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Asymptotically autonomous robustness of random attractors for a class of weakly dissipative stochastic wave equations on unbounded domains

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This paper is concerned with the asymptotic behaviour of solutions to a class of non-autonomous stochastic nonlinear wave equations with dispersive and viscosity dissipative terms driven by operator-type noise defined on the entire space $\mathbb{R}^n$. The existence, uniqueness, time-semi-uniform compactness and *asymptotically autonomous robustness* of pullback random attractors are proved in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ when the growth rate of the nonlinearity has a subcritical range, the density of the noise is suitably controllable, and the time-dependent force converges to a time-independent function in some sense. The main difficulty to establish the *time-semi-uniform* pullback asymptotic compactness of the solutions in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ is caused by the lack of compact Sobolev embeddings on $\mathbb{R}^n$, as well as the weak dissipativeness of the equations is surmounted at light of the idea of uniform tail-estimates and a spectral decomposition approach. The measurability of random attractors is proved by using an argument which considers two attracting universes developed by Wang and Li (Phys. D **382**: 46–57, 2018).

Keywords: Weakly dissipative wave equation; pullback random attractors; asymptotically autonomous robustness; time-semi-uniform compactness; operator-type noise

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1. Introduction

In this article we investigate existence, uniqueness, *time-semi-uniform compactness* as well as *asymptotically autonomous robustness* of pullback random attractors of the following non-autonomous stochastic dispersive-dissipative wave equations perturbed by *operate-type noise* define on $\mathbb{R}^n$:

$$\begin{cases} u_{tt} + \alpha u_t - \Delta u_t - \beta\Delta u_{tt} + \lambda u - \Delta u + f(x,u) = g(t,x) + \varepsilon \mathcal{S} u \circ \frac{\mathrm{d}W}{\mathrm{d}t}, \\ u(\tau,x) = u_\tau(x), \quad u_t(\tau,x) = u_{\tau,1}(x),\ x \in \mathbb{R}^n,\ t > \tau,\ \tau \in \mathbb{R}, \end{cases} \tag{1.1}$$

where $n \in \mathbb{N}$ is arbitrary, α, β and λ are positive constants, $\varepsilon > 0$ is the density of noise, $\mathcal{S} = I - \beta\Delta$, W is a two-sided real-valued Wiener process defined on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, $g \in L^2_{\mathrm{loc}}(\mathbb{R}, L^2(\mathbb{R}^n))$, and the nonlinear function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ has a subcritical growth rate in its second argument. The symbol $\circ$ means that the stochastic equation is interpreted in the sense of Stratonovich integration. The two terms $(-\Delta)^s u_{tt}$ and $(-\Delta)^s u_t$ are referred to as the dispersive and viscosity dissipative terms respectively due to their own physical background.

For deterministic version of (1.1) defined on bounded domains, the well-posedness and existence of global attractors have been investigated by Carvalho and Cholewa [**11**] as well as Sun *et al.* [**38**].

For additive white noise driven version of (1.1) defined on unbounded domain $\mathbb{R}^n$ for $n = 1, 2, 3$, the existence of random attractors was recently examined by Jones and Wang [**22**] when the force g is time-independent, and the stochastic term $\varepsilon \mathcal{S} u \circ (\mathrm{d}W/\mathrm{d}t)$ is replaced by $h(\mathrm{d}W/\mathrm{d}t)$ with $h \in L^2(\mathbb{R}^n)$ being a known function.

As far as the authors are concerned, up to now, the existence of random attractors remains open for the non-autonomous stochastic version of problem (1.1) even for $n = 1, 2, 3$ and the bounded domain case. The main reason here is that we can only transform the additive noise driven version of (1.1) into a pathwise random equation, but cannot transform the multiplicative noise driven version of (1.1) with $\mathcal{S} = I$ into a pathwise random one due to the dispersive and dissipative $(-\Delta)^s u_{tt}$ and $(-\Delta)^s u_t$. This *essentially* distinguishes from the damped (or strong damped) wave equations as considered by many authors, see e.g., [**21**, **37**, **40**, **42**, **48**, **49**, **54**, **56**–**58**]. Nevertheless, in this paper we are able to convert problem (1.1) with $\mathcal{S} = I - \beta\Delta$ into a pathwise deterministic one, and hence study the random attractors of stochastic (1.1) with $\mathcal{S} = I - \beta\Delta$.

As is well known, the concept of attractors investigated by many authors, see e.g., Robinson *et al.* [**5**, **6**, **12**, **13**, **27**, **28**, **34**–**36**], plays an important role in the study of asymptotic behaviour of solutions to differential equations. Autonomous random attractors proposed by Brzeźniak *et al.* [**2**], Crauel and Flandoli [**14**], Crauel et al. [**15**], Flandoli and Schmalfuß [**18**] and Caraballo and Langa [**8**] can be viewed as a generation of global attractors form deterministic to random. Non-autonomous random attractors developed by Caraballo *et al.* [**7**], Caraballo and Langa [**3**] and Wang [**41**, **42**] can be regarded as an extension of autonomous random attractors form autonomous to non-autonomous. In light of those theoretical frameworks, random attractors have reached a flourishing development in recent years, see e.g., [1, 4, 9, 10, 19, 23, 29, 40, 49, 58] and [16, 20, 33, 46, 48, 50, 52, 53, 57, 58] for autonomous and non-autonomous PDEs, respectively.

In general, a non-autonomous random attractor typically takes the form $\mathcal{A}_\alpha = \{\mathcal{A}_\alpha(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, where α is an external parameter that comes from various perturbations. Notice that, in the literature aforementioned, many properties of $\mathcal{A}_\alpha$ such as compactness, attraction, regularity as well as finite fractal dimension were usually discussed for each fixed time-section $\mathcal{A}_\alpha(\tau,\omega)$, and the robustness of $\mathcal{A}_\alpha(\tau,\omega)$ was only investigated with respect to the external parameter α but not the internal parameter τ. This kind of researches are just analogous to the autonomous case, and thereby the time-dependence character related to non-autonomous random attractors are not well-understood.

In the present paper we will not only establish the existence and uniqueness but also the time-semi-uniform compactness as well as asymptotically autonomous robustness of non-autonomous random attractors of problem (1.1). Our first aim is to prove that problem (1.1) admits a unique pullback random attractor $\mathcal{A} = \{\mathcal{A}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ under the theoretical framework established in [41]. In order to achieve the goal, as usual, we must prove the *usual* pullback asymptotic compactness of solutions to (1.1) in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. There are three difficulties we need to surmount.

1. The compact Sobolev embeddings on unbounded domain $\mathbb{R}^n$ are not available.
2. Equation (1.1) is a weakly dissipative one due to the dispersive and dissipative terms $(-\Delta)^s u_{tt}$ and $(-\Delta)^s u_t$, which is essentially different from the damped wave equations as widely considered in the literature aforementioned.
3. The uniform estimates of solutions to (1.1) cannot be derived by differentiating the equation with respect to time since the Wiener process is almost surely nowhere differentiable with respect to time variable.

We combine the ideal of uniform tail-estimates developed by Wang [39] and a spectral decomposition approach to overcome the three difficulties, and hence establish the desired pullback asymptotic compactness in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for our purpose.

Another significant goal of this article is to prove the following time-semi-uniform compactness of $\mathcal{A}$:

$$\bigcup_{s\leqslant\tau} \mathcal{A}(s,\omega) \text{ is precompact in } H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \tag{1.2}$$

for each $(\tau,\omega) \in \mathbb{R} \times \Omega$ as well as the following **asymptotically autonomous robustness** of the time-section $\mathcal{A}(\tau,\omega)$ of $\mathcal{A}$ as time τ goes to negative infinity:

$$\lim_{\tau\to-\infty} \mathrm{dist}_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}(\mathcal{A}(\tau,\omega), \mathcal{A}_\infty(\omega)) = 0, \quad \mathbb{P}\text{-a.s. } \omega \in \Omega, \tag{1.3}$$

where $\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega) : \omega \in \Omega\}$ is a random attractor of the autonomous version of problem (1.1). Furthermore, we prove that such a robustness is basically uniform in the probability space Ω for discrete time sequence, and we also prove that for

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any discrete time sequence $\tau_n \to -\infty$, there exist $\{\tau_{n_k}\}_{k=1}^{\infty}$ of $\{\tau_n\}_{n=1}^{\infty}$ such that

$$\lim_{k\to\infty} \mathrm{dist}_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}(\mathcal{A}(\tau_{n_k},\theta_{\tau_{n_k}}\omega),\mathcal{A}_\infty(\theta_{\tau_{n_k}}\omega)) = 0, \quad \mathbb{P}\text{-a.s. } \omega\in\Omega.$$

Notice that the *usual* pullback asymptotic compactness of solutions to (1.1) is no longer useful (or say not enough) to establish (1.2) and (1.3). To solve this problem, at present, we first introduce a time-semi-uniform attracting universe (see (2.20)) that is indeed small than the usual attracting universe (see (2.19)), and then derive the *time-semi-uniform* pullback asymptotic compactness of solutions to (1.1) in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$. Based upon this we are able to prove (1.2) and (1.3). It is worth mentioning that the radii $\sup_{s\leqslant\tau} R(s,\omega)$ of the absorbing set in the case is taken the supremum over an uncountable set $(-\infty,\tau]$. This introduces difficulties to prove the measurability of attractors. Our idea to solve this issue is to prove that the two attractors with respect to the two different universes are equal, see theorem 4.3.

We remark that the time-semi-uniform compactness of non-autonomous attractors and kernel sections has been recently investigated in [**30**, **32**, **55**, **56**] and [**43**, **44**] for deterministic and stochastic PDEs, respectively. The asymptotically autonomous robustness of non-autonomous attractors was studied for deterministic equations [**24**–**26**, **31**, **45**]. In this paper we study both time-semi-uniform compactness and asymptotically autonomous robustness of non-autonomous random attractors of stochastic equation (1.1).

The structure of the paper is as follows. In the next section we define a non-autonomous cocycle for (1.1). In § 3 we derive two types of long time uniform estimates. In § 4 we establish the existence, uniqueness and time-semi-uniform compactness of random attractors. In the last section we discuss the asymptotically autonomous robustness of random attractors. In Appendix we provide the proof of measurability of the solution operators.

2. Non-autonomous cocycle generated by stochastic wave equations

In this section we consider the following wave equation perturbed by *operate-type noise* on unbounded domain $\mathbb{R}^n$:

$$\begin{cases} u_{tt} + \alpha u_t - \Delta u_t - \beta\Delta u_{tt} + \lambda u - \Delta u + f(x,u) = g(t,x) + \varepsilon\mathcal{S}u \circ \dfrac{\mathrm{d}W}{\mathrm{d}t}, \\ u(\tau,x) = u_\tau(x), \quad u_t(\tau,x) = u_{\tau,1}(x),\ x\in\mathbb{R}^n,\ t>\tau,\ \tau\in\mathbb{R}, \end{cases} \tag{2.1}$$

where $\alpha,\beta,\lambda,\varepsilon>0$, $\mathcal{S}=I-\beta\Delta$, $g\in L^2_{\mathrm{loc}}(\mathbb{R},L^2(\mathbb{R}^n))$, the nonlinearity f will be specified later, and the two-sided real-valued Wiener process W is defined on the probability space $(\Omega,\mathcal{F},\mathbb{P})$, where $\Omega=\{\omega\in C(\mathbb{R},\mathbb{R}):\ \omega(0)=0\}$ equipped with the compact-open topology, $\mathcal{F}=\mathfrak{B}(\Omega)$ is the Borel sigma-algebra of Ω, and $\mathbb{P}$ is the Wiener measure. Define a family of shift operators $\{\theta_t\}_{t\in\mathbb{R}}$ acting on Ω defined by $\theta_t\omega(\cdot)=\omega(\cdot+t)-\omega(t)$ for $(\omega,t)\in\Omega\times\mathbb{R}$. Then $(\Omega,\mathcal{F},\mathbb{P},\{\theta_t\}_{t\in\mathbb{R}})$ forms a metric dynamical system.

2.1. First-order random wave equations

Denote by $z := u_t + \delta u$ with $\delta > 0$ to be determined latter, then we have the following equivalent system:

$$\begin{cases} u_t = -\delta u + z, \\ z_t - \beta\Delta z_t + \delta_1 z - \delta_2 \Delta z + \delta_3 u - \delta_4 \Delta u + f(x,u) = g(t,x) + \varepsilon \mathcal{S} u \circ \dfrac{\mathrm{d}W}{\mathrm{d}t}, \\ u(\tau,x) = u_\tau(x), \quad z(\tau,x) = u_{\tau,1}(x) + \delta u_\tau(x), \end{cases} \tag{2.2}$$

where $\delta_1 := \alpha - \delta$, $\delta_2 := 1 - \beta\delta$, $\delta_3 := \lambda - \alpha\delta + \delta^2$ and $\delta_4 := 1 - \delta + \beta\delta^2$. Let

$$v(t,\tau,\omega,v_\tau) := z(t,\tau,\omega,z_\tau) - \varepsilon y(\theta_t\omega) u(t,\tau,\omega,u_\tau), \tag{2.3}$$

where $v_\tau = z_\tau - \varepsilon y(\theta_\tau\omega)u_\tau$ and $y(\theta_t\omega) = -\delta\int_{-\infty}^0 e^{\delta\tau}(\theta_t\omega)(\tau)\,\mathrm{d}\tau$ is the stationary solution of the one-dimensional Ornstein–Uhlenbeck equation $\mathrm{d}y + \delta y \mathrm{d}t = \mathrm{d}W(t)$. By Fan [**17**], Caraballo and Langa [**8**], and Wang and Zhou [**47**], there exists $\{\theta_t\}_{t\in\mathbb{R}}$-invariant subset which we still denoted by Ω of full measure such that

$$\lim_{t\to\pm\infty} \frac{y(\theta_t\omega)}{t} = \lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^0 y(\theta_s\omega)\,\mathrm{d}s = 0, \quad \text{for every } \omega\in\Omega, \tag{2.4}$$

$$\lim_{t\to\pm\infty} \frac{1}{t}\int_{-t}^0 |y(\theta_s\omega)|^m\,\mathrm{d}s = \frac{\Gamma(\frac{1+m}{2})}{\sqrt{\pi\delta^m}}, \quad \text{for every } \omega\in\Omega \text{ and } m>0. \tag{2.5}$$

Then one can find that $\varphi = (u,v)$ solves the following random system:

$$\begin{cases} u_t = (\varepsilon y(\theta_t\omega) - \delta)u + v, \\ v_t - \beta\Delta v_t + \delta_1 v - \delta_2\Delta v + \delta_3 u - \delta_4\Delta u + f(x,u) \\ \quad = g(t,x) - \varepsilon y(\theta_t\omega)v + \varepsilon\beta y(\theta_t\omega)\Delta v - \big(\varepsilon\delta_5 y(\theta_t\omega) + \varepsilon^2 y^2(\theta_t\omega)\big)u \\ \qquad + \big(\varepsilon\delta_6 y(\theta_t\omega) + \varepsilon^2\beta y^2(\theta_t\omega)\big)\Delta u, \\ u(\tau,x) = u_\tau(x), \quad v(\tau,x) = v_\tau(x) = u_{\tau,1}(x) + \delta u_\tau(x) - \varepsilon y(\theta_\tau\omega)u_\tau(x), \end{cases} \tag{2.6}$$

where $\delta_5 := \alpha - 3\delta$, $\delta_6 := 1 - 3\beta\delta$.

2.2. Assumptions

Next, we list the hypotheses on the nonlinearity, on the density of noise and on the time-dependent force.

Hypothesis F. The smooth function $f : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}$ has a *subcritical* growth range such that

$$|f(x,s)| \leqslant \gamma_1|s|^p + \phi_1(x), \qquad \phi_1 \in L^2(\mathbb{R}^n),\ \gamma_1 > 0, \tag{2.7a}$$

$$f(x,s)s \geqslant \gamma_2 F(x,s) + \phi_2(x), \qquad \phi_2 \in L^1(\mathbb{R}^n),\ \gamma_2 > 0, \tag{2.7b}$$

$$F(x,s) \geqslant \gamma_3|s|^{p+1} - \phi_3(x), \qquad \phi_3 \in L^1(\mathbb{R}^n),\ \gamma_3 > 0, \tag{2.7c}$$

$$\left|\frac{\partial f}{\partial s}(x,s)\right| \leqslant \gamma_4|s|^{p-1} + \phi_4(x), \qquad \phi_4 \in L^2(\mathbb{R}^n),\ \gamma_4 > 0, \tag{2.7d}$$

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where $F(x,s)=\int_0^s f(x,\sigma)\,\mathrm{d}\sigma$, and

$$1\leqslant p<\infty \text{ for } n=1,2;\quad 1\leqslant p<\frac{n+2}{n-2}\ \text{ for } n\geqslant 3. \tag{2.8}$$

By (2.7a)–(2.7d), for all $(x,s)\in\mathbb{R}^n\times\mathbb{R}$, there is a constant c, independent of x and s, such that

$$F(x,s)\leqslant c(1+|s|^{p+1}+|\phi_1(x)|^2+|\phi_2(x)|). \tag{2.9}$$

Let $\delta>0$ be small enough such that the constants $\delta_i>0$ $(i=1,2,3,4,5,6)$, and denote by

$$\kappa_1 := \min\left\{\delta_1,\frac{\delta_2}{\beta},\delta,\delta\gamma_2\right\}, \tag{2.10a}$$

$$\kappa_2 := \max\left\{2(\delta_5+1),\frac{2(\delta_5+1)}{\delta_3},\frac{2(\delta_6+\beta)}{\beta},\frac{2(\delta_6+\beta)}{\delta_4},\frac{\gamma_1}{\gamma_3},4\right\}. \tag{2.10b}$$

Hypothesis S. The size of the noise is suitably controllable:

$$\varepsilon\in(0,\varepsilon_0]\ \text{ with }\ \varepsilon_0:=\min\left\{1,\frac{\kappa_1}{2(p+1)^2(\frac{2}{\sqrt{\pi\delta}}+\frac{1}{\delta})\kappa_2}\right\}. \tag{2.11}$$

The following lemma is useful when establishing the existence of pullback random attractors.

LEMMA 2.1. *Let (2.11) hold. Then for each $\omega\in\Omega$, there are $T_0(\omega)>0$ and $C_0(\omega)>0$ such that*

$$|y(\theta_{-t}\omega)|+\sup_{\varepsilon\in(0,\varepsilon_0]}\varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,d\sigma\leqslant\frac{\kappa_1 t}{(p+1)^2},\quad \forall t\geqslant T_0(\omega), \tag{2.12a}$$

$$|y(\theta_{-t}\omega)|+\sup_{\varepsilon\in(0,\varepsilon_0]}\varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\mathrm{d}\sigma\leqslant\frac{\kappa_1 t}{(p+1)^2}+C_0(\omega),\quad \forall t\geqslant 0, \tag{2.12b}$$

where $Y(\theta_\sigma\omega)=|y(\theta_\sigma\omega)|+|y(\theta_\sigma\omega)|^2$ for $\sigma\in\mathbb{R}$.

Proof. By (2.11) and (2.4) and (2.5) with $m=1,2$, there exists $T_0(\omega)>0$ such that for all $t\geqslant T_0$,

$$\begin{aligned}|y(\theta_{-t}\omega)|&+\sup_{\varepsilon\in(0,\varepsilon_0]}\varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma\leqslant\frac{1}{2(p+1)^2}\kappa_1 t+\varepsilon_0\kappa_2\left(\frac{2\Gamma(1)}{\sqrt{\pi\delta}}+\frac{2\Gamma(3/2)}{\sqrt{\pi}\delta}\right)t\\ &\leqslant\frac{1}{2(p+1)^2}\kappa_1 t+\frac{\kappa_1\kappa_2}{2(p+1)^2(\frac{2}{\sqrt{\pi\delta}}+\frac{1}{\delta})\kappa_2}\left(\frac{2}{\sqrt{\pi\delta}}+\frac{1}{\delta}\right)t=\frac{1}{(p+1)^2}\kappa_1 t,\end{aligned} \tag{2.13}$$

which implies (2.12a). Take $C_0(\omega)=\sup_{t\in[0,T_0]}|y(\theta_{-t}\omega)|+\varepsilon_0\kappa_2\int_{-T_0}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma$. Then we have (2.12b). □

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Throughout this paper, the inner product and norm of $L^2(\mathbb{R}^n)$ are written as $(\cdot,\cdot)$ and $\|\cdot\|$ respectively. The norm of $L^p(\mathbb{R}^n)$ for $p\geqslant 1$ is written as $\|\cdot\|_p$. The letter $c>0$ denotes a generic constant which may change its values from line to line or even in the same line.

Hypothesis G. The time-dependent function $g\in L^2_{\rm loc}(\mathbb{R},L^2(\mathbb{R}^n))$ converges to a time-independent function $g_\infty\in L^2(\mathbb{R}^n)$ in the sense that

$$\lim_{\tau\to-\infty}\int_{-\infty}^{\tau}\|g(r)-g_\infty\|^2\,\mathrm{d}r=0. \tag{2.14}$$

A typical and simple example for the functions g and g_∞ which satisfy condition (2.14), and illustrates that the new condition used here is reasonable is the following. Choose a function $g_0\in L^2(\mathbb{R}^n)$, we set $g(t,x)=(e^t+1)g_0(x)$ and $g_\infty(x)=g_0(x)$ for $(t,x)\in\mathbb{R}\times\mathbb{R}^n$. Then we have

$$\lim_{\tau\to-\infty}\int_{-\infty}^{\tau}\|g(r)-g_\infty\|^2\,\mathrm{d}r=\|g_0\|^2\lim_{\tau\to-\infty}\int_{-\infty}^{\tau}e^{2r}\,\mathrm{d}r=0.$$

We remark that, under hypothesis **G** only, we can not only show the convergence of solutions to (2.2) from nonautonomous to autonomous, but also can show the following properties on the time-dependent force, which are important to discuss the existence and time-semi-uniform compactness of the random attractors.

PROPOSITION 2.2. *Let hypothesis **G** hold. Then we have the following assertions.*

(i) *g is κ-**integrable:***

$$\int_{-\infty}^{\tau}e^{\kappa(r-\tau)}\|g(r)\|^2\,dr<+\infty,\quad \textit{for all } \kappa>0 \textit{ and } \tau\in\mathbb{R}. \tag{2.15}$$

(ii) *g is κ-**tail-small:***

$$\lim_{k\to\infty}\int_{-\infty}^{\tau}e^{\kappa(r-\tau)}\int_{|x|\geqslant k}|g(r,x)|^2dxdr<+\infty,\quad \textit{for all } \kappa>0 \textit{ and } \tau\in\mathbb{R}. \tag{2.16}$$

(iii) *g is **time-semi-uniformly** κ-**integrable:***

$$\sup_{s\leqslant\tau}\int_{-\infty}^{s}e^{\kappa(r-s)}\|g(r)\|^2\,dr<+\infty,\quad \textit{for all } \kappa>0 \textit{ and } \tau\in\mathbb{R}. \tag{2.17}$$

(iv) *g is **time-semi-uniformly** κ-**tail-small:***

$$\lim_{k\to\infty}\sup_{s\leqslant\tau}\int_{-\infty}^{s}e^{\kappa(r-s)}\int_{|x|\geqslant k}|g(r,x)|^2dxdr<+\infty,\quad \textit{for all } \kappa>0 \textit{ and } \tau\in\mathbb{R}. \tag{2.18}$$

Proof. The proof is similar to that as considered in [**45**], we do not repeat it again. □

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REMARK 2.3. Conditions (2.15) and (2.16) are used to ensure the existence of pullback attractors of PDEs defined on unbounded domains, see e.g.,[**41**, **42**]. Condition (2.17) is used to ensure the existence of *time-semi-uniformly compact* pullback attractors of PDEs defined on bounded domains, see [**32**, **55**]. Both (2.17) and (2.18) are used in [**30**, **56**] to ensure the existence of *time-semi-uniformly compact* pullback attractors of PDEs defined on unbounded domains.

2.3. Non-autonomous cocycle

In this article we consider the energy space $E = H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ equipped with the equivalent norm:

$$\|\varphi\|_E = (\|v\|^2 + \beta\|\nabla v\|^2 + \delta_3\|u\|^2 + \delta_4\|\nabla u\|^2)^{1/2}, \quad \forall \varphi = (u, v) \in E.$$

By Carvalho and Cholewa [**11**], we are able to show that for each $(\tau, \omega) \in \mathbb{R} \times \Omega$ and $\varphi_\tau = (u_\tau, v_\tau) \in E$, problem (2.6) has a unique solution $\varphi(\cdot, \tau, \omega, \varphi_\tau) = (u(\cdot, \tau, \omega, u_\tau), v(\cdot, \tau, \omega, v_\tau)) \in C([\tau, \infty), E)$ such that the solution continuously depends on $\varphi_\tau \in E$. In addition, we can also prove the $(\mathcal{F}, \mathcal{B}(H^s(\mathbb{R}^n)))$-measurability of the solutions. Then we find that $\Phi : \mathbb{R}^+ \times \mathbb{R} \times \Omega \times E \to E$ given by

$$\Phi(t, \tau, \omega, (u_\tau, z_\tau)) = (u(t + \tau, \tau, \theta_{-\tau}\omega, u_\tau), z(t + \tau, \tau, \theta_{-\tau}\omega, z_\tau))$$

is a continuous cocycle over $\mathbb{R}$ and $(\Omega, \mathcal{F}, \mathbb{P}, \ \{\theta_t\}_{t\in\mathbb{R}})$ in the sense of [**41**, Def. 1.1]. Let $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ be a family of bounded nonempty subsets in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying

$$\lim_{t\to+\infty} e^{-\kappa_1 t/(p+1)^2} \|\mathcal{D}(\tau - t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} = 0. \tag{2.19}$$

Let $\mathfrak{D} = \{\mathcal{D} = \{\emptyset \neq \mathcal{D}(\tau, \omega) \subseteq E : \tau \in \mathbb{R}, \omega \in \Omega\} : \mathcal{D}$ satisfies (2.19)$\}$. We also introduce $\mathcal{B} = \{\mathcal{B}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$, which is a family of bounded nonempty subsets of $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying

$$\lim_{t\to+\infty} e^{-\kappa_1 t/(p+1)^2} \sup_{s\leqslant\tau} \|\mathcal{B}(s - t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} = 0. \tag{2.20}$$

Denote by $\mathfrak{B} = \{\mathcal{B} = \{\emptyset \neq \mathcal{B}(\tau, \omega) \subseteq E : \tau \in \mathbb{R}, \omega \in \Omega\} : \mathcal{B}$ satisfies (2.20)$\}$.

3. Long time $(\mathcal{B}, \mathcal{D})$-uniform estimates

This section is devoted to several kinds of long time $(\mathcal{B}, \mathcal{D})$-uniform estimates of solutions to problem (2.6).

3.1. $(\mathcal{B}, \mathcal{D})$-uniform estimates in the entire space

Let us start with the following long time $(\mathcal{B}, \mathcal{D})$-uniform estimates of solutions of problem (2.6) in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

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LEMMA 3.1. *Let hypotheses* **F**, **S** *and* **G** *be satisfied. Then for each* $(\tau,\omega,\mathcal{B},\mathcal{D})\in\mathbb{R}\times\Omega\times\mathfrak{B}\times\mathfrak{D}$*, there are* $T_{\mathcal{B}}=T(\tau,\omega)>0$ *and* $T_{\mathcal{D}}=T(\tau,\omega)>0$ *such that for all* $t\geqslant T_{\mathcal{B}}$ *and* $t\geqslant T_{\mathcal{D}}$*,*

$$\sup_{s\leqslant\tau}\|(u(s,s-t,\theta_{-s}\omega,u_{s-t}),z(s,s-t,\theta_{-s}\omega,z_{s-t}))\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}$$
$$\leqslant Me^{|y(\omega)|}(1+\sup_{s\leqslant\tau}R(s,\omega)), \tag{3.1}$$

and

$$\|(u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}),z(\tau,\tau-t,\theta_{-\tau}\omega,z_{\tau-t}))\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}$$
$$\leqslant Me^{|y(\omega)|}(1+R(\tau,\omega)), \tag{3.2}$$

where $(u_{s-t},z_{s-t})\in\mathcal{B}(s-t,\theta_{-t}\omega)$ *for* $s\leqslant\tau$*,* $(u_{\tau-t},z_{\tau-t})\in\mathcal{D}(\tau-t,\theta_{-t}\omega)$*,* $M>0$ *is a constant independent of* τ*,* ω*,* $\mathcal{B}$ *and* $\mathcal{D}$*, and* $R(s,\omega)$ *is given by*

$$R(s,\omega):=\int_{-\infty}^{0}e^{\kappa_1 r+|y(\theta_r\omega)|+\varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}(1+\|g(r+s)\|^2)dr. \tag{3.3}$$

Proof. Taking the inner product of the second equation of (2.6) with v in $L^2(\mathbb{R}^n)$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}(\|v\|^2+\beta\|\nabla v\|^2)+2\delta_1\|v\|^2+2\delta_2\|\nabla v\|^2+2\delta_3(u,v)-2\delta_4(\Delta u,v)+2(f(x,u),v)\\&\quad=2(g(t),v)-2\varepsilon y\|v\|^2-2\varepsilon\beta\|\nabla v\|^2-2(\varepsilon\delta_5 y+\varepsilon^2y^2)(u,v)\\&\qquad+2(\varepsilon\delta_6 y+\varepsilon^2\beta y^2)(\Delta u,v),\end{aligned} \tag{3.4}$$

where $y:=y(\theta_t\omega)$. Thanks to (2.10a) we see from (3.4) that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi\|_E^2+2\int_{\mathbb{R}^n}F(x,u)\mathrm{d}x\right)+2\kappa_1\|\varphi\|_E^2+2\delta(f(x,u),u)\\&\quad\leqslant 2\varepsilon y(f(x,u),u)+2(g(t),v)-2\varepsilon y\|v\|^2-2\varepsilon\beta y\|\nabla v\|^2+2\varepsilon\delta_3 y\|u\|^2\\&\qquad+2\varepsilon\delta_4 y\|\nabla u\|^2-2(\varepsilon\delta_5 y+\varepsilon^2y^2)(u,v)+2(\varepsilon\delta_6 y+\varepsilon^2\beta y^2)(\Delta u,v).\end{aligned} \tag{3.5}$$

By Young's inequality, we see

$$2(g(t),v)\leqslant 2\|v\|\|g(t)\|\leqslant c\|\varphi\|_E\|g(t)\|\leqslant\frac{1}{2}\kappa_1\|\varphi\|_E^2+c\|g(t)\|^2. \tag{3.6}$$

By (2.11) and (2.10b), we have

$$\begin{aligned}&-2(\varepsilon\delta_5 y+\varepsilon^2y^2)(u,v)+2(\varepsilon\delta_6 y+\varepsilon^2\beta y^2)(\Delta u,v)\\&\quad\leqslant\varepsilon(\delta_5+1)(|y|+|y|^2)(\|v\|^2+\|u\|^2)+\varepsilon(\delta_6+\beta)(|y|+|y|^2)(\|\nabla v\|^2+\|\nabla u\|^2)\\&\quad\leqslant\frac{1}{2}\varepsilon\kappa_2(|y|+|y|^2)(\|v\|^2+\delta_3\|u\|^2+\beta\|\nabla v\|^2+\delta_4\|\nabla u\|^2)=\frac{1}{2}\varepsilon\kappa_2 Y(\theta_t\omega)\|\varphi\|_E^2.\end{aligned} \tag{3.7}$$

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where $Y(\theta_t\omega) = |y(\theta_t\omega)| + |y(\theta_t\omega)|^2$. By the same argument,

$$-2\varepsilon y\|v\|^2 - 2\varepsilon\beta|y|\|\nabla v\|^2 + 2\varepsilon\delta_3|y|\|u\|^2 + 2\varepsilon\delta_4|y|\|\nabla u\|^2 \leqslant \frac{1}{2}\varepsilon\kappa_2 Y(\theta_t\omega)\|\varphi\|_E^2. \tag{3.8}$$

By (2.10a) and (2.7c), we know $\delta\gamma_2 \geqslant \kappa_1$ and $F + \phi_3 \geqslant 0$. Then we find from (2.7b) that

$$2\delta(f(x,u),u) \geqslant 2\kappa_1 \int_{\mathbb{R}^n} F(x,u)\,\mathrm{d}x + 2(\kappa_1 - \delta\gamma_2)\int_{\mathbb{R}^n}\phi_3(x)\,\mathrm{d}x + 2\delta\int_{\mathbb{R}^n}\phi_2(x)\,\mathrm{d}x. \tag{3.9}$$

By (2.10b), we have $\gamma_1/\gamma_3 \leqslant \kappa_2$. Then we see from (2.7a) and (2.7c) that

$$\begin{aligned} 2\varepsilon y(f(x,u),u) &\leqslant 2\varepsilon\gamma_1|y|\int_{\mathbb{R}^n}|u|^{p+1}\mathrm{d}x + 2\varepsilon|y|\|\phi_1\|\|u\| \\ &\leqslant 2\varepsilon\kappa_2 Y(\theta_t\omega)\int_{\mathbb{R}^n}\gamma_3|u|^{p+1}\mathrm{d}x + 2\varepsilon|y|\|\phi_1\|\|u\| \\ &\leqslant 2\varepsilon\kappa_2 Y(\theta_t\omega)\int_{\mathbb{R}^n}F(x,u)\,\mathrm{d}x \\ &\quad + \frac{1}{2}\kappa_1\|\varphi\|_E^2 + \varepsilon c(|y| + |y|^2). \end{aligned} \tag{3.10}$$

Substituting (3.6)–(3.10) into (3.5) we find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi\|_E^2 + 2\int_{\mathbb{R}^n}F(x,u)\,\mathrm{d}x\right) + \big(\kappa_1 - \varepsilon\kappa_2 Y(\theta_t\omega)\big)\left(\|\varphi\|_E^2 + 2\int_{\mathbb{R}^n}F(x,u)\,\mathrm{d}x\right) \\ &\quad\leqslant ce^{|y(\theta_t\omega)|}(1 + \|g(t)\|^2). \end{aligned}$$

Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we obtain from $\int_{\mathbb{R}^n}F(x,u)\,\mathrm{d}x \geqslant -\int_{\mathbb{R}^n}\phi_3\,\mathrm{d}x$ that for $\varsigma \geqslant s-t$, $s \leqslant \tau$ and $t \geqslant 0$,

$$\begin{aligned} \|\varphi(\varsigma, s-t, \theta_{-s}\omega, \varphi_{s-t})\|_E^2 &\leqslant e^{-\kappa_1(\varsigma - s + t) + \varepsilon\kappa_2\int_{-t}^{\varsigma - s}Y(\theta_\sigma\omega)\mathrm{d}\sigma} \\ &\quad\times\left(\|\varphi_{s-t}\|_E^2 + 2\int_{\mathbb{R}^n}F(x,u_{s-t})\,\mathrm{d}x\right) \\ &\quad + c\int_{-t}^{\varsigma - s}e^{\kappa_1(r+s-\varsigma) + |y(\theta_r\omega)| + \varepsilon\kappa_2\int_r^{\varsigma - s}Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\ &\quad\times(1 + \|g(r+s)\|^2)\mathrm{d}r + c. \end{aligned} \tag{3.11}$$

In particular, we let $\varsigma = s$ and then take the supremum over $s \in (-\infty, \tau]$ in (3.11) to obtain

$$\begin{aligned} &\sup_{s\leqslant\tau}\|\varphi(s, s-t, \theta_{-s}\omega, \varphi_{s-t})\|_E^2 \leqslant e^{-\kappa_1 t + \varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\ &\quad\times\sup_{s\leqslant\tau}\left(\|\varphi_{s-t}\|_E^2 + 2\int_{\mathbb{R}^n}F(x,u_{s-t})\,\mathrm{d}x\right) + c\sup_{s\leqslant\tau}R(s,\omega) + c, \end{aligned} \tag{3.12}$$

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where the function $\sup_{s\leqslant\tau} R(s,\omega)$ is well-defined due to (2.12a) and (iii) of proposition 2.2. By (2.3), we have, for all $s \leqslant \tau$,

$$v(s, s-t, \theta_{-s}\omega, v_{s-t}) = z(s, s-t, \theta_{-s}\omega, z_{s-t}) - \varepsilon y(\omega) u(s, s-t, \theta_{-s}\omega, u_{s-t})), \tag{3.13}$$

where $v_{s-t} = z_{s-t} - \varepsilon y(\theta_{-t}\omega) u_{s-t}$. Then we see from (3.12) and (3.13) that

$$\begin{aligned}
&\sup_{s\leqslant\tau} \|(u(s, s-t, \theta_{-s}\omega, u_{s-t}), z(s, s-t, \theta_{-s}\omega, u_{s-t}))\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \\
&\quad\leqslant c e^{|y(\omega)|} e^{-\kappa_1 t + |y(\theta_{-t}\omega)| + \varepsilon\kappa_2 \int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\
&\qquad\times \sup_{s\leqslant\tau}\left(1 + \|u_{s-t}\|^2_{H^1(\mathbb{R}^n)} + \|z_{s-t}\|^2_{H^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n} (F(x, u_{s-t}) + \phi_3)\,\mathrm{d}x\right) \\
&\qquad + c e^{|y(\omega)|} \sup_{s\leqslant\tau} R(s,\omega) + c e^{|y(\omega)|}.
\end{aligned} \tag{3.14}$$

Note that $(u_{s-t}, z_{s-t}) \in \mathcal{B}(s-t, \theta_{-t}\omega)$ for $s \leqslant \tau$ and $\mathcal{B} \in \mathfrak{B}$. Then by (2.12a) and (2.9), as $t \to +\infty$,

$$\begin{aligned}
&e^{-\kappa_1 t + |y(\theta_{-t}\omega)| + \varepsilon\kappa_2 \int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\
&\quad\times \sup_{s\leqslant\tau}\left(1 + \|u_{s-t}\|^2_{H^1(\mathbb{R}^n)} \|z_{s-t}\|^2_{H^1(\mathbb{R}^n)} + \int_{\mathbb{R}^n} (F(x, u_{s-t}) + \phi_3)\,\mathrm{d}x\right) \\
&\qquad\leqslant c e^{-\kappa_1 t + |y(\theta_{-t}\omega)| + \varepsilon\kappa_2 \int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \left(\sup_{s\leqslant\tau} \|(u_{s-t}, z_{s-t})\|^{p+1}_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} + 1\right) \\
&\qquad\leqslant c\left(e^{\frac{-\kappa_1 t}{(p+1)^2}} \sup_{s\leqslant\tau} \|\mathcal{B}(s-t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\right)^{p+1} + c e^{-\frac{p^2+2p}{(p+1)^2}\kappa_1 t} \to 0,
\end{aligned} \tag{3.15}$$

which along with (3.14) implies (3.1). Notice that (3.2) can be proved in the same way. □

3.2. $(\mathcal{B}, \mathcal{D})$-uniform estimates outsider a large ball

In this subsection we derive $(\mathcal{B}, \mathcal{D})$-uniform tail-estimates of solutions to (2.2) outsider a large ball. To that end, we need the following auxiliary estimates.

LEMMA 3.2. *Let hypothesis* ***F*** *be satisfied. Then for every* $\tau \in \mathbb{R}$, $\omega \in \Omega$ *and* $(u_\tau, v_\tau) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$*, the derivative of the solution* $\varphi = (u, v)$ *of* (2.6) *satisfies*

$$\|\varphi_t(t, \tau, \omega, \varphi_\tau)\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \leqslant c e^{|y(\theta_t\omega)|} \big(1 + \|\varphi(t, \tau, \omega, \varphi_\tau)\|^{2p}_E + \|g(t)\|^2\big).$$

Proof. The proof is similar to the autonomous case as in [**51**, lemma 3] and so omitted here. □

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Let $\rho : \mathbb{R} \mapsto [0,1]$ be a smooth function satisfying

$$\rho(s) = \begin{cases} 0, & \text{if } \ 0 \leqslant s \leqslant 1, \\ 1, & \text{if } \ s \geqslant 2. \end{cases} \tag{3.16}$$

For $k \in \mathbb{N}$ and $x \in \mathbb{R}^n$, we denote by $\rho_k(x) := \rho(|x|^2/k^2)$, $\mathcal{O}_k = \{x \in \mathbb{R}^n : |x| \geqslant k\}$ and $\mathcal{O}_k^c = \mathbb{R}^n \backslash \mathcal{O}_k$. Then we derive the following $(\mathcal{B}, \mathcal{D})$-uniform tail-estimates.

LEMMA 3.3. *Let hypotheses* ***F****,* ***S*** *and* ***G*** *be satisfied. Then for each* $(\tau, \omega, \mathcal{B}, \mathcal{D}) \in \mathbb{R} \times \Omega \times \mathfrak{B} \times \mathfrak{D}$*, we have*

$$\lim_{t,k \to +\infty} \sup_{s \leqslant \tau} \|(u(s, s-t, \theta_{-s}\omega, u_{s-t}), z(s, s-t, \theta_{-s}\omega, z_{s-t})\|_{H^1(\mathcal{O}_k^c) \times H^1(\mathcal{O}_k^c)} = 0, \tag{3.17}$$

for all $(u_{s-t}, z_{s-t}) \in \mathcal{B}(s-t, \theta_{-t}\omega)$*, and*

$$\lim_{t,k \to +\infty} \|(u(\tau, \tau-t, \theta_{-\tau}\omega, u_{\tau-t}), z(\tau, \tau-t, \theta_{-\tau}\omega, z_{\tau-t})\|_{H^1(\mathcal{O}_k^c) \times H^1(\mathcal{O}_k^c)} = 0, \tag{3.18}$$

for all $(u_{\tau-t}, z_{\tau-t}) \in \mathcal{D}(\tau-t, \theta_{-t}\omega)$*.*

Proof. Taking the inner product of the second equation of (2.6) with $\rho_k v$ in $L^2(\mathbb{R}^n)$ we find

$$\begin{aligned} &\int_{\mathbb{R}^n} \rho_k v v_t \, \mathrm{d}x - \beta \int_{\mathbb{R}^n} \rho_k v \Delta v_t \, \mathrm{d}x + \delta_1 \int_{\mathbb{R}^n} \rho_k |v|^2 \, \mathrm{d}x - \delta_2 \int_{\mathbb{R}^n} \rho_k v \Delta v \, \mathrm{d}x \\ &+ \delta_3 \int_{\mathbb{R}^n} \rho_k v u \, \mathrm{d}x - \delta_4 \int_{\mathbb{R}^n} \rho_k v \Delta u \, \mathrm{d}x + \int_{\mathbb{R}^n} \rho_k f(x,u) v \, \mathrm{d}x \\ &\quad = \int_{\mathbb{R}^n} \rho_k v g(t) \, \mathrm{d}x - \varepsilon y \int_{\mathbb{R}^n} \rho_k |v|^2 + \varepsilon \beta y \int_{\mathbb{R}^n} \rho_k v \Delta v \, \mathrm{d}x \\ &\quad\quad - (\varepsilon \delta_5 y + \varepsilon^2 y^2) \int_{\mathbb{R}^n} \rho_k v u \, \mathrm{d}x + (\varepsilon \delta_6 y + \varepsilon^2 \beta y^2) \int_{\mathbb{R}^n} \rho_k v \Delta u \, \mathrm{d}x, \end{aligned} \tag{3.19}$$

where $y := y(\theta_t \omega)$. Denote by $|\varphi| := (|v|^2 + \beta |\nabla v|^2 + \delta_1 |u|^2 + \delta_2 |\nabla u|^2)^{1/2}$. As in the autonomous case, see [**51**], by (2.10a) and (3.19), we see

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t} \int_{\mathbb{R}^n} \rho_k (|\varphi|^2 + 2F(x,u)) \, \mathrm{d}x + 2\kappa_1 \int_{\mathbb{R}^n} \rho_k |\varphi|^2 \, \mathrm{d}x + 2(\delta - \varepsilon y) \int_{\mathbb{R}^n} \rho_k f(x,u) u \, \mathrm{d}x \\ &\leqslant -2\varepsilon y \int_{\mathbb{R}^3} \rho_k |v|^2 - 2\varepsilon \beta y \int_{\mathbb{R}^n} \rho_k |\nabla v| dx + 2\varepsilon \delta_3 y \int_{\mathbb{R}^n} \rho_k |u|^2 \, \mathrm{d}x \\ &\quad + 2\varepsilon \delta_4 y \int_{\mathbb{R}^n} \rho_k |\nabla u|^2 \, \mathrm{d}x - 2(\varepsilon \delta_5 y + \varepsilon^2 y^2) \int_{\mathbb{R}^n} \rho_k v u \, \mathrm{d}x \\ &\quad + 2(\varepsilon \delta_6 y + \varepsilon^2 \beta y^2) \int_{\mathbb{R}^n} \rho_k v \Delta u \, \mathrm{d}x + 2 \int_{\mathbb{R}^n} \rho_k v g(t) \, \mathrm{d}x \end{aligned}$$

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$$- 2\delta_4 \int_{\mathbb{R}^n} u_t(\nabla u \cdot \nabla \rho_k)\,\mathrm{d}x - 2\delta_4(\delta - \varepsilon y) \int_{\mathbb{R}^n} u(\nabla u \cdot \nabla \rho_k)\,\mathrm{d}x$$
$$- 2\beta \int_{\mathbb{R}^n} v(\nabla v_t \cdot \nabla \rho_k)\,\mathrm{d}x - 2(\delta_2 + \varepsilon\beta y) \int_{\mathbb{R}^n} v(\nabla v \cdot \nabla \rho_k)\,\mathrm{d}x. \tag{3.20}$$

As in lemma 3.1, we have

$$\begin{aligned} &- 2\varepsilon y \int_{\mathbb{R}^n} \rho_k|v|^2 - 2\varepsilon\beta y \int_{\mathbb{R}^n} \rho_k|\nabla v|^2\,\mathrm{d}x \\ &+ 2\varepsilon\delta_3 y \int_{\mathbb{R}^n} \rho_k|u|^2\,\mathrm{d}x + 2\varepsilon\delta_4 y \int_{\mathbb{R}^n} \rho_k|\nabla u|^2\,\mathrm{d}x \\ &\leqslant 2\varepsilon|y| \int_{\mathbb{R}^n} \rho_k|\varphi|^2\,\mathrm{d}x \leqslant \frac{1}{2}\varepsilon\kappa_2 Y(\theta_t\omega) \int_{\mathbb{R}^n} \rho_k|\varphi|^2\,\mathrm{d}x, \end{aligned} \tag{3.21}$$

where $Y(\theta_t\omega) = |y(\theta_t\omega)| + |y(\theta_t\omega)|^2$. As in lemma 3.1, we get

$$\begin{aligned} &- 2(\varepsilon\delta_5 y + \varepsilon^2 y^2) \int_{\mathbb{R}^n} \rho_k vu\,\mathrm{d}x + 2(\varepsilon\delta_6 y + \varepsilon^2\beta y^2) \int_{\mathbb{R}^n} \rho_k v\Delta u\,\mathrm{d}x \\ &\leqslant \frac{1}{2}\varepsilon\kappa_2 Y(\theta_t\omega) \int_{\mathbb{R}^n} \rho_k|\varphi|^2\,\mathrm{d}x. \end{aligned} \tag{3.22}$$

Note that

$$2\int_{\mathbb{R}^n} \rho_k v g(t)\,\mathrm{d}x \leqslant \frac{1}{2}\kappa_1 \int_{\mathbb{R}^n} \rho_k|\varphi|^2\,\mathrm{d}x + c\int_{\mathbb{R}^n} \rho_k|g(t)|^2\,\mathrm{d}x. \tag{3.23}$$

By (3.16) we know $\|\nabla\rho_k\|_\infty \leqslant c/k$. Then we see from lemma 3.2 that

$$\begin{aligned} &- 2\delta_4 \int_{\mathbb{R}^n} u_t(\nabla u \cdot \nabla \rho_k)\,\mathrm{d}x - 2\delta_4(\delta - \varepsilon y) \int_{\mathbb{R}^n} u(\nabla u \cdot \nabla \rho_k)\,\mathrm{d}x \\ &- 2\beta \int_{\mathbb{R}^n} v(\nabla v_t \cdot \nabla \rho_k)\,\mathrm{d}x - 2(\delta_2 + \varepsilon\beta y) \int_{\mathbb{R}^n} v(\nabla v \cdot \nabla \rho_k)\,\mathrm{d}x \\ &\leqslant \frac{c}{k} e^{|y|}(1 + \|\varphi\|_E^{2p} + \|g(t)\|^2). \end{aligned} \tag{3.24}$$

For the nonlinear term in (3.20), we see from (2.10a) and (2.7b) and (2.7c) that

$$\begin{aligned} 2\delta \int_{\mathbb{R}^n} \rho_k f(x,u)u\,\mathrm{d}x \geqslant 2\kappa_1 \int_{\mathbb{R}^n} \rho_k F(x,u)\,\mathrm{d}x + 2(\kappa_1 - \delta\gamma_2) \int_{\mathbb{R}^3} \rho_k \phi_3(x)\,\mathrm{d}x \\ + 2\delta \int_{\mathbb{R}^n} \rho_k \phi_2(x)\,\mathrm{d}x. \end{aligned} \tag{3.25}$$

As in lemma 3.1, it yields from (2.7a) and (2.7c) and (2.10b) that

$$\begin{aligned} 2\varepsilon y \int_{\mathbb{R}^n} \rho_k f(x,u)u\,\mathrm{d}x \leqslant 2\varepsilon\kappa_2 Y(\theta_t\omega) \int_{\mathbb{R}^n} \rho_k F(x,u)\,\mathrm{d}x + \frac{1}{2}\kappa_1 \int_{\mathbb{R}^n} \rho_k|\varphi|^2\,\mathrm{d}x \\ + ce^{|y|} \int_{\mathbb{R}^n} \rho_k(|\phi_3| + |\phi_1|^2)\,\mathrm{d}x. \end{aligned} \tag{3.26}$$

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Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we substitute (3.21)–(3.26) into (3.20) to find that $\varphi(\varsigma) := \varphi(\varsigma, s-t, \theta_{-s}\omega, \varphi_{s-t})$ with $\varsigma \geqslant s-t$, $s \leqslant \tau$ and $t \geqslant 0$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}\int_{\mathbb{R}^n} \rho_k(|\varphi|^2 + 2F(x,u))\,\mathrm{d}x + (\kappa_1 - \varepsilon\kappa_2 Y(\theta_{\varsigma-s}\omega))\int_{\mathbb{R}^n} \rho_k(|\varphi|^2 + 2F(x,u))\,\mathrm{d}x$$
$$\leqslant c r_k e^{|y(\theta_{\varsigma-s}\omega)|}(1 + \|g(\varsigma)\|^2 + \|\varphi\|_E^{2p}) + c\int_{\mathcal{O}_k^c} |g(\varsigma)|^2\,\mathrm{d}x. \tag{3.27}$$

where $r_k := (1/k) + \int_{\mathcal{O}_k^c}(|\phi_2| + |\phi_3| + |\phi_1|^2)\,\mathrm{d}x \to 0$ as $k \to \infty$. Multiplying (3.27) by $e^{\int_{s-t}^{\varsigma}(\kappa_1 - \varepsilon\kappa_2 Y(\theta_{\sigma-s}\omega))\,\mathrm{d}\sigma}$ and integrating over $(s-t, s)$ with $s \leqslant \tau$, we obtain from (2.7c) and (3.13) that

$$\begin{aligned}
&\int_{\mathbb{R}^n} \rho_k\big(|u(s, s-t, \theta_{-s}\omega, u_{s-t})|^2 + |\nabla u(s, s-t, \theta_{-s}\omega, u_{s-t})|^2\big)\,\mathrm{d}x \\
&\quad + \int_{\mathbb{R}^n} \rho_k\big(|z(s, s-t, \theta_{-s}\omega, z_{s-t})|^2 + |\nabla z(s, s-t, \theta_{-s}\omega, z_{s-t})|^2\big)\,\mathrm{d}x \\
&\quad\leqslant c e^{|y(\omega)|} e^{-\kappa_1 t + |y(\theta_{-t}\omega)| + \varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \int_{\mathbb{R}^n} \rho_k(|u_{s-t}|^2 \\
&\qquad + |\nabla u_{s-t}|^2 + |z_{s-t}|^2 + |\nabla z_{s-t}|^2 + F(x, u_{s-t}) + \phi_3 + |\phi_3|)\,\mathrm{d}x \\
&\qquad + c r_k e^{|y(\omega)|}\int_{-\infty}^0 e^{\kappa_1 r + |y(\theta_r\omega)| + \varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}(1 + \|g(r+s)\|^2)\,\mathrm{d}r \\
&\qquad + c e^{|y(\omega)|}\int_{-\infty}^0 e^{\kappa_1 r + \varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\int_{\mathcal{O}_k^c} |g(r+s, x)|^2\,\mathrm{d}x\,\mathrm{d}r \\
&\qquad + c r_k e^{|y(\omega)|}\int_{s-t}^s e^{\kappa_1(r-s) + |y(\theta_{r-s}\omega)| + \varepsilon\kappa_2\int_{r-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\
&\qquad \times \|\varphi(r, s-t, \theta_{-s}\omega, \varphi_{s-t}\|_E^{2p}\,\mathrm{d}r + c e^{|y(\omega)|}\int_{\mathcal{O}_k^c} |\phi_3|\,\mathrm{d}x.
\end{aligned} \tag{3.28}$$

Note that $\|\rho_k\|_\infty \leqslant c$ and $(u_{s-t}, z_{s-t}) \in \mathcal{B}(s-t, \theta_{-t}\omega)$ for $s \leqslant \tau$ and $\mathcal{B} \in \mathfrak{B}$. Then by the argument of (3.15), we have, as $t \to +\infty$,

$$\begin{aligned}
&e^{-\kappa_1 t + |y(\theta_{-t}\omega)| + \varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \sup_{s\leqslant\tau}\int_{\mathbb{R}^n} \rho_k(|u_{s-t}|^2 \\
&\quad + |\nabla u_{s-t}|^2 + |z_{s-t}|^2 + |\nabla z_{s-t}|^2 + F(x, u_{s-t}) + \phi_3 + |\phi_3|)\,\mathrm{d}x \\
&\quad\leqslant c\Big(e^{\frac{-\kappa_1 t}{(p+1)^2}} \sup_{s\leqslant\tau}\|\mathcal{B}(s-t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\Big)^{p+1} + c e^{-\frac{p^2+2p}{(p+1)^2}\kappa_1 t} \to 0.
\end{aligned} \tag{3.29}$$

By (2.12b) and (iii) of proposition 2.2, we know

$$\sup_{s\leqslant\tau}\int_{-\infty}^0 e^{\kappa_1 r + |y(\theta_r\omega)| + \varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}(1 + \|g(r+s)\|^2)\,\mathrm{d}r < \infty. \tag{3.30}$$

By (2.12b) and (iv) of proposition 2.2, we know

$$\sup_{s\leqslant\tau}\int_{-\infty}^{0} e^{\kappa_1 r+\varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\int_{\mathcal{O}_k^c}|g(r+s,x)|^2\,\mathrm{d}x\,\mathrm{d}r\to 0 \quad \text{as } k\to\infty. \tag{3.31}$$

In order to consider the remaining term on the right-hand side of (3.28), we infer from (3.11) that for all $\varsigma\in[s-t,s]$ with $s\leqslant\tau$ and $t\geqslant 0$,

$$\begin{aligned}\|\varphi(\varsigma,s-t,\theta_{-s}\omega,\varphi_{s-t})\|_E^2 &\leqslant c e^{-(((\kappa_1(\varsigma-s+t))/(p+(1/(p+3))))+\varepsilon\kappa_2\int_{-t}^{\varsigma-s}Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\ &\quad\times\left(1+\|\varphi_{s-t}\|_E^2+\int_{\mathbb{R}^n}(F(x,u_{s-t})+\phi_3(x))\,\mathrm{d}x\right) \\ &\quad + c e^{-(1/(p+(1/(p+1)))\kappa(\varsigma-s)}(1+\hat{R}(s,\omega)),\end{aligned} \tag{3.32}$$

where

$$\hat{R}(s,\omega):=\int_{-\infty}^{0} e^{\frac{1}{p+\frac{1}{p+3}}\kappa_1 r+|y(\theta_r\omega)|+\varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}(1+\|g(r+s)\|^2)\,\mathrm{d}r.$$

By (2.12b) and proposition 2.2 we find $\sup_{s\leqslant\tau}\hat{R}(s,\omega)<\infty$. Taking the p-th power of (3.32), multiplying by $e^{\kappa_1(\varsigma-s)+|y(\theta_{\varsigma-s}\omega)|+\varepsilon\kappa_2\int_{\varsigma-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}$ and integrating over $(s-t,s)$ with $t\geqslant 0$, we obtain

$$\begin{aligned}&\int_{s-t}^{s} e^{\kappa_1(\varsigma-s)+|y(\theta_{\varsigma-s}\omega)|+\varepsilon\kappa_2\int_{\varsigma-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\|\varphi(\varsigma,s-t,\theta_{-s}\omega,\varphi_{s-t})\|_E^{2p}\,\mathrm{d}\varsigma \\ &\quad\leqslant c\int_{s-t}^{s} e^{\kappa_1(\varsigma-s)+|y(\theta_{\varsigma-s}\omega)|+\varepsilon\kappa_2\int_{\varsigma-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma-\frac{p}{p+\frac{1}{p+3}}\kappa_1(\varsigma-s+t)+p\varepsilon\kappa_2\int_{-t}^{\varsigma-s}Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\ &\quad\quad\times\left(1+\|\varphi_{s-t}\|_E^2+\int_{\mathbb{R}^n}(F(x,u_{s-t})+\phi_3(x))\,\mathrm{d}x\right)^p\mathrm{d}\varsigma \\ &\quad\quad + c(1+\hat{R}(s,\omega))^p\int_{s-t}^{s} e^{\kappa_1(\varsigma-s)+|y(\theta_{\varsigma-s}\omega)|+\varepsilon\kappa_2\int_{\varsigma-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}e^{-\frac{p}{p+\frac{1}{p+1}}\kappa(\varsigma-s)}\,\mathrm{d}\varsigma.\end{aligned} \tag{3.33}$$

Now, we let the first and second terms on the right-hand side of (3.33) as I_1 and I_2, respectively. In addition, we set

$$r_1(\omega):=\int_{-\infty}^{0} e^{(1-(p/(p+(1/(p+3)))))\kappa_1 r+|y(\theta_r\omega)|}\,\mathrm{d}r<\infty.$$

Note that

$$-\frac{p}{p+\frac{1}{p+3}}+\frac{p}{(p+1)^2}\leqslant-\frac{p}{p+1}$$

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for all $p \geqslant 1$. Then by $(u_{s-t}, z_{s-t}) \in \mathcal{B}(s-t, \theta_{-t}\omega)$ for $s \leqslant \tau$ and $\mathcal{B} \in \mathfrak{B}$ we see from (2.9), (3.13) and (2.12a) that as $t \to +\infty$,

$$\begin{aligned}
I_1 &\leqslant c r_1(\omega) e^{-\frac{p}{p+\frac{1}{p+3}}\kappa_1 t + p\varepsilon\kappa_2 \int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\
&\quad \times \left(1 + \|\varphi_{s-t}\|_E^2 + \int_{\mathbb{R}^n} (F(x, u_{s-t}) + \phi_3(x))\,\mathrm{d}x\right)^p \\
&\leqslant c r_1(\omega) e^{-\frac{p}{p+\frac{1}{p+3}}\kappa_1 t + p\varepsilon\kappa_2 \int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \left(\|u_{s-t}\|_{H^1(\mathbb{R}^n)}^{p(p+1)} + \|v_{s-t}\|_{H^1(\mathbb{R}^n)}^{p(p+1)} + 1\right) \\
&\leqslant c r_1(\omega) e^{-\frac{p}{p+\frac{1}{p+3}}\kappa_1 t + p|y(\theta_{-t}\omega)| + p\varepsilon\kappa_2 \int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \\
&\quad \times \left(\|(u_{s-t}, z_{s-t})\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}^{p(p+1)} + 1\right) \\
&\leqslant c r_1(\omega) e^{-\frac{p}{p+\frac{1}{p+3}}\kappa_1 t + \frac{p}{(p+1)^2}\kappa_1 t} \left(\|\sup_{s\leqslant\tau} \|\mathcal{B}(s-t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}^{p(p+1)} + 1\right) \\
&\leqslant c r_1(\omega) \left(e^{-\frac{1}{(p+1)^2}\kappa_1 t} \sup_{s\leqslant\tau} \|\mathcal{B}(s-t, \theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\right)^{p(p+1)} \\
&\qquad + c e^{-\frac{p}{p+1}\kappa_1 t} \to 0.
\end{aligned} \tag{3.34}$$

Note that

$$1 - \frac{p}{p + \frac{1}{p+1}} - \frac{1}{(p+1)^2} > 0$$

for all $p \geqslant 1$. Then by (2.12b) we see

$$I_2 \leqslant c\left(1 + \sup_{s\leqslant\tau} \hat{R}^p(s, \omega)\right) \int_{-\infty}^0 e^{\left(1 - \frac{p}{p+\frac{1}{p+1}}\right)\kappa_1 r + |y(\theta_r\omega)| + \varepsilon\kappa_2 \int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\,\mathrm{d}r < \infty. \tag{3.35}$$

By (3.33)–(3.35), the remaining term on the right-hand side satisfies, as $t, k \to +\infty$,

$$r_k \sup_{s\leqslant\tau} \int_{s-t}^s e^{\kappa_1(r-s) + |y(\theta_{r-s}\omega)| + \varepsilon\kappa_2 \int_{r-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma} \|\varphi(r, s-t, \theta_{-s}\omega, \varphi_{s-t}\|_E^{2p}\,\mathrm{d}r \to 0. \tag{3.36}$$

Finally, we take the supremum over $s \in (-\infty, \tau]$ in (3.28), then the desired result (3.17) follows from (3.29)–(3.31) and (3.36). By the same argument, we can show (3.18), the details are omitted. □

3.3. $(\mathcal{B}, \mathcal{D})$-uniform estimates inside a large ball

In this subsection we derive $(\mathcal{B}, \mathcal{D})$-uniform estimates of solutions to problem (2.2) on bounded domains. Denote by $\xi_k(x) := 1 - \rho_k(x)$ for $k \in \mathbb{N}$ with ρ given in (3.16). Let $\bar{\varphi} = (\bar{u}, \bar{v}) := \xi_k\varphi = (\xi_k u, \xi_k v)$, where $\varphi = (u, v)$ is the solution of

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problem (2.6). Multiplying (2.6) by ξ_k we find

$$\begin{cases} \bar{u}_t = (\varepsilon y - \delta)\bar{u} + \bar{v}, \\ \bar{v}_t - \beta\Delta\bar{v}_t + \delta_1\bar{v} - \delta_2\Delta\bar{v} + \delta_3\bar{u} - \delta_4\Delta\bar{u} + \xi_k f(x,u) \\ = -\varepsilon y\bar{v} + \varepsilon\beta y\Delta\bar{v} - (\varepsilon\delta_5 y + \varepsilon^2 y^2)\bar{u} + (\varepsilon\delta_6 y + \varepsilon^2\beta y^2)\Delta\bar{u} + J, \end{cases} \tag{3.37}$$

where $y = y(\theta_t\omega)$ and the remaining terms are given by

$$\begin{aligned} J &:= \xi_k g(t) - \beta v_t\Delta\xi_k - 2\beta\nabla\xi_k\cdot\nabla v_t - \delta_2 v\Delta\xi_k - 2\delta_2\nabla\xi_k\cdot\nabla v \\ &\quad - \delta_2 u\Delta\xi_k - 2\delta_2\nabla\xi_k\cdot\nabla u - \varepsilon\beta y v\Delta\xi_k - 2\varepsilon\beta y\nabla\xi_k\cdot\nabla v \\ &\quad - (\varepsilon\delta_4 y + \varepsilon^2\beta y^2)u\Delta\xi_k - 2(\varepsilon\delta_4 y + \varepsilon^2\beta y^2)\nabla\xi_k\cdot\nabla u. \end{aligned} \tag{3.38}$$

Note that the eigenvalue problem: $-\Delta u = \lambda u$ in $\mathcal{O}_{2k}$ with $u|_{\partial\mathcal{O}_{2k}} = 0$ has a family of eigenvalues $\{\lambda_i\}_{i=1}^{\infty}$ such that $0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_i \to \infty$ as $i \to \infty$ and the corresponding eigenfunctions $\{e_i\}_{i=1}^{\infty}$ in $H^1(\mathcal{O}_{2k})$ form an orthonormal basis of $L^2(\mathcal{O}_{2k})$. Let $\mathcal{P}_i : L^2(\mathcal{O}_{2k}) \to \mathrm{span}\{e_1, e_2 \ldots, e_i\}$ be the canonical projection.

Lemma 3.4. *Let hypotheses* ***F***, ***S*** *and* ***G*** *be satisfied. Then for each* $(k,\tau,\omega,\mathcal{B},\mathcal{D}) \in \mathbb{N}\times\mathbb{R}\times\Omega\times\mathfrak{B}\times\mathfrak{D}$,

$$\begin{aligned} &\lim_{t,i\to+\infty}\sup_{s\leqslant\tau}\sup_{(u_{s-t},z_{s-t})\in\mathcal{B}(s-t,\theta_{-t}\omega)} \big\|\big((I-\mathcal{P}_i)\xi_k u(s,s-t,\theta_{-s}\omega,u_{s-t}), \\ &\quad (I-\mathcal{P}_i)\xi_k z(s,s-t,\theta_{-s}\omega,z_{s-t})\big)\big\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} = 0, \end{aligned} \tag{3.39}$$

and

$$\begin{aligned} &\lim_{t,i\to+\infty}\sup_{(u_{\tau-t},z_{\tau-t})\in\mathcal{D}(\tau-t,\theta_{-t}\omega)} \big\|\big((I-\mathcal{P}_i)\xi_k u(\tau,\tau-t,\theta_{-\tau}\omega,u_{\tau-t}), \\ &\quad (I-\mathcal{P}_i)\xi_k z(\tau,\tau-t,\theta_{-\tau}\omega,z_{\tau-t})\big)\big\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} = 0. \end{aligned} \tag{3.40}$$

Proof. Let $\bar{u}_i = (I-\mathcal{P}_i)\xi_k u$, $\bar{z}_i = (I-\mathcal{P}_i)\xi_k z$ and $\bar{v}_i = (I-\mathcal{P}_i)\xi_k v$. Applying $I - P_i$ to the second equation of (3.37) and taking the inner product of the resulting equation with $\bar{v}_i$ in $L^2(\mathcal{O}_{2k})$, we find

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}(\|\bar{v}_i\|^2 + \beta\|\nabla\bar{v}_i\|^2) + 2\delta_1\|\bar{v}_i\|^2 + 2\delta_2\|\nabla\bar{v}_i\|^2 + 2\delta_3(\bar{u}_i,\bar{v}_i) \\ &\quad - 2\delta_4(\Delta\bar{u}_i,\bar{v}_i) + 2(\xi_k f(x,u),\bar{v}_i) = -2\varepsilon y\|\bar{v}_i\|^2 - 2\varepsilon\beta y\|\nabla\bar{v}_i\|^2 \\ &\quad - 2(\varepsilon\delta_5 y + \varepsilon^2 y^2)(\bar{u},\bar{v}_i) + 2(\varepsilon\delta_6 y + \varepsilon^2\beta y^2)(\Delta\bar{u},\bar{v}_i) + 2\big(J,\bar{v}_i\big). \end{aligned}$$

Let $\bar{\varphi}_i = (\bar{u}_i,\bar{v}_i)$, and then we find that

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}t}\|\bar{\varphi}_i\|_E^2 + 2\kappa_1\|\bar{\varphi}_i\|_E^2 \leqslant -2(\xi_k f(x,u),\bar{v}_i) - 2\varepsilon y\|\bar{v}_i\|^2 \\ &\quad - 2\varepsilon\beta y\|\nabla\bar{v}_i\|^2 + 2\varepsilon\delta_3 y\|\bar{u}_i\|^2 + 2\varepsilon\delta_4 y\|\nabla\bar{u}_i\|^2 \\ &\quad - 2(\varepsilon\delta_5 y + \varepsilon^2 y^2)(\bar{u},\bar{v}_i) + 2(\varepsilon\delta_6 y + \varepsilon^2\beta y^2)(\Delta\bar{u},\bar{v}_i) + 2(J,\bar{v}_i). \end{aligned} \tag{3.41}$$

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Note that $\mu := ((np-n)/(2p+2)) \in [0,1)$ for all $n \geqslant 1$ due to (2.8). Then we see from (2.7a) and the Gagliardo–Nirenberg inequality that

$$\begin{aligned}
-2(\xi_k f(x,u),\bar{v}_i) &\leqslant c\|u\|_{p+1}^p\|\bar{v}_i\|_{p+1} + c\|\phi_1\|\|\bar{v}_i\| \\
&\leqslant c\|u\|_{p+1}^p\|\nabla\bar{v}_i\|^{\mu}\|\bar{v}_i\|^{1-\mu} + c\|\bar{v}_i\| \\
&\leqslant c\lambda_{i+1}^{\frac{\mu-1}{2}}\|u\|_{H^1(\mathbb{R}^n)}^p\|\nabla\bar{v}_i\| + c\lambda_{i+1}^{-\frac{1}{2}}\|\|\nabla\bar{v}_i\| \\
&\leqslant c\lambda_{i+1}^{\frac{\mu-1}{2}}\|\varphi\|_E^p\|\bar{\varphi}_i\|_E + c\lambda_{i+1}^{-\frac{1}{2}}\|\bar{\varphi}_i\|_E \\
&\leqslant \frac{1}{2}\kappa_1\|\bar{\varphi}_i\|_E^2 + c\lambda_{i+1}^{\mu-1}\|\varphi\|_E^{2p} + c\lambda_{i+1}^{-1}. \qquad (3.42)
\end{aligned}$$

Note that

$$\begin{aligned}
&-2\varepsilon y\|\bar{v}_i\|^2 - 2\varepsilon\beta y\|\nabla\bar{v}_i\|^2 + 2\varepsilon\delta_3 y\|\bar{u}_i\|^2 + 2\varepsilon\delta_4 y\|\nabla\bar{u}_i\|^2 \\
&\quad\leqslant 2\varepsilon|y|\|\bar{\varphi}_i\|_E^2 \leqslant \frac{1}{2}\varepsilon\kappa_2 Y(\theta_t\omega)\|\bar{\varphi}_i\|_E^2. \qquad (3.43)
\end{aligned}$$

By Young's inequality, (2.11) and (2.10b), we have

$$\begin{aligned}
&-2(\varepsilon\delta_5 y + \varepsilon^2 y^2)(\bar{u},\bar{v}_i) + 2(\varepsilon\delta_6 y + \varepsilon^2\beta y^2)(\Delta\bar{u},\bar{v}_i) \\
&\quad\leqslant \varepsilon(\delta_5|y| + |y|^2)(\|\bar{u}_i\|^2 + \|\bar{v}_i\|^2) + \varepsilon(\delta_6|y| + \beta|y|^2)(\|\nabla\bar{u}_i\|^2 + \|\nabla\bar{v}_i\|^2) \\
&\quad\leqslant \frac{1}{2}\varepsilon\kappa_2 Y(\theta_t\omega)\|\bar{\varphi}_i\|_E^2. \qquad (3.44)
\end{aligned}$$

Finally, by lemma 3.2, we see

$$(J,\bar{v}_i) \leqslant \|J\|\|\bar{v}_i\| \leqslant c\lambda_{i+1}^{-\frac{1}{2}}\|J\|\|\bar{\varphi}_i\|_E \leqslant \frac{1}{4}\kappa_1\|\bar{\varphi}_i\|_E^2 + c\lambda_{i+1}^{-1}e^{|y|}(1 + \|g(t)\|^2 + \|\varphi\|_E^{2p}). \qquad (3.45)$$

Given $\tau \in \mathbb{R}$ and $\omega \in \Omega$, we substitute (3.42)–(3.45) into (3.41) to find that $\varphi_i(\varsigma) := \varphi_i(\varsigma, s-t, \theta_{-s}\omega)$ with $\varsigma \geqslant s-t$, $s \leqslant \tau$ and $t \geqslant 0$ satisfies the following energy inequality:

$$\frac{\mathrm{d}}{\mathrm{d}\varsigma}\|\varphi_i\|_E^2 + (\kappa_1 - \varepsilon\kappa_2 Y(\theta_{\varsigma-s}\omega))\|\varphi_i\|_E^2 \leqslant c\theta_i e^{|y(\theta_{\varsigma-s}\omega)|}(1 + \|g(\varsigma)\|^2 + \|\varphi\|_E^{2p}). \qquad (3.46)$$

where $\theta_i := \lambda_{i+1}^{\mu-1} + \lambda_{i+1}^{-1} \to 0$ as $i \to +\infty$. Multiplying (3.46) by $e^{\int_{s-t}^{\varsigma}(\kappa_1 - \varepsilon\kappa_2 Y(\theta_{\sigma-s}\omega))\,\mathrm{d}\sigma}$ and integrating over $(s-t, s)$, then we take the supremum

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over $s \in (-\infty, \tau]$, finally, we obtain (3.13) from that

$$\begin{aligned}
&\sup_{s\leqslant\tau} \big\|\big(u_i(s-t,\theta_{-s}\omega,(I-\mathcal{P}_i)(\xi_k u_{s-t})),\\
&\quad \times z_i(s-t,\theta_{-s}\omega,(I-\mathcal{P}_i)(\xi_k z_{s-t}))\big)\big\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\\
&\qquad \leqslant c e^{|y(\omega)|} e^{-\kappa_1 t+|y(\theta_{-t}\omega)|+\varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\\
&\qquad\quad \times \sup_{s\leqslant\tau}\|((I-\mathcal{P}_i)\xi_k u_{s-t},(I-\mathcal{P}_i)\xi_k z_{s-t})\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\\
&\qquad\quad + c\theta_i e^{|y(\omega)|}\sup_{s\leqslant\tau}\int_{-\infty}^0 e^{\kappa_1 r+|y(\theta_r\omega)|+\varepsilon\kappa_2\int_r^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}(1+\|g(r+s)\|^2)dr\\
&\qquad\quad + c\theta_i e^{|y(\omega)|}\sup_{s\leqslant\tau}\int_{s-t}^s e^{\kappa_1(r-s)+|y(\theta_{r-s}\omega)|+\varepsilon\kappa_2\int_{r-s}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\\
&\qquad\quad \times\|\varphi(r,s-t,\theta_{-s}\omega,\varphi_{s-t}\|^{2p}_E\,\mathrm{d}r. \qquad (3.47)
\end{aligned}$$

By $\|I-\mathcal{P}_i\|\leqslant 1$, $\|\xi_k\|_\infty\leqslant 1$ and $(u_{s-t},z_{s-t})\in\mathcal{B}(s-t,\theta_{-t}\omega)$ for all $s\leqslant\tau$, we find from (2.12a) that as $t\to+\infty$,

$$\begin{aligned}
&e^{-\kappa_1 t+|y(\theta_{-t}\omega)|+\varepsilon\kappa_2\int_{-t}^0 Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}\sup_{s\leqslant\tau}\|(I-\mathcal{P}_i)(\xi_k u_{s-t},\xi_k z_{s-t})\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\\
&\leqslant c\Big(e^{-\frac{1}{(p+1)^2}\kappa_1 t}\sup_{s\leqslant\tau}\|\mathcal{B}(s-t,\theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\Big)^2\to 0.
\end{aligned}$$

By (3.30) and (3.36) we find that the remaining terms on the right-hand side of (3.47) go to zero as $i,t\to+\infty$, and hence we have (3.39). By the same argument, we can show (3.40), the details are omitted here. □

4. Existence, uniqueness and semi-uniform compactness of pullback random attractors

In this section we establish the semi-uniform compactness of pullback random attractors of the cocycle Φ generated by the stochastic wave equation (2.2).

4.1. Existence of $(\mathfrak{B},\mathfrak{D})$-pullback absorbing set

First, we establish the existence of $(\mathfrak{B},\mathfrak{D})$-pullback absorbing set in $H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)$.

Lemma 4.1. *Assume hypotheses* ***F***, ***S*** *and* ***G*** *hold. Then we have the following two conclusions for the cocycle* Φ *generated by problem* (2.2)*:*

(i) Φ *has a* $\mathfrak{B}$*-pullback absorbing set* $\mathcal{K}_{\mathfrak{B}}=\{\mathcal{K}_{\mathfrak{B}}(\tau,\omega):\tau\in\mathbb{R},\omega\in\Omega\}\in\mathfrak{B}$, *which is given by, for* $(\tau,\omega)\in\mathbb{R}\times\Omega$,

$$\begin{aligned}
\mathcal{K}_{\mathfrak{B}}(\tau,\omega)=\Big\{&(u,z)\in H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n):\|(u,z)\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}\\
&\leqslant Me^{|y(\omega)|}\Big(1+\sup_{s\leqslant\tau}R(s,\omega)\Big)\Big\}. \qquad (4.1)
\end{aligned}$$

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(ii) *Φ has a $\mathfrak{D}$-pullback **random** absorbing set $\mathcal{K}_{\mathfrak{D}} = \{\mathcal{K}_{\mathfrak{D}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$, which is given by, for every $(\tau,\omega) \in \mathbb{R} \times \Omega$,*

$$\mathcal{K}_{\mathfrak{D}}(\tau,\omega) = \Big\{(u,z) \in H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) : \|(u,z)\|^2_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \leqslant M e^{|y(\omega)|}(1 + R(\tau,\omega))\Big\}, \tag{4.2}$$

where M and $R(s,\omega)$ are the same as given in lemma 3.1.

Proof. (i) By (i) lemma 3.1, for each $(\tau,\omega,\mathcal{B}) \in \mathbb{R} \times \Omega \times \mathfrak{B}$, there exists $T_{\mathcal{B}} := T_{\mathcal{B}}(\tau,\omega) > 0$ such that $\bigcup_{t \geqslant T_{\mathcal{B}}} \bigcup_{s \leqslant \tau} \Phi(t, s-t, \theta_{-t}\omega)\mathcal{B}(s-t,\theta_{-t}\omega) \subset \mathcal{K}_{\mathfrak{B}}(\tau,\omega)$. Next, we show $\mathcal{K}_{\mathfrak{B}} \in \mathfrak{B}$. For every $(\tau,\omega) \in \mathbb{R} \times \Omega$, we deduce from (2.12b) as well as (iii) of proposition 2.2 that as $t \to +\infty$,

$$\begin{aligned} &e^{-\frac{2}{(p+1)^2}\kappa_1 t} \sup_{s \leqslant \tau} R(s,\theta_{-t}\omega) \\ &\quad = e^{-\frac{2}{(p+1)^2}\kappa_1 t} \sup_{s \leqslant \tau} \int_{-\infty}^{-t} e^{\kappa_1(r+t) + |y(\theta_r\omega)| + \varepsilon\kappa_2 \int_r^{-t} Y(\theta_\sigma\omega)\,\mathrm{d}\sigma}(1 + \|g(r+s)\|^2)\,\mathrm{d}r \\ &\quad \leqslant e^{\frac{-\kappa_1 t}{2(p+1)^2}} \sup_{s \leqslant \tau} \int_{-\infty}^{0} e^{\frac{\kappa_1 r}{2(p+1)^2}}(1 + \|g(r+s)\|^2)\,\mathrm{d}r \to 0. \end{aligned}$$

Therefore, we have, as $t \to +\infty$,

$$\begin{aligned} &e^{-\frac{1}{(p+1)^2}\kappa_1 t} \sup_{s \leqslant \tau} \|\mathcal{K}_{\mathcal{B}}(s-t,\theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \\ &\qquad \leqslant c\Big(e^{|y(\theta_{-t}\omega)|} e^{-\frac{2}{(p+1)^2}\kappa_1 t}\Big(1 + \sup_{s \leqslant \tau} R(s,\omega)\Big)\Big)^{1/2} \to 0, \end{aligned}$$

which shows that $\mathfrak{B}$ is a $\mathfrak{B}$-pullback absorbing set of Φ.

(ii) Since the mapping $\omega \to R(\tau,\omega)$ is $\mathcal{F}$-measurable and $\mathcal{K}_{\mathfrak{D}} \subseteq \mathcal{K}_{\mathfrak{B}} \in \mathfrak{B} \subseteq \mathfrak{D}$. Then by (ii) of lemma 3.1, we find that $\mathcal{K}_{\mathfrak{D}}$ is a $\mathfrak{D}$-pullback random absorbing set for Φ.

□

4.2. $(\mathfrak{B},\mathfrak{D})$-pullback asymptotic compactness

Then, we establish the $(\mathfrak{B},\mathfrak{D})$-pullback asymptotic compactness of Φ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

LEMMA 4.2. *Assume hypotheses **F**, **S** and **G** hold. Then we have the following two conclusions for the cocycle Φ generated by (2.2):*

(i) *Φ is $\mathfrak{B}$-pullback **time-semi-uniformly** asymptotically compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{B} = \{\mathcal{B}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega)\} \in \mathfrak{B}$, the sequence $\{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (u_{0,n}, z_{0,n})\}_{n=1}^{\infty}$ is pre-compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ whenever $t_n \to +\infty$, $(u_{0,n}, z_{0,n}) \in \mathcal{B}(s_n - t_n, \theta_{-t_n}\omega)$ and $s_n \leqslant \tau$.*

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(ii) *Φ is $\mathfrak{D}$-pullback asymptotically compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, that is, for every $\tau \in \mathbb{R}$, $\omega \in \Omega$ and $\mathcal{D} = \{\mathcal{D}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega)\} \in \mathfrak{D}$, the sequence $\{\Phi(t_n, \tau - t_n, \theta_{-t_n}\omega, (u_{0,n}, z_{0,n}))\}_{n=1}^{\infty}$ is pre-compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ whenever $t_n \to +\infty$ and $(u_{0,n}, z_{0,n}) \in \mathcal{D}(\tau - t_n, \theta_{-t_n}\omega)$.*

Proof. (i) Let $\varepsilon > 0$ be an arbitrary number, we want to show that the sequence

$$\{(U_n, Z_n)\}_{n=1}^{\infty} := \{\Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (u_{0,n}, z_{0,n}))\}_{n=1}^{\infty}$$

has a finite open cover with radiis less then ε in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ provided $t_n \to +\infty$, $(u_{0,n}, z_{0,n}) \in \mathcal{B}(s_n - t_n, \theta_{-t_n}\omega)$ and $s_n \leqslant \tau$. By (3.1), there are $N_1 = N_1(\tau, \omega, \mathcal{B}) \geqslant 1$ and $c_1 = c_1(\tau, \omega) > 0$ such that for all $n \geqslant N_1$,

$$\|(U_n, Z_n)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \leqslant c_1. \tag{4.3}$$

By (3.17), there are $N_2 = N_2(\tau, \omega, \mathcal{B}, \varepsilon) \geqslant N_1$ and $k = k(\tau, \omega, \mathcal{B}, \varepsilon) \geqslant 1$ such that for all $n \geqslant N_2$,

$$\|(U_n, Z_n)\|_{H^1(\mathbb{R}^n\setminus\mathcal{O}_k)\times H^1(\mathbb{R}^n\setminus\mathcal{O}_k)} < \frac{\varepsilon}{2}. \tag{4.4}$$

Recall that $\xi_k(x) = 1 - \rho(|x|/k)$ with the function ρ as given in (3.16). By (3.39), there are $N_3 = N_3(\tau, \omega, \mathcal{B}, \varepsilon) \geqslant N_2$ and $i = i(\tau, \omega, \mathcal{B}, \varepsilon) \geqslant 1$ such that for all $n \geqslant N_3$,

$$\|((I - \mathcal{P}_i)\xi_k U_n, (I - \mathcal{P}_i)\xi_k Z_n)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} < \frac{\varepsilon}{4}, \tag{4.5}$$

Then by (4.3), $\|\xi_k\|_\infty \leqslant 1$ and the finite-dimensional range of $\mathcal{P}_i$, we know that $\{(\mathcal{P}_i\xi_k U_n, \mathcal{P}_i\xi_k Z_n)\}_{n=1}^{\infty}$ is pre-compact in $\mathcal{P}_{m_0}(H^1(\mathcal{O}_{2k}) \times H^1(\mathcal{O}_{2k}))$, which along with (4.5) implies that $\{(\xi_k U_n, \xi_k Z_n)\}_{n=1}^{\infty}$ has a finite open cover with radiis less then $\frac{1}{2}\varepsilon$ in $H^1(\mathcal{O}_{2k}) \times H^1(\mathcal{O}_{2k})$. This along with the fact that $(U_n(x), Z_n(x)) = (\xi_k(x)U_n(x), \xi_k(x)Z_n(x))$ for all $x \in \mathcal{O}_{k_0}$ implies that the sequence $\{(U_n, Z_n)\}_{n=1}^{\infty}$ has a finite open cover with radiis less then $\frac{1}{2}\varepsilon$ in $H^1(\mathcal{O}_{k_0}) \times H^1(\mathcal{O}_{k_0})$. This together with (4.4) further implies that the sequence $\{(U_n, Z_n)\}_{n=1}^{\infty}$ has a finite open cover with radiis less then ε in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

(ii) By (3.2), (3.18) and (3.40), we can similarly prove (ii), the details are omitted here.

□

4.3. Existence and time-semi-uniform compactness of pullback random attractors

We are in the position to establish the existence, uniqueness and **time-semi-uniform compactness** of pullback random attractors of Φ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. This will be used to discuss the **asymptotically autonomous robustness** of the pullback random attractors in the next section.

THEOREM 4.3. *Let hypotheses **F**, **S** and **G** be satisfied. Then the following two conclusions hold for the cocycle Φ generated by problem (2.2):*

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(i) *Φ has a $\mathfrak{B}$-pullback attractor $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{B}$, which is given by*

$$\mathcal{A}_{\mathfrak{B}}(\tau,\omega) = \bigcap_{t_0>0} \overline{\bigcup_{t\geqslant t_0} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{K}_{\mathfrak{B}}(\tau-t,\theta_{-t}\omega)}^{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}. \quad (4.6)$$

(ii) *$\mathcal{A}_{\mathfrak{B}}$ is* ***time-semi-uniformly compact*** *in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ in the sense that the union $\bigcup_{s\leqslant\tau} \mathcal{A}_{\mathfrak{B}}(s,\omega)$ is pre-compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for every $(\tau,\omega) \in \mathbb{R} \times \Omega$.*

(iii) *Φ has a $\mathfrak{D}$-pullback* ***random*** *attractor $\mathcal{A}_{\mathfrak{D}} = \{\mathcal{A}_{\mathfrak{D}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$, which is given by*

$$\mathcal{A}_{\mathfrak{D}}(\tau,\omega) = \bigcap_{t_0>0} \overline{\bigcup_{t\geqslant t_0} \Phi(t,\tau-t,\theta_{-t}\omega)\mathcal{K}_{\mathfrak{D}}(\tau-t,\theta_{-t}\omega)}^{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)}. \quad (4.7)$$

(iv) *$\mathcal{A}_{\mathfrak{B}} = \mathcal{A}_{\mathfrak{D}}$, and thus Φ has a unique pullback random attractor which is time-semi-uniformly compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.*

Proof. (i) By (i) of lemma 4.1 we find that $\mathcal{K}_{\mathfrak{B}} = \{\mathcal{K}_{\mathfrak{B}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{B}$ is a $\mathfrak{B}$-pullback absorbing set for Φ. By (i) of lemma 4.2 we know that Φ is $\mathfrak{B}$-pullback asymptotically compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. Hence by the abstract result as given by Wang [**41**, proposition 3.8] we know that Φ has a unique $\mathfrak{B}$-pullback attractor $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{B}$ given by (4.6) in the sense of [**41**, definition 2.15]. However, we remark that only the $\mathcal{F}$-measurability of $\mathcal{A}_{\mathfrak{B}}$ is unknown, that is why we here say $\mathcal{A}_{\mathfrak{B}}$ is a $\mathfrak{B}$-pullback attractor but not a $\mathfrak{B}$-pullback random attractor.

(ii) It suffices to show that $\cup_{s\leqslant\tau}\mathcal{A}_{\mathfrak{B}}(s,\omega)$ is pre-compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ for each $\tau \in \mathbb{R}$ and $\omega \in \Omega$. Let $\{(U_n, Z_n)\}_{n=1}^{\infty}$ be an arbitrary sequence taken from $\cup_{s\leqslant\tau}\mathcal{A}_{\mathfrak{B}}(s,\omega)$. Then there exists $s_n \leqslant \tau$ such that $(U_n, Z_n) \in \mathcal{A}_{\mathfrak{B}}(s_n,\omega)$ for each $n \in \mathbb{N}$. Now, we let $t_n \to \infty$, and by the invariance of $\mathcal{A}_{\mathfrak{B}}$ we have $(U_n, Z_n) = \Phi(t_n, s_n - t_n, \theta_{-t_n}\omega)\mathcal{A}_{\mathfrak{B}}(s_n - t_n, \theta_{-t_n}\omega)$, which implies that there exists $(u_{0,n}, z_{0,n}) \in \mathcal{A}_{\mathfrak{B}}(s_n - t_n, \theta_{-t_n}\omega)$ such that $(U_n, Z_n) = \Phi(t_n, s_n - t_n, \theta_{-t_n}\omega, (u_{0,n}, z_{0,n}))$. Note that $(u_{0,n}, z_{0,n}) \in \mathcal{A}_{\mathfrak{B}}(s_n - t_n, \theta_{-t_n}\omega) \subseteq \mathcal{K}_{\mathfrak{B}}(s_n - t_n, \theta_{-t_n}\omega)$ with $s_n \leqslant \tau$ and $\mathcal{K}_{\mathfrak{B}} \in \mathfrak{B}$, then by the $\mathfrak{B}$-pullback time-semi-uniform asymptotic compactness of Φ as proved in lemma 4.2 we know that the sequence $\{(U_n, Z_n)\}_{n=1}^{\infty}$ is pre-compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, which means that $\cup_{s\leqslant\tau}\mathcal{A}_{\mathfrak{B}}(s,\omega)$ is pre-compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

(iii) By (i) of lemma 4.1 we know that $\mathcal{K}_{\mathfrak{D}} = \{\mathcal{K}_{\mathfrak{D}}(\tau,\omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{D}$ is a $\mathfrak{D}$-pullback random absorbing set for Φ. By (ii) of lemma 4.2 we find that Φ is $\mathfrak{D}$-pullback asymptotically compact in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. And therefore, by the abstract result established by Wang [**41**, definition 2.15] and [**42**], we know that Φ has a $\mathfrak{D}$-pullback **random** attractor $\mathcal{A}_{\mathfrak{D}} \in \mathfrak{D}$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$, which is given by (4.7).

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(iv) Given $(\tau, \omega) \in \mathbb{R} \times \Omega$. Note that by the construction of $\mathcal{K}_{\mathfrak{B}}$ and $\mathcal{K}_{\mathfrak{D}}$ we find $\mathcal{K}_{\mathfrak{B}}(\tau, \omega) \supseteq \mathcal{K}_{\mathfrak{D}}(\tau, \omega)$. Hence, by (4.6) and (4.7), we have $\mathcal{A}_{\mathfrak{B}}(\tau, \omega) \subseteq \mathcal{A}_{\mathfrak{D}}(\tau, \omega)$. On the other hand, since $\mathcal{A}_{\mathfrak{B}} \in \mathfrak{B} \subseteq \mathfrak{D}$, then by the invariance of $\mathcal{A}_{\mathfrak{B}}$ as well as the attraction of $\mathcal{A}_{\mathfrak{D}}$ we have, as $t \to +\infty$,

$$\begin{aligned} &\operatorname{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau, \omega), \mathcal{A}_{\mathfrak{D}}(\tau, \omega)) \\ &\quad = \operatorname{dist}_E(\Phi(t, \tau - t, \theta_{-t}\omega)\mathcal{A}_{\mathfrak{B}}(\tau - t, \theta_{-t}\omega), \mathcal{A}_{\mathfrak{D}}(\tau, \omega)) \to 0, \end{aligned}$$

which implies $\mathcal{A}_{\mathfrak{B}}(\tau, \omega) \subset \overline{\mathcal{A}_{\mathfrak{D}}(\tau, \omega)}^E = \mathcal{A}_{\mathfrak{D}}(\tau, \omega)$. Hence we have $\mathcal{A}_{\mathfrak{B}} = \mathcal{A}_{\mathfrak{D}}$, which along with the $\mathcal{F}$-measurability of $\mathcal{A}_{\mathfrak{D}}$ implies the $\mathcal{F}$-measurability of $\mathcal{A}_{\mathfrak{B}}$.

□

5. Asymptotically autonomous robustness of pullback random attractors

In this section we discuss the asymptotically autonomous robustness of the time-section $\mathcal{A}_{\mathfrak{B}}(\tau, \omega)$ of pullback random attractor $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ as time τ goes to negative infinity.

5.1. Random attractors of autonomous stochastic wave equations

To be more specific, we also consider an autonomous version of problem (2.1):

$$\begin{cases} \hat{u}_{tt} + \alpha\hat{u}_t - \Delta\hat{u}_t - \beta\Delta\hat{u}_{tt} + \lambda\hat{u} - \Delta\hat{u} + f(x, \hat{u}) = g_\infty(x) + \varepsilon\mathcal{S}\hat{u} \circ \dfrac{\mathrm{d}W}{\mathrm{d}t}, \\ \hat{u}(0, x) = \hat{u}_0(x), \quad \hat{u}_t(0, x) = \hat{u}_1(x), \ x \in \mathbb{R}^n, \ t > 0, \end{cases} \tag{5.1}$$

where $g_\infty \in L^2(\mathbb{R}^n)$ is the same function as in (2.14).

Denote by $\hat{z} := \hat{u}_t + \delta\hat{u}$ with same $\delta > 0$ as given in above sections. Then we have the following equivalent system:

$$\begin{cases} \hat{u}_t = -\delta\hat{u} + \hat{z}, \\ \hat{z}_t - \beta\Delta\hat{z}_t + \delta_1\hat{z} - \delta_2\Delta\hat{z} + \delta_3\hat{u} - \delta_4\Delta\hat{u} + f(x, \hat{u}) = g_\infty(x) + \varepsilon\mathcal{S}\hat{u} \circ \dfrac{\mathrm{d}W}{\mathrm{d}t}, \\ \hat{u}(0, x) = \hat{u}_0(x), \quad \hat{z}(0, x) = \hat{u}_1(x) + \delta\hat{u}_0(x). \end{cases} \tag{5.2}$$

Let $\hat{v} := \hat{z} - \varepsilon y(\theta_t\omega)\hat{u}$ to find that $\hat{\varphi} = (\hat{u}, \hat{v})$ satisfies the following random system

$$\begin{cases} \hat{u}_t = (\varepsilon y(\theta_t\omega) - \delta)\hat{u} + \hat{v}, \\ \hat{v}_t - \beta\Delta\hat{v}_t + \delta_1\hat{v} - \delta_2\Delta\hat{v} + \delta_3\hat{u} - \delta_4\Delta\hat{u} + f(x, \hat{u}) \ = g_\infty(x) - \varepsilon y(\theta_t\omega)\hat{v} \\ \quad + \varepsilon\beta y(\theta_t\omega)\Delta\hat{v} - \big(\varepsilon\delta_5 y(\theta_t\omega) + \varepsilon^2 y^2(\theta_t\omega)\big)\hat{u} + \big(\varepsilon\delta_6 y(\theta_t\omega) + \varepsilon^2\beta y^2(\theta_t\omega)\big)\Delta\hat{u}, \\ \hat{u}(0, x) = \hat{u}_0(x), \quad \hat{v}(0, x) = \hat{v}_0(x) = \hat{u}_1(x) + \delta\hat{u}_0(x) - \varepsilon y(\theta_\tau\omega)\hat{u}_0(x). \end{cases} \tag{5.3}$$

In fact, the well-posedness of problem (5.3) permits us to define an autonomous cocycle $\Phi_\infty : \mathbb{R}^+ \times \Omega \times H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n) \to H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ given by, for every

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$t \geqslant 0$ and $\omega \in \Omega$,

$$\Phi_\infty(t,\omega,(\hat{z}_0,\hat{v}_0)) = \big(\hat{u}(t,\omega,\hat{u}_0),\hat{z}(t,\omega,\hat{z}_0)\big)$$
$$= \big(\hat{u}(t,\omega,\hat{u}_0),\varepsilon y(\theta_t\omega)\hat{u}(t,\omega,\hat{u}_0)+\hat{v}(t,\omega,\hat{v}_0)\big). \tag{5.4}$$

Denote by $\mathcal{D}_\infty = \{\mathcal{D}_\infty(\omega) : \omega \in \Omega\}$ a family of bounded nonempty subsets of $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying

$$\lim_{t\to+\infty} e^{(-\kappa_1 t)/((p+1)^2)} \|\mathcal{D}_\infty(\theta_{-t}\omega)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} = 0. \tag{5.5}$$

Let $\mathfrak{D}_\infty$ be the universe of all families of bounded nonempty random subsets of $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ satisfying (5.5). By the standard method as in [**51**], it is not difficult to prove that Φ_∞ has a unique $\mathfrak{D}_\infty$-pullback random attractor $\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega) : \omega \in \Omega\} \in \mathfrak{D}_\infty$. The main goal of this section is to prove that the time-section $\mathcal{A}_{\mathfrak{B}}(\tau,\omega)$ of $\mathcal{A}_{\mathfrak{B}}$ is upper semi-continuous to $\mathcal{A}_\infty(\omega)$ as $\tau \to -\infty$ in the sense of the Hausdorff semi-distance of $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

5.2. Asymptotically autonomous convergence of stochastic wave equations

In this subsection we establish the asymptotically autonomous convergence of solutions to problem (2.6) in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$.

LEMMA 5.1. *Let hypotheses **F** and **G** be satisfied. Then the solutions to the non-autonomous equations of* (2.6) *converge to the solutions of autonomous equations* (5.3) *in the sense that for every* $(t,\omega) \in \mathbb{R}^+ \times \Omega$,

$$\lim_{\tau\to-\infty} \|\Phi(t,\tau,\omega,(u_\tau,z_\tau)) - \Phi_\infty(t,\omega,(\hat{u}_0,\hat{z}_0))\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} = 0,$$

whenever $\|(u_\tau-\hat{u}_0, z_\tau-\hat{z}_0)\|_{H^1(\mathbb{R}^n)\times H^1(\mathbb{R}^n)} \to 0$ *as* $\tau \to -\infty$.

Proof. Given $T>0$, for $t \in (0,T)$, we let $\mathbf{u}(t) := u(t+\tau,\tau,\theta_{-\tau}\omega,u_\tau) - \hat{u}(t,\omega,\hat{u}_0)$, $\mathbf{z}(t) := z(t+\tau,\tau,\theta_{-\tau}\omega,z_\tau) - \hat{z}(t,\omega,\hat{z}_0)$ and $\mathbf{v}(t) := v(t+\tau,\tau,\theta_{-\tau}\omega,v_\tau) - \hat{v}(t,\omega,\hat{v}_0)$ By (2.6)–(2.10a) and (2.7a) and (2.7b) we have $\mathbf{u}_t = \mathbf{v} - \delta\mathbf{u} + \varepsilon y\mathbf{u}$ and

$$\mathbf{v}_t - \beta\Delta\mathbf{v}_t + \delta_1\mathbf{v} - \delta_2\Delta\mathbf{v} + \delta_3\mathbf{u} - \delta_4\Delta\mathbf{u} = f(x,\hat{u}(t)) - f(x,u(t+\tau))$$
$$+ g(t+\tau) - g_\infty - \varepsilon y\mathbf{v} + \varepsilon\beta y\Delta\mathbf{v} - (\varepsilon\delta_5 y + \varepsilon^2 y^2)\mathbf{u} + (\varepsilon\delta_6 y + \varepsilon^2\beta y^2)\Delta\mathbf{u}, \tag{5.6}$$

where $y := y(\theta_t\omega)$. Taking the inner product of (5.6) with $\mathbf{v}$ in $L^2(\mathbb{R}^n)$ we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}(\|\mathbf{v}\|^2 + \beta\|\nabla\mathbf{v}\|^2) + \delta_1\|\mathbf{v}\|^2 + \delta_2\|\nabla\mathbf{v}\|^2 + \delta_3(\mathbf{u},\mathbf{v}) - \delta_4(\Delta\mathbf{u},\mathbf{v})$$
$$= (f(x,\hat{u}(t)) - f(x,u(t+\tau)),\mathbf{v}) + (g(t+\tau)-g_\infty,\mathbf{v}) - \varepsilon y\|\mathbf{v}\|^2 - \varepsilon\beta y\|\nabla\mathbf{v}\|^2$$
$$- (\varepsilon\delta_5 y + \varepsilon^2 y^2)(\mathbf{u},\mathbf{v}) + (\varepsilon\delta_6 y + \varepsilon^2\beta y^2)(\Delta\mathbf{u},\mathbf{v}). \tag{5.7}$$

Let $\Psi(t) = (\mathbf{u}(t), \mathbf{v}(t))$, then we find from (5.7) that

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi\|_E^2 \leqslant (f(x,\hat{u}(t)) - f(x,u(t+\tau)),\mathbf{v}) + (g(t+\tau) - g_\infty, \mathbf{v}) \\ + \varepsilon\delta_3 y\|\mathbf{u}\|^2 + \varepsilon\delta_4 y\|\nabla\mathbf{u}\|^2 - \varepsilon y\|\mathbf{v}\|^2 - \varepsilon\beta y\|\nabla\mathbf{v}\|^2 \\ - (\varepsilon\delta_5 y + \varepsilon^2 y^2)(\mathbf{u},\mathbf{v}) + (\varepsilon\delta_6 y + \varepsilon^2\beta y^2)(\Delta\mathbf{u},\mathbf{v}). \tag{5.8}$$

Let $\hat{f}(x,s) = (\partial/\partial s)f(x,s)$, then by hypothesis **F**, $H^1(\mathbb{R}^n) \hookrightarrow L^{p+1}(\mathbb{R}^n)$ and the Hölder inequality, we have

$$(f(x,\hat{u}(t)) - f(x,u(t+\tau)),\mathbf{v}) \leqslant c\big(1 + \|u(t+\tau)\|_{H^1(\mathbb{R}^n)}^{p-1} + \|\hat{u}(t)\|_{H^1(\mathbb{R}^n)}^{p-1}\big)\|\Psi\|_E^2. \tag{5.9}$$

Note that the remaining terms on the right-hand side of (5.8) are bounded by

$$(g(t+\tau) - g_\infty, \mathbf{v}) + \varepsilon\delta_3 y\|\mathbf{u}\|^2 + \varepsilon\delta_4 y\|\nabla\mathbf{u}\|^2 - \varepsilon y\|\mathbf{v}\|^2 \\ - \varepsilon\beta y\|\nabla\mathbf{v}\|^2 - (\varepsilon\delta_5 y + \varepsilon^2 y^2)(\mathbf{u},\mathbf{v}) + (\varepsilon\delta_6 y + \varepsilon^2\beta y^2)(\Delta\mathbf{u},\mathbf{v}) \\ \leqslant c(1 + |y| + |y|^2)\|\Psi\|_E^2 + c\|g(t+\tau) - g_\infty\|^2. \tag{5.10}$$

Substituting (5.9) and (5.10) into (5.8), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\Psi\|_E^2 \leqslant c\big(e^{|y(\theta_t\omega)|} + \|u(t+\tau)\|_{H^1(\mathbb{R}^n)}^{p-1} \\ + \|\hat{u}(t)\|_{H^1(\mathbb{R}^n)}^{p-1}\big)\|\Psi\|_E^2 + c\|g(t+\tau) - g_\infty\|^2. \tag{5.11}$$

Applying the Gronwall inequality to (5.11) over $(0,t)$, we have

$$\|\Psi(t)\|_E^2 \leqslant ce^{J(\tau,\omega)}\left(\|\Psi(0)\|_E^2 + \int_0^T \|g(r+\tau) - g_\infty\|^2\,\mathrm{d}r\right),$$

where $J(\tau,\omega) := c\int_0^T e^{c|y(\theta_r\omega)|} + \|\hat{u}(r)\|_{H^1(\mathbb{R}^n)} + \|u(r+\tau)\|_{H^1(\mathbb{R}^n)}\,\mathrm{d}r$. Then we have

$$\|\mathbf{u}(t)\|_{H^1(\mathbb{R}^n)} + \|\mathbf{z}(t)\|_{H^1(\mathbb{R}^n)} \leqslant ce^{|y(\theta_t\omega)|+J(\tau,\omega)}(\|(u_\tau - \hat{u}_0, z_\tau - \hat{z}_0)\|_{H^1\times H^1} \\ + \int_0^T \|g(r+\tau) - g_\infty\|^2\,\mathrm{d}r).$$

By hypothesis **G**, we find

$$\int_0^T \|g(r+\tau) - g_\infty\|^2\,\mathrm{d}r \leqslant \int_{-\infty}^{\tau+T} \|g(r) - g_\infty\|^2\,\mathrm{d}r \to 0 \text{ as } \tau \to -\infty. \tag{5.12}$$

Note that $\|u_\tau - \hat{u}_0\|_{H^1(\mathbb{R}^n)} + \|z_\tau - \hat{z}_0\|_{H^1(\mathbb{R}^n)} \to 0$ as $\tau \to -\infty$. Then it suffices to show that $J(\tau,\omega)$ is bounded as $\tau \to -\infty$. Note that $\varphi = (u,v)$ satisfies

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\varphi(t+\tau)\|_E^2 + 2\int_{\mathbb{R}^n} F(x,u(t+\tau))\,\mathrm{d}x\right) \\ \leqslant c_1\left(\|\varphi\|_E^2 + 2\int_{\mathbb{R}^n} F(x,u)\,\mathrm{d}x\right) + c_2\|g(t+\tau)\|^2 + c_2.$$

where c_1 and c_2 are positive constants independent of τ. Then by (2.7a) and (2.7b) and (2.7c), we find, for all $t \in (0, T)$,

$$\|u(t+\tau, \tau, \theta_{-\tau}\omega, u_\tau)\|^2_{H^1(\mathbb{R}^n)} \leqslant c\bigg(1 + \|u_\tau\|^{p+1}_{H^1(\mathbb{R}^n)} + \|v_\tau\|^{p+1}_{H^1(\mathbb{R}^n)} + \|g_\infty\|^2 + \int_0^T \|g(r+\tau) - g_\infty\|^2 \,\mathrm{d}r\bigg),$$

which is bounded as $\tau \to -\infty$ due to (5.12). By the same argument we can show that $\|\hat{u}(t, \omega, \hat{u}_0)\|^2_{H^1(\mathbb{R}^n)}$ is also bounded for all $t \in (0, T)$. Hence, the function $J(\tau, \omega)$ is bounded as $\tau \to -\infty$. The proof is completed. □

5.3. Asymptotically autonomous convergence of random attractors

In this subsection we establish the asymptotically autonomous robustness of the time-section of the pullback random attractor $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\}$ in $H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$ as time $\tau \to -\infty$. Furthermore, two different versions of such robustness are also discussed for discrete time sequence $\tau_n \to -\infty$ as $n \to \infty$.

THEOREM 5.2. *Let hypotheses **F**, **S** and **G** be satisfied. Then the non-autonomous random attractor $\mathcal{A}_{\mathfrak{B}} = \{\mathcal{A}_{\mathfrak{B}}(\tau, \omega) : \tau \in \mathbb{R}, \omega \in \Omega\} \in \mathfrak{B}$ of Φ is **asymptotically autonomous** to the autonomous random attractor $\mathcal{A}_\infty = \{\mathcal{A}_\infty(\omega) : \omega \in \Omega\}$ of Φ_∞ in the following sense:*

$$\lim_{\tau \to -\infty} dist_{H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)}(\mathcal{A}_{\mathfrak{B}}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0, \quad \mathbb{P}\text{-}a.s. \omega \in \Omega. \tag{5.13}$$

Furthermore, for any sequence $\tau_n \to -\infty$, there exists $\{\tau_{n_k}\}_{k=1}^\infty$ such that

$$\lim_{k \to \infty} dist_{H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)}(\mathcal{A}_{\mathfrak{B}}(\tau_{n_k}, \theta_{\tau_{n_k}}\omega), \mathcal{A}_\infty(\theta_{\tau_{n_k}}\omega)) = 0, \quad \mathbb{P}\text{-}a.s.\ \omega \in \Omega. \tag{5.14}$$

In addition, for any $\varepsilon > 0$ and sequence $\tau_n \to -\infty$, there exists $\{\tau_{n_k}\}_{k=1}^\infty$, $\Omega_\varepsilon \in \mathcal{F}$ with $\mathbb{P}(\Omega_\varepsilon) > 1 - \varepsilon$ such that

$$\lim_{n \to \infty} \sup_{\omega \in \Omega_\varepsilon} dist_{H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)}(\mathcal{A}_{\mathfrak{B}}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) = 0. \tag{5.15}$$

Proof. We first show (5.13). Denote by

$$\Omega_1 = \bigg\{\omega \in \Omega : \lim_{\tau \to -\infty} \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau, \omega), \mathcal{A}_\infty(\omega)) = 0\bigg\}.$$

where $E = H^1(\mathbb{R}^n) \times H^1(\mathbb{R}^n)$. Then it suffices to prove $\mathbb{P}(\Omega_1) = 1$. Let $\Omega_2 = \Omega \backslash \Omega_1$. If $\mathbb{P}(\Omega_1) < 1$, then $\Omega_2 \neq \emptyset$, and hence there exists $\omega \in \Omega_2$. This implies $\omega \notin \Omega_1$.

Based upon this fact we find that there exist $\varepsilon_0 > 0$ and $-\infty \leqslant \tau_n \downarrow< 0$ such that

$$\mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau,\omega), \mathcal{A}_\infty(\omega)) \geqslant 3\varepsilon_0, \ \forall n \in \mathbb{N}.$$

By the compactness of $\mathcal{A}_{\mathfrak{B}}(\tau_n,\omega)$ we can take a sequence $\{x_n\}_{n=1}^\infty$ from $\mathcal{A}(\tau_n,\omega)$ such that

$$\mathrm{dist}_E(x_n, \mathcal{A}_\infty(\omega)) = \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau_n,\omega), \mathcal{A}_\infty(\omega)) \geqslant 3\varepsilon_0. \tag{5.16}$$

By (4.1) we find that the $\mathcal{K}_{\mathfrak{B}}(\tau,\omega)$ is increasing in $\tau \in \mathbb{R}$, and hence

$$\mathcal{A}_{\mathfrak{B},0}(\omega) := \bigcup_{s\leqslant 0} \mathcal{A}_{\mathfrak{B}}(s,\omega) \subset \bigcup_{s\leqslant 0} \mathcal{K}_{\mathfrak{B}}(s,\omega) = \mathcal{K}_{\mathfrak{B}}(0,\omega).$$

By the same argument of (i) of lemma 4.1, it is not difficult to show $\mathcal{A}_{\mathfrak{B},0} \in \mathfrak{D}_\infty$. Therefore, the set $\mathcal{A}_{\mathfrak{B},0} = \{\mathcal{A}_{\mathfrak{B},0} : \omega \in \Omega\}$ can be attracted by the attractor $\mathcal{A}_\infty$. This means that there exists $n_0 \in \mathbb{N}$ such that

$$\mathrm{dist}_E(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\mathcal{A}_{\mathfrak{B},0}(\theta_{\tau_{n_0}}\omega), \ \mathcal{A}_\infty(\omega)) \leqslant \varepsilon_0,$$

which along with the continuity of $\Phi_\infty : E \to E$ implies

$$\mathrm{dist}_E(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\overline{\mathcal{A}_{\mathfrak{B},0}(\theta_{\tau_{n_0}}\omega)}^E, \ \mathcal{A}_\infty(\omega)) \leqslant \varepsilon_0. \tag{5.17}$$

On the other hand, we see from the invariance of $\mathcal{A}_{\mathfrak{B}}$ that

$$\mathcal{A}_{\mathfrak{B}}(\tau_n,\omega) = \Phi(|\tau_{n_0}|, \tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\mathcal{A}_{\mathfrak{B}}(\tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega).$$

This permits us to rewrite $x_n \in \mathcal{A}_{\mathfrak{B}}(\tau_n,\omega)$ as

$$x_n = \Phi(|\tau_{n_0}|, \tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_n \text{ for some } y_n \in \mathcal{A}_{\mathfrak{B}}(\tau_n - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega).$$

Note that $\tau_n - |\tau_{n_0}| \leqslant \tau_n \leqslant \tau_{n_0} \leqslant 0$ for all $n \geqslant n_0$, then we have

$$\{y_n : \ n \geqslant n_0\} \subset \bigcup_{s\leqslant 0} \mathcal{A}_{\mathfrak{B}}(s, \theta_{\tau_{n_0}}\omega) = \mathcal{A}_{\mathfrak{B},0}(\theta_{\tau_{n_0}}\omega).$$

This together with (ii) of theorem 4.3 implies that the set $\mathcal{A}_{\mathfrak{B},0}(\theta_{\tau_{n_0}}\omega)$ is precompact in E. Then the sequence $\{y_n\}_{n=1}^\infty$ has a convergent subsequence $\{y_{n_k}\}_{k=1}^\infty$ in E such that

$$y_{n_k} \to y_0 \text{ as } k \to \infty \text{ for some } y_0 \in \overline{\mathcal{A}_{\mathfrak{B},0}(\theta_{\tau_{n_0}}\omega)}^E.$$

Note that $\theta_{\tau_{n_0}}\omega \in \theta_{\tau_{n_0}}\Omega \subseteq \Omega$ due to the $\{\theta_t\}_{t\in\mathbb{R}}$-invariance of Ω. This fact permits us to apply the asymptotically autonomous convergence for Φ as proved in lemma 5.1 for the sample $\theta_{\tau_{n_0}}\omega$, $t = |\tau_{n_0}|$ as well as $\tau = \tau_{n_k} - |\tau_{n_0}| \to -\infty$ as

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$k \to \infty$ to find, there exists a large enough $k \in \mathbb{N}$ such that

$$\|x_{n_k} - \Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0\|_E = \|\Phi(|\tau_{n_0}|, \tau_{n_k} - |\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_{n_k} \\ - \Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0\|_E \leqslant \varepsilon_0.$$

This along with (5.17) implies

$$\begin{aligned} \mathrm{dist}_E(x_{n_k}, \mathcal{A}_\infty(\omega)) &\leqslant \|x_{n_k} - \Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0\|_E \\ &\quad + \mathrm{dist}_E(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)y_0, \mathcal{A}_\infty(\omega)) \\ &\leqslant \varepsilon_0 + \mathrm{dist}_E(\Phi_\infty(|\tau_{n_0}|, \theta_{\tau_{n_0}}\omega)\overline{\mathcal{A}_{\mathfrak{B},0}(\theta_{\tau_{n_0}}\omega)}^E, \mathcal{A}_\infty(\omega)) \leqslant 2\varepsilon_0, \end{aligned}$$

which indeed is a contradiction to (5.16), and thus (5.13) is proved.

We then prove (5.19). For any sequence $\tau_n \to -\infty$ as $n \to \infty$, by (5.13) we have

$$\lim_{n\to\infty} \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) = 0, \tag{5.18}$$

which implies, for any $\varepsilon > 0$,

$$\lim_{n\to\infty} \mathbb{P}\{\omega \in \Omega : \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau_n, \omega), \mathcal{A}_\infty(\omega)) \geqslant \varepsilon\} = 0. \tag{5.19}$$

Since $\{\theta_t\}_{t\in\mathbb{R}}$ is measure preserving, then we have

$$\begin{aligned} &\mathbb{P}\{\omega \in \Omega : \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau_n, \theta_{\tau_n}\omega, \mathcal{A}_\infty(\theta_{\tau_n}\omega)) \geqslant \varepsilon\} \\ &\quad = \mathbb{P}\theta_{\tau_n}\{\omega \in \Omega : \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau_n, \theta_{\tau_n}\omega, \mathcal{A}_\infty(\theta_{\tau_n}\omega)) \geqslant \varepsilon\} \\ &\quad = \mathbb{P}\{\omega \in \Omega : \mathrm{dist}_E(\mathcal{A}_{\mathfrak{B}}(\tau_n, \omega, \mathcal{A}_\infty(\omega)) \geqslant \varepsilon\}. \end{aligned}$$

This along with (5.19) as well as Riesz theorem implies (5.19).

The proof of (5.15) is similar to that of [**52**, theorem 4.8], we omit the details here. □

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