma_2^{B_1}(u_2v_2) = \sigma_2^{B_1}(u_2)\sigma_2^{B_1}(v_2) = u_1u_1u_1u_1v_1u_1.$$

We refer the reader to [1] to see more examples of ordered premorphisms.

4. Rank of factors

In this section we prove Theorem 1.1. To be prepared we firstly mention some observations and facts about *microscoping* of ordered Bratteli diagrams.

The following ‘packing lemma’ is useful for reducing the rank of an ordered Bratteli diagram. Also, it is of interest in its own right. As it is a form of *microscoping of an ordered Bratteli diagram*, one may conclude that from the results of *S*-adic sequences of morphisms [3]. Here, not losing the consistency of our literature, we make a proof for that using the ‘traditional’ language of Bratteli diagrams that is in fact for the sake of completeness.

LEMMA 4.1. *Let $B = ((V_n)_{n \geq 0}, (E_n)_{n \geq 1}, \geq)$ be an ordered Bratteli diagram for which there exist some $k \geq 1$ and a set of words $W \subseteq V_{k-1}^*$ such that*

$$\sigma_k^B(v) \in W^* \quad \text{for every } v \in V_k. \tag{4.1}$$

Suppose that W is a minimal subset of V_{k-1}^ (with respect to the inclusion relation) satisfying (4.1). Then there is an ordered Bratteli diagram B' isomorphic to B which is constructed from B by adding W as a set of vertices between levels V_{k-1} and V_k .*

Proof. Let $W = \{w_1, \dots, w_s\}$, where the w_i are distinct. Define the set of vertices of $B' = ((V'_n)_{n \geq 0}, (E'_n)_{n \geq 1}, \geq')$ by

$$V'_n = V_n \quad \text{for } 0 \leq n < k, \quad V'_k = W, \quad \text{and} \quad V'_n = V_{n-1} \quad \text{for } n > k.$$

For the set of edges of B' , first we set

$$E'_n = E_n \quad \text{for } 1 \leq n < k \quad \text{and} \quad E'_n = E_{n-1} \quad \text{for } n \geq k + 2.$$

It remains to define E'_k and E'_{k+1} . Since every $w \in W$ is a word in V_{k-1}^* , we can define (uniquely) a partially ordered set of edges E'_k from $V'_{k-1} = V_{k-1}$ to $V'_k = W$ such that $\sigma_k^{B'}(w) = w$ for every $w \in W$. To define E'_{k+1} , first note that for every $v \in V'_{k+1} = V_k$ we have

$$\sigma_k^B(v) = w_{i_1}w_{i_2} \cdots w_{i_r} \tag{4.2}$$

for some $w_{i_1}, w_{i_2}, \dots, w_{i_r}$ in W depending on v . (Note that this representation of $\sigma_k^B(v)$ in terms of the words of W is not necessarily unique but we fix one representation.) Then we can define a partially ordered set of edges E'_{k+1} from $V'_k = W$ to $V'_{k+1} = V_k$ such that $\sigma_{k+1}^{B'}(v) = w_{i_1}w_{i_2} \cdots w_{i_r}$, where $w_{i_1}, w_{i_2}, \dots, w_{i_r}$ are considered as vertices of V'_k here. The minimality of W guarantees that for very $w \in W$ there is at least one edge in E'_{k+1} with source w . The resulting ordered Bratteli diagram B' is isomorphic to B since $E'_k \circ E'_{k+1}$ is order isomorphic to E_k . In fact, if $v \in V_k$ and if we consider the representation in (4.2)

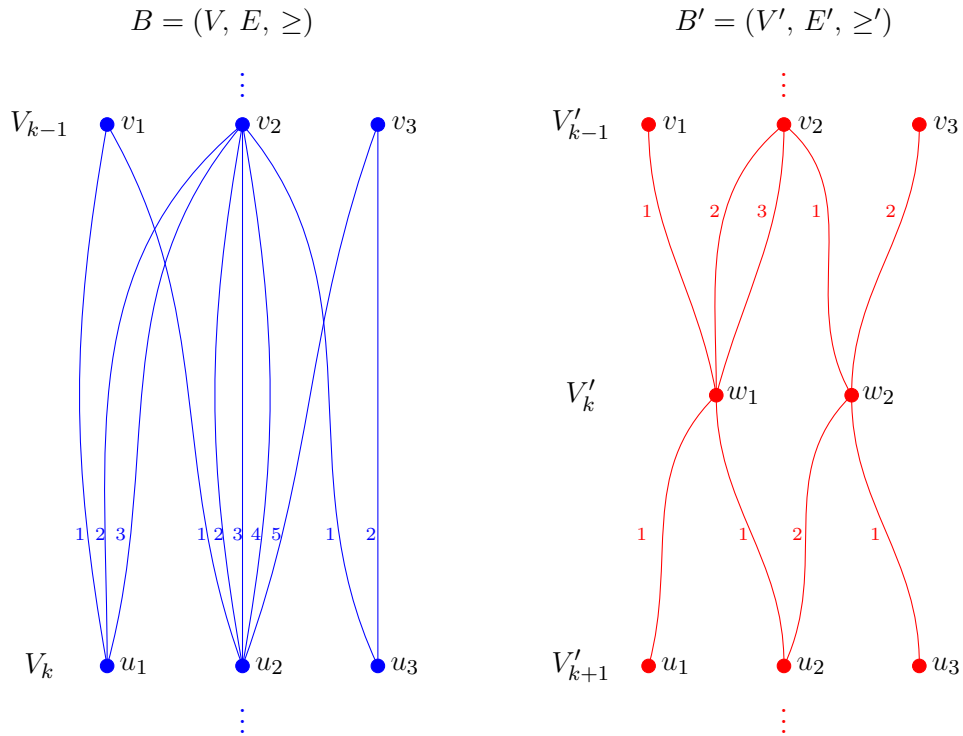


FIGURE 2. Applying Lemma 4.1 to the left-hand diagram will make the right-hand diagram.

for $\sigma_k^B(v)$, then

$$\sigma_{[k-1, k+1]}^{B'}(v) = \sigma_k^{B'}(\sigma_{k+1}^{B'}(v)) = \sigma_k^{B'}(w_{i_1} w_{i_2} \cdots w_{i_r}) = w_{i_1} w_{i_2} \cdots w_{i_r} = \sigma_k^B(v).$$

It follows that B' is isomorphic to B . □

Example 4.2. Let $B = (V, E, \geq)$ be an ordered Bratteli diagram in which the levels $V_{k-1} = \{v_1, v_2, v_3\}$ and $V_k = \{u_1, u_2, u_3\}$ are as in the left-hand diagram in Figure 2.

So,

$$\sigma_k^B(u_1) = v_1 v_2 v_2, \quad \sigma_k^B(u_2) = v_1 v_2 v_2 v_2 v_3, \quad \sigma_k^B(u_3) = v_2 v_3.$$

The described procedure in the proof of the previous lemma to construct the diagram $B' = (V', E', \geq')$ can be implemented by letting

$$w_1 = v_1 v_2 v_2, \quad w_2 = v_2 v_3$$

that is shown in the right-hand diagram in Figure 2. Hence,

$$\sigma_{k+1}^{B'}(u_1) = w_1, \quad \sigma_{k+1}^{B'}(u_2) = w_1 w_2, \quad \sigma_{k+1}^{B'}(u_3) = w_2.$$

As it is easy to check, the telescoping of the right-hand diagram between levels V'_{k-1} and V'_{k+1} is the same as the left-hand diagram.

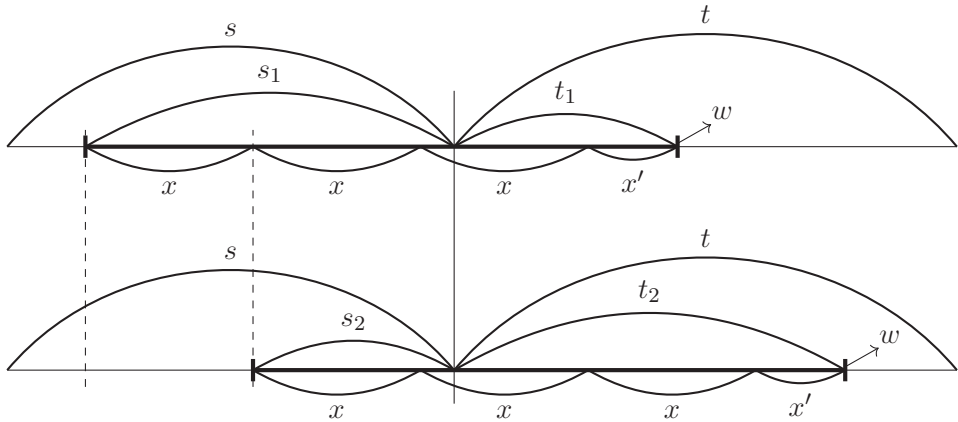


FIGURE 3. The two thick line segments represent the periodic word w .

The following lemma enables us to control the size of the generating set of a certain set of words. It can be considered as a generalization of the classical Fine–Wilf theorem [14].

LEMMA 4.3. *Let A be a finite alphabet and let $p \in \mathbb{N}$. Let $s_1, \dots, s_p, t_1, \dots, t_p$, and w be words in A^* such that*

$$w = s_1 t_1 = s_2 t_2 = \dots = s_p t_p.$$

Suppose that there are two words s and t in A^ with $|s|, |t| \geq |w|$ such that for any $1 \leq i \leq p$, s_i is a suffix of s and t_i is a prefix of t . Then there exists a set of words $B \subseteq A^*$ such that:*

- (1) $\text{card}(B) \leq 3$;
- (2) $s_i, t_i \in B^*$ for every $1 \leq i \leq p$.

Proof. We may assume that each s_i and each t_i is non-empty. Also, we may assume that $s_i \neq s_j$ for $i \neq j$. If $p = 1$, we take $B = \{s_1, t_1\}$. Hence, in the following we assume that $p \geq 2$. Note that for every $1 \leq i, j \leq p$, since both s_i and s_j are suffixes of s , either s_i is a suffix of s_j or s_j is a suffix of s_i . Suppose that the s_i are sorted by their lengths:

$$|s_1| > |s_2| > \dots > |s_p|. \tag{4.3}$$

This implies that $|t_1| < |t_2| < \dots < |t_p|$ (since $|s_i| + |t_i| = |w|$). We claim that w is periodic with period $|s_1| - |s_2|$. To prove the claim, first note that both s_1 and s_2 are suffixes of s and $|s_1| > |s_2|$. Hence, there is a non-empty word x such that $s_1 = x s_2$. See Figure 3.

Similarly, since t_1 and t_2 are prefixes of t and $|t_1| < |t_2|$, there is a non-empty word y such that $t_2 = t_1 y$. Note that $|x| < |s_1| < |w|$ and $|y| = |t_2| - |t_1| = |s_1| - |s_2| = |x|$. We

have

$$wy = s_1t_1y = s_1t_2 = xs_2t_2 = xw. \tag{4.4}$$

Let $|w| = k|x| + r$, where $k \in \mathbb{N}$ and $0 \leq r < |x|$. It follows from (4.4) that $wy^k = x^k w$. Since $|y^k| = |x^k| = k|x| \leq |w|$, we see that y^k is a suffix of w . So, there is a word x' such that $w = x'y^k$. Then $wy^k = x^k w = x^k x'y^k$ and so $w = x^k x'$. Note that $|x'| = |w| - k|x| = r < |x|$. Moreover, $x'y^k = w = x^k x'$ implies that x' is a prefix of x . Therefore, w is periodic with period $|x|$. The claim is proved.

Using $w = s_i t_i = s_{i+1} t_{i+1}$, a similar argument shows that w is periodic with period $|s_i| - |s_{i+1}|$ for all $1 \leq i < p$. We have

$$\sum_{i=1}^{p-1} (|s_i| - |s_{i+1}|) = |s_1| - |s_p| < |w|.$$

Applying Lemma 2.5, it follows that w is periodic with period

$$h = \gcd(|s_1| - |s_2|, \dots, |s_{p-1}| - |s_p|).$$

Thus, we can write $w = u^k u'$ for some $u, u' \in A^*$ and $k \in \mathbb{N}$, where $|u'| < |u| = h$ and u' is a prefix of u . See Figure 4.

Since each s_i is a prefix of w , we can write $s_i = u^{k_i} u_i$, where $k_i \geq 0$, $|u_i| < |u|$, and u_i is a prefix of u . We claim that $u_1 = u_2 = \dots = u_p$. To see this, let $2 \leq i \leq p$. Since s_i is a suffix of s_1 and h divides $|s_1| - |s_i|$, we can write $s_1 = u^\ell s_i$ for some $\ell \geq 1$. Hence,

$$u^{k_1} u_1 = s_1 = u^\ell s_i = u^\ell u^{k_i} u_i = u^{\ell+k_i} u_i.$$

Since $|u_1|, |u_i| < |u|$, this implies that $u_1 = u_i$ and the claim is proved. Since u_1 is a prefix of u , there is a word u'_1 such that $u = u_1 u'_1$. See Figure 4.

We consider the following three cases.

Case I: u_1 is the empty word. Then $s_i = u^{k_i}$ and hence $t_i = u^{\ell_i} u'$ for some $\ell_i \geq 0$. We set $B = \{u, u'\} \setminus \{\emptyset\}$. (Note that u' may be the empty word.) Then B generates all the s_i and t_i .

Case II: u_1 is non-empty and $|u'_1| > |t_1|$. This is equivalent to $|u_1 t_1| < |u|$ since $u = u_1 u'_1$. Then $u' = u_1 t_1$ and, for every $2 \leq i \leq p$, since t_1 is a prefix of t_i , there exists some $\ell_i \geq 0$ such that $t_i = u'_1 u^{\ell_i} u'$. All together, these imply that $B = \{u_1, u'_1, t_1\}$ generates all the s_i and t_i .

Case III: u_1 is non-empty and $|u'_1| \leq |t_1|$. This means that $|u_1 t_1| > |u|$. It follows that $t_1 = u'_1 u^{\ell_1} u'$ for some $\ell_1 \geq 0$ and, for every $2 \leq i \leq p$, there exists $\ell_i \geq 1$ such that $t_i = u'_1 u^{\ell_i} u'$. Hence, $B = \{u_1, u'_1, u'\} \setminus \{\emptyset\}$ generates all the s_i and t_i .

Therefore, in each case we obtained a set of words B satisfying Conditions (1) and (2). This finishes the proof. □

Remark 4.4. In the preceding lemma, the upper bound 3 for the cardinality of the set B is sharp. For example, let $A = \{x, y, z, w\}$, $w = xyzwxyzwxyzwx$, and

$$\begin{aligned} s_1 &= xyzwxyzwxy, & t_1 &= zwx, & s_2 &= xyzwxy, & t_2 &= zwxxyzwx, \\ s_3 &= xy, & t_3 &= zwxxyzwxyzwx, & s &= s_1, & t &= t_3. \end{aligned}$$

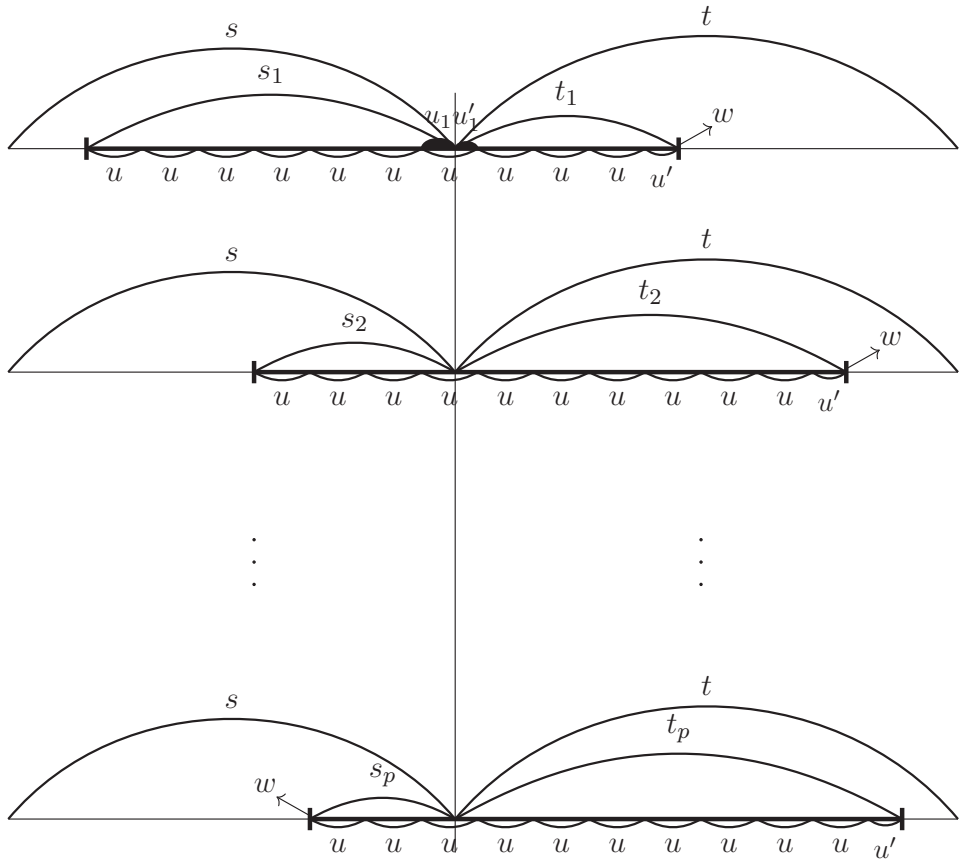


FIGURE 4. A depiction of the idea of the proof of Lemma 4.3.

Then the argument in the proof of the preceding lemma gives the set of words $B = \{u_1, u'_1, u'\} = \{xy, zw, x\}$ generating all the s_i and t_i . However, it is not hard to see that there is no generating set B' with $\text{card}(B') < 3$.

Remark 4.5. Let B be a properly ordered Bratteli diagram. Then there exists a telescoping of B , say $B' = ((V_k)_{k \geq 0}, (E_k)_{k \geq 1}, \geq)$, such that for each $k \geq 0$ there are (necessarily unique) vertices v_{\min}^k and v_{\max}^k in V_k such that for every $v \in V_{k+1}$, $\sigma_{k+1}^{B'}(v)$ starts with v_{\min}^k and ends with v_{\max}^k , that is, the min edge in E_{k+1} to v comes from v_{\min}^k and the max edge to v comes from v_{\max}^k . This simple fact follows easily from an argument using König's lemma similar to the argument showing that every ordered Bratteli diagram has at least one min infinite path.

PROPOSITION 4.6. *Let $f: B_1 \rightarrow B_2$ be an ordered premorphism between two properly ordered Bratteli diagrams such that B_1 is simple. Consider the Vershik system on B_1 .*

Then

$$\text{rank}_{\text{top}}(X_{B_1}, T_{B_1}) \leq 3 \text{rank}(B_2).$$

Proof. Let $B_1 = (V, E, \geq)$, $B_2 = (W, S, \geq)$, and $f = (F, (f_n)_{n=0}^\infty, \geq)$. Let $V = \bigcup_{n=0}^\infty V_n$ and $W = \bigcup_{n=0}^\infty W_n$ be the canonical decompositions of V and W , respectively. Also, let the morphisms τ_n associated to the ordered premorphism f be as in Definition 3.3.

If $\text{rank}_{\text{top}}(X_{B_1}, T_{B_1}) = 1$, then there is nothing to prove. Thus, we suppose that $\text{rank}_{\text{top}}(X_{B_1}, T_{B_1}) \geq 2$. In particular, B_1 has infinitely many levels each of which has at least two vertices.

By making telescoping of the two diagrams along appropriate subsequences if necessary, we can (and do) make the following assumptions.

- (1) For every $n \in \mathbb{N}$, every $v \in V_n$, and every $v' \in V_{n+1}$, there is an edge in E_{n+1} with source v and range v' (since B_1 is simple).
- (2) For every $n \geq 0$, there are vertices v_{\min}^n and v_{\max}^n in V_n such that for every $v \in V_{n+1}$, $\sigma_{n+1}^{B_1}(v)$ starts with v_{\min}^n and ends with v_{\max}^n (by Remark 4.5).
- (3) $\text{card}(V_n) \geq 2$ for all $n \in \mathbb{N}$ (by the preceding paragraph).
- (4) For every $n \geq 0$, $f_n = n$, and for every $v \in V_n$ and every $w \in W_n$, there is an edge in F with source v and range w . (This follows from (1) and Definition 3.3 and an appropriate telescoping of B_2 .)

The following claim contains the main part of the proof.

Claim. For every $n \geq 1$, there exist some $\ell > n$ and a set of words $C_n \subseteq V_n^*$ such that:

- (5) $\sigma_{[n,\ell]}^{B_1}(v) \in C_n^*$ for all $v \in V_\ell$;
- (6) $\text{card}(C_n) \leq 3 \text{card}(W_n)$.

To prove the claim, fix $n \geq 1$. First, using (1) and (3), there is $\ell > n + 1$ such that:

- (7) $|\sigma_{[n,\ell-1]}^{B_1}(v_{\min}^{\ell-1})|, |\sigma_{[n,\ell-1]}^{B_1}(v_{\max}^{\ell-1})| > \max\{|\tau_n(w)| : w \in W_n\}$.

Fix an arbitrary vertex $w_0 \in W_\ell$. Suppose that $\tau_\ell(w_0) = v_1 v_2 \cdots v_m$, where $v_1, v_2, \dots, v_m \in V_\ell$. By (4), every vertex of V_ℓ appears at least one time in the word $\tau_\ell(w_0)$ and hence $V_\ell = \{v_1, v_2, \dots, v_m\}$ as sets. As $\text{card}(V_\ell) \geq 2$, we see that $m \geq 2$. Suppose that $\sigma_{[n,\ell]}^{B_2}(w_0) = w_1 w_2 \cdots w_r$, where $w_1, w_2, \dots, w_r \in W_n$. Set $z_j = \tau_n(w_j)$ for all $1 \leq j \leq r$. Using the ordered commutativity of f at the second step (see equality (3.1)), we get

$$\begin{aligned} \sigma_{[n,\ell]}^{B_1}(v_1) \sigma_{[n,\ell]}^{B_1}(v_2) \cdots \sigma_{[n,\ell]}^{B_1}(v_m) &= \sigma_{[n,\ell]}^{B_1}(\tau_\ell(w_0)) \\ &= \tau_n(\sigma_{[n,\ell]}^{B_2}(w_0)) \\ &= \tau_n(w_1 w_2 \cdots w_r) \\ &= z_1 z_2 \cdots z_r. \end{aligned} \tag{4.5}$$

For all $1 \leq k \leq m$, by (2), the word $\sigma_\ell^{B_1}(v_k)$ starts with $v_{\min}^{\ell-1}$ and ends with $v_{\max}^{\ell-1}$. Hence, for all $1 \leq k \leq m$:

- (8) $\sigma_{[n,\ell]}^{B_1}(v_k)$ starts with $\sigma_{[n,\ell-1]}^{B_1}(v_{\min}^{\ell-1})$ and ends with $\sigma_{[n,\ell-1]}^{B_1}(v_{\max}^{\ell-1})$.

In particular, by (7), $|\sigma_{[n,\ell]}^{B_1}(v_k)| > |z_j|$ for all $1 \leq j \leq r$ and all $1 \leq k \leq m$. Using this and (4.5), it follows that there are words s_1, s_2, \dots, s_{m-1} and t_1, t_2, \dots, t_{m-1} in V_n^* and words T_1, T_2, \dots, T_m in $\{z_1, \dots, z_r\}^*$ such that

$$\sigma_{[n,\ell]}^{B_1}(v_1) = T_1 s_1, \sigma_{[n,\ell]}^{B_1}(v_k) = t_{k-1} T_k s_k, \sigma_{[n,\ell]}^{B_1}(v_m) = t_{m-1} T_m \tag{4.6}$$

for all $1 < k < m$, where, for every $i = 1, \dots, m - 1$, $s_i = t_i = \emptyset$ or both s_i and t_i are non-empty and $s_i t_i = z_j$ for some $1 \leq j \leq r$. This means that we can write

$$\sigma_{[n,\ell]}^{B_1}(\tau_\ell(w_0)) = T_1 s_1 t_1 T_2 s_2 t_2 T_3 s_3 \cdots s_{m-1} t_{m-1} T_m = z_1 z_2 \cdots z_r.$$

Now we want to analyze the s_i and t_i to see how one can generate all of them by a set of words C_n of cardinality less than or equal to $3 \text{ card}(W_n)$. Set

$$s = \sigma_{[n,\ell-1]}^{B_1}(v_{\max}^{\ell-1}) \text{ and } t = \sigma_{[n,\ell-1]}^{B_1}(v_{\min}^{\ell-1}).$$

By (7), $|s|, |t| > |z_j|$ for all $j = 1, \dots, r$. Also, by (8), every $\sigma_{[n,\ell]}^{B_1}(v_k)$ starts with t and ends with s . Then (4.6) implies that each s_i is a proper suffix of s and each t_i is a proper prefix of t . Note that we have

$$\{s_i t_i \mid 1 \leq i < m\} \setminus \{\emptyset\} \subseteq \{z_j \mid 1 \leq j \leq r\} \subseteq \{\tau_n(w) \mid w \in W_n\}.$$

Let $w \in W_n$. If there is no i with $s_i t_i = \tau_n(w)$, then we simply set $C_w = \{\tau_n(w)\}$. Otherwise, we apply Lemma 4.3 with $\tau_n(w)$ in place of w , with s and t as above, and with the s_i and t_i satisfying $s_i t_i = \tau_n(w)$ to obtain a set of words $C_w \subseteq V_n^*$ such that:

- (9) $\text{card}(C_w) \leq 3$;
- (10) $s_i, t_i \in C_w^*$ for every $1 \leq i < m$ with $s_i t_i = \tau_n(w)$.

Now consider the set of words

$$C_n = \bigcup_{w \in W_n} C_w \subseteq V_n^*.$$

First note that $\tau_n(w) \in C_n^*$ for every $w \in W_n$ and hence $z_1, \dots, z_r \in C_n^*$. Moreover, $s_i, t_i \in C_n^*$ for every $1 \leq i < m$, because if s_i and t_i are non-empty then there is some j with $s_i t_i = z_j = \tau_n(w_j)$ and hence $s_i, t_i \in C_{w_j}^*$ by (10). Therefore, $\sigma_{[n,\ell]}^{B_1}(v_k) \in C_n^*$ for all $1 \leq k \leq m$. Since $V_\ell = \{v_1, v_2, \dots, v_m\}$, this implies that $\sigma_{[n,\ell]}^{B_1}(v) \in C_n^*$ for all $v \in V_\ell$. This is (5). Also, (6) follows from (9). This finishes the proof of the claim.

To complete the proof of the proposition, we will find an ordered Bratteli diagram B_1'' equivalent to B_1 such that $\text{rank}(B_1'') \leq 3 \text{ rank}(B_2)$. For this, first we put $\ell_1 = 1$ and we apply the claim with $n = \ell_1$ to obtain a natural number $\ell_2 > \ell_1$ and a set of words $C_{\ell_1} \subseteq V_{\ell_1}^*$ such that $\sigma_{[\ell_1,\ell_2]}^{B_1}(v) \in C_{\ell_1}^*$ for all $v \in V_{\ell_2}$ and $\text{card}(C_{\ell_1}) \leq 3 \text{ card}(W_{\ell_1})$. Then we apply the claim with $n = \ell_2$ to obtain $\ell_3 > \ell_2$ and C_{ℓ_2} . Continuing this procedure, we obtain a strictly increasing sequence $(\ell_k)_{k=1}^\infty$ and a sequence $(C_{\ell_k})_{k=1}^\infty$ such that $C_{\ell_k} \subseteq V_{\ell_k}^*$, $\sigma_{[\ell_k,\ell_{k+1}]}^{B_1}(V_{\ell_{k+1}}) \subseteq C_{\ell_k}^*$, and $\text{card}(C_{\ell_k}) \leq 3 \text{ card}(W_{\ell_k})$. By passing to a subset of C_{ℓ_k} if necessary, we may assume that C_{ℓ_k} is a minimal subset of $V_{\ell_k}^*$ (with respect to the inclusion relation) with the property $\sigma_{[\ell_k,\ell_{k+1}]}^{B_1}(V_{\ell_{k+1}}) \subseteq C_{\ell_k}^*$.

Now define an ordered Bratteli diagram $B'_1 = (V', E', \geq')$ (which will be a microscoping of a telescoping of B_1) as follows. Set $\ell_0 = 0$ and $C_{\ell_0} = V_0$. The vertices of B'_1 are defined by

$$V'_{2k} = C_{\ell_k} \text{ and } V'_{2k+1} = V_{\ell_{k+1}}, \quad k \geq 0.$$

For the set $E' = \bigcup_{k=1}^{\infty} E'_k$ of edges of B'_1 , first we set $E'_1 = E_1$. Next, let $k \geq 1$. Applying Lemma 4.1 to the telescoping of B_1 along the sequence $(\ell_n)_{n=0}^{\infty}$ (with $V_{\ell_k}, V_{\ell_{k+1}}, C_{\ell_k}$ in place of V_{k-1}, V_k, W in that lemma, respectively), we obtain two sets E'_{2k} and E'_{2k+1} of edges, one from V_{ℓ_k} to C_{ℓ_k} and the other from C_{ℓ_k} to $V_{\ell_{k+1}}$, such that $E'_{2k} \circ E'_{2k+1}$ is order isomorphic to $E_{\ell_k, \ell_{k+1}}$ (see the proof of Lemma 4.1). Thus, E'_{2k} is a set of edges from V'_{2k-1} to V'_{2k} and E'_{2k+1} is a set of edges from V'_{2k} to V'_{2k+1} . In this way, we obtain an ordered Bratteli diagram $B'_1 = (V', E', \geq')$.

Let $B''_1 = (V'', E'', \geq'')$ be the telescoping of B'_1 along the even levels. Thus, $V''_k = C_{\ell_k}$ for every $k \geq 0$. Since the telescoping of B'_1 along the odd levels is isomorphic to the telescoping of B_1 along the sequence $(\ell_k)_{k=0}^{\infty}$, it follows that B'_1 is simple and properly ordered and hence so is B''_1 . Moreover, B_1, B'_1 and B''_1 are isomorphic as ordered Bratteli diagrams, so their associated Vershik systems are conjugate. Therefore,

$$\begin{aligned} \text{rank}_{\text{top}}(X_{B_1}, T_{B_1}) &= \text{rank}_{\text{top}}(X_{B''_1}, T_{B''_1}) \\ &\leq \text{rank}(B''_1) \\ &= \sup\{\text{card}(C_{\ell_k}) \mid k \geq 0\} \\ &\leq \sup\{3 \text{ card}(W_{\ell_k}) \mid k \geq 0\} \\ &\leq 3 \text{ rank}(B_2). \end{aligned}$$

This finishes the proof. □

Now we have all the tools for the proof of our main result.

Proof of Theorem 1.1. Choose a point $x \in X$ so that the properly ordered Bratteli diagram B_2 associated to (X, T) with base on $x_{\min} := x$ realizes the topological rank of (X, T) , that is, $\text{rank}_{\text{top}}(X, T) = \text{rank}(B_2)$. Suppose that $y := \alpha(x) \in Y$ and let B_1 be the simple properly ordered Bratteli diagram associated to (Y, S) based on the point $y_{\min} := y$. Then by Proposition 3.2 there exists an ordered premorphism $f : B_1 \rightarrow B_2$. Using the conjugacy of (Y, S) and (X_{B_1}, T_{B_1}) and Proposition 4.6, one can conclude that

$$\text{rank}_{\text{top}}(Y, S) = \text{rank}_{\text{top}}(X_{B_1}, T_{B_1}) \leq 3 \text{ rank}(B_2) = 3 \text{ rank}(X, T),$$

as desired. □

Remark 4.7. In Theorem 1.1, instead of $\text{rank}_{\text{top}}(Y, S) \leq 3 \text{ rank}_{\text{top}}(X, T)$, it is likely that $\text{rank}_{\text{top}}(Y, S) \leq \text{rank}_{\text{top}}(X, T)$. For example, it is known that this is the case if (X, T) is an odometer. Our proof guarantees this inequality in some special cases. For instance, assuming the notation in the proof of Proposition 4.6, if there is an ordered premorphism $f : B_1 \rightarrow B_2$ for the factor map $\alpha : X \rightarrow Y$ (as in the proof of Theorem 1.1) such that

$$v_{\max}^n v_{\min}^n \text{ is not a subword of } \tau_n(w) \quad \text{for all } n \geq 1 \text{ and all } w \in W_n, \tag{4.7}$$

then $s_i = t_i = \emptyset$ for all i (since every non-empty s_i ends with v_{\max}^n and every non-empty t_i starts with v_{\min}^n , and $s_i t_i = \tau_n(w)$ for some $w \in W_n$) and hence we can take $C_n = \tau_n(W_n)$. (Recall that v_{\max}^n and v_{\min}^n are the vertices of V_n carrying the max and min infinite paths, respectively.) Therefore, $\text{rank}_{\text{top}}(X_{B_1}, T_{B_1}) \leq \text{rank}(B_2)$ and so $\text{rank}_{\text{top}}(Y, S) \leq \text{rank}_{\text{top}}(X, T)$. In Example 5.3, we will see that Condition (4.7) holds for the premorphism f as $v_{\max}^n v_{\min}^n = z_n x_n$ is not a subword of $\tau_n(u_n)$ and $\tau_n(v_n)$ for all n .

Another wide class of systems (Y, S) that the inequality $\text{rank}_{\text{top}}(Y, S) \leq \text{rank}_{\text{top}}(X, T)$ holds for them, when (X, T) is a finite-rank subshift extension of them, is the class of minimal subshifts. This is the result of Espinoza [13], who improved his previous result in [12] and made the proof of the above inequality for the case that (Y, S) is a minimal subshift. In this recent work, besides other nice results, Espinoza has translated the notion of ordered premorphism into the S -adic language.

5. Ordered premorphisms and conjugacy

In this section we obtain a combinatorial criterion for conjugacy of two Cantor minimal systems in terms of the aforesaid premorphisms. Suppose that (X, T) and (Y, S) are Cantor minimal systems (or, more generally, essentially minimal systems) for which there are Bratteli–Vershik models B_2 and B_1 , respectively, with an ordered premorphism $f: B_1 \rightarrow B_2$. This gives a factor map $\alpha: X \rightarrow Y$, by Proposition 3.2. However, if f satisfies the conditions of the following proposition, then B_1 is equivalent to B_2 and hence (X, T) is conjugate to (Y, S) .

We need the following notion in the following.

Definition 5.1. Let A be an alphabet. We say that a set of words $B \subseteq A^*$ is a *code* if every word in B^* has a unique representation in terms of the elements of B .

For example, if no word in B is a prefix of another word, then B is a code.

PROPOSITION 5.2. *Let $f: B_1 \rightarrow B_2$ be an ordered premorphism between two ordered Bratteli diagrams. Assume the notation in Definitions 3.1 and 3.3. Suppose that for infinitely many $n \in \mathbb{N}$ the following hold:*

- (1) *the set $D_n = \{\tau_n(w) \mid sw \in W_{f_n}\} \subseteq V_n^*$ is a code;*
- (2) *there is $\ell > n$ such that $\sigma_{[n,\ell]}^{B_1}(v) \in D_n^*$, for all $v \in V_\ell$, and D_n is a minimal subset of V_n^* (with respect to inclusion) having this property.*

Then B_1 is equivalent to B_2 .

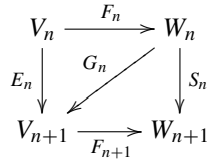
Proof. By passing to appropriate telescoping of B_1 and B_2 , we may assume that $f_n = n$ for any $n \geq 0$, (1) and (2) hold for every $n \in \mathbb{N}$, and $\ell = n + 1$ in (2). To prove that B_1 is equivalent to B_2 , we will construct an ordered Bratteli diagram B whose telescoping along the odd (respectively, even) levels is equivalent to B_1 (respectively, B_2).

In the following, by an ordered set G_n of edges from W_n to V_{n+1} we mean a finite non-empty set of edges with a pair of source and range maps $s: G_n \rightarrow W_n$ and $r: G_n \rightarrow V_{n+1}$, respectively, and with a partial ordering on G_n such that $g, g' \in G_n$ are comparable if and only if $r(g) = r(g')$, the restriction of this ordering to each $r^{-1}\{v\}$, $v \in V_{n+1}$, is a linear ordering, and $r^{-1}\{v\}$ and $s^{-1}\{w\}$ are non-empty for all $v \in V_{n+1}$ and $w \in W_n$.

We claim that for every $n \in \mathbb{N}$ there is an ordered set G_n of edges from W_n to V_{n+1} such that:

- (3) $F_n \circ G_n$ is order isomorphic to E_n ;
- (4) $G_n \circ F_{n+1}$ is order isomorphic to S_n ;
- (5) for every $w \in W_n$ and every $v \in V_{n+1}$, there are $g, g' \in G_n$ with $s(g) = w$ and $r(g') = v$.

In fact, (3) and (4) say that in the following diagram, the two triangles are ordered commutative.



To prove the claim, first note that by (1) and (2) (recall that $\ell = n + 1$ in (2)), for every $v \in V_{n+1}$ the word $\sigma_{n+1}^{B_1}(v)$ has a unique representation with respect to D_n , which is

$$\sigma_{n+1}^{B_1}(v) = \tau_n(w_{i_1})\tau_n(w_{i_2}) \cdots \tau_n(w_{i_k}) \tag{5.1}$$

for some $w_{i_1}, w_{i_2}, \dots, w_{i_k}$ in W_n . (Note that, since D_n is a code, we have implicitly assumed that $\tau_n(w) \neq \tau_n(w')$ for $w \neq w'$.) Hence, there is a (necessarily unique) ordered set of edges G_n from W_n to V_{n+1} that induces a morphism $\sigma_n: V_{n+1} \rightarrow W_n^*$ such that for every $v \in V_{n+1}$ (by (5.1)),

$$\sigma_n(v) = w_{i_1} w_{i_2} \cdots w_{i_k}. \tag{5.2}$$

Moreover, since D_n is minimal, for every $w \in W_n$ there exists some $v \in V_{n+1}$ such that w occurs as a letter of $\sigma_n(v)$. Thus, (5) holds.

To see (3), it is enough to show that the morphisms $\tau_n \circ \sigma_n$ and $\sigma_{n+1}^{B_1}$ (associated to $F_n \circ G_n$ and E_n , respectively) are the same. Let $v \in V_{n+1}$ satisfy (5.1). Then

$$\sigma_{n+1}^{B_1}(v) = \tau_n(w_{i_1}) \cdots \tau_n(w_{i_k}) = \tau_n(w_{i_1} \cdots w_{i_k}) = \tau_n(\sigma_n(v)).$$

Thus, $\sigma_{n+1}^{B_1} = \tau_n \circ \sigma_n$, proving (3). For (4), first by (3) and the ordered commutativity of f , we have

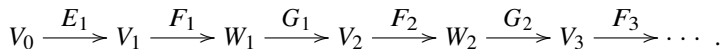
$$F_n \circ G_n \circ F_{n+1} \cong E_n \circ F_{n+1} \cong F_n \circ S_n \tag{5.3}$$

(where \cong means ordered isomorphism between the edge sets preserving the source and range maps). Then by the assumption (1) one can eliminate F_n from (5.3) to get $G_n \circ F_{n+1} \cong S_n$. In fact, for every $w \in W_{n+1}$, by (5.3), we have

$$\tau_n(\sigma_n(\tau_{n+1}(w))) = \tau_n(\sigma_{n+1}^{B_2}(w))$$

and the latter (by having codes by the assumption) implies that $\sigma_n(\tau_{n+1}(w)) = \sigma_{n+1}^{B_2}(w)$. This gives (3) and finishes the proof of the claim.

Now consider the following ordered Bratteli diagram B .



By (3), the telescoping of B along the odd levels (starting with V_0 as the zeroth level) is equivalent to B_1 and, by (4), the telescoping along the even levels is equivalent to B_2 (note that $E_1 \circ F_1 \cong F_0 \circ S_1 \cong S_1$ by the ordered commutativity of f). Therefore, B_1 is equivalent to B_2 . \square

It is worth noting that if the condition of Proposition 5.2 holds, then the ordered premorphism $f : B_1 \rightarrow B_2$ is invertible in the sense of [1]. More precisely, using the G_n constructed in the proof of Proposition 5.2, we obtain an ordered premorphism $g : B_2 \rightarrow B_1$ such that $f \circ g \sim \text{id}_{B_2}$ and $g \circ f \sim \text{id}_{B_1}$ (by (3) and (4)), where \sim denotes the equivalence of ordered premorphisms (see [1, Proposition 2.19] and the remarks preceding it). In particular, if (X, T) , (Y, S) , f , and α are as in the first paragraph of this section, where f satisfies the condition of the preceding proposition, then the factor map $\alpha : X \rightarrow Y$ is in fact a conjugacy.

Here is an example.

Example 5.3. Consider the ordered premorphism $f : B \rightarrow C'$ of [1, Example 2.11]. Its diagram is drawn in Figure 3 of [1] and we draw it here in Figure 5 for the convenience of the reader. Let us recall that C' is a telescoping of the Bratteli diagram of the Chacon system and B is a simple properly ordered Bratteli diagram. By Proposition 3.2, f induces a factor map $\alpha : X_{C'} \rightarrow X_B$. We use Proposition 5.2 to show that $X_{C'}$ is conjugate to X_B . (In [1], another argument is given to show that these systems are conjugate by constructing the inverse of f .) We label the vertices of the i th level of B by x_i, y_i, z_i and those of C' by u_i, v_i for $i \geq 1$. Let $(\tau_i)_{i \geq 0}$ be the sequence of morphisms induced by f according to Definition 3.3. For every $i \geq 1$, we have

$$\tau_i(u_i) = x_i y_i x_i y_i z_i \text{ and } \tau_i(v_i) = x_i y_i z_i z_i.$$

Put $D_i = \{\tau_i(u_i), \tau_i(v_i)\}$. It is easy to check that D_i is a code (since no word in D_i is a prefix of another word). Moreover, for every $i \geq 1$,

$$\sigma_{i+1}^B(x_{i+1}) = x_i y_i, \quad \sigma_{i+1}^B(y_{i+1}) = x_i y_i z_i, \text{ and } \sigma_{i+1}^B(z_{i+1}) = x_i y_i z_i z_i.$$

We see that D_i does not generate all three words above. So, we go one level down in the diagram B and compute

$$\begin{aligned} \sigma_{[i,i+2]}^B(x_{i+2}) &= x_i y_i x_i y_i z_i, \quad \sigma_{[i,i+2]}^B(y_{i+2}) = x_i y_i x_i y_i z_i x_i y_i z_i z_i, \text{ and} \\ \sigma_{[i,i+2]}^B(z_{i+2}) &= x_i y_i x_i y_i z_i x_i y_i z_i z_i x_i y_i z_i z_i. \end{aligned}$$

These words are generated by D_i as $\sigma_{[i,i+2]}^B(x_{i+2}) = \tau_i(u_i)$, $\sigma_{[i,i+2]}^B(y_{i+2}) = \tau_i(u_i)\tau_i(v_i)$, and $\sigma_{[i,i+2]}^B(z_{i+2}) = \tau_i(u_i)\tau_i(v_i)\tau_i(v_i)$. Also, D_i is a minimal subset of $\{x_i, y_i, z_i\}^*$ having this property. Now, Proposition 5.2 implies that B is equivalent to C' and so X_B is conjugate to $X_{C'}$.

Acknowledgements. We would like to thank Fabien Durand for drawing our attention to this problem and for kind discussions. Our special thanks go to Samuel Petite and Thierry Giordano for reading the first version of our manuscript and making very helpful comments. We also appreciate the anonymous referee’s insightful comments and

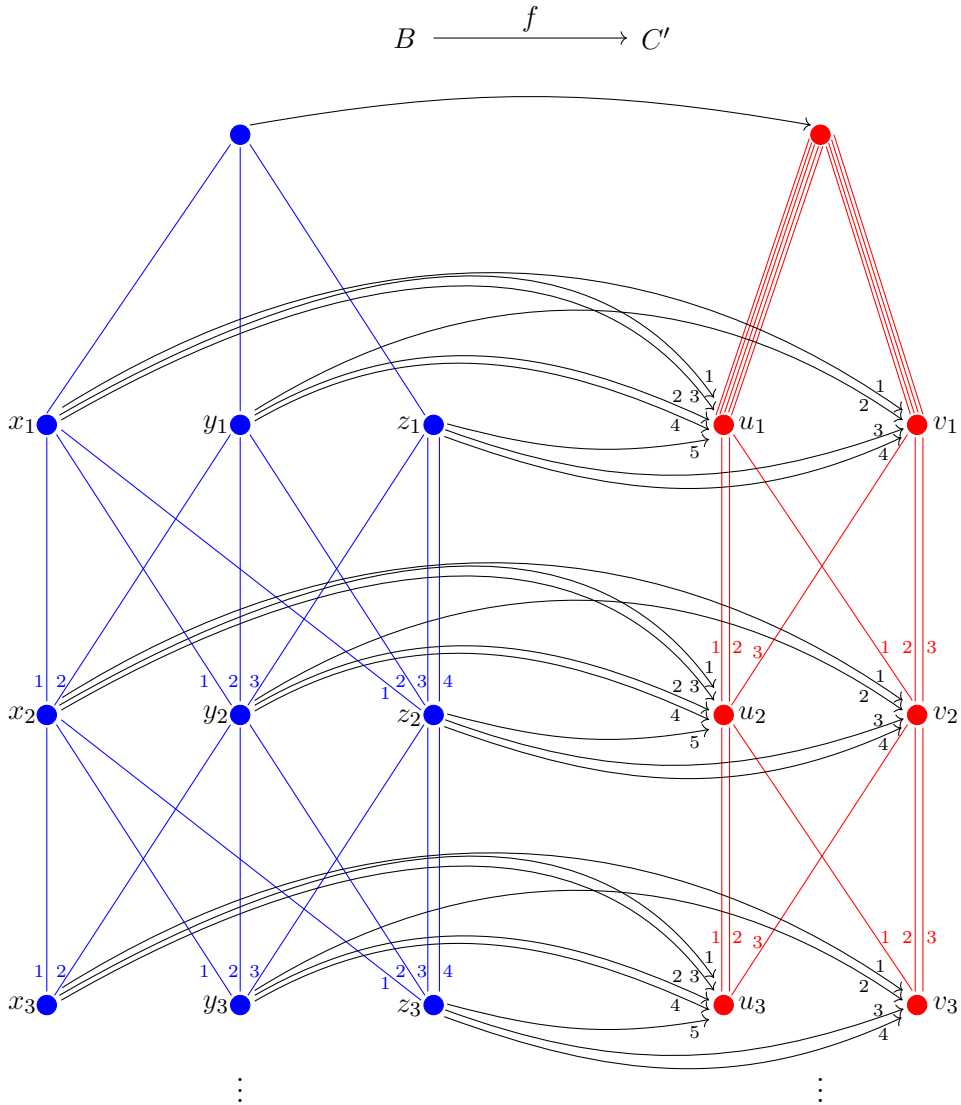


FIGURE 5. The ordered premorphism $f : B \rightarrow C'$ of Example 5.3.

suggestions. The research of the first author was in part supported by a grant from IPM (No. 1400460120).

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