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Global and exponential attractor of the repulsive Keller–Segel model with logarithmic sensitivity†

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We consider a Keller–Segel model that describes the cellular chemotactic movement away from repulsive chemical subject to logarithmic sensitivity function over a confined region in $\mathbb{R}^n$, $n \leq 2$. This sensitivity function describes the empirically tested Weber–Fecher's law of living organism's perception of a physical stimulus. We prove that, regardless of chemotaxis strength and initial data, this repulsive system is globally well-posed and the constant solution is the global and exponential in time attractor. Our results confirm the 'folklore' that chemorepulsion inhibits the formation of non-trivial steady states within the logarithmic chemotaxis model, hence preventing cellular aggregation therein.

Key words: Global existence, chemorepulsion, Keller–Segel, logarithmic sensitivity, exponential decay

2020 Mathematics Subject Classification: 35B35, 35K51, 35K57, 92C17

1 Introduction

Chemotaxis is the mechanism by which unicellular or multicellular organisms direct their movements in response to a chemical stimulus gradient in the environment. Bacterial chemotaxis was first documented by T. Engelmann [14] and W. Pfeffer [39] in the 1880s, and thorough quantitative and biochemical studies were started by J. Adler on *Escherichia coli* [1, 2] in the 1960s. Over one century's research has illustrated the importance of chemotaxis in many physiological processes, such as the recruitment of inflammatory cells to sites of injury or infection, cell–cell interactions in the immune system, the development and organisation of tissues and organs during embryogenesis, progression and metastasis in many diseases and operation in each crucial step of tumour cell dissemination. See the reviews in Refs. [20, 42].

Chemotactic bacteria, such as *E. coli*, typically have 4–10 flagella per cell, and they can help the bacterium to swim in a straight line or tumble in place. In an environment absence of or with a uniform chemical stimulus, *E. coli* 'runs' in a straight line for several seconds and then it

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'tumbles' or 'twiddles' (abruptly changes its direction) for a fraction of a second; then it again swims in a straight line, but in a new and randomly chosen direction [1]. This movement looks like a random walk with relatively straight swims interrupted by random tumbles that reorient the bacterium.

In the presence of an increasing concentration of a chemical attractant, the bacterium will direct its overall motion based on the attractant gradient by tumbling less frequently and running longer compared with the unstimulated state. If the bacterium senses the environment's improvement, it will keep swimming in a directed and straight line for a longer time before tumbling, while it will tumble sooner and try a new direction at random when the chemical concentration decreases.

1.1 Keller–Segel model

To model bacterial movement due to random noise and chemical stimulus in the environment, E. Keller and L. Segel proposed in the 1970s [24, 25, 26] continuum models to describe the spatial-temporal change of the cell population distribution and chemical concentration. The general form of Keller–Segel model consists of two strongly coupled parabolic equations of $(u, v) = (u(x,t), v(x,t))$ in the following system:

$$\begin{cases} u_t = \nabla \cdot (\overbrace{\mu \nabla u}^{\text{random (flux)}} - \overbrace{\chi u \phi(v) \nabla v}^{\text{chemotactic (flux)}}), & x \in \Omega, t > 0, \\ v_t = \overbrace{d \Delta v}^{\text{chemical diffusion}} + \overbrace{-\alpha v + \beta u}^{\text{chemical creation/consumption}}, & x \in \Omega, t > 0, \end{cases} \tag{1.1}$$

given non-negative initial data $u(x,0), v(x,0) \geq, \not\equiv 0$ over a spatial region Ω, which is usually taken to be the whole space $\mathbb{R}^n$, $n \geq 1$, or a bounded domain with additional non-flux boundary conditions imposed on u and v in an enclosed environment. Here, (u, v) denotes the cellular population density and chemical concentration at space–time location (x, t). Cell motility ($\mu > 0$), chemotaxis rate ($\chi > 0$) and chemical diffusion ($d > 0$) are assumed to be constants. Function ϕ measures the change of chemotactic sensitivity due to the variation of chemical concentration.

Although bacteria may behave independently, their populations exhibit collective behaviours. One of the most impressive experimental findings in bacterial chemotaxis is the self-organised cellular aggregation that during starvation, initially evenly distributed cells release a diffusing chemical to attract each other and then group into one or several small regions of space. The Keller–Segel model, premised on the simplest possible assumptions in 1D, admits solutions such that cells *aggregate in several 'collecting points' or centres. At each centre, a slug forms, migrates and eventually forms a multicellular fruiting body* [26]. Moreover, this intuitively simple model can be used to model the pulsing cellular movement or traveling bands of the bacteria when placed in one end of a capillary tube containing oxygen and an energy source. Therefore, system (1.1) has quickly achieved an enormous academic success after the proposal, and it now serves as one of the foundation stones to model chemotactic movement.

The choice of sensitivity function $\phi(v)$ depends on the biological significance that one tries to model, and one limitation on bacterial ability to follow chemical gradients is imposed by saturation of the sensory system at high ligand concentrations, in light of which $\phi(v) = \ln v$ is a natural candidate. This is the case, for example, in the sensory system of human beings according to the empirically tested Weber–Fechner law[1] which states that people feel the logarithmic value of the magnitude of a physical stimulus. Experiments studying the density distribution of the swimming bacteria subject to the different spatial ligand profiles [5, 11, 23] suggest that bacteria also obey this law and the bacterial chemotactic sensitivity is proportional to the change of the logarithmic ligand concentration. We would like to mention that the singularity of $\ln v$ at $v = 0$ brings challenges and complexity to the theoretical analysis of (1.1); however, Refs. [26, 53] analytically argued that this is indispensable for the formation of its traveling waves.

There are two well-accepted methodologies to model the aforementioned cellular aggregation phenomenon. The first one is to show that the time-dependent system 'blows up' within finite or infinite time, with cellular population density u collapsing into a single or a combination of several δ functions, plus a regular part to tune the v-equation [34, 9, 19, 21]. Needless to say, the concentrating structure of $\delta(x)$ can serve as a natural candidate to describe the cellular aggregation; however, the blow-up profile suggests that a population density approaches infinity at single points, which is unrealistic from the viewpoint of mathematical modelling. It also brings challenges to the theoretical and numerical analysis of the aggregation close to the collapsing time and the model itself loses the rationale after the blow-up time.

An alternative method is to show that the time-dependent model prohibits blow-up and converges to a stationary system in a long time. Then, the stationary solutions with spatial heterogeneity, in particular, accumulating or concentrating structures, can be used to model the cellular aggregation. This approach was originally adopted by Keller and Segel [24] in 1D with homogeneous Neumann boundary conditions, which established a necessary condition, as the constant solution turns unstable, for the formation of stable spatially heterogeneous solutions. Schaaf [41] analysed the stationary solutions of system (1.1) over higher dimensions via the Crandall–Rabinowitz bifurcation techniques [10, 40]. In particular, she considered the model with generally non-linear diffusion and sensitivity functions and provided a criterion for the emergence of stable nonhomogeneous aggregation patterns as small perturbations from the constant solution. There had been no analytical results on the stationary solution to system (1.1) with large amplitude (i.e., away from the constant solution) until those of Lin, Ni, and Takagi around the 1990s [30, 36, 37].

Lin, Ni and Takagi [30] considered the stationary system of problem (1.1) with logarithmic sensitivity over a multi-dimensional bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 1$

$$\begin{cases} \nabla \cdot (\mu \nabla u - \chi u \nabla \ln v) = 0, & x \in \Omega, \\ d\Delta v - v + u = 0, & x \in \Omega, \\ \partial_\nu u = \partial_\nu v = 0, & x \in \partial\Omega, \\ \int_\Omega u = \int_\Omega v = \bar{u}|\Omega|, \end{cases} \tag{1.2}$$

where ν is the unit outer normal on $\partial\Omega$, and $\bar{u}$ is the fixed average population density out of the conserved total population $\int_\Omega u(x,t)dx = \int_\Omega u_0(x)dx = \bar{u}|\Omega|$ in the time-dependent system. By the maximum principles, one can easily show that any non-negative solution (u, v) of (1.2)

[1]This law was named after Ernst Heinrich Weber (1795—1878) who first quantitatively studied the human response to a physical stimulus published in 1860 in *Elemente der Psychophysik*, and Gustav Theodor Fechner (1801—1887) who later offered an elaborate theoretical interpretation of Weber's findings.

must be strictly positive in $\bar{\Omega}$. Multiplying the u-equation by $\mu \ln u - \chi \ln v$ and then integrating it over Ω by parts, one has

$$\int_\Omega u|\nabla(\mu \ln u - \chi \ln v)|^2 dx = 0,$$

which readily implies that $\mu \ln u - \chi \ln v$ equals some constant in Ω, or $u = Cv^p$, $p := \frac{\chi}{\mu}$, $C \in \mathbb{R}^+$. Therefore, one converts solving system (1.2) into finding solutions of the following Neumann boundary value problem:

$$\begin{cases} d\Delta w - w + w^p = 0, & x \in \Omega, \\ \partial_\nu w = 0, & x \in \partial\Omega, \end{cases} \tag{1.3}$$

because if w is a positive solution of (1.3), the pair (u, v) given by

$$u := \left(\frac{1}{\bar{u}|\Omega|}\int_\Omega w(x)^p dx\right)^{-1} w^p, \; v := \left(\frac{1}{\bar{u}|\Omega|}\int_\Omega w(x) dx\right)^{-1} w \tag{1.4}$$

is a solution of system (1.2).

Assuming that $p \in (1, \infty)$ for $n = 1, 2$ and $p \in (1, (n+2)/(n-2))$ for $n \geq 3$, they proved in Ref. [30] that, equation (1.3) has only constant solution if d is large, and it admits non-trivial solutions if d is small, which are critical points of the energy functional

$$\mathcal{J}_d(w) := \frac{1}{2}\int_\Omega \left(d|\nabla w|^2 + w^2\right) dx - \frac{1}{p+1}\int_\Omega w_+^{p+1} dx,$$

in certain Sobolev space. Moreover, they proceeded to prove in Refs. [36, 37] that, if d is sufficiently small, then the least energy solution w_d must achieve its unique local (hence global) maximum at a single boundary point $x_d \in \partial\Omega$; furthermore, as $d \to 0^+$, $x_d \to x_0 \in \partial\Omega$, where the mean curvature of the boundary achieves its maximum. Since then equation (1.3) has been extensively studied by various authors, and we refer the reader to Refs. [35, 13, 12, 18, 31, 46] and references therein for its recent development. We want to mention that this approach heavily depends on the smallness of the chemical diffusion rate d and hence requires a primitive understanding of the 'ground-state' of the counterpart of equation (1.3) in the whole space. Moreover, this method is not applicable in general when cellular growth is considered since system (1.2) plus a cell growth cannot be converted into a single equation anymore.

Wang [51] initiated a completely different approach to directly tackle this model in 1D without converting it into a single equation. With the aid of the global bifurcation theories [40, 38, 43], this technique is further developed and successfully applied to a wide class of Keller–Segel models in Refs. [8, 51, 52]. They take χ as a bifurcation parameter and show that the first bifurcation branch must extend to right infinity without intersecting with the χ–axis, which implies the existence of non-constant steady states whenever χ surpasses a critical threshold value, given explicitly in terms of system parameters. Moreover, by Helly's compactness theorem, they obtained the spiky transition layer structures of the steady states when the chemotaxis rate is large (compared to the cell motility rate). The stability and dynamics of these spiky solutions are investigated in Refs. [7, 55]. We would like to mention that Li adopted this approach in Ref. [29] to investigate the steady states of (1.2) in 1D and obtained a refined asymptotic profile of this spiky structure for $u(x)$ as $\chi \to \infty$. In contrast to Ref. [30], this bifurcation method can tackle (1.1) and a wide class of its variants with cellular growth [17, 48, 49, 50].

1.2 Motivations and main result

Motile bacteria are attracted by certain chemicals, and they are repelled by others and can move away from unfavourable environments. This process is called negative chemotaxis or chemorepulsion, and it was discovered along with positive chemotaxis. For example, *E. coli* cells swim towards amino acids, sugars and oxygen, but away from potentially noxious chemicals, such as alcohols and fatty acids. For repellents, bacteria encounter an increasing concentration tumble more often, while a decreasing concentration suppresses tumbling. In 1974, Tso and Adler [45] proposed several methods to detect or measure and study negative chemotaxis in *E. coli* and its mechanism. Their experiments showed that although most of the repellents are harmful compounds, they demonstrated that harmfulness is neither necessary nor sufficient to make a compound a repellent. Repellents at very low concentrations are not repellents, and attractants at very high concentrations are not attractants. Moreover importantly, *in vivo* experiments suggest no aggregation in chemorepulsion systems.

In this paper, we perform theoretical studies to confirm the experimental findings of chemorepulsion model with logarithmic sensitivity function by considering the following system:

$$\begin{cases} u_t = \nabla\cdot(\mu\nabla u + \chi u\nabla\ln v), & x\in\Omega, t>0,\\ v_t = d\Delta v - v + u, & x\in\Omega, t>0,\\ \partial_\nu u = \partial_\nu v = 0, & x\in\partial\Omega, t>0,\\ u(x,0)=u_0(x)\geq, \not\equiv 0, v(x,0)=v_0(x)>0, & x\in\Omega. \end{cases} \tag{1.5}$$

We are concerned with the effect of chemorepulsion on the pattern formation and spatial-temporal dynamics of this model. Before presenting our main result, let us illustrate the motivations of this piece of work by first studying its steady states, that is, non-negative solutions (u, v) of the following problem:

$$\begin{cases} 0 = \nabla\cdot(\mu\nabla u + \chi u\nabla\ln v), & x\in\Omega,\\ 0 = d\Delta v - v + u, & x\in\Omega,\\ \partial_\nu u = \partial_\nu v = 0, & x\in\partial\Omega,\\ \int_\Omega u = \int_\Omega v = \bar{u}|\Omega|, \end{cases} \tag{1.6}$$

where $\bar{u} = \bar{v} := \frac{1}{|\Omega|}\int_\Omega u$. Similarly as in Ref. [30], we see that (1.6) leads us to the single equation

$$\begin{cases} d\Delta\hat{w} - \hat{w} + \hat{w}^{-p} = 0, x\in\Omega,\\ \partial_\nu\hat{w} = 0, & x\in\partial\Omega, \end{cases} \tag{1.7}$$

and the pair (u, v) given by $\hat{w}$ as in equation (1.4) is a solution of system (1.6). We claim that equation (1.7) has only the constant solution $(\bar{u}, \bar{v})$. To see this, we find that

$$\begin{aligned} d\int_\Omega |\nabla\hat{w}|^2 dx &= \int_\Omega \big(\hat{w}-1\big)\big(-\hat{w}+\hat{w}^{-p}\big)dx\\ &= \int_{\{\Omega|\hat{w}(x)\geq 1\}} \big(\hat{w}-1\big)\big(-\hat{w}+\hat{w}^{-p}\big)dx + \int_{\{\Omega|0<\hat{w}(x)<1\}} \big(\hat{w}-1\big)\big(-\hat{w}+\hat{w}^{-p}\big)dx \leq 0, \end{aligned}$$

hence $\hat{w}\equiv 1$ in Ω and $(\bar{u}, \bar{v})$ is the only solution to (1.6).

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Now that (1.6) has only the constant solution $(\bar{u}, \bar{v})$, a natural question arises as if this trivial solution is the global attractor of the time-dependent chemorepulsive system (1.5) with logarithmic sensitivity. To this end, we need to rule out the possibility of time-periodic dynamics or blow-ups within (1.5). Therefore, in this paper, we study its global dynamics, and in particular, we shall show that $(\bar{u}, \bar{v})$ is its global attractor. Our main result can be stated as follows.

Theorem 1.1 *Let Ω be a bounded convex domain in $\mathbb{R}^n$, $n = 1, 2$, and the constants $\mu, \chi, d > 0$ be arbitrary. Then for any initial data $(u_0, v_0) \in W^{1,\infty}(\Omega) \times W^{2,\infty}(\Omega)$, $u_0 \geq 0, \not\equiv 0$ and $\inf_{x\in\Omega} v_0 > 0$, (1.5) is globally well-posed and its unique solution (u, v) is classical and uniformly bounded in time; moreover, there exist positive constants C and $\delta > 0$ such that*

$$\|u(\cdot, t) - \bar{u}\|_{L^\infty(\Omega)} + \|v(\cdot, t) - \bar{v}\|_{W^{1,\infty}(\Omega)} \leq Ce^{-\delta t}, \forall t > 0. \tag{1.8}$$

This result shows that $(\bar{u}, \bar{v})$ stabilises system (1.5) and chemorepulsion inhibits the formation of any non-trivial patterns to the Keller–Segel model (1.5).

Remark 1.1 *We would like to mention that global well-posedness of the chemoattraction model has been investigated by various authors in Refs. [16, 27, 28, 47]. In summary, these attraction models are globally well-posedness if χ is not too large. No blow-up in models with logarithmic sensitivity is known to the best of our knowledge.*

Since we are concerned with the qualitative large-time dynamics in (1.5), throughout this paper, we denote C as a generic time-independent constant that may vary from line to line with confusing the reader.

2 Local existence and preliminary results

To study the spatial-temporal dynamics of (1.5), we first establish its global well-posedness, while its local well-posedness and extension criterion are more or less standard in the literature thanks to the well-established theories of Amann [3]: Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$. Then for any initial data $(u_0, v_0) \in C^0(\bar{\Omega}) \times W^{1,p}(\Omega)$, $p > n$, system (1.5) has a unique solution $(u(x, t), v(x, t))$ defined on $\bar{\Omega} \times [0, T_{\max})$ with $0 < T_{\max} \leq \infty$ such that $(u(\cdot, t), v(\cdot, t)) \in C^0(\bar{\Omega} \times [0, T_{\max})) \times C^0(\bar{\Omega} \times [0, T_{\max}))$ and $(u, v) \in C^{2,1}(\bar{\Omega} \times (0, T_{\max})) \times C^{2,1}(\bar{\Omega} \times (0, T_{\max}))$. If $\sup_{s\in(0,t)} \|(u, v)(\cdot, s)\|_{L^\infty}$ is bounded for $t \in (0, T_{\max})$, then $T_{\max} = \infty$, that is, (u, v) is a global solution to (1.5). Furthermore, (u, v) is a classical solution such that, $(u, v) \in C^\alpha((0, \infty), C^{2(1-\beta)}(\bar{\Omega}) \times C^{2(1-\beta)}(\bar{\Omega}))$ for any $0 \leq \alpha \leq \beta \leq 1$.

As we mentioned earlier, the singularity of $\ln v$ at $v = 0$ in (1.5) brings a challenge to its theoretical analysis; however, one can apply the maximum principle to show that $v(x, t)$ has a positive lower bound for any $t > 0$ as follows.

Lemma 2.1 (Lemma 2.2 in Refs. [15, 27]) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded domain with smooth boundary $\partial\Omega$. Suppose that $u_0(x) \geq, \not\equiv 0$ and $\inf_{x\in\Omega} v_0 > 0$, then there exists $\eta > 0$ such that*

$$v(x, t) \geq \eta > 0 \text{ for all } (x, t) \in \bar{\Omega} \times [0, T_{\max}).$$

We next estimate $\|\nabla v\|_{L^q}$ by $\|u\|_{L^p}$ for $p > 1$ through the v-equation. To this end, we shall employ the well-known smoothing properties of the operator $-\Delta + 1$ and embeddings between the semi-groups generated by $\{e^{t\Delta}\}_{t\geq 0}$. Applying ∇ to the v-equation in (1.5), we have the following result.

Lemma 2.2 *Let (u, v) be the solution of (1.5) over $(0, T_{\max})$, $T_{\max} \leq \infty$. There exists a positive constant C depending on $\|\nabla v_0\|_{L^q(\Omega)}$ and $|\Omega|$ such that*

$$\|v(\cdot, t)\|_{W^{1,q}} \leq C\Big(1 + \sup_{s\in(0,t)} \|u(\cdot, s)\|_{L^p}\Big), \forall t \in (0, T_{\max}), \tag{2.1}$$

where $q \in [1, \frac{np}{n-p})$ if $p \in [1, n)$, $q \in [1, \infty)$ if $p = n$ and $q = \infty$ if $p > n$.

Proof Write v-equation into the following abstract form:

$$v(\cdot, t) = e^{d(\Delta-1)t}v_0 + \int_0^t e^{d(\Delta-1)(t-s)}u(\cdot, s)ds. \tag{2.2}$$

Applying Lemma 1.3 of Ref. [54] on (2.2) for $1 \leq p, q \leq \infty$, we find that there exists a positive constant C such that

$$\|v(\cdot, t)\|_{W^{1,q}} \leq C \left(1 + \int_0^t e^{-d\nu_1(t-s)}(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}\|u(\cdot, s)\|_{L^p} ds\right), \tag{2.3}$$

where ν_1 is the first Neumann eigenvalue of $-\Delta$. On the other hand, we see that

$$\sup_{t\in(0,\infty)} \int_0^t e^{-d\nu_1(t-s)}(t-s)^{-\frac{1}{2}-\frac{n}{2}(\frac{1}{p}-\frac{1}{q})}ds < \infty,$$

with $q \in [1, \frac{np}{n-p})$ if $p \in [1, n)$, $q \in [1, \infty)$ if $p = n$ and $q = \infty$ if $p > n$, then (2.1) follows from (2.3). □

Let us conclude this section by recording the well-known Gagliardo–Nirenberg inequality and its fractional variant in the following two lemmas for future reference.

Lemma 2.3 (e.g., Lemma 2.5 in Ref. [27]) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded smooth domain. Let $j \geq 0$, $k \geq 0$ be integers and $p, q, r, s > 1$. There is a constant $C > 0$ such that for any function $w \in L^q(\Omega) \cap L^s(\Omega)$ with $D^k w \in L^r(\Omega)$,*

$$\|D^j w\|_{L^p(\Omega)} \leq C\|D^k w\|^{\alpha}_{L^r(\Omega)}\|w\|^{1-\alpha}_{L^q(\Omega)} + C\|w\|_{L^s(\Omega)},$$

whenever $\frac{1}{p} = \frac{j}{n} + (\frac{1}{r} - \frac{k}{n})\alpha + \frac{1-\alpha}{q}$ and $\frac{j}{k} \leq \alpha < 1$.

Lemma 2.4 (e.g., Lemma 2.5 in Ref. [22]) *Let $\Omega \subset \mathbb{R}^n$, $n \geq 1$, be a bounded smooth domain. Let $q, s \geq 1$, $p > 0$ and $\alpha \in (0, 1)$. There is a constant $C > 0$ such that for any function $w \in W^{p,2}(\Omega) \cap L^{\frac{s}{q}}(\Omega)$*

$$\|w\|_{W^{p,2}(\Omega)} \leq C\|\nabla w\|^{\alpha}_{L^2(\Omega)}\|w\|^{1-\alpha}_{L^{\frac{s}{q}}(\Omega)} + C\|w\|_{L^{\frac{s}{q}}(\Omega)},$$

whenever $\frac{1}{2} - \frac{p}{n} = (\frac{1}{2} - \frac{1}{n})\alpha + (1-\alpha)\frac{q}{s}$ and $p \leq \alpha < 1$.

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3 Lyapunov functional and global well-posedness

The main vehicle in the proof of our main results is the following energy functional

$$\mathcal{F}(u,v) := \int_\Omega u \ln\Big(\frac{u}{\bar{u}}\Big)dx + 2\chi \int_\Omega |\nabla\sqrt{v}|^2 dx, \, t \in (0, T_{\max}), \tag{3.1}$$

which is non-increasing along the solution trajectories of (1.5). To verify this for (3.1), we give the following important result to begin with.

Lemma 3.1 (Lemma 3.4 in Ref. [27]) *For any positive $w \in C^2(\bar{\Omega})$ satisfying $\partial_\nu w = 0$ on $\partial\Omega$,*

$$\int_\Omega \Big(\frac{|\Delta w|^2}{w} - \frac{1}{2}\frac{|\nabla w|^2}{w^2}\Delta w\Big)dx = \int_\Omega w|D^2 \ln w|^2 dx - \frac{1}{2}\int_{\partial\Omega} \frac{\partial_\nu |\nabla w|^2}{w} dS; \tag{3.2}$$

moreover, if the bounded domain Ω is convex, then

$$\int_\Omega \Big(\frac{|\Delta w|^2}{w} - \frac{1}{2}\frac{|\nabla w|^2}{w^2}\Delta w\Big)dx \geq 0. \tag{3.3}$$

Proof This Lemma follows from Lemma 3.4 in Ref. [27] with slight modifications made for our later reference and we represent its verification here. We shall only prove (3.2), and (3.3) follows from the fact that $\partial_\nu |\nabla w|^2 \leq 0$ on $\partial\Omega$ if Ω is convex [35]. To this end, we calculate the pointwise identity

$$\begin{aligned} w|D^2 \ln w|^2 =& w\Big|D\Big(\frac{\nabla w}{w}\Big)\Big|^2 \\ =& w\Big| -\frac{1}{w^2}\nabla w(\nabla w)^T + \frac{1}{w}D^2 w\Big|^2 \\ =& \frac{|\nabla w|^4}{w^3} + \frac{|D^2 w|^2}{w} - \frac{2(D^2 w \nabla w)\cdot \nabla w}{w^2} \\ =& \frac{|\nabla w|^4}{w^3} + \frac{|D^2 w|^2}{w} - \frac{\nabla |\nabla w|^2 \cdot \nabla w}{w^2}, \end{aligned} \tag{3.4}$$

as in the proof of Lemma 3.2 in Ref. [27]. Integrating (3.4) over Ω by parts with 'dx' skipped, we have

$$\begin{aligned} \int_\Omega w|D^2 \ln w|^2 =& \int_\Omega \frac{|\nabla w|^4}{w^3} + \int_\Omega \frac{|D^2 w|^2}{w} \overbrace{- \int_\Omega \frac{\nabla |\nabla w|^2 \cdot \nabla w}{w^2}}^{I} \\ =& \int_\Omega \frac{|\nabla w|^4}{w^3} + \int_\Omega \frac{|D^2 w|^2}{w} \overbrace{+ \int_\Omega \frac{|\nabla w|^2}{w^2}\Delta w - 2\int_\Omega \frac{|\nabla w|^4}{w^3}}^{\text{integration by parts from } I \text{ using } \partial_\nu w = 0 \text{ on } \partial\Omega}. \end{aligned} \tag{3.5}$$

One can obtain by straightforward calculations that

$$-\int_\Omega \frac{|\Delta w|^2}{w} = -\int_\Omega \frac{|D^2 w|^2}{w} - \frac{3}{2}\int_\Omega \frac{|\nabla w|^2 \Delta w}{w^2} + \int_\Omega \frac{|\nabla w|^4}{w^3} + \frac{1}{2}\int_{\partial\Omega} \frac{\partial_\nu |\nabla w|^2}{w}, \tag{3.6}$$

as is in [27]. Combining (3.5) and (3.6) gives us (3.2). □

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3.1 Lyapunov functional

We shall show that the Lyapunov functional in (3.1) is a free energy functional. To this end, we need the following.

Lemma 3.2 *Suppose Ω is a bounded convex domain in $\mathbb{R}^n$, $n=1,2$, and let (u,v) be the solution of (1.5) obtained above, then its energy functional $\mathcal{F}$ given by (3.1) is non-increasing along the solution trajectory and its dissipation rate satisfies*

$$\frac{d}{dt}\mathcal{F}(u,v)(t) \le -\mu\int_\Omega \frac{|\nabla u|^2}{u}dx - \frac{\chi d}{n}\int_\Omega v|\Delta \ln v|^2 dx - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v}dx \le 0, \forall t\in(0,T_{\max}). \tag{3.7}$$

Proof Through direct calculations and integration by parts, we find by using the u-equation

$$\begin{aligned}
\frac{d\mathcal{F}}{dt} &= \int_\Omega \left(\ln\left(\frac{u}{\bar{u}}\right)+1\right)u_t + 4\chi\int_\Omega \nabla\sqrt{v}\cdot(\nabla\sqrt{v})_t \\
&= -\int_\Omega \frac{\nabla u}{u}\cdot\left(\mu\nabla u + \chi\frac{u}{v}\nabla v\right) + 4\chi\int_\Omega \nabla\sqrt{v}\cdot\nabla\left(\frac{v_t}{2\sqrt{v}}\right) \\
&= -\int_\Omega \frac{\nabla u}{u}\cdot\left(\mu\nabla u + \chi\frac{u}{v}\nabla v\right) + 4\chi\int_\Omega \frac{\nabla v}{2\sqrt{v}}\cdot\left(\frac{\nabla v_t}{2\sqrt{v}} - \frac{v_t\nabla v}{4v^{\frac{3}{2}}}\right) \\
&= -\mu\int_\Omega \frac{|\nabla u|^2}{u} - \chi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} + \overbrace{\chi\int_\Omega \frac{\nabla v\cdot\nabla v_t}{v}}^{I_1} \overbrace{-\frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2 v_t}{v^2}}^{I_2},
\end{aligned}$$

which, after we apply the v-equation, leads to

$$\begin{aligned}
\frac{d\mathcal{F}}{dt} &= -\mu\int_\Omega \frac{|\nabla u|^2}{u} - \chi\int_\Omega \frac{\nabla u\cdot\nabla v}{v} + \overbrace{\chi d\int_\Omega \frac{\nabla v\cdot\nabla\Delta v}{v} - \chi\int_\Omega \frac{|\nabla v|^2}{v} + \chi\int_\Omega \frac{\nabla u\cdot\nabla v}{v}}^{\text{substituting } v\text{-equation into } I_1} \\
&\quad \overbrace{-\frac{\chi d}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v + \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v} - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}u}^{\text{substituting } v\text{-equation into } I_2} \\
&= -\mu\int_\Omega \frac{|\nabla u|^2}{u} + \overbrace{\chi d\int_\Omega \frac{\nabla v\cdot\nabla\Delta v}{v}}^{I_3} - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v} - \frac{\chi d}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}u \\
&= -\mu\int_\Omega \frac{|\nabla u|^2}{u} \overbrace{-\chi d\int_\Omega \frac{(\Delta v)^2}{v} + \chi d\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v}^{\text{integration by parts from } I_3} - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v} - \frac{\chi d}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}\Delta v \\
&\quad - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}u \\
&= -\mu\int_\Omega \frac{|\nabla u|^2}{u} - \chi d\int_\Omega \left(\frac{(\Delta v)^2}{v} - \frac{1}{2}\frac{|\nabla v|^2}{v^2}\Delta v\right) - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v} - \frac{\chi}{2}\int_\Omega \frac{|\nabla v|^2}{v^2}u,
\end{aligned}$$

which, after we use Lemma 3.1, becomes

$$\frac{d\mathcal{F}}{dt} = -\mu\int_\Omega \frac{|\nabla u|^2}{u} - \chi d\int_\Omega v|D^2\ln v|^2 + \frac{\chi d}{2}\overbrace{\int_{\partial\Omega}\frac{\partial_\nu|\nabla v|^2}{v}dS}^{\leq 0 \text{ thanks to convex domain}}$$
$$-\frac{\chi}{2}\int_\Omega\frac{|\nabla v|^2}{v} - \frac{\chi}{2}\overbrace{\int_\Omega\frac{|\nabla v|^2}{v^2}u}^{\geq 0}$$
$$\leq -\mu\int_\Omega\frac{|\nabla u|^2}{u} - \overbrace{\frac{\chi d}{n}\int_\Omega v|\Delta\ln v|^2}^{\text{thanks to } \frac{1}{n}|\Delta\ln v|^2\leq|D^2\ln v|^2} - \frac{\chi}{2}\int_\Omega\frac{|\nabla v|^2}{v},$$

from which (3.7) readily follows. □

Corollary 1 *There exist a positive constant C such that the classical solution of (1.5) satisfies*

$$\int_0^t\int_\Omega\left\{\frac{|\nabla u|^2}{u} + v(\Delta\ln v)^2\right\}dxds + \int_\Omega\left(u\ln u + |\nabla v|^2\right)dx \leq C, \text{ for all } t\in(0,T_{\max}). \tag{3.8}$$

Proof First of all, the boundedness of $\int_\Omega u\ln u$ readily follows from the fact that $\mathcal{F}(u,v)\leq\mathcal{F}(u_0,v_0)$. To prove for the rest, we integrate (3.7) over $(0,t)$ for any $t\in(0,T_{\max})$ and have

$$2\chi\int_\Omega|\nabla\sqrt{v}|^2 + \mu\int_0^t\int_\Omega\frac{|\nabla u|^2}{u} + \frac{\chi d}{n}\int_0^t\int_\Omega v(\Delta\ln v)^2 \leq \mathcal{F}(u_0,v_0) - \int_\Omega u\ln\left(\frac{u}{\bar{u}}\right).$$

Since $\int_\Omega u\ln\bar{u}$ is a constant and $-\xi\ln\xi\leq\frac{1}{e}$ for all $\xi>0$, there exists a constant $C>0$ such that

$$\int_0^t\int_\Omega\left\{\frac{|\nabla u|^2}{u} + v(\Delta\ln v)^2\right\}dxds \leq C, \forall t\in(0,T_{\max}).$$

Therefore, we are left to show the boundedness of $\int_\Omega|\nabla v|^2dx$ to finish this proof. We test the v-equation against $-\Delta v$ over Ω and by using the Cauchy's inequality to obtain

$$\frac{1}{2}\frac{d}{dt}\int_\Omega|\nabla v|^2 \leq -\frac{d}{2}\int_\Omega(\Delta v)^2 - \int_\Omega|\nabla v|^2 + \frac{1}{2d}\int_\Omega u^2, \forall t\in(0,T_{\max}).$$

If we denote $y:=\int_\Omega|\nabla v|^2$ and $f:=\frac{1}{d}\int_\Omega u^2$, then the inequality above can be rewritten as

$$y' + 2y\leq f, \forall t\in(0,T_{\max}),$$

and solving this inequality gives us

$$y(t)\leq y_0e^{-2t} + \int_0^t e^{-2(t-s)}f(s)ds.$$

Hence, it is sufficient to prove that f is integrable in $(0,t)$, or equivalently $\int_0^t\int_\Omega u^2dxds<\infty$ for all $t\in(0,T_{\max})$.

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To prove this, we apply Lemma 2.3 with $j=0$, $k=1$, $q=r=2$, $\alpha=\frac{1}{2}$ and find that there exists a constant $C>0$ such that for any $t\in(0,T_{\max})$

$$\int_0^t\int_\Omega u^2=\int_0^t\|\sqrt{u}\|^4_{L^4(\Omega)}\le\int_0^t\left(C\|\nabla\sqrt{u}\|^{\alpha}_{L^2(\Omega)}\|\sqrt{u}\|^{1-\alpha}_{L^2(\Omega)}+C\|\sqrt{u}\|_{L^2(\Omega)}\right)^4$$
$$=\int_0^t\left(C\left(\int_\Omega\frac{|\nabla u|^2}{4u}\right)^{\frac{1}{4}}\left(\int_\Omega u\right)^{\frac{1}{4}}+C\left(\int_\Omega u\right)^{\frac{1}{2}}\right)^4.$$

We already know from above that both $\int_\Omega udx$ and $\int_0^t\int_\Omega\frac{|\nabla u|^2}{u}dxds$ are bounded, therefore

$$\int_0^t\int_\Omega u^2dxds\le C,\forall t\in(0,T_{\max}),$$

as expected. We finish the proof of (3.8). □

3.2 Global existence and uniform boundedness

To show the global convergence of (1.5) to $(\bar{u},\bar{v})$, it is necessary to prove the global in time existence and uniform boundedness of its solutions as we shall do in this section. The proof in 1D is very simple and that in 2D is also standard thanks to (3.8) or more precisely

$$\int_\Omega\left(u\ln u+|\nabla v|^2\right)dx\le C,\forall t\in(0,T_{\max}).$$

The existence and uniform boundedness of the global solution can be summarised as follows.

Proposition 3.1 *Assume that all the conditions in Theorem 1.1 hold. Then, system (1.5) has a unique global solution (u,v) for all $t\in(0,\infty)$; moreover, this solution is uniformly bounded such that there exists a constant $C>0$*

$$\|u(\cdot,t)\|_{L^\infty}+\|v(\cdot,t)\|_{W^{1,\infty}}\le C,\forall t\le\infty.$$

Proof The proof of the boundedness of u in L^∞ in 2D by the fact that $\int_\Omega u\ln u<C$ for Keller–Segel chemoattraction model is rather standard and has been developed to be a user-friendly Lemma by a few authors (e.g., [4, 33, 32]). The argument and conclusion therein apparently apply to the chemorepulsion model (1.5), and we present the proof here for the sake of completeness.

Step 1: We estimate the L^2 of u by

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega u^2&=-\mu\int_\Omega|\nabla u|^2-\chi\int_\Omega u\nabla u\cdot\nabla\ln v\\&=-\mu\int_\Omega|\nabla u|^2-\chi\int_\Omega u\nabla u\cdot\frac{\nabla v}{v}\\&\le-\mu\int_\Omega|\nabla u|^2+\frac{\chi}{\eta}\int_\Omega u|\nabla u|\cdot|\nabla v|,\end{aligned}\tag{3.9}$$

where the last inequality follows from the fact that $v(x,t)\ge\eta$ for all time $t>0$.

To further estimate in (3.9), we have from Young's inequality that

$$\frac{\chi}{\eta}\int_\Omega u|\nabla u|\cdot|\nabla v|\le\frac{\mu}{2}\int_\Omega|\nabla u|^2+\frac{\chi^2}{2\mu\eta^2}\int_\Omega u^2|\nabla v|^2.$$

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Therefore, we have derived the inequality

$$\frac{d}{dt}\int_\Omega u^2 + \mu\int_\Omega |\nabla u|^2 \le \frac{\chi^2}{\mu\eta^2}\int_\Omega u^2|\nabla v|^2. \tag{3.10}$$

Moreover, thanks to Gagliardo–Nirenberg inequality, one finds that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\int_\Omega u^2 \le \epsilon\int_\Omega |\nabla u|^2 + C_\epsilon. \tag{3.11}$$

Using (3.11), we infer from (3.10) that

$$\frac{d}{dt}\int_\Omega u^2 + \int_\Omega u^2 + \frac{\mu}{2}\int_\Omega |\nabla u|^2 \le \frac{\chi^2}{\mu\eta^2}\int_\Omega u^2|\nabla v|^2 + C, \tag{3.12}$$

where $C > 0$ is a constant. On the other hand, we have from the v-equation that

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 &= \int_\Omega |\nabla v|^2\nabla v\cdot\nabla(d\Delta v - v + u)\\ &= d\int_\Omega |\nabla v|^2\nabla v\cdot\nabla\Delta v - \int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^2\nabla v\cdot\nabla u.\end{aligned}$$

Note that the identity $\nabla v\cdot\nabla\Delta v = \frac{1}{2}\Delta|\nabla v|^2 - |D^2 v|^2$ holds. Then, one finds

$$\begin{aligned}\frac{1}{4}\frac{d}{dt}\int_\Omega |\nabla v|^4 = &-\frac{d}{2}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{d}{2}\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} - d\int_\Omega |\nabla v|^2|D^2v|^2\\ &-\int_\Omega |\nabla v|^4 - \int_\Omega u|\nabla v|^2\Delta v - \int_\Omega u\nabla v\cdot\nabla|\nabla v|^2.\end{aligned}$$

Furthermore, thanks to Young's inequality, we obtain

$$\begin{aligned}-\int_\Omega u|\nabla v|^2\Delta v - \int_\Omega u\nabla v\cdot\nabla|\nabla v|^2 &\le \frac{d}{3}\int_\Omega |\nabla v|^2|\Delta v|^2 + \frac{d}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{7}{4d}\int_\Omega u^2|\nabla v|^2\\ &\le d\int_\Omega |\nabla v|^2|D^2v|^2 + \frac{d}{4}\int_\Omega |\nabla|\nabla v|^2|^2 + \frac{7}{4d}\int_\Omega u^2|\nabla v|^2,\end{aligned}$$

where $|\Delta v|^2 \le 3|D^2 v|^2$ from the Cauchy–Schwarz inequality is applied to the second inequality. Then, we continue to find that

$$\frac{d}{dt}\int_\Omega |\nabla v|^4 + \int_\Omega |\nabla v|^4 + d\int_\Omega |\nabla|\nabla v|^2|^2 \le 2d\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} + \frac{7}{d}\int_\Omega u^2|\nabla v|^2. \tag{3.13}$$

To estimate $\int_{\partial\Omega}|\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu}$ in (3.13), we apply inequality (2.4) in [22] that $\frac{\partial|\nabla v|^2}{\partial\nu} \le C|\nabla v|^2$ with some constant $C > 0$ to obtain

$$\int_{\partial\Omega} |\nabla v|^2\frac{\partial|\nabla v|^2}{\partial\nu} \le C\int_{\partial\Omega} |\nabla v|^4 = C\||\nabla v|^2\|^2_{L^2(\partial\Omega)}; \tag{3.14}$$

moreover, by applying the Sobolev trace embedding (1.9) in Ref. [22] (or Lemma 2.3 and Lemma 2.4 there), one has that there exists a constant $C > 0$ such that for arbitrary $r \in (0, \frac{1}{2})$

$$\||\nabla v|^2\|_{L^2(\partial\Omega)} \le C\||\nabla v|^2\|_{W^{r+\frac{1}{2},2}(\Omega)}. \tag{3.15}$$

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Choosing $p = r + \frac{1}{2}$, $s = q = 2$ and $\alpha = \frac{3}{4} + \frac{r}{2} \in (r + \frac{1}{2}, 1)$ in Lemma 2.4, we obtain

$$\begin{aligned} \||\nabla v|^2\|_{W^{r+\frac{1}{2},2}(\Omega)} &\leq C\|\nabla|\nabla v|^2\|^{\alpha}_{L^2(\Omega)}\||\nabla v|^2\|^{1-\alpha}_{L^1(\Omega)} + C\||\nabla v|^2\|_{L^1(\Omega)} \\ &\leq C\||\nabla|\nabla v|^2\|^{\alpha}_{L^2(\Omega)} + C, \end{aligned} \tag{3.16}$$

where the second inequality holds thanks to the boundedness of $\int_\Omega |\nabla v|^2$. In view of (3.14), (3.15), (3.16) and the fact that $\alpha < 1$, we use Young's inequality to find that for any $\epsilon > 0$, there exists a constant $C_\epsilon > 0$ such that

$$\begin{aligned} \int_{\partial\Omega} |\nabla v|^2 \left|\frac{\partial|\nabla v|^2}{\partial\nu}\right| &\leq C\left(\left(\int_\Omega |\nabla|\nabla v|^2|^2\right)^{\alpha} + 1\right) \\ &\leq \epsilon \int_\Omega |\nabla|\nabla v|^2|^2 + C_\epsilon. \end{aligned} \tag{3.17}$$

To estimate $\int_\Omega u^2|\nabla v|^2$ in (3.13), we apply Young's inequality with sufficiently small $\epsilon > 0$

$$\int_\Omega u^2|\nabla v|^2 \leq \frac{\epsilon}{3}\int_\Omega |\nabla v|^6 + \frac{2}{3\sqrt{\epsilon}}\int_\Omega u^3. \tag{3.18}$$

Then, we have from (22) in Ref. [4] (or Lemma 3.5 in Ref. [33]) and the fact $\int u \ln u < C$ that there exists $C_\epsilon > 0$ such that

$$\begin{aligned} \int_\Omega u^3 = \|u\|^3_{L^3(\Omega)} &\leq \epsilon\|\nabla u\|^2_{L^2(\Omega)}\|u\ln u\|_{L^1(\Omega)} + C\|u\|^3_{L^1(\Omega)} + C_\epsilon \\ &\leq \epsilon\int_\Omega |\nabla u|^2 + C_\epsilon, \end{aligned} \tag{3.19}$$

and on the other hand, Lemma 2.3 with $j = 0$, $k = 1$, $p = 3$, $r = 2$, $q = s = 1$ and $\alpha = \frac{2}{3}$ implies

$$\begin{aligned} \int_\Omega |\nabla v|^6 = \||\nabla v|^2\|^3_{L^3(\Omega)} &\leq (C\|\nabla|\nabla v|^2\|^{\alpha}_{L^2(\Omega)}\||\nabla v|^2\|^{1-\alpha}_{L^1(\Omega)} + C\||\nabla v|^2\|_{L^1(\Omega)})^3 \\ &\leq C\int_\Omega |\nabla|\nabla v|^2|^2 + C, \end{aligned} \tag{3.20}$$

where the second inequality is due to the boundedness of $\||\nabla v|^2\|_{L^1(\Omega)}$. Then, we conclude from (3.18), (3.19) and (3.20) that

$$\int_\Omega u^2|\nabla v|^2 \leq \sqrt{\epsilon}\int_\Omega |\nabla u|^2 + \epsilon\int_\Omega |\nabla|\nabla v|^2|^2 + C_\epsilon. \tag{3.21}$$

Collecting (3.17) and (3.21), we conclude from (3.12) and (3.13) with $\epsilon > 0$ small that

$$\frac{d}{dt}\left(\int_\Omega u^2 + \int_\Omega |\nabla v|^4\right) + \int_\Omega u^2 + \int_\Omega |\nabla v|^4 \leq C, \forall t \in (0, T_{\max}).$$

Solving this differential inequality gives rise to the uniform boundedness of $\int_\Omega u^2$ and $\int_\Omega |\nabla v|^4$.

Step 2. We estimate $\int_\Omega u^p$ for any fixed $p > 1$. One can first have

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p + \frac{2(p-1)\mu}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 \leq 2\int_\Omega u^{p+1} + C_p\int_\Omega |\nabla v|^{2(p+1)} + C_p. \tag{3.22}$$

Since $\int_\Omega u^2 dx < \infty$ (from Step 1), one finds from inequality (2.1) that

$$\int_\Omega |\nabla v|^{2(p+1)} \leq C_p, \forall t \in (0, T_{\max}).$$

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Then using the following inequality from Galiardo–Nirenberg's inequality

$$2\int_\Omega u^{p+1} \le \frac{2(p-1)\mu}{p}\int_\Omega |\nabla u^{\frac{p}{2}}|^2 + C_p, \forall t \in (0, T_{\max}), \tag{3.23}$$

we obtain from inequalities (3.22) and (3.23) that

$$\frac{d}{dt}\int_\Omega u^p + \int_\Omega u^p \le C_p, \forall t \in (0, T_{\max}).$$

Step 3. Now that $p > 1$ is arbitrary, one concludes from (2.1) that $\|\nabla v\|_{L^\infty}$ is uniformly bounded, which in turn implies the boundedness of u in L^∞ thanks to the standard Moser–L^p iteration. Then, we have that $T_{\max} = \infty$ and the global well-posedness follows. □

4 Exponential stabilisation in L^∞

We proceed to show that the solution to (1.5) converges to $(\bar{u}, \bar{v})$ in L^∞ exponentially as $t \to \infty$, regardless of the initial data. To this end, we first show that the energy functional (3.1) decays to zero exponentially.

Lemma 4.1 *Let Ω be a bounded convex domain in $\mathbb{R}^n$, $n = 1, 2$. Then, the functional $\mathcal{F}$ given by (3.1) is non-negative and decays exponentially as there exists a constant $\alpha_0 > 0$ such that*

$$0 \le \mathcal{F}(u, v) \le \mathcal{F}(u_0, v_0)e^{-\alpha_0 t}, \forall t \in (0, \infty). \tag{4.1}$$

Proof First of all, we apply Jensen's inequality to obtain that

$$\int_\Omega u \ln\Big(\frac{u}{\bar{u}}\Big) = \bar{u}\int_\Omega \frac{u}{\bar{u}} \ln\Big(\frac{u}{\bar{u}}\Big) \ge \bar{u}\Big(\int_\Omega \frac{u}{\bar{u}}\Big) \ln\Big(\int_\Omega \frac{u}{\bar{u}}\Big) = 0,$$

which readily shows that $\mathcal{F} \ge 0$.

To show the exponential decay, we have from inequality (3.7) that

$$\frac{d}{dt}\mathcal{F}(u, v) \le -\min\{\mu, 2\chi\}\mathcal{E}(u, v), \tag{4.2}$$

where

$$\mathcal{E}(u, v) := \int_\Omega \frac{|\nabla u|^2}{u} + \int_\Omega |\nabla\sqrt{v}|^2.$$

Moreover, in light of $\int_\Omega (u/\bar{u} - 1) \equiv 0$, we have that

$$\int_\Omega u \ln\Big(\frac{u}{\bar{u}}\Big) = \bar{u}\int_\Omega \Big(\frac{u}{\bar{u}} \ln\Big(\frac{u}{\bar{u}}\Big) - (\frac{u}{\bar{u}} - 1)\Big),$$

then applying the preliminary inequality (Lemma 3.7 in Ref. [44])

$$r\ln r \le \begin{cases} 0, & 0 \le r < 1, \\ r - 1 + \frac{1}{2}(r-1)^2, & r \ge 1, \end{cases}$$

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with $r = u/\bar{u}$, and using Poincare's inequality, we find $C > 0$ such that

$$\bar{u}\int_\Omega \left(\frac{u}{\bar{u}}\ln\left(\frac{u}{\bar{u}}\right) - \left(\frac{u}{\bar{u}} - 1\right)\right) \le \bar{u}\int_\Omega \frac{1}{2}\left(\frac{u}{\bar{u}} - 1\right)^2 = \frac{1}{2\bar{u}}\int_\Omega (u-\bar{u})^2 \le C\int_\Omega |\nabla u|^2,$$

and note that $\|u(\cdot,t)\|_{L^\infty}$ is bounded uniformly, we obtain

$$\int_\Omega u\ln\left(\frac{u}{\bar{u}}\right) \le C_\Omega \int_\Omega \frac{|\nabla u|^2}{u}, \forall t \in (0,\infty),$$

where C_Ω is a positive constant, therefore

$$\mathcal{F}(u,v) \le \max\{C_\Omega, 2\chi\}\mathcal{E}(u,v). \tag{4.3}$$

Denoting $\alpha_0 := \frac{\min\{\mu,2\chi\}}{\max\{C_\Omega,2\chi\}} > 0$, we conclude from (4.2) and (4.3) that

$$\frac{d}{dt}\mathcal{F}(u,v) \le -\alpha_0\mathcal{F}(u,v), \forall t \in (0,\infty),$$

from which inequality (4.1) follows. □

The decay of $\|u-\bar{u}\|_{L^1}$ follows from inequality (4.1) and the Csiszár–Kullback–Pinsker inequality [6] that

$$\|u-\bar{u}\|_{L^1(\Omega)} \le C\int_\Omega u\ln\left(\frac{u}{\bar{u}}\right).$$

Corollary 2 *Assume that all the conditions in Theorem 1.1 hold. Then, there exist positive constants C and α such that*

$$\|u-\bar{u}\|_{L^1(\Omega)} \le Ce^{-\alpha t}, \forall t \in (0,\infty). \tag{4.4}$$

Before presenting the exponential convergence of (u,v) to $(\bar{u},\bar{v})$ in L^∞ within (1.5), we estimate $\int_\Omega |\nabla u|^4$ in the following lemma.

Lemma 4.2 *Assume that all the conditions in Theorem 1.1 hold. Then, the solution (u,v) to (1.5) satisfies*

$$\int_\Omega |\nabla u|^4 \le C, \forall t \in (0,\infty). \tag{4.5}$$

Proof First of all, we test the first equation in (1.5) against $-\Delta u$ to obtain

$$\begin{aligned}\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \mu\int_\Omega (\Delta u)^2 &= -\chi\int_\Omega \nabla\cdot(u\nabla\ln v)\Delta u\\ &\le \frac{\mu}{2}\int_\Omega(\Delta u)^2 + \frac{\chi^2}{2\mu}\int_\Omega(\nabla u\cdot\nabla\ln v + u\Delta\ln v)^2. \end{aligned} \tag{4.6}$$

In light of the boundedness of u and v in L^∞ and the fact that $v(x,t) \ge \eta > 0$ for all $(x,t) \in \bar{\Omega}\times[0,T_{\max})$, we apply Lemma 2.2 to obtain the boundedness of $\|\nabla\ln v\|_{L^\infty(\Omega)}$, therefore

$$\int_\Omega(\nabla u\cdot\nabla\ln v + u\Delta\ln v)^2 \le C\int_\Omega(|\nabla u|^2 + |\Delta\ln v|^2),$$

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for some positive constant C, and then inequality (4.6) implies that

$$\frac{1}{2}\frac{d}{dt}\int_\Omega |\nabla u|^2 + \frac{\mu}{2}\int_\Omega (\Delta u)^2 \leq C\int_\Omega |\nabla u|^2 + C\int_\Omega |\Delta \ln v|^2, \forall t > 0,$$

one can integrate it over $(0, t)$ to have that

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla u_0|^2 \leq C\int_0^t\int_\Omega |\nabla u|^2 + C\int_0^t\int_\Omega |\Delta \ln v|^2, \forall t > 0,$$

from which inequality (3.8) implies

$$\frac{1}{2}\int_\Omega |\nabla u|^2 - \frac{1}{2}\int_\Omega |\nabla u_0|^2 \leq C, \forall t > 0.$$

Then one readily obtains the boundedness of $\|\nabla u(\cdot, t)\|_{L^2}$ since $u_0 \in W^{1,\infty}(\Omega)$. In light of this fact, applying the same arguments for Lemma 2.2 on $\nabla v_t = d\nabla\Delta v - \nabla v + \nabla u$ implies the boundedness of $\|\nabla v\|_{W^{1,p}}$, $p = \infty$ for $n = 1$ and $p \in (1, \infty)$ for $n = 2$.

We continue to find by using the abstract form of u-equation that

$$\nabla u(\cdot, t) = \nabla e^{t\mu\Delta}u_0 + \chi\int_0^t \nabla e^{(t-s)\mu\Delta}\nabla\cdot\left(\frac{u(\cdot, s)}{v(\cdot, s)}\nabla v\right)ds, \forall t > 0. \tag{4.7}$$

After applying Lemma 1.3 of Ref. [54] on (4.7), we find a constant $C > 0$ such that

$$\begin{aligned}\|\nabla u\|_{L^4(\Omega)} &\leq C\left(1 + \int_0^t e^{-\mu\nu_1(t-s)}(t-s)^{-\frac{3}{4}}\|\nabla u\cdot\nabla \ln v + u\Delta \ln v\|_{L^2(\Omega)}ds\right)\\ &\leq C\left(1 + \int_0^t e^{-\mu\nu_1(t-s)}(t-s)^{-\frac{3}{4}}(\|\nabla u\|_{L^2(\Omega)} + \|\Delta \ln v\|_{L^2(\Omega)})ds\right),\end{aligned} \tag{4.8}$$

where ν_1 represents the first Neumann eigenvalue of $-\Delta$ and the last inequality follows from the uniform boundedness of $\|u\|_{L^\infty}$ and $\|\nabla \ln v\|_{L^\infty}$.

To further estimate (4.8), we deduce from the boundedness of $\|\nabla u\|_{L^2}$ and $\|\Delta v\|_{L^2}$ that

$$\sup_{t\in(0,\infty)}\int_0^t e^{-\mu\nu_1(t-s)}(t-s)^{-\frac{3}{4}}(\|\nabla u\|_{L^2} + \|\Delta \ln v\|_{L^2})ds \leq C,$$

and then one obtains the boundedness of $\|\nabla u\|_{L^4}$. This concludes the proof. □

Now we prove the main result as follows.

Proof of Theorem 1.1 We shall only prove the exponential convergence property in inequality (1.8), while the global well-posedness and uniform boundedness are already verified in Proposition 3.1.

In one dimension, choosing $j = 0$, $k = 1$, $p = \infty$, $r = 2$, $q = s = 1$ and $\alpha = \frac{2}{3}$ in Lemma 2.3 gives us

$$\|u - \bar{u}\|_{L^\infty(\Omega)} \leq C\|\nabla u\|^{\frac{2}{3}}_{L^2(\Omega)}\|u - \bar{u}\|^{\frac{1}{3}}_{L^1(\Omega)} + C\|u - \bar{u}\|_{L^1(\Omega)},$$

from which (1.8) readily follows thanks to (4.4), (4.5) and the standard parabolic comparison principle.

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In two dimensions, by using Lemma 2.3, we choose $j=0$, $k=1$, $p=r=2$, $q=s=1$ and $\alpha=\frac{1}{2}$ to find

$$\|u-\bar{u}\|_{L^2(\Omega)} \leq C\|\nabla u\|_{L^2(\Omega)}^{\frac{1}{2}}\|u-\bar{u}\|_{L^1(\Omega)}^{\frac{1}{2}} + C\|u-\bar{u}\|_{L^1(\Omega)},$$

which implies the exponential decay of $\|u-\bar{u}\|_{L^2(\Omega)}$. Then, we employ Lemma 2.3 again with $j=0$, $k=1$, $p=\infty$, $r=4$, $\alpha=\frac{2}{3}$ and $q=s=2$ to obtain

$$\|u-\bar{u}\|_{L^\infty(\Omega)} \leq C\|\nabla u\|_{L^4(\Omega)}^{\frac{2}{3}}\|u-\bar{u}\|_{L^2(\Omega)}^{\frac{1}{3}} + C\|u-\bar{u}\|_{L^2(\Omega)},$$

which completes the proof of Theorem 1.1 thanks to the exponential decay of $\|u-\bar{u}\|_{L^2(\Omega)}$ and (4.5) as above. □

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