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Homogenizations of integro-differential equations with Lévy operators with asymmetric and degenerate densities

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We consider periodic homogenization problems for Lévy operators with asymmetric Lévy densities. The formal asymptotic expansion used for the α-stable (symmetric) Lévy operators ($\alpha \in (0,2)$) is not directly applicable to such asymmetric cases. We rescale the asymmetric densities and extract the most singular parts of the measures, which average out the microscopic dependencies in the homogenization procedures. We give two conditions, (A) and (B), that characterize such a class of asymmetric densities under which the above 'rescaled' homogenization is available.

1. Introduction

We are interested in the following homogenization problems involving the Lévy operator

$$u_\varepsilon(x) - a\left(\frac{x}{\varepsilon}\right)\int_{\mathbb{R}^M}[u_\varepsilon(x+\beta(z)) - u_\varepsilon(x) - \mathbf{1}_{|z|\leqslant 1}\langle\nabla u_\varepsilon(x), \beta(z)\rangle]\,\mathrm{d}q(z) - f\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } \Omega, \tag{1.1}$$

$$u = \phi(x) \quad \text{in } \Omega^c, \tag{1.2}$$

and

$$u_\varepsilon(x) - a\left(\frac{x}{\varepsilon}\right)\int_{\mathbb{R}^M}[u_\varepsilon(x+\beta(z)) - u_\varepsilon(x)]\,\mathrm{d}q(z) - f\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } \Omega, \tag{1.3}$$

with (1.2). Here, Ω is an open bounded domain in $\mathbb{R}^N$, $M \leqslant N$, β is a positively homogeneous, continuous function from $\mathbb{R}^M$ to $\mathbb{R}^N$ such that

$$\beta(cz) = c\beta(z),\ \forall c > 0, \qquad |\beta(z)| \leqslant B_1|z|,\ \forall z \in \mathbb{R}^M, \tag{1.4}$$

where $B_1 > 0$ is a constant, $\mathrm{d}q(z) = q(z)\,\mathrm{d}z$ is a positive Radon measure on $\mathbb{R}^M$ which satisfies

$$\int_{|z|<1} |z|^\gamma\,\mathrm{d}q(z) + \int_{|z|\geqslant 1} |z|^{\gamma-1}\,\mathrm{d}q(z) < \infty, \tag{1.5}$$

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with $\gamma = 2$ in the case of (1.1), and with $\gamma = 1$ in the case of (1.3), where a and f are real-valued continuous functions defined in $\mathbb{R}^N$, periodic in $\boldsymbol{T}^N = [0,1]^N$, such that there exist constants θ_1, $\theta_2 \in (0,1]$, $L > 0$, $a_0 > 0$ with which the following hold:

$$a(\cdot) \geqslant \exists a_0 > 0, \quad |a(y) - a(y')| \leqslant L|y - y'|^{\theta_1}, \quad y, y' \in \mathbb{R}^N, \tag{1.6}$$

$$|f(y) - f(y')| \leqslant L|y - y'|^{\theta_2}, \quad y, y' \in \mathbb{R}^N, \tag{1.7}$$

and ϕ is a real-valued bounded continuous function defined in Ω^c.

For any $\varepsilon > 0$, there exists a unique solution u_ε of (1.1) and (1.2), and of (1.3) and (1.2), respectively, in the framework of the viscosity solution (see Appendix A for the definition, see Arisawa [2, 3, 8] and Barles and Imbert [10] for existence and uniqueness results, and see Crandall *et al*. [13] for the general theory of viscosity solutions). As ε goes to zero, the sequence of functions $\{u_\varepsilon\}$ converges locally uniformly to a limit $\bar{u}$, and we are interested in finding an effective non-local equation which characterizes $\bar{u}$.

Such a homogenization problem was solved in the case where the Lévy measure is α-stable (see Arisawa [6, 7]):

$$\mathrm{d}q(z) = \frac{1}{|z|^{N+\alpha}}\,\mathrm{d}z, \quad z \in \mathbb{R}^M, \ \alpha \in (0,2) \text{ a fixed number,}$$

by using the formal asymptotic expansion

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon^\alpha v\left(\frac{x}{\varepsilon}\right) + o(\varepsilon^\alpha), \quad x \in \mathbb{R}^N, \tag{1.8}$$

where $\bar{u} = \lim_{\varepsilon \to 0} u_\varepsilon$ and v is a periodic function defined in $\mathbb{R}^N$ called a corrector. The above expansion leads to the so-called ergodic cell problem, which gives the effective equation for $\bar{u}$. We refer the interested reader to Bensoussan *et al*. [12] for a detailed discussion of this method. In the framework of the viscosity solution, the formal argument can be justified rigorously using the perturbed test function method established by Evans [14, 15] (see also Lions *et al*. [18]). However, as we shall see in examples 1.2–1.5, the above formal expansion cannot be employed directly if the measure $\mathrm{d}q(z)$ is asymmetric. Here, we assume that the Lévy measure satisfies condition (A) below.

(A) Let $S = \mathrm{supp}(\mathrm{d}q(z)) \subset \mathbb{R}^M$. There exists a constant $\alpha \in (0,2)$ such that

$$\varepsilon^{M+\alpha} q(\varepsilon z) \leqslant C_1 |z|^{-(M+\alpha)}, \quad \forall \varepsilon \in (0,1), \ \forall z \in \mathbb{R}^M, \tag{1.9}$$

where $C_1 > 0$ is a constant independent on ε, a subset $S_0 \subset S$ and a positive function $q_0(z)$ ($z \in \mathbb{R}^M$) such that

$$\lim_{\varepsilon \downarrow 0} \varepsilon^{M+\alpha} q(\varepsilon z) = q_0(z), \ \forall z \in S_0, \qquad \lim_{\varepsilon \downarrow 0} \varepsilon^{M+\alpha} q(\varepsilon z) = 0, \ \forall z \in \mathbb{R}^M \setminus S_0. \tag{1.10}$$

We define a new measure

$$\mathrm{d}q_0(z) = q_0(z)\,\mathrm{d}z, \ \forall z \in S_0, \qquad \mathrm{d}q_0(z) = 0\,\mathrm{d}z, \ \forall z \in \mathbb{R}^M \setminus S_0. \tag{1.11}$$

The following property holds for this rescaled measure $\mathrm{d}q_0(z)$.

Lemma 1.1. *Assume the Radon measure* $\mathrm{d}q(z)$ *satisfies (1.5) and condition* (A). *Then,* S_0 *is a positive cone, i.e.*

$$sS_0 \subset S_0, \quad \forall s > 0.$$

Moreover, $s^{M+\alpha} q_0(sz) = q_0(z)$, $\forall s > 0$, $\forall z \in S_0$, *and*

$$q_0(z) = |z|^{M+\alpha} \bar{q}_0(\arg z), \quad \forall z \in \mathbb{R}^M, \tag{1.12}$$

where $\bar{q}_0(\theta)$, $\theta \in [0, 2\pi)$ *is a bounded real-valued function.*

Proof. Let $z \in S_0$. For any $s \in (0,1)$, from condition (A) and (1.10),

$$\begin{aligned} \lim_{\varepsilon \to 0} \varepsilon^{M+\alpha} q(\varepsilon s z) &= s^{-(M+\alpha)} \lim_{\varepsilon \to 0} (\varepsilon s)^{M+\alpha} q(\varepsilon s z) \\ &= s^{-(M+\alpha)} \lim_{\varepsilon' \to 0} \varepsilon'^{M+\alpha} q(\varepsilon' z) \\ &= s^{-(M+\alpha)} q_0(z) \\ &> 0. \end{aligned}$$

Thus, $sz \in S_0$, and

$$q_0(sz) = s^{-(M+\alpha)} q_0(z), \quad \forall s \in (0,1), \ \forall z \in S_0.$$

Therefore, $q_0(z) = |z|^{M+\alpha} q_0(z/|z|)$, and from condition (A), (1.12) is proved. □

The following examples satisfy condition (A).

Example 1.2. Let $M = N$, $\beta(z) = z$ and, for $\alpha \in (1,2)$ and $\alpha \in (0,1)$, respectively,

$$\mathrm{d}q(z) = |z|^{-(M+\alpha)} \,\mathrm{d}z, \ z \in \mathbb{R}^M_+, \qquad \mathrm{d}q(z) = 0, \ z \in (\mathbb{R}^M_+)^c,$$

where $\mathbb{R}^M_+ = \{z = (z_1, \dots, z_M) \mid z_i > 0, \ 1 \leqslant \forall i \leqslant M\}$. In this case, for $S = S_0 = \mathbb{R}^M_+$, we have

$$q(\varepsilon z)\varepsilon^{M+\alpha} = |z|^{-(M+\alpha)} = q_0(z), \ \forall z \in S_0, \qquad q(\varepsilon z)\varepsilon^{M+\alpha} = 0, \ \forall z \in S_0^c, \ \forall \varepsilon > 0,$$

and condition (A) is satisfied for

$$\mathrm{d}q_0(z) = |z|^{-(M+\alpha)} \,\mathrm{d}z, \ z \in \mathbb{R}^M_+, \qquad \mathrm{d}q_0(z) = 0 \,\mathrm{d}z, \ z \in (\mathbb{R}^M_+)^c.$$

Both $\mathrm{d}q(z)$ and $\mathrm{d}q_0(z)$ satisfy (1.5) with $\gamma = 2$ (respectively, $\gamma = 1$).

Example 1.3. Let $M = N = 1$, $\beta(z) = z$ and, for $1 < \alpha_1 < \alpha_2 < 2$,

$$\mathrm{d}q(z) = |z|^{-(1+\alpha_1)} \,\mathrm{d}z, \ z \leqslant -1, \ z > 0, \qquad \mathrm{d}q(z) = |z|^{-(1+\alpha_2)} \,\mathrm{d}z, \ -1 < z < 0.$$

In this case, for $\alpha = \alpha_2$, $S = \mathbb{R}$, $S_0 = \{z \in \mathbb{R} \mid z < 0\}$, we have

$$\lim_{\varepsilon \to 0} q(\varepsilon z)\varepsilon^{1+\alpha} = |z|^{-(1+\alpha_2)} = q_0(z), \ \forall z \in S_0, \qquad \lim_{\varepsilon \to 0} q(\varepsilon z)\varepsilon^{1+\alpha} = 0, \ \forall z \in S_0^c,$$

and condition (A) is satisfied for

$$\mathrm{d}q_0(z) = 0 \,\mathrm{d}z, \ z > 0, \qquad \mathrm{d}q_0(z) = |z|^{-(1+\alpha)} \,\mathrm{d}z, \ z < 0.$$

Both $\mathrm{d}q(z)$ and $\mathrm{d}q_0(z)$ satisfy (1.5) with $\gamma = 2$.

Example 1.4. Let $M = 1$, $N = 2$, $\beta(z) = (z, \xi z)$, where $\xi > 0$ is an irrational number and, for $\alpha \in (1, 2)$ (respectively, $\alpha \in (0, 1)$),

$$\mathrm{d}q(z) = |z|^{-(1+\alpha)}\,\mathrm{d}z, \quad z \in \mathbb{R}.$$

In this case, for $S = S_0 = \mathbb{R}$, we have

$$q(\varepsilon z)\varepsilon^{1+\alpha} = |z|^{-(1+\alpha)} = q_0(z), \quad \forall z \in S = S_0,\ \forall \varepsilon > 0,$$

and condition (A) is satisfied for

$$\mathrm{d}q_0(z) = |z|^{-(1+\alpha)}\,\mathrm{d}z, \quad z \in \mathbb{R}.$$

Both $\mathrm{d}q(z)$ and $\mathrm{d}q_0(z)$ satisfy (1.5) with $\gamma = 2$ (respectively, $\gamma = 1$).

Example 1.5. Let $M = N$, $\beta(z) = z$ and, for $\gamma > 0$, $1 < \alpha < 2$,

$$\mathrm{d}q(z) = \exp(-\gamma|z|)|z|^{-(M+\alpha)}\,\mathrm{d}z, \quad z \in \mathbb{R}^M.$$

In this case, for $g(s) = s^\alpha$, $S = S_0 = \mathbb{R}^M$, we have

$$\lim_{\varepsilon\to 0} q(\varepsilon z)\varepsilon^{M+\alpha} = \lim_{\varepsilon\to 0} \exp(-\varepsilon\gamma|z|)|z|^{-(M+\alpha)} = |z|^{-(M+\alpha)}, \quad \forall z \in S = S_0,$$

and condition (A) is satisfied for

$$\mathrm{d}q_0(z) = |z|^{-(M+\alpha)}\,\mathrm{d}z, \quad z \in \mathbb{R}^M.$$

Both $\mathrm{d}q(z)$ and $\mathrm{d}q_0(z)$ satisfy (1.5) with $\gamma = 2$.

In examples 1.2–1.4, the Lévy measures are either asymmetric or degenerate (in the sense that S or S_0 does not contain an open ball centred at the origin in $\mathbb{R}^M$). Example 1.4 corresponds to the jump process satisfying the non-resonance condition [9]. At first sight, the formal asymptotic expansion (1.8) used for the α-stable Lévy operator seems to be unapplicable for the measures in examples 1.2–1.5. However, by using the constant α in condition (A), we can still use the expansion (1.8):

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon^\alpha v\left(\frac{x}{\varepsilon}\right) + o(\varepsilon^\alpha), \quad x \in \mathbb{R}^N.$$

We introduce the formal derivatives of u_ε into (1.1) (respectively, (1.3)). From condition (A) ((1.9) and (1.10)), we note that

$$\lim_{\varepsilon\to 0} \int_{|z|\leqslant 1} |z|^\gamma \varepsilon^{M+\alpha} q(\varepsilon z)\,\mathrm{d}z = \int_{|z|\leqslant 1} |z|^\gamma\,\mathrm{d}q_0(z), \tag{1.13}$$

$$\lim_{\varepsilon\to 0} \int_{|z|>1} |z|^{\gamma-1} \varepsilon^{M+\alpha} q(\varepsilon z)\,\mathrm{d}z = \int_{|z|>1} |z|^{\gamma-1}\,\mathrm{d}q_0(z), \tag{1.14}$$

for $\alpha \in (1, 2)$ with $\gamma = 2$ (respectively, $\alpha \in (0, 1)$ with $\gamma = 1$). We formally obtain the following ergodic cell problem. For any fixed $x \in \Omega$, and for the given

$$I_1 = \int_{\mathbb{R}^M} [\bar{u}(x + \beta(z)) - \bar{u}(x) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\bar{u}(x), \beta(z)\rangle]\,\mathrm{d}q(z),$$

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and, respectively,

$$I_2 = \int_{\mathbb{R}^M} [\bar{u}(x + \beta(z)) - \bar{u}(x)] \, \mathrm{d}q(z),$$

find a unique number d_{I_1} (and respectively, d_{I_2}) such that the following problem has at least one periodic viscosity solution $v(y)$:

$$d_{I_1} - a(y) \int_{\mathbb{R}^M} [v(y + \beta(z)) - v(y) - \langle \nabla v(y), \beta(z) \rangle] \, \mathrm{d}q_0(z) \\ - a(y) I_1 - f(y) = 0 \quad \text{in } \boldsymbol{T}^N, \quad (1.15)$$

and, respectively,

$$d_{I_2} - a(y) \int_{\mathbb{R}^M} [v(y + \beta(z)) - v(y)] \, \mathrm{d}q_0(z) - a(y) I_2 - f(y) = 0 \quad \text{in } \boldsymbol{T}^N, \quad (1.16)$$

provided that $\mathrm{d}q_0(z)$ (the rescaled measure defined in (1.11)) satisfies (1.5) with $\gamma = 2$ (respectively $\gamma = 1$). In some cases, we can only find the unique number d_{I_1} (respectively, d_{I_2}) that satisfies the following weaker property. For the case of (1.15), d_{I_1} is the unique number such that, for any $\delta > 0$, there exist a subsolution v_δ and a supersolution v^δ of

$$d_{I_1} - a(y) \int_{\mathbb{R}^M} [v_\delta(y + \beta(z)) - v_\delta(y) - \langle \nabla v_\delta(y), \beta(z) \rangle] \, \mathrm{d}q_0(z) \\ - a(y) I_1 - f(y) \leqslant \delta \qquad \text{in } \boldsymbol{T}^N$$

and

$$d_{I_1} - a(y) \int_{\mathbb{R}^M} [v^\delta(y + \beta(z)) - v^\delta(y) - \langle \nabla v^\delta(y), \beta(z) \rangle] \, \mathrm{d}q_0(z) \\ - a(y) I_1 - f(y) \geqslant -\delta \quad \text{in } \boldsymbol{T}^N.$$

The weaker version of (1.16) will be stated in § 4. As noted in [9] for the case of partial differential equations, the existence of the unique number d_{I_1} (respectively, d_{I_2}) is shown by the strong maximum principle (SMP) for the Lévy operator. Since the Lévy density $\mathrm{d}q_0(z)$ in (1.15) (respectively, (1.16)) is possibly degenerate, we must establish a new SMP for our present purposes. We shall give a general sufficient condition for the SMP in § 2 (condition (B)), in terms of the controllability of the jump process: $x \to x + \beta(z)$, $z \in S_0$.

Although we have stated our problem in linear cases, for reasons of simplicity, the present method is also applicable to nonlinear homogenization problems.

Example 1.6. Let $\Omega \subset \mathbb{R}^3$ be an open domain, and let $\beta_1 \colon \mathbb{R} \to \mathbb{R}^3$, $\beta_2 \colon \mathbb{R}^2 \to \mathbb{R}^3$ be such that

$$\beta_1(z') = (0, 0, z'), \ \forall z' \in \mathbb{R}, \qquad \beta_2(z'') = (z''_1, z''_2, 0), \ \forall z'' = (z''_1, z''_2) \in \mathbb{R}^2.$$

Consider

$$u_\varepsilon(x) + \max\left\{-a\left(\frac{x}{\varepsilon}\right)\int_{\mathbb{R}}[u_\varepsilon(x+\beta_1(z')) - u_\varepsilon(x) - \mathbf{1}_{|z'|\leqslant 1}\langle\nabla u_\varepsilon(x), \beta_1(z')\rangle]\,\mathrm{d}q_1(z') \right.$$
$$- a\left(\frac{x}{\varepsilon}\right)\int_{\mathbb{R}^2}[u_\varepsilon(x+\beta_2(z'')) - u_\varepsilon(x)$$
$$\left. - \mathbf{1}_{|z''|\leqslant 1}\langle\nabla u_\varepsilon(x), \beta_2(z'')\rangle]\,\mathrm{d}q_2(z'')\right\}$$
$$- f\left(\frac{x}{\varepsilon}\right) = 0 \quad \text{in } \Omega, \tag{1.17}$$

with the Dirichlet condition (1.2). Here, $\mathrm{d}q_1(z')$, $\mathrm{d}q_2(z'')$ are, respectively, a one-dimensional Lévy measure and a two-dimensional Lévy measure. More detailed assumptions will be given later. We shall give the effective equation for this homogenization problem in § 5.

This paper is organized as follows. In § 2 we state the SMP for Lévy operators with degenerate densities satisfying a fairly general condition (B) given below. In § 3, under condition (B), we solve the ergodic cell problems (1.15) and (1.16). In § 4 the homogenization problem (1.1), (1.3) is solved rigorously. In § 5 a generalization to nonlinear problems, such as example 1.6, is given. In Appendix A the definitions of viscosity solutions for integro-differential equations with Lévy operators are reviewed for the reader's benefit. Throughout the paper, by subsolution and supersolution we take to mean the viscosity subsolution and the viscosity supersolution, respectively. We denote by $\mathrm{USC}(\mathbb{R}^N)$ and by $\mathrm{LSC}(\mathbb{R}^N)$ the set of all upper semicontinuous functions on $\mathbb{R}^N$, and the set of all lower semicontinuous functions on $\mathbb{R}^N$, respectively. For $x \in \mathbb{R}^N$ we denote by $B_r(x)$ a ball centred at x with radius $r > 0$.

2. Strong maximum principle in $\boldsymbol{T}^N$

In this section we establish the SMP for Lévy operators with asymmetric, degenerate densities. We use this result to solve the ergodic cell problem in § 3. Our presentation is slightly more general than necessary. Let $H(y, p)$ be a continuous real-valued function defined in $\mathbb{R}^N \times \mathbb{R}^N$, periodic in y with the period $\boldsymbol{T}^N$, satisfying

$$H(y, 0) \geqslant 0, \quad \forall y \in \boldsymbol{T}^N. \tag{2.1}$$

We consider

$$H(y, \nabla u) - a(y)\int_{\mathbb{R}^M}[u(y+\beta(z)) - u(y) - \langle\nabla u(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) = 0 \quad \text{in } \boldsymbol{T}^N \tag{2.2}$$

and

$$H(y, \nabla u) - a(y)\int_{\mathbb{R}^M}[u(y+\beta(z)) - u(y)]\,\mathrm{d}q_0(z) = 0 \quad \text{in } \boldsymbol{T}^N, \tag{2.3}$$

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where $\beta(z)$ satisfies (1.4), $a(y)$ satisfies (1.6), and $\mathrm{d}q_0(z)$ satisfies (1.5) with $\gamma = 2$ in the case of (2.2), and $\gamma = 1$ in the case of (2.3). We assume the following condition.

(B) For any two points $y, y' \in \boldsymbol{T}^N$, there exist a finite number of points $y_1, \dots, y_m \in \boldsymbol{T}^N$ such that $y_1 = y$, $y_m = y'$, and, for any m positive numbers $\varepsilon_i > 0$. $1 \leqslant i \leqslant m$, we can take subsets $J_i \subset S_0 = \operatorname{supp}(\mathrm{d}q_0(z))$, $1 \leqslant \forall i \leqslant m-1$, satisfying

$$y_i + \beta(z) \in B_{\varepsilon_i}(y_{i+1}), \ \forall z \in J_i, \qquad \int_{J_i} 1 \mathrm{d}q_0(z) > 0, \ 1 \leqslant \forall i \leqslant m. \tag{2.4}$$

Condition (B) describes the controllability of the jump process $y \to y+\beta(z)$, $z \in S_0$.

Theorem 2.1. *Let $u \in \mathrm{USC}(\mathbb{R}^N)$ be a viscosity subsolution of (2.2) (respectively, (2.3)). Assume that (1.4), (1.6) and (2.1) hold, and that $\mathrm{d}q_0(z)$ satisfies condition* (B) *and (1.5) with $\gamma = 2$ (respectively, $\gamma = 1$). If u attains a maximum at $\bar{y}$ in $\boldsymbol{T}^N$, then u is constant in $\boldsymbol{T}^N$.*

Proof. Let $u(\bar{y}) = M$, and set $\Omega_0 = \{y \in \boldsymbol{T}^N \mid u \equiv M\}$. Assume that $\Omega_0^c \neq \emptyset$, which should lead to a contradiction. Take a point $y' \in \Omega_0^c$, and note that $u(y') < M$. From condition (B), we can take a finite number of points, $y_1, \dots, y_m \in \boldsymbol{T}^N$ such that $y_1 = \bar{y}$, $y_m = y'$, m positive numbers ε_i, $1 \leqslant i \leqslant m$, and $m-1$ subsets $J_i \subset S_0$ that satisfy (2.4). There exists a number k such that $1 \leqslant k < m$, with which $y_k \in \Omega_0$ and $y_{k+1} \in \Omega_0^c$. Since Ω_0^c is open, we can take sufficiently small $\varepsilon_k > 0$ such that $B_{\varepsilon_k}(y_{k+1}) \subset \Omega_0^c$. From condition (B), there exists $J_k \subset S_0 = \operatorname{supp}(\mathrm{d}q_0(z))$ such that $\int_{J_k} 1 \, \mathrm{d}q_0(z) > 0$, and

$$y_k + \beta(z) \in U_{\varepsilon_k}(y_{k+1}), \quad \forall z \in J_k.$$

Thus, we can take $\delta_k > 0$ such that

$$u(y_k + \beta(z)) < M - \delta_k, \quad \forall z \in J_k. \tag{2.5}$$

For the constant function $\phi(y) \equiv M$, $y \in \boldsymbol{T}^N$, since $u - \phi$ takes a maximum at y_k, from the definition of the viscosity subsolution (see definition A.3), by using $\nabla\phi(y_k) = 0$, we have

$$H(y_k, 0) - a(y_k) \int_{\mathbb{R}^M} [u(y_k + \beta(z)) - u(y_k) - \langle 0, \beta(z) \rangle] \, \mathrm{d}q_0(z) \leqslant 0$$

and, respectively,

$$H(y_k, 0) - a(y_k) \int_{\mathbb{R}^M} [u(y_k + \beta(z)) - u(y_k)] \, \mathrm{d}q_0(z) \leqslant 0.$$

From (1.6) and (2.1), and from the fact that $u(y_k) = M > u(y_k + \beta(z))$ for any $z \in \operatorname{supp}(\mathrm{d}q_0(z))$, the above leads to

$$-\int_{J_k} [u(y_k + \beta(z)) - u(y_k)] \, \mathrm{d}q_0(z) \leqslant 0.$$

However, from condition (B), this contradicts (2.5), since

$$-\int_{J_k} [u(y_k + \beta(z)) - u(y_k)] \, \mathrm{d}q_0(z) \geqslant \delta_k \int_{J_k} 1 \, \mathrm{d}q_0(z) > 0.$$

Therefore, $\Omega_0^c = \emptyset$ must hold. □

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Remark 2.2.

1. Consider the jump process $y \to y + \beta(z)$, $z \in S_0 = \mathrm{supp}(\mathrm{d}q_0(z))$, in $\boldsymbol{T}^N$, where $\mathrm{d}q_0(z)$ is any of the measures defined in examples 1.2–1.5. Then, it is easy to see that condition (B) is satisfied by each of the measures $\mathrm{d}q_0(z)$. (Note that, in example 1.4, for fixed $y \in \boldsymbol{T}^2$, the set $\{y + (z, \xi z) \mid z \in \mathbb{R} = S_0\}$ is dense in $\boldsymbol{T}^2$ for $\xi > 0$ is irrational.)

2. Let $M = N$ and $\beta(z) = z$. If, for some $r > 0$, $B_r(0) \subset \mathrm{d}q_0(z)$, then condition (B) is satisfied.

3. The SMP in theorem 2.1 can be stated in parallel for a supersolution $u \in \mathrm{LSC}(\mathbb{R}^N)$ of (2.2) (respectively, (2.3)), i.e. if u attains a minimum at $\bar{y} \in \boldsymbol{T}^N$, then u is a constant function.

4. Let us replace the Lévy operator in (2.2) to the following:

$$\int_{\mathbb{R}^M} [u(y + \beta(z)) - u(y) - \mathbf{1}_{|z| \leqslant 1} \langle \nabla u(y), \beta(z) \rangle] \,\mathrm{d}q_0(z),$$

   where $\mathrm{d}q_0(z)$ satisfies (1.5) with $\gamma = 2$. Then the SMP also holds for the above operator under condition (B).

3. Ergodic problem

In this section we study the ergodic problem of the jump process $x \to x + \beta(z)$, $z \in \mathrm{supp}(\mathrm{d}q_0(z))$. For $\lambda > 0$, we consider

$$\lambda v_\lambda(y) - a(y) \int_{\mathbb{R}^M} [v_\lambda(y + \beta(z)) - v_\lambda(y) - \langle \nabla v_\lambda(y), \beta(z) \rangle] \,\mathrm{d}q_0(z) - f_0(y) = 0 \quad \text{in } \boldsymbol{T}^N \tag{3.1}$$

and, respectively,

$$\lambda v_\lambda(y) - a(y) \int_{\mathbb{R}^M} [v_\lambda(y + \beta(z)) - v_\lambda(y)] \,\mathrm{d}q_0(z) - f_0(y) = 0 \quad \text{in } \boldsymbol{T}^N. \tag{3.2}$$

In (3.1), $\langle \nabla v_\lambda(y), \beta(z) \rangle$ is used instead of the usual term $\mathbf{1}_{|z| \leqslant 1} \langle \nabla v_\lambda(y), \beta(z) \rangle$ in the Lévy operator studied in [2, 3, 10]. However, condition (1.5) compensates it in the integral, and the comparison of solutions for (3.1) (respectively, (3.2)) holds similarly to [2, 3, 10]. Thus, there exists a unique periodic viscosity solution v_λ of (3.1) (respectively, (3.2)).

Theorem 3.1. *Let v_λ be a viscosity solution of (3.1) (respectively, (3.2)). Assume that (1.4) and (1.6) hold, that f_0 satisfies (1.7), that $\mathrm{d}q_0(z)$ satisfies condition* (B) *and (1.5) with $\gamma = 2$ (respectively, $\gamma = 1$). Then there exists a unique real number d such that*

$$\lim_{\lambda \to 0} \lambda v_\lambda(y) = d \quad \textit{uniformly in } \boldsymbol{T}^N. \tag{3.3}$$

The number d is characterized by the property that, for any $\delta > 0$, there exists a subsolution v_δ and a supersolution v^δ of

$$d - a(y)\int_{\mathbb{R}^M}[v_\delta(y+\beta(z)) - v_\delta(y) - \langle \nabla v_\delta(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) - f_0(y) \leqslant \delta, \tag{3.4}$$

$$d - a(y)\int_{\mathbb{R}^M}[v^\delta(y+\beta(z)) - v^\delta(y) - \langle \nabla v^\delta(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) - f_0(y) \geqslant -\delta, \tag{3.5}$$

and, respectively,

$$d - a(y)\int_{\mathbb{R}^M}[v_\delta(y+\beta(z)) - v_\delta(y)]\,\mathrm{d}q_0(z) - f_0(y) \leqslant \delta, \tag{3.6}$$

$$d - a(y)\int_{\mathbb{R}^M}[v^\delta(y+\beta(z)) - v^\delta(y)]\,\mathrm{d}q_0(z) - f_0(y) \geqslant -\delta, \tag{3.7}$$

in $\boldsymbol{T}^N$.

Proof. We prove (3.3) for the problem (3.1). The proof of (3.2) is similar and we do not write it here. We multiply (3.1) by $\lambda > 0$, and set $m_\lambda = \lambda v_\lambda$. We have

$$\lambda m_\lambda(y) - a(y)\int_{\mathbb{R}^M}[m_\lambda(y+\beta(z)) - m_\lambda(y) - \langle \nabla m_\lambda(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) - \lambda f_0(y) = 0 \quad \text{in } \boldsymbol{T}^N. \tag{3.8}$$

We claim that the following holds.

Lemma 3.2. *Let the assumptions in theorem 3.1 hold.*

(i) *There exists a constant $M > 0$ such that the following hold:*

$$|m_\lambda|_{L^\infty} \leqslant M, \quad \forall \lambda \in (0,1). \tag{3.9}$$

(ii) *For any $\theta \in (0, \min\{\theta_1, \theta_2\})$, there exists a constant $C_\theta > 0$ such that*

$$|m_\lambda(y) - m_\lambda(y')| \leqslant C_\theta |y - y'|^\theta, \quad \forall y, y' \in \boldsymbol{T}^N,\ \forall \lambda \in (0,1). \tag{3.10}$$

The constants M, $C_\theta > 0$ are independent on $\lambda \in (0,1)$.

We admit the above estimates for a while, which we shall prove later. Following lemma 3.2 ($m_\lambda = \lambda v_\lambda$), from the Ascoli–Arzelá lemma we can take a sequence $\lambda' \to 0$ such that

$$\lambda' v_{\lambda'}(y) \to \exists d(y) \quad \text{as } \lambda' \to 0 \text{ uniformly in } \boldsymbol{T}^N,$$

where $d(y)$ is a Hölder continuous, periodic function satisfying (3.10). To see that $d(y)$ is constant, we multiply (3.1) by $\lambda' > 0$, and tend λ' to zero. By using (3.9), from the stability of viscosity solutions we obtain

$$-\int_{\mathbb{R}^M}[d(y+\beta(z)) - d(y) - \langle \nabla d(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) \leqslant 0 \quad \text{in } \boldsymbol{T}^N.$$

Hence, from the SMP in theorem 2.1, $d(y)$ is constant, i.e. $d(y) \equiv d$ for some real number d. Next, assume that there exists another sequence $\lambda'' \to 0$ and another number d' such that

$$\lambda'' v_{\lambda''}(y) \to d' \quad \text{as } \lambda'' \to 0 \text{ uniformly in } \boldsymbol{T}^N.$$

Without loss of generality, we may assume that $d' < d$. For arbitrary small $\mu > 0$, by taking $\lambda' > 0$ and sufficiently small $\lambda'' > 0$, we have the following two inequalities:

$$d - a(y) \int_{\mathbb{R}^M} [v_{\lambda'}(y + \beta(z)) - v_{\lambda'}(y) - \langle \nabla v_{\lambda'}(y), \beta(z)\rangle] \, \mathrm{d}q_0(z) - f_0(y) \leqslant \tfrac{1}{2}\mu,$$

$$d' - a(y) \int_{\mathbb{R}^M} [v_{\lambda''}(y + \beta(z)) - v_{\lambda''}(y) - \langle \nabla v_{\lambda''}(y), \beta(z)\rangle] \, \mathrm{d}q_0(z) - f_0(y) \geqslant -\tfrac{1}{2}\mu.$$

We shall write $\underline{w} = v_{\lambda'}$, $\bar{w} = v_{\lambda''}$. By adding a constant if necessary, we may assume that

$$\underline{w}(y) > \bar{w}(y), \quad \forall y \in \boldsymbol{T}^N. \tag{3.11}$$

We take sufficiently small $\lambda > 0$ such that $|\lambda \underline{w}|_{L^\infty}$, $|\lambda \bar{w}|_{L^\infty} < \frac{1}{2}\mu$. Then $\underline{w}$ and $\bar{w}$ respectively satisfy

$$\lambda \underline{w}(y) - a(y) \int_{\mathbb{R}^M} [\underline{w}(y + \beta(z)) - \underline{w}(y) - \langle \nabla \underline{w}(y), \beta(z)\rangle] \, \mathrm{d}q_0(z) + d - f_0(y) \leqslant \mu,$$

$$\lambda \bar{w}(y) - a(y) \int_{\mathbb{R}^M} [\bar{w}(y + \beta(z)) - \bar{w}(y) - \langle \nabla \bar{w}(y), \beta(z)\rangle] \, \mathrm{d}q_0(z) + d' - f_0(y) \geqslant -\mu.$$

From the comparison principle [2, 3, 10] we obtain

$$\lambda(\underline{w}(y) - \bar{w}(y)) \leqslant d' - d + 2\mu, \quad \forall y \in \boldsymbol{T}^N,$$

which contradicts (3.11) for sufficiently small $\mu > 0$. Therefore, $d = d'$ should hold, and the claim is proved. □

Proof of lemma 3.2. (i) The uniform bound for $|m_\lambda|_{L^\infty}$, $\forall \lambda \in (0, 1)$, is clear from the comparison principle for (3.8), i.e. $|\lambda m_\lambda|_{L^\infty} \leqslant |\lambda f_0|_{L^\infty}$.

(ii) We prove the inequality using a contradiction argument. Let $r > 0$ be a fixed number to be determined later. Set

$$C_\theta = \frac{2M}{r^\theta}. \tag{3.12}$$

Assume that there exist $\bar{y}$, $\bar{y}' \in \boldsymbol{T}^N$ such that

$$|m_\lambda(\bar{y}) - m_\lambda(\bar{y}')| > C_\theta |\bar{y} - \bar{y}'|^\theta, \tag{3.13}$$

which leads to a contradiction. Note that $|\bar{y} - \bar{y}'| < r$ must hold. Set

$$\Phi(y, y') = m_\lambda(y) - m_\lambda(y') - C_\theta |y - y'|^\theta, \quad y, y' \in \boldsymbol{T}^N.$$

Let $(\hat{y}, \hat{y}')$ be a maximum point of Φ in $\boldsymbol{T}^N$. We may assume that $\Phi(\hat{y}, \hat{y}')$ is the strict maximum. Set $\phi(y, y') = C_\theta |y - y'|^\theta$, $p = \nabla_y \phi(\hat{y}, \hat{y}')$, $Q = \nabla_y^2 \phi(\hat{y}, \hat{y}')$. From

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the definition of the viscosity solution, we obtain

$$\lambda m_\lambda(\hat{y}) - a(\hat{y}) \int_{\mathbb{R}^M} [m_\lambda(\hat{y} + \beta(z)) - m_\lambda(\hat{y}) - \langle p, \beta(z)\rangle] \, dq_0(z) \leqslant \lambda f_0(\hat{y}),$$

$$\lambda m_\lambda(\hat{y}') - a(\hat{y}') \int_{\mathbb{R}^M} [m_\lambda(\hat{y}' + \beta(z)) - m_\lambda(\hat{y}') - \langle p, \beta(z)\rangle] \, dq_0(z) \geqslant \lambda f_0(\hat{y}').$$

By dividing the above two inequalities by $a(\hat{y})$ and $a(\hat{y}')$, respectively, then taking the difference, we have

$$\frac{\lambda m_\lambda(\hat{y})}{a(\hat{y})} - \frac{\lambda m_\lambda(\hat{y}')}{a(\hat{y}')} - \int_{\mathbb{R}^M} [m_\lambda(\hat{y} + \beta(z)) - m_\lambda(\hat{y}) - m_\lambda(\hat{y}' + \beta(z)) + m_\lambda(\hat{y}')] \, dq_0(z) \leqslant \frac{\lambda f_0(\hat{y})}{a(\hat{y})} - \frac{\lambda f_0(\hat{y}')}{a(\hat{y}')}.$$

Since, for any $z \in \mathbb{R}^M$,

$$m_\lambda(\hat{y}) - m_\lambda(\hat{y}') - C_\theta|\hat{y} - \hat{y}'|^\theta \geqslant m_\lambda(\hat{y} + \beta(z)) - m_\lambda(\hat{y}' + \beta(z)) - C_\theta|\hat{y} - \hat{y}'|^\theta,$$

the preceding inequality leads to

$$\lambda a(\hat{y}') m_\lambda(\hat{y}) - \lambda a(\hat{y}) m_\lambda(\hat{y}') \leqslant \lambda a(\hat{y}') f_0(\hat{y}) - \lambda a(\hat{y}) f_0(\hat{y}'),$$

which leads to

$$a(\hat{y}')(m_\lambda(\hat{y}) - m_\lambda(\hat{y}')) \leqslant (a(\hat{y}) - a(\hat{y}')) m_\lambda(\hat{y}') + a(\hat{y}')(f_0(\hat{y}) - f_0(\hat{y}')) + (a(\hat{y}') - a(\hat{y})) f_0(\hat{y}').$$

Thus, from (1.6), (1.7) and (3.13), since $(\hat{y}, \hat{y}')$ is the maximum point of Φ, the above leads to

$$C_\theta|\hat{y} - \hat{y}'|^\theta \leqslant L'(|\hat{y} - \hat{y}'|^{\theta_1} + |\hat{y} - \hat{y}'|^{\theta_2}),$$

where $L' = a_0^{-1} L(M + \|a\|_{L^\infty(\boldsymbol{T}^N)} + \|f_0\|_{L^\infty(\boldsymbol{T}^N)})$. Therefore, from (3.12), since $\theta \in (0, \min\{\theta_1, \theta_2\})$ and $|\hat{x} - \hat{x}'| < r$,

$$2M \leqslant L'(|\hat{y} - \hat{y}'|^{\theta_1 - \theta} r^\theta + |\hat{y} - \hat{y}'|^{\theta_2 - \theta} r^\theta) \leqslant L'(r^{\theta_1} + r^{\theta_2}).$$

By taking $r > 0$ sufficiently small such that $r^{\theta_1} + r^{\theta_2} < 2ML'^{-1}$, we obtain the desired contradiction. This shows the existence of $C_\theta > 0$ such that (ii) holds. Moreover, the constant C_θ does not depend on $\lambda \in (0, 1)$. □

Corollary 3.3.

(i) *Let v_λ be the solution of (3.1) with* $dq_0(z)$ *and* $\beta(z)$ *given in either example 1.2 with* $\alpha \in (1, 2)$*, examples 1.3 and 1.4 with* $\alpha \in (1, 2)$*, or example 1.5 with* $\alpha \in (1, 2)$*. Then there exists a unique constant d such that (3.3) holds.*

(ii) *Let v_λ be the solution of (3.2) with* $dq_0(z)$ *and* $\beta(z)$ *given in either example 1.2 with* $\alpha \in (0, 1)$*, example 1.4 with* $\alpha \in (0, 1)$*, or example 1.5 with* $\alpha \in (0, 1)$*. Then there exists a unique constant d such that (3.3) holds.*

Proof. As we have seen in remark 2.2 each of the measures $dq_0(z)$ in examples 1.2–1.5 satisfies condition (B). Hence, the claim follows from theorem 3.1. □

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Remark 3.4.

1. The SMP (theorem 2.1) is essential to prove the existence of the ergodic number d in theorem 3.1.

2. We can generalize theorem 3.1 by adding a fully nonlinear degenerate elliptic second-order operator $F(x, \nabla u, \nabla^2 u)$ to (3.1) (respectively, (3.2)) (see [4] for an outline of the proof).

4. Homogenizations

In this section we give our main results of the homogenization problems (1.1) and (1.2), and (1.3) and (1.2) in theorems 4.7 and 4.9, respectively. Throughout this section we assume that condition (A) holds. Let u_ε be the solution of (1.1) and (1.2) (respectively, (1.3) and (1.2)). By introducing the formal asymptotic expansion (1.8),

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon^\alpha v\left(\frac{x}{\varepsilon}\right) + o(\varepsilon^\alpha), \quad x \in \mathbb{R}^N,$$

into (1.1) (respectively, (1.3)), by using the homogeneity of β in (1.4) and by noting that (1.13) and (1.14) hold, we obtain the following cell problem (1.15):

$$d_{I_1} - a(y)\int_{\mathbb{R}^M} [v(y+\beta(z)) - v(y) - \langle \nabla v(y), \beta(z)\rangle]\, \mathrm{d}q_0(z) - a(y)I_1 - f(y) = 0 \quad \text{in } \boldsymbol{T}^N,$$

where

$$I_1 = I_1[\bar{u}](x) = \int_{\mathbb{R}^M} [\bar{u}(x+\beta(z)) - \bar{u}(x) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla \bar{u}(x), \beta(z)\rangle]\, \mathrm{d}q(z),$$

respectively, we obtain (1.16):

$$d_{I_2} - a(y)\int_{\mathbb{R}^M} [v(y+\beta(z)) - v(y)]\, \mathrm{d}q_0(z) - a(y)I_2 - f(y) = 0 \quad \text{in } \boldsymbol{T}^N,$$

where

$$I_2 = I_2[\bar{u}](x) = \int_{\mathbb{R}^M} [\bar{u}(x+\beta(z)) - \bar{u}(x)]\, \mathrm{d}q(z),$$

provided that $\mathrm{d}q_0(z)$ satisfies (1.5) with $\gamma = 2$ (respectively, $\gamma = 1$). Note that, according to condition (A), the Lévy measure $\mathrm{d}q(z)$ in (1.1) (respectively, (1.3)) is transformed to $\mathrm{d}q_0(z)$ in the cell problem (1.15) (respectively, (1.16)). For any $I_1 \in \mathbb{R}$ (respectively, $I_2 \in \mathbb{R}$), from theorem 3.1 (with $f_0(y) = a(y)I_i + f(y)$, $i = 1, 2$), there exists a unique number d_{I_1} (respectively, d_{I_2}) such that, for any $\delta > 0$, there exist v_δ a periodic subsolution and v^δ a periodic supersolution of

$$\begin{aligned} d_{I_1} - a(y)\int_{\mathbb{R}^M} [v_\delta(y+\beta(z)) - v_\delta(y) - \langle \nabla v_\delta(y), \beta(z)\rangle]\, \mathrm{d}q_0(z) \\ - a(y)I_1 - f(y) \leqslant \tfrac{1}{2}\delta \quad \text{in } \boldsymbol{T}^N, \\ d_{I_1} - a(y)\int_{\mathbb{R}^M} [v^\delta(y+\beta(z)) - v^\delta(y) - \langle \nabla v_\delta(y), \beta(z)\rangle]\, \mathrm{d}q_0(z) \\ - a(y)I_1 - f(y) \geqslant -\tfrac{1}{2}\delta \quad \text{in } \boldsymbol{T}^N, \end{aligned}$$

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and, respectively,

$$d_{I_2} - a(y)\int_{\mathbb{R}^M} [v_\delta(y+\beta(z)) - v_\delta(y)]\,\mathrm{d}q_0(z) - a(y)I_2 - f(y) \leqslant \tfrac{1}{2}\delta \quad \text{in } \boldsymbol{T}^N,$$

$$d_{I_2} - a(y)\int_{\mathbb{R}^M} [v^\delta(y+\beta(z)) - v^\delta(y)]\,\mathrm{d}q_0(z) - a(y)I_2 - f(y) \geqslant -\tfrac{1}{2}\delta \quad \text{in } \boldsymbol{T}^N.$$

For reasons that will be explained later, let us regularize v_δ and v^δ. For $\nu > 0$, define

$$\text{(sup convolution)}\quad v_\delta^\nu(x) = \sup_{|y-x|\leqslant\nu}\left\{v_\delta(y) - \frac{1}{\nu^2}|y-x|^2\right\},$$

$$\text{(inf convolution)}\quad v_\nu^\delta(x) = \inf_{|y-x|\leqslant\nu}\left\{v^\delta(y) + \frac{1}{\nu^2}|y-x|^2\right\}.$$

Set $\underline{v} = v_\delta^\nu$, $\bar{v} = v_\nu^\delta$. It is known that $\underline{v}$ is semiconvex, $\bar{v}$ is semiconcave, and both are Lipschitz continuous [13, 16]. Moreover, since

$$\lim_{\nu\to 0} v_\delta^\nu = v_\delta, \qquad \lim_{\nu\to 0} v_\nu^\delta = v^\delta$$

uniformly in $\boldsymbol{T}^N$, for any $\delta > 0$, we can take $\nu > 0$ such that $\underline{v}$ and $\bar{v}$ are, respectively, a subsolution and a supersolution of the following:

$$d_{I_1} - a(y)\int_{\mathbb{R}^M} [\underline{v}(y+\beta(z)) - \underline{v}(y) - \langle\nabla\underline{v}(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) - a(y)I_1 - f(y) \leqslant \delta \quad \text{in } \boldsymbol{T}^N, \tag{4.1}$$

$$d_{I_1} - a(y)\int_{\mathbb{R}^M} [\bar{v}(y+\beta(z)) - \bar{v}(y) - \langle\nabla\bar{v}(y), \beta(z)\rangle]\,\mathrm{d}q_0(z) - a(y)I_1 - f(y) \geqslant -\delta \quad \text{in } \boldsymbol{T}^N \tag{4.2}$$

and, respectively,

$$d_{I_2} - a(y)\int_{\mathbb{R}^M} [\underline{v}(y+\beta(z)) - \underline{v}(y)]\,\mathrm{d}q_0(z) - a(y)I_2 - f(y) \leqslant \delta \quad \text{in } \boldsymbol{T}^N, \tag{4.3}$$

$$d_{I_2} - a(y)\int_{\mathbb{R}^M} [\bar{v}(y+\beta(z)) - \bar{v}(y)]\,\mathrm{d}q_0(z) - a(y)I_2 - f(y) \geqslant -\delta \quad \text{in } \boldsymbol{T}^N \tag{4.4}$$

(see, for example, [3, 16]). We use the above approximated cell problem in place of (1.15) in the following argument. Define

$$\bar{I}_1(I_1) = -d_{I_1},\ \forall I_1 \in \mathbb{R} \quad \text{and, respectively, } \bar{I}_2(I_2) = -d_{I_2},\ \forall I_2 \in \mathbb{R}, \tag{4.5}$$

where the right-hand side is a unique number such that, for any $\delta > 0$, (4.1) and (4.2) (respectively, (4.3) and (4.4)) have a subsolution and a supersolution, respectively. We now prepare some lemmas which we will use later in the paper.

LEMMA 4.1 (Arisawa [6]). *Assume that (1.4), (1.6) and (1.7) hold, and that* $\mathrm{d}q_0(z)$ *satisfies condition* (B) *and (1.5) with* $\gamma = 2$ (*respectively,* $\gamma = 1$). *Then the function* $\bar{I}_1$ (*respectively,* $\bar{I}_2$) *defined in (4.5) is continuous and satisfies the following*

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property. There exists $\Theta > 0$ *such that*

$$\bar{I}_1(I + I') - \bar{I}_1(I) \leqslant -\Theta I', \quad \forall I \in \mathbb{R}, \ \forall I' \geqslant 0, \tag{4.6}$$

and, respectively,

$$\bar{I}_2(I + I') - \bar{I}_2(I) \leqslant -\Theta I', \quad \forall I \in \mathbb{R}, \ \forall I' \geqslant 0. \tag{4.7}$$

The above result was presented in [6], which was originally given in [14] for the partial differential equation case. The proof does not differ much from [6, 14], so we omit it here.

Remark 4.2. Let $u \in C^2(\mathbb{R}^N)$. Then, by putting

$$I_1 = I_1[u](x) = \int_{\mathbb{R}^M} [u(x + \beta(z)) - u(x) - \mathbf{1}_{|z| \leqslant 1} \langle \nabla u(x), \beta(z) \rangle] \, \mathrm{d}q(z),$$

$$I_2 = I_2[u](x) = \int_{\mathbb{R}^M} [u(x + \beta(z)) - u(x)] \, \mathrm{d}q(z)$$

into $\bar{I}_1$ (respectively, $\bar{I}_2$), the map $u \to \bar{I}_1(I_1[u](x))$ (respectively, $u \to \bar{I}_2(I_2[u](x))$) can be regarded as an integro-differential operator. The property (4.7) implies that $\bar{I}_1(I_1[u](x))$ (respectively, $\bar{I}_2(I_2[u](x))$) is subelliptic [6].

Lemma 4.3. *Let* $\bar{I}_1$ (*respectively,* $\bar{I}_2$) *be the functions defined in (4.5). Consider*

$$u + \bar{I}_1(I_1[u](x)) = 0 \quad \text{and, respectively, } u + \bar{I}_2(I_2[u](x)) = 0 \quad \text{in } \Omega, \tag{4.8}$$

with (1.2). Let u, v *be a subsolution and a supersolution, respectively, of (4.8) and (1.2). Then,* $u \leqslant v$ *in* Ω. *Moreover, there exists a unique viscosity solution* u *of (4.8) and (1.2).*

Proof. The comparison principle can be shown by the usual contradiction argument, from the subellipticities (4.6) and (4.7). The existence of the solution can be obtained using Perron's method. This argument was used in [2, 3, 6, 10] and we do not repeat it here. □

We give the following result in the convex analysis, which we cite without proof (see [13, 16] for details). For an upper or a lower semicontinuous function Φ defined in an open subset $\mathcal{O}$ in $\mathbb{R}^n$, for $\rho > 0$, put

$$M_\rho = \{\bar{x} \in \mathcal{O} \mid \exists p \in \mathbb{R}^n \text{ such that } |p| \leqslant \rho, \ \Phi(x) \leqslant \Phi(\bar{x}) + \langle p, \ x - \bar{x} \rangle, \ \forall x \in \mathcal{O}\}.$$

Lemma 4.4 (Crandall *et al*. [13], Fleming and Soner [16]). *Let* Φ *be a semiconvex function in an open domain* $\mathcal{O}$, *and let* x' *be a maximizer of* Φ *in* $\mathcal{O}$ *such that*

$$\mu = \sup_{\mathcal{O}} \Phi(x) - \sup_{\partial \mathcal{O}} \Phi(x) = \Phi(x') - \sup_{\partial \mathcal{O}} \Phi(x) > 0.$$

Then the following hold.

(i) Φ *is differentiable at* x' *and* $\nabla \Phi(x') = 0$.

(ii) *For any* $m \in \boldsymbol{N}$, *there exists* $x_m \in M_{1/m}$ *such that* Φ *is twice differentiable at* x_m, $\lim_{m\to\infty} x_m = x'$, $\nabla^2\Phi(x_m) \leqslant O$, $|\nabla\Phi(x_m)| \leqslant 1/m$. *For* $p_m = \nabla\Phi(x_m)$, *the function*

$$\Phi_m(x) = \Phi(x) - \langle p_m, x\rangle$$

takes a maximum at $x = x_m$.

Lemma 4.5. *Let* $\bar{v}(y)$ *be a periodic semiconcave function defined in* $\boldsymbol{T}^N$. *Assume that, for a function* $\Psi(x) \in C^2(\mathbb{R}^N)$, $\Psi(x) + \varepsilon^\alpha \bar{v}(x/\varepsilon)$ *takes a global minimum at* $\bar{x}$. *Then, the following hold for any* $z \in \mathbb{R}^M$, *with a constant* $C > 0$ *independent on* $\varepsilon > 0$ *and* $\bar{x}$*:*

(i)

$$-C\varepsilon^{2-\alpha}|z|^2 \leqslant \bar{v}\left(\frac{\bar{x}}{\varepsilon} + \beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z)\right\rangle \leqslant C|z|^2; \quad (4.9)$$

(ii)

$$-C\varepsilon^{1-\alpha}|z| \leqslant \bar{v}\left(\frac{\bar{x}}{\varepsilon} + \beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) \leqslant C|z|. \quad (4.10)$$

Proof of lemma 4.5. (i) The second inequality comes from the semiconcavity of $\bar{v}$ and (1.4). The first inequality is derived from the fact that $\Psi(x) + \varepsilon^\alpha \bar{v}(x/\varepsilon)$ takes a global minimum at $\bar{x}$. In fact, since $\Psi(x) + \varepsilon^\alpha \bar{v}(x/\varepsilon)$ is semiconcave, it is differentiable at $\bar{x}$ and $\nabla\Psi(\bar{x}) + \varepsilon^{-1+\alpha}\nabla_y \bar{v}(\bar{x}/\varepsilon) = 0$,

$$\Psi(\bar{x}) + \varepsilon^\alpha \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) \leqslant \Psi(\bar{x} + \varepsilon\beta(z)) + \varepsilon^\alpha \bar{v}\left(\frac{\bar{x}}{\varepsilon} + \beta(z)\right), \quad \forall z \in \mathbb{R}^M,$$

for any $\varepsilon > 0$. Thus, we obtain

$$\begin{aligned}\varepsilon^\alpha\left(\bar{v}\left(\frac{\bar{x}}{\varepsilon} + \beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \left\langle \varepsilon^{-1}\nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \varepsilon\beta(z)\right\rangle\right)& \\ \geqslant -(\Psi(\bar{x} + \varepsilon\beta(z)) - \Psi(\bar{x}) - \langle \nabla\Psi(\bar{x}), \varepsilon\beta(z)\rangle)& \\ \geqslant -\varepsilon^2|\beta(z)|^2|\nabla^2\Psi(\bar{x} + \mu\varepsilon\beta(z))|,&\end{aligned}$$

where $\mu \in (0, 1)$. From (1.4), the first inequality holds with a constant $C > 0$ independent on $\varepsilon > 0$ and $\bar{x}$.

(ii) The second inequality comes from the Lipschitz continuity of $\bar{v}$ and (1.4). The first inequality is proved in a similar way to (i). □

Lemma 4.6. *Let* $\bar{v}(y)$ *be a periodic semiconcave function defined in* $\boldsymbol{T}^N$. *Assume that, for a function* $\Psi(x) \in C^2(\mathbb{R}^N)$, $\Psi(x) + g(\varepsilon)\bar{v}(x/\varepsilon)$ *takes a minimum at* $\bar{x}$. *Then the following hold.*

(i) *If* $\mathrm{d}q_0(z)$ *satisfies (1.5) with* $\gamma = 2$,

$$\begin{aligned}\varepsilon^\alpha \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x} + \beta(z)}{\varepsilon}\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \mathbf{1}_{|z|\leqslant 1}\left\langle \varepsilon^{-1}\nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z)\right\rangle\right] \mathrm{d}q(z)& \\ = \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon} + \beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z)\right\rangle\right] \mathrm{d}q_0(z) + O(\varepsilon).&\end{aligned} \quad (4.11)$$

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(ii) *If* $\mathrm{d}q_0(z)$ *satisfies (1.5) with* $\gamma = 1$,

$$\varepsilon^{\alpha} \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}+\beta(z)}{\varepsilon}\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right)\right] \mathrm{d}q(z) = \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon}+\beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right)\right] \mathrm{d}q_0(z) + O(\varepsilon). \quad (4.12)$$

Proof. (i) From (1.4) (i.e. $\varepsilon^{-1}\beta(z) = \beta(z/\varepsilon)$), we have

$$\begin{aligned}
&\varepsilon^{\alpha} \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}+\beta(z)}{\varepsilon}\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \mathbf{1}_{|z|\leqslant 1}\left\langle \varepsilon^{-1}\nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z)\right\rangle\right] \mathrm{d}q(z) \\
&\quad = \varepsilon^{\alpha} \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon}+\beta\left(\frac{z}{\varepsilon}\right)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \mathbf{1}_{|z|\leqslant 1}\left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta\left(\frac{z}{\varepsilon}\right)\right\rangle\right] \mathrm{d}q(z) \\
&\quad = \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon}+\beta(z')\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \mathbf{1}_{|\varepsilon z'|\leqslant 1}\left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z')\right\rangle\right] \varepsilon^{M+\alpha} q(\varepsilon z')\,\mathrm{d}z'.
\end{aligned}$$

Then, by condition (A), and (1.5) with $\gamma = 2$,

$$\begin{aligned}
&\left| \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon}+\beta(z')\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \mathbf{1}_{|\varepsilon z'|\leqslant 1}\left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z')\right\rangle\right] \varepsilon^{M+\alpha} q(\varepsilon z')\,\mathrm{d}z' \right.\\
&\qquad \left. - \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon}+\beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z)\right\rangle\right] \mathrm{d}q_0(z) \right| \\
&\quad \leqslant C \int_{|z|\leqslant 1} \left[\bar{v}\left(\frac{\bar{x}}{\varepsilon}+\beta(z)\right) - \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right) - \left\langle \nabla_y \bar{v}\left(\frac{\bar{x}}{\varepsilon}\right), \beta(z)\right\rangle\right] |\varepsilon^{M+\alpha} q(\varepsilon z) - q_0(z)|\,\mathrm{d}z \\
&\qquad + C \int_{|z|>1} |z|\,|\varepsilon^{M+\alpha} q(\varepsilon z) - q_0(z)|\,\mathrm{d}z \\
&\quad \leqslant C' \left(\int_{|z|\leqslant 1} |z|^2 |\varepsilon^{M+\alpha} q(\varepsilon z) - q_0(z)|\,\mathrm{d}z + \int_{|z|>1} |z|\,|\varepsilon^{M+\alpha} q(\varepsilon z) - q_0(z)|\,\mathrm{d}z \right) \\
&\quad = O(\varepsilon),
\end{aligned}$$

where we used part (i) of lemma 4.5 to obtain the last estimate.

(ii) The proof is similar to that of (i), while we use (1.5) with $\gamma = 1$ and part (ii) of lemma 4.5. □

We now state the main result of the paper.

THEOREM 4.7. *Let* u_ε *be the solution of (1.1) and (1.2). Assume that (1.4), (1.5)* (*with* $\gamma = 2$), *(1.6), (1.7), and conditions* (A) *and* (B) *hold. Assume also that* $\mathrm{d}q_0(z)$ *defined in (1.11) satisfies (1.5) with* $\gamma = 2$. *Then there exists a unique function*

$$\bar{u}(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x), \quad \forall x \in \mathbb{R}^N,$$

which is a unique viscosity solution of

$$\bar{u}(x) + \bar{I}_1[I_1[\bar{u}](x)] = 0 \quad \text{in } \Omega, \quad (4.13)$$

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and (1.2), where $\bar{I}_1$ *is given by (4.5) with*

$$I_1[\bar{u}](x) = \int_{\mathbb{R}^M} [\bar{u}(x+\beta(z)) - \bar{u}(x) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla \bar{u}(x), \beta(z)\rangle]\, \mathrm{d}q(z).$$

Proof of theorem 4.7. We use the perturbed test function method introduced in [14] (see also [18]), which is now standard, to solve homogenization problems in the framework of viscosity solutions. Here, we must take extra care to treat the difference between the original Lévy measure $\mathrm{d}q(z)$ and the rescaled measure $\mathrm{d}q_0(z)$ in the cell problem (1.15) (and (4.1) and (4.2)). Let

$$u^*(x) = \lim \sup_{\varepsilon\to 0, y\to x} u_\varepsilon(y), \qquad u_*(x) = \lim \inf_{\varepsilon\to 0, y\to x} u_\varepsilon(y), \quad \forall x \in \mathbb{R}^N.$$

In the following, we divide our argument into two steps.

1. We show that u^* is a subsolution of (4.13). By assuming that u^* is not the subsolution of (4.13), we obtain a contradiction. So, assume that, for a function $\phi(x) \in C^2(\mathbb{R}^N)$, $u^* - \phi$ takes a global strict maximum at $\bar{x}$, $u^*(\bar{x}) = \phi(\bar{x})$, and for some $\gamma > 0$, the following holds:

$$\phi(\bar{x}) + \bar{I}_1\biggl[\int_{\mathbb{R}^M} [\phi(\bar{x}+\beta(z)) - \phi(\bar{x}) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\phi(\bar{x}), \beta(z)\rangle]\, \mathrm{d}q(z)\biggr] = 3\gamma > 0.$$

Then, from the continuities of $\bar{I}_1$ (lemma 4.1) and ϕ, for sufficiently small $r > 0$,

$$\phi(x) + \bar{I}_1[I_1[\phi](x)] > 2\gamma \quad \text{in } B_r(\bar{x}), \tag{4.14}$$

where

$$I_1[\phi](x) = \int_{\mathbb{R}^M} [\phi(x+\beta(z)) - \phi(x) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\phi(x), \beta(z)\rangle]\, \mathrm{d}q(z).$$

From (4.2), for $\delta > 0$ and $I_1 = I_1[\phi](\bar{x})$, we know that there exists a periodic, semiconcave, Lipschitz continuous function $\bar{v}$ which satisfies

$$d_{I_1[\phi](\bar{x})} - a(y)\int_{\mathbb{R}^M} [\bar{v}(y+\beta(z)) - \bar{v}(y) - \langle \nabla\bar{v}(y), \beta(z)\rangle]\, \mathrm{d}q_0(z) \\ - a(y)I_1[\phi](\bar{x}) - f(y) \geqslant -\tfrac{1}{2}\delta \quad \text{in } \boldsymbol{T}^N. \tag{4.15}$$

To continue the proof of theorem 4.7 we need the following lemma.

Lemma 4.8. *Let* $\phi_\varepsilon(x) = \phi(x) + \varepsilon^\alpha \bar{v}(x/\varepsilon)$. *The function* ϕ_ε *is a viscosity supersolution of*

$$\phi_\varepsilon(x) - a\biggl(\frac{x}{\varepsilon}\biggr)\int_{\mathbb{R}^M} [\phi_\varepsilon(x+\beta(z)) - \phi_\varepsilon(x) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\phi_\varepsilon(x), \beta(z)\rangle]\, \mathrm{d}q(z) \\ - f\biggl(\frac{x}{\varepsilon}\biggr) \geqslant \gamma \quad \textit{in } B_r(\bar{x}), \tag{4.16}$$

where the Lévy density $\mathrm{d}q(z)$ *is the one in (1.1).*

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Proof of lemma 4.8. To confirm (4.16) in the sense of viscosity solutions, assume that, for some $\psi \in C^2(\mathbb{R}^N)$, $\phi_\varepsilon - \psi$ takes a strict minimum at $x = x'$ and $\phi_\varepsilon(x') = \psi(x')$. From definition A.2 we must show that

$$\phi_\varepsilon(x') - a\left(\frac{x'}{\varepsilon}\right) \int_{\mathbb{R}^M} [\psi(x' + \beta(z)) - \psi(x') - \mathbf{1}_{|z| \leqslant 1}\langle \nabla\psi(x'), \beta(z)\rangle] \, \mathrm{d}q(z) - f\left(\frac{x'}{\varepsilon}\right) \geqslant \gamma. \tag{4.17}$$

Since $-(\phi_\varepsilon - \psi)$ is semiconvex, from lemma 4.4, we can take a sequence $x'_m \in \Omega$ such that $x'_m \to x'$ as $m \to \infty$, $\phi_\varepsilon - \psi$ is twice differentiable at x'_m, $\nabla^2(\phi_\varepsilon - \psi)(x'_m) \geqslant O$, $|\nabla(\phi_\varepsilon - \psi)(x'_m)| \leqslant 1/m$. Furthermore, by setting

$$p_m = \nabla(\phi_\varepsilon - \psi)(x'_m),$$

$(\phi_\varepsilon - \psi)(x) - \langle p_m, x\rangle$ takes a minimum at x'_m. Put $\psi_m(x) = \psi(x) + \langle p_m, x\rangle$. To see (4.17), we first prove

$$\phi_\varepsilon(x'_m) - a\left(\frac{x'_m}{\varepsilon}\right) \int_{\mathbb{R}^M} [\psi_m(x'_m + \beta(z)) - \psi_m(x'_m) - \mathbf{1}_{|z| \leqslant 1}\langle \nabla\psi_m(x'_m), \beta(z)\rangle] \, \mathrm{d}q(z) - f\left(\frac{x'_m}{\varepsilon}\right) \geqslant \gamma \tag{4.18}$$

for any sufficiently large $m \in \boldsymbol{N}$. By noting that $\phi_\varepsilon - \psi_m$ is twice differentiable at x'_m and that $\psi_m \in C^2$, we know that ϕ_ε is twice differentiable at x'_m. Since ϕ_ε is semiconcave and Lipschitz, from (1.5) we obtain

$$\phi_\varepsilon(x'_m + \beta(z)) - \phi_\varepsilon(x'_m) - \mathbf{1}_{|z| \leqslant 1}\langle \nabla\phi_\varepsilon(x'_m), \beta(z)\rangle \in L^1(\mathbb{R}^M, \mathrm{d}q(z)).$$

We can show that

$$\phi_\varepsilon(x'_m) - a\left(\frac{x'_m}{\varepsilon}\right) \int_{\mathbb{R}^M} [\phi_\varepsilon(x'_m + \beta(z)) - \phi_\varepsilon(x'_m) - \mathbf{1}_{|z| \leqslant 1}\langle \nabla\phi_\varepsilon(x'_m), \beta(z)\rangle] \, \mathrm{d}q(z) - f\left(\frac{x'_m}{\varepsilon}\right) \geqslant \gamma, \tag{4.19}$$

in the classical sense, for any sufficiently large $m \in \boldsymbol{N}$. To see (4.19), we use part (i) of lemma 4.6 (4.11) for $\Psi = \phi - \psi_m$, $\bar{x} = x'_m$ to obtain

$$\begin{aligned} \varepsilon^\alpha \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{x'_m + \beta(z)}{\varepsilon}\right) - \bar{v}\left(\frac{x'_m}{\varepsilon}\right) - \mathbf{1}_{|z| \leqslant 1}\left\langle \varepsilon^{-1}\nabla_y \bar{v}\left(\frac{x'_m}{\varepsilon}\right), \beta(z)\right\rangle\right] \mathrm{d}q(z) \\ = \int_{\mathbb{R}^M} \left[\bar{v}\left(\frac{x'_m}{\varepsilon} + \beta(z)\right) - \bar{v}\left(\frac{x'_m}{\varepsilon}\right) - \left\langle \nabla_y \bar{v}\left(\frac{x'_m}{\varepsilon}\right), \beta(z)\right\rangle\right] \mathrm{d}q_0(z) + O(\varepsilon). \end{aligned} \tag{4.20}$$

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Thus, from (4.15), with $y = x'_m/\varepsilon$, for sufficiently small $\varepsilon > 0$,

$$
\begin{aligned}
d_{I_1[\phi](\bar{x})} - a\left(\frac{x'_m}{\varepsilon}\right)\varepsilon^\alpha \int_{\mathbb{R}^M} \bigg[\bar{v}\left(\frac{x'_m + \beta(z)}{\varepsilon}\right) - \bar{v}\left(\frac{x'_m}{\varepsilon}\right) \\
- \mathbf{1}_{|z|\leqslant 1}\left\langle \varepsilon^{-1}\nabla_y \bar{v}\left(\frac{x'_m}{\varepsilon}\right), \beta(z)\right\rangle\bigg]\,\mathrm{d}q(z) \\
- a\left(\frac{x'_m}{\varepsilon}\right) I_1[\phi](\bar{x}) - f\left(\frac{x'_m}{\varepsilon}\right) \geqslant -\delta.
\end{aligned}
$$

We introduce this into (4.14) (for $x = x'_m \in B_r(\bar{x})$):

$$
\phi(x'_m) + \bar{I}_1\bigg[\int_{\mathbb{R}^M} [\phi(x'_m + \beta(z)) - \phi(x'_m) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\phi(x'_m), \beta(z)\rangle]\,\mathrm{d}q(z)\bigg] > 2\gamma.
$$

By taking sufficiently small $\varepsilon > 0$, $\delta > 0$ such that

$$
\delta + \left|\varepsilon^\alpha \bar{v}\left(\frac{x'_m}{\varepsilon}\right)\right| \leqslant \tfrac{1}{4}\gamma,
$$

by noting that $d_{I_1[\phi](\bar{x})} = -\bar{I}_1(I_1[\phi](\bar{x}))$, and from the continuities of $\bar{I}_1$, ϕ, for sufficiently small $r > 0$ we obtain

$$
\begin{aligned}
&\phi(x'_m) + \varepsilon^\alpha \bar{v}\left(\frac{x'_m}{\varepsilon}\right) \\
&\quad - a\left(\frac{x'_m}{\varepsilon}\right)\int_{\mathbb{R}^M}\bigg[\left(\phi(x'_m + \beta(z)) + \varepsilon^\alpha \bar{v}\left(\frac{x'_m + \beta(z)}{\varepsilon}\right)\right) - \left(\phi(x'_m) + \varepsilon^\alpha \bar{v}\left(\frac{x'_m}{\varepsilon}\right)\right) \\
&\qquad - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\phi(x'_m) + \varepsilon^{\alpha-1}\nabla_y \bar{v}\left(\frac{x'_m}{\varepsilon}\right), \beta(z)\rangle\bigg]\,\mathrm{d}q(z) \\
&\qquad - f\left(\frac{x'_m}{\varepsilon}\right) \geqslant \gamma.
\end{aligned}
$$

Thus, (4.19) is proved. From $\nabla\phi_\varepsilon(x'_m) = \nabla\psi_m(x'_m)$ and

$$
(\phi_\varepsilon - \psi_m)(x'_m) \leqslant (\phi_\varepsilon - \psi_m)(x'_m + \beta(z)), \quad \forall z \in \mathbb{R}^M,
$$

(4.19) leads to (4.18):

$$
\begin{aligned}
\phi_\varepsilon(x'_m) - a\left(\frac{x'_m}{\varepsilon}\right)\int_{\mathbb{R}^M}[\psi_m(x'_m + \beta(z)) - \psi_m(x'_m) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\psi_m(x'_m), \beta(z)\rangle]\,\mathrm{d}q(z) \\
- f\left(\frac{x'_m}{\varepsilon}\right) \geqslant \gamma.
\end{aligned}
$$

From (4.18), since $|p_m| \leqslant 1/m$, and since

$$
\begin{aligned}
&\psi_m(x'_m + \beta(z)) - \psi_m(x'_m) - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\psi_m(x'_m), \beta(z)\rangle \\
&\quad \to \psi(x' + \beta(z)) - \psi(x') - \mathbf{1}_{|z|\leqslant 1}\langle \nabla\psi(x'), \beta(z)\rangle \in L^1(\mathbb{R}^M, \mathrm{d}q(z))
\end{aligned}
$$

as $m \to \infty$, we have shown (4.17), and thus lemma 4.8 is proved. □

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We continue the proof of theorem 4.7. Now, the comparison principle for (1.1) and (4.16) leads to

$$\sup_{x\in U_r(\bar{x})}\{u_\varepsilon(x)-\phi_\varepsilon(x)\}\leqslant \sup_{x\in U_r(\bar{x})^c}\{u_\varepsilon(x)-\phi_\varepsilon(x)\}+\gamma.$$

By letting $\varepsilon\to 0$, since $\gamma>0$ is arbitrary,

$$\sup_{x\in U_r(\bar{x})}\{u^*(x)-\phi(x)\}\leqslant \sup_{x\in U_r(\bar{x})^c}\{u^*(x)-\phi(x)\}.$$

However, this contradicts the fact that $\bar{x}$ is the strict global maximum of $u^*-\phi$. Therefore, u^* must be a viscosity subsolution of (4.13).

2. By the parallel argument, we can prove that u_* is a viscosity supersolution of (4.13), which we do not repeat here. Now, from the definition of u_* and u^*, we have

$$u_*\leqslant u_\varepsilon\leqslant u^*,\quad \forall\varepsilon>0.$$

From the comparison principle for the viscosity solution of (4.13) and (1.2) in lemma 4.3, we have

$$u^*\leqslant u_*\quad\text{in }\bar{\Omega}.$$

Thus, there exists a limit

$$\bar{u}=\lim_{\varepsilon\to 0}u_\varepsilon=u_*=u^*$$

that is the unique viscosity solution of (4.13) and (1.2). □

Our second result is the following.

Theorem 4.9. *Let u_ε be the solution of (1.3) and (1.2). Assume that (1.4), (1.5)* (*with* $\gamma=1$), *(1.6) and (1.7) hold, and that conditions* (A) *and* (B) *hold. Assume also that* $\mathrm{d}q_0(z)$ *defined in (1.11) satisfies (1.5) with* $\gamma=1$. *Then there exists a unique function*

$$\bar{u}(x)=\lim_{\varepsilon\to 0}u_\varepsilon(x),\quad\forall x\in\mathbb{R}^N,$$

which is a unique viscosity solution of

$$\bar{u}(x)+\bar{I}_2[I_2[\bar{u}](x)]=0\quad\text{in }\Omega \tag{4.21}$$

and (1.2), where $\bar{I}_2$ is given by (4.5) and

$$I_2[\bar{u}](x)=\int_{\mathbb{R}^M}[\bar{u}(x+\beta(z))-\bar{u}(x)]\,\mathrm{d}q(z).$$

Proof of theorem 4.9. The proof is similar to that of theorem 4.7 (in fact, it is simpler because there is no term $\mathbf{1}_{|z|\leqslant 1}\langle\nabla u(x),\beta(z)\rangle$ in the integral). We use part (ii) of lemma 4.6 instead of part (i).

Corollary 4.10.

(i) *Let u_ε be the solution of (1.1) and (1.2). Assume that (1.6) and (1.7) hold, and that* $\mathrm{d}q(z)$ *and* $\beta(z)$ *are given by either example 1.2 with* $\alpha\in(1,2)$,

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examples 1.3 and 1.4 with $\alpha \in (1,2)$, or example 1.5 with $\alpha \in (1,2)$. Then there exists a unique function

$$\bar{u}(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x), \quad \forall x \in \mathbb{R}^N,$$

which is a unique viscosity solution of (4.13) and (1.2).

(ii) *Let u_ε be the solution of (1.3) and (1.2). Assume that (1.6) and (1.7) hold, and that $\mathrm{d}q(z)$ and $\beta(z)$ are given by either example 1.2 with $\alpha \in (0,1)$, example 1.4 with $\alpha \in (0,1)$, or example 1.5 with $\alpha \in (0,1)$. Then there exists a unique function*

$$\bar{u}(x) = \lim_{\varepsilon \to 0} u_\varepsilon(x), \quad \forall x \in \mathbb{R}^N,$$

which is a unique viscosity solution of (4.21) and (1.2).

The claims follow from corollary 3.3 and theorems 4.7 and 4.9. □

Remark 4.11. The present argument can be generalized to the following type of homogenization problem:

$$u_\varepsilon(x) + \sup_{\tilde{\alpha} \in \mathcal{A}} \biggl\{ -a\Bigl(\frac{x}{\varepsilon}\Bigr) \int_{\mathbb{R}^M} [u_\varepsilon(x + \beta(z,\tilde{\alpha})) - u_\varepsilon(x) - \mathbf{1}_{|z| \leqslant 1} \langle \nabla u_\varepsilon(x), \beta(z,\tilde{\alpha}) \rangle] \, \mathrm{d}q(z) - f\Bigl(\frac{x}{\varepsilon}, \tilde{\alpha}\Bigr) \biggr\} = 0 \quad \text{in } \Omega,$$

with (1.2), where $\mathcal{A}$ is a compact metric set (control set), $\beta(z,\alpha)$ is a continuous function in $\mathbb{R}^M \times \mathcal{A}$ with values in $\mathbb{R}^N$ satisfying (1.4) uniformly in $\mathcal{A}$, and $f(y,\alpha)$ is a real-valued continuous function in $\boldsymbol{T}^N \times \mathcal{A}$ satisfying (1.7) uniformly in $\mathcal{A}$. We leave the detail to the reader.

5. A nonlinear problem

In this section we show how the present method can be applied to more general nonlinear problems. We consider example 1.6. Let u_ε be the unique viscosity solution of (1.17).

Assume that there exist two positive numbers $\alpha_l \in (0,2)$, $l = 1,2$, subsets $S_0^l \subset S^l = \operatorname{supp}(\mathrm{d}q_l(z))$, $l = 1,2$, and positive functions $q_0^l(z)$, $l = 1,2$, such that condition (A) is satisfied:

$$\left.\begin{aligned} &\lim_{\varepsilon \to 0} q_l(\varepsilon z)\varepsilon^{l+\alpha_l} \, \mathrm{d}z = q_0^l(z) \, \mathrm{d}z, && \forall z \in S_0^l, \ l = 1,2, \\ &\lim_{\varepsilon \to 0} q_l(\varepsilon z)\varepsilon^{l+\alpha_l} \, \mathrm{d}z = 0 \, \mathrm{d}z, && \forall z \in \mathbb{R}^l / S_0^l, \ l = 1,2, \end{aligned}\right\} \tag{5.1}$$

and

$$|\varepsilon^{l+\alpha_l} q_l(\varepsilon z)| \leqslant C|z|^{-(l+\alpha_l)}, \ \forall \varepsilon \in (0,1), \ \forall z \in \mathbb{R}^l, \tag{5.2}$$

where $\mathrm{d}q_l(z) = q_l(z) \, \mathrm{d}z$, $l = 1,2$, and $C > 0$ is a constant. We define the following new measures:

$$\mathrm{d}q_0^l(z) = q_0^l(z) \, \mathrm{d}z, \ \forall z \in S_0^l, \qquad \mathrm{d}q_0^l(z) = 0 \, \mathrm{d}z, \ \forall z \in \mathbb{R}^l / S_0^l, \ l = 1,2. \tag{5.3}$$

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Here, we further assume that $\alpha_1 = \alpha_2 = \alpha$ (otherwise, a different problem which is not concerned with the present interest of the non-local problem arises). We use the formal asymptotic expansion

$$u_\varepsilon(x) = \bar{u}(x) + \varepsilon^\alpha v\left(\frac{x_1}{\varepsilon}, \frac{x_2}{\varepsilon}, \frac{x_3}{\varepsilon}\right), \quad x \in \mathbb{R}^3, \tag{5.4}$$

and obtain the following ergodic cell problem. For any given I', $I'' \in \mathbb{R}$, find a unique number $d_{I',I''}$ with which the following problem has a periodic viscosity solution v:

$$\begin{aligned} d_{I',I''} + \max\bigg\{ &-a(y)\int_{\mathbb{R}} [v(y+\beta_1(z')) - v(y) - \langle \beta_1(z'), \nabla v(y)\rangle]\,\mathrm{d}q_0^1(z') \\ &- a(y)I', -a(y)\int_{\mathbb{R}^2} [v(y+\beta_2(z'')) - v(y) - \langle \beta_2(z''), \nabla v(y)\rangle]\,\mathrm{d}q_0^2(z'') \\ &- a(y)I'' \bigg\} - f(y) = 0 \quad \text{in } \boldsymbol{T}^3, \end{aligned} \tag{5.5}$$

where

$$I' = I'[\bar{u}](x) = \int_{\mathbb{R}} [\bar{u}(x+\beta_1(z')) - \bar{u}(x) - \mathbf{1}_{|z'|\leqslant 1}\langle \beta_1(z'), \nabla \bar{u}(x)\rangle]\,\mathrm{d}q_1(z'),$$

$$I'' = I''[\bar{u}](x) = \int_{\mathbb{R}^2} [\bar{u}(x+\beta_2(z'')) - \bar{u}(x) - \mathbf{1}_{|z''|\leqslant 1}\langle \beta_2(z''), \nabla \bar{u}(x)\rangle]\,\mathrm{d}q_2(z'').$$

As in § 3, the existence of the unique number $d_{I',I''}$ in (5.5) comes from the SMP of the integro-differential equation

$$\begin{aligned} H(y,\nabla v) + \max\bigg\{ &-\int_{\mathbb{R}} [v(y+\beta_1(z')) - v(y) - \langle \beta_1(z'), \nabla v(y)\rangle]\,\mathrm{d}q_0^1(z'), \\ &-\int_{\mathbb{R}^2} [v(y+\beta_2(z'')) - v(y) - \langle \beta_2(z''), \nabla v(y)\rangle]\,\mathrm{d}q_0^2(z'') \bigg\} = 0 \quad \text{in } \boldsymbol{T}^3. \end{aligned} \tag{5.6}$$

In order to establish the SMP for (5.6), we need to generalize condition (B) of theorem 2.1 to the following.

(B′) For any two points $y, y' \in \boldsymbol{T}^3$, there exist a finite number of points $y_1, \dots, y_m \in \boldsymbol{T}^3$ such that $y_1 = y$, $y_m = y'$, and for any m positive numbers $\varepsilon_i > 0$, $1 \leqslant i \leqslant m$, we can take subsets J_i, $1 \leqslant \forall i \leqslant m$, either $J_i \subset S_0^1$ or $J_i \subset S_0^2$, such that if $J_i \subset S_0^l$, $l = 1, 2$,

$$\int_{J_i} 1\,\mathrm{d}q_0^l(z) > 0, \quad y_i + \beta_l(z) \in B_{\varepsilon_i}(y_{i+1}), \quad \forall z \in J_i,$$

for any $1 \leqslant i \leqslant m$.

Theorem 5.1. *Let $u \in \mathrm{USC}(\mathbb{R}^3)$ be a viscosity subsolution of (5.6). Assume that β_l, $l = 1, 2$, satisfy (1.4), that $\mathrm{d}q_0^l$, $l = 1, 2$, satisfy (1.5) and condition* (B′)*, and that (2.1) holds. If u attains a maximum at $\bar{y}$ in $\boldsymbol{T}^3$, then u is constant in $\boldsymbol{T}^3$.*

The proof of theorem 5.1 is similar to theorem 2.1, which we do not reproduce here. By using theorem 5.1, the existence of the unique number $d_{I',I''}$ in (5.5) can be shown by using a similar argument in § 3. In this way, we can define

$$\bar{I}(I', I'') = -d_{I',I''}, \quad \forall (I', I'') \in \mathbb{R}^2.$$

Then, the effective integro-differential equation for $\bar{u} = \lim_{\varepsilon\to 0} u_\varepsilon$ is the following:

$$\bar{u} + \bar{I}(I'[\bar{u}](x), I''[\bar{u}](x)) = 0, \quad x \in \Omega,$$

associated with the Dirichlet condition (1.2), where $I'[\bar{u}](x)$ and $I''[\bar{u}](x)$ are as given previously. This formal argument can be confirmed by the perturbed test function method used in § 4. Since the argument is similar, we just give the outline here.

Appendix A.

In this section, by following [5], we note three types of equivalent definitions of the viscosity solutions for a class of integro-differential equations, which includes (1.1). The comparison and the existence of viscosity solutions in this framework can be found in [1, 3, 8, 10, 11] and the references therein. The equivalence of these definitions was shown in [5]. We consider the following problem:

$$\begin{aligned} &F(x, u(x), \nabla u(x), \nabla^2 u(x)) \\ &\quad - \int_{\boldsymbol{R}^M} [u(x+\beta(z)) - u(x) - \mathbf{1}_{|z|\leqslant 1}\langle \beta(z), \nabla u(x)\rangle]\,\mathrm{d}q(z) = 0 \quad \text{in } \Omega, \qquad \text{(A 1)} \end{aligned}$$

where F is a real-valued continuous function defined in $\Omega \times \mathbb{R} \times \mathbb{R}^N \times \boldsymbol{S}^N$, which satisfies the degenerate ellipticity (see [13] for this concept). We say that, for $u \in \mathrm{USC}(\mathbb{R}^N)$ (respectively, $\mathrm{LSC}(\mathbb{R}^N)$), $(p, X) \in \mathbb{R}^N \times \boldsymbol{S}^{\boldsymbol{N}}$ is a superdifferential (respectively, subdifferential) of u at $x \in \Omega$ if, for any small $\mu > 0$, there exists $\nu > 0$ such that the following holds:

$$u(x+z) - u(x) \leqslant \langle p, z\rangle + \tfrac{1}{2}\langle Xz, z\rangle + \mu|z|^2, \quad \forall |z| \leqslant \nu,\ z \in \mathbb{R}^N,$$

and, respectively,

$$u(x+z) - u(x) \geqslant \langle p, z\rangle + \tfrac{1}{2}\langle Xz, z\rangle - \mu|z|^2, \quad \forall |z| \leqslant \nu,\ z \in \mathbb{R}^N,$$

We denote the set of all subdifferentials (respectively, superdifferentials) of $u \in \mathrm{USC}(\mathbb{R}^N)$ (respectively, $\mathrm{LSC}(\mathbb{R}^N)$) at $x \in \Omega$ by $J^{2,+}_{\Omega} u(x)$ (respectively, $J^{2,-}_{\Omega} u(x)$). We say that $(p, X) \in \mathbb{R}^N \times \boldsymbol{S}^N$ belongs to $\bar{J}^{2,+}_{\Omega} u(x)$ (respectively, $\bar{J}^{2,-}_{\Omega} u(x)$), if there exist a sequence of points $x_n \in \Omega$ and $(p_n, X_n) \in J^{2,+}_{\Omega} u(x_n)$ (respectively, $J^{2,-}_{\Omega} u(x_n)$) such that

$$\lim_{n\to\infty} x_n = x, \qquad \lim_{n\to\infty} (p_n, X_n) = (p, X).$$

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From (1.4), for $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$), if $(p,X) \in J_\Omega^{2,+}u(x)$ ($(p,X) \in J_\Omega^{2,-}u(x)$), we can take a pair of positive numbers (ν,μ) such that

$$
\begin{aligned}
&u(x+\beta(z)) - u(x) \leqslant \langle p,\beta(z)\rangle + \tfrac{1}{2}\langle X\beta(z),\beta(z)\rangle + \mu|\beta(z)|^2, \quad \forall |z| \leqslant \nu,\ z \in \mathbb{R}^M,\\
&(u(x+\beta(z)) - u(x) \geqslant \langle p,\beta(z)\rangle + \tfrac{1}{2}\langle X\beta(z),\beta(z)\rangle - \mu|\beta(z)|^2, \quad \forall |z| \leqslant \nu,\ z \in \mathbb{R}^M).
\end{aligned}
\tag{A 2}
$$

Definition A.1 (Arisawa [2]). Let $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$). We say that u is a viscosity subsolution (supersolution) of (A 1) if, for any $\hat{x} \in \Omega$, any $(p,X) \in J_\Omega^{2,+}u(\hat{x})$ ($(p,X) \in J_\Omega^{2,-}u(\hat{x})$), and any pair of numbers (ν,μ) satisfying (A 2), the following holds:

$$
\begin{aligned}
F(\hat{x},u(\hat{x}),p,X) &- \int_{|z|<\nu} \tfrac{1}{2}\langle (X+2\mu I)\beta(z),\beta(z)\rangle \,\mathrm{d}q(z)\\
&- \int_{|z|\geqslant\nu} [u(\hat{x}+\beta(z)) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle \beta(z),p\rangle] \,\mathrm{d}q(z) \leqslant 0\\
\biggl(F(\hat{x},u(\hat{x}),p,X) &- \int_{|z|<\nu} \tfrac{1}{2}\langle (X-2\mu I)\beta(z),\beta(z)\rangle \,\mathrm{d}q(z)\\
&- \int_{|z|\geqslant\nu} [u(\hat{x}+\beta(z)) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle \beta(z),p\rangle] \,\mathrm{d}q(z) \geqslant 0\biggr).
\end{aligned}
$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Definition A.2 (Barles [11], Barles and Imbert [10], Jacobsen and Karlsen [17]). Let $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$). We say that u is a viscosity subsolution (supersolution) of (A 1) if, for any $\hat{x} \in \Omega$, and for any $\phi \in C^2(\mathbb{R}^N)$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a maximum (minimum) at $\hat{x}$, the following holds:

$$
\begin{aligned}
&F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))\\
&\qquad - \int_{\mathbb{R}^M} [\phi(\hat{x}+\beta(z)) - \phi(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle \beta(z),\nabla\phi(\hat{x})\rangle] \,\mathrm{d}q(z) \leqslant 0\\
&\biggl(F(\hat{x},u(\hat{x}),\nabla\phi(\hat{x}),\nabla^2\phi(\hat{x}))\\
&\qquad - \int_{\mathbb{R}^M} [\phi(\hat{x}+\beta(z)) - \phi(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle \beta(z),\nabla\phi(\hat{x})\rangle] \,\mathrm{d}q(z) \geqslant 0\biggr).
\end{aligned}
$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Definition A.3 (Arisawa [5]). Let $u \in \mathrm{USC}(\mathbb{R}^N)$ ($\mathrm{LSC}(\mathbb{R}^N)$). We say that u is a viscosity subsolution (supersolution) of (A 1) if, for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbb{R}^N)$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (respectively, minimum) at $\hat{x}$:

$$
h(z) = u(\hat{x}+z) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle \beta(z),\nabla\phi(\hat{x})\rangle \in L^1(\boldsymbol{R}^M,\mathrm{d}q(z))
$$

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and

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z\in\boldsymbol{R}^M} [u(\hat{x}+\beta(z)) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle\beta(z), \nabla\phi(\hat{x})\rangle]\,\mathrm{d}q(z) \leqslant 0,$$

$$\biggl(F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z\in\boldsymbol{R}^M} [u(\hat{x}+\beta(z)) - u(\hat{x}) - \mathbf{1}_{|z|\leqslant 1}\langle\beta(z), \nabla\phi(\hat{x})\rangle]\,\mathrm{d}q(z) \geqslant 0\biggr).$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

THEOREM A.4. *Definitions A.1, A.2 and A.3 are equivalent.*

The claim was proved for the case $M = N$ and $\beta(z) = z$ in [5], and for the case β depending also on $x \in \mathbb{R}^N$ in [8]. The present case is contained in [8], which is similar to [5]. Thus, we do not reproduce the proof here.

We next modify the above definitions to treat the following:

$$F(x, u(x), \nabla u(x), \nabla^2 u(x)) - \int_{z\in\boldsymbol{R}^M} [u(x+\beta(z)) - u(x)]\,\mathrm{d}q(z) = 0 \quad \text{in } \Omega, \tag{A 3}$$

which includes (1.3), where $\mathrm{d}q(z)$ satisfies (1.5) with $\gamma = 1$. Note that, from (1.4), for $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$), if $(p, X) \in J^{2,+}_{\Omega}u(x)$ ($(p, X) \in J^{2,-}_{\Omega}u(x)$), for any $\mu > 0$, we can take $\nu > 0$ such that

$$u(x+\beta(z)) - u(x) \leqslant \langle p, \beta(z)\rangle + \mu|\beta(z)|^2, \quad \forall |z| \leqslant \nu,\ z \in \mathbb{R}^M$$

$$(u(x+\beta(z)) - u(x) \geqslant \langle p, \beta(z)\rangle - \mu|\beta(z)|^2, \quad \forall |z| \leqslant \nu,\ z \in \mathbb{R}^M). \tag{A 4}$$

DEFINITION A.5. Let $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$). We say that u is a viscosity subsolution (supersolution) of (A 3) if, for any $\hat{x} \in \Omega$, any $(p, X) \in J^{2,+}_{\Omega}u(\hat{x})$ ($(p, X) \in J^{2,-}_{\Omega}v(\hat{x})$), and any pair of positive numbers (ν, μ) satisfying (A 4), the following holds:

$$F(\hat{x}, u(\hat{x}), p, X) - \int_{|z|<\nu} \langle p + \mu\beta(z), \beta(z)\rangle\,\mathrm{d}q(z) - \int_{|z|\geqslant\nu} [u(\hat{x}+\beta(z)) - u(\hat{x})]\,\mathrm{d}q(z) \leqslant 0$$

$$\biggl(F(\hat{x}, u(\hat{x}), p, X) - \int_{|z|<\nu} \langle p - \mu\beta(z), \beta(z)\rangle\,\mathrm{d}q(z) - \int_{|z|\geqslant\nu} [u(\hat{x}+\beta(z)) - u(\hat{x})]\,\mathrm{d}q(z) \geqslant 0\biggr).$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Definition A.6. Let $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$). We say that u is a viscosity subsolution (supersolution) of (A 3) if, for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbb{R}^N)$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a maximum (minimum) at $\hat{x}$, and for any $\nu > 0$,

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{\mathbb{R}^M} [\phi(\hat{x} + \beta(z)) - \phi(\hat{x})] \, \mathrm{d}q(z) \leqslant 0$$

$$\left(F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{\mathbb{R}^M} [\phi(\hat{x} + \beta(z)) - \phi(\hat{x})] \, \mathrm{d}q(z) \geqslant 0\right).$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Definition A.7. Let $u \in \mathrm{USC}(\mathbb{R}^N)$ ($u \in \mathrm{LSC}(\mathbb{R}^N)$). We say that u is a viscosity subsolution (supersolution) of (A 1) if, for any $\hat{x} \in \Omega$ and for any $\phi \in C^2(\mathbb{R}^N)$ such that $u(\hat{x}) = \phi(\hat{x})$ and $u - \phi$ takes a global maximum (minimum) at $\hat{x}$,

$$h(z) = u(\hat{x} + z) - u(\hat{x}) \in L^1(\boldsymbol{R}^M, \mathrm{d}q(z)),$$

and

$$F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z \in \boldsymbol{R}^M} [u(\hat{x} + \beta(z)) - u(\hat{x})] \, \mathrm{d}q(z) \leqslant 0$$

$$\left(F(\hat{x}, u(\hat{x}), \nabla\phi(\hat{x}), \nabla^2\phi(\hat{x})) - \int_{z \in \boldsymbol{R}^M} [u(\hat{x} + \beta(z)) - u(\hat{x})] \, \mathrm{d}q(z) \geqslant 0\right).$$

If u is both a viscosity subsolution and a viscosity supersolution, it is called a viscosity solution.

Theorem A.8. *Definitions A.5, A.6 and A.7 are equivalent.*

Proof. Theorem A.8 can be proved in the same way as [5]. We omit it here to avoid redundancy. □

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