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NON-ZERO RADIAL SOLUTIONS FOR ELLIPTIC SYSTEMS WITH COUPLED FUNCTIONAL BCS IN EXTERIOR DOMAINS

FILOMENA CIANCIARUSO, GENNARO INFANTE AND PAOLAMARIA PIETRAMALA

Dipartimento di Matematica e Informatica, Università della Calabria, 87036 Arcavacata di Rende, Cosenza, Italy (cianciaruso@unical.it; gennaro.infante@unical.it; pietramala@unical.it)

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Abstract We prove new results on the existence, non-existence, localization and multiplicity of non-trivial radial solutions of a system of elliptic boundary value problems on exterior domains subject to non-local, nonlinear, functional boundary conditions. Our approach relies on fixed point index theory. As a by-product of our theory we provide an answer to an open question posed by do Ó, Lorca, Sánchez and Ubilla. We include some examples with explicit nonlinearities in order to illustrate our theory.

Keywords: elliptic system; fixed point index; cone; non-trivial solution; nonlinear functional boundary conditions

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1. Introduction

The existence of solutions of elliptic systems on exterior domains has been studied by a number of authors. Two interesting papers in this direction are those by do Ó and co-authors [**14**, **15**], where results on the existence, non-existence and multiplicity of positive solutions of the elliptic system with *non-homogenous* boundary conditions (BCs):

$$\begin{cases} \Delta u + f_1(|x|, u, v) = 0, \ |x| \in [1, +\infty), \\ \Delta v + f_2(|x|, u, v) = 0, \ |x| \in [1, +\infty), \\ u(x) = a \quad \text{for } x \in \partial B_1, \quad \lim\limits_{|x|\to+\infty} u(|x|) = 0, \\ v(x) = b \quad \text{for } x \in \partial B_1, \quad \lim\limits_{|x|\to+\infty} v(|x|) = 0, \end{cases} \tag{1.1}$$

were given. The methodology in [**14**, **15**] relies on a careful use of the Krasnosel'skiĭ–Guo theorem on cone compressions and cone expansions, combined with the upper-lower solutions method and the fixed point index theory. These papers follow earlier works by do Ó *et al.* [**11**] on annular domains with non-homogenous BCs, and by Lee [**39**] on

Dirichlet BCs on exterior domains. In [15] (see Open Problem 3) the authors posed an interesting question regarding the existence of multiple positive solutions of the elliptic system (1.1) under more general BCs.

Here we study the existence and the multiplicity of *non-zero* solutions of the system of nonlinear elliptic boundary value problems with *non-local* and *functional* BCs

$$\begin{cases} \Delta u + h_1(|x|)f_1(u,v) = 0, & |x| \in [R_1, +\infty), \\ \Delta v + h_2(|x|)f_2(u,v) = 0, & |x| \in [R_1, +\infty), \\ u(R_1 x) = \beta_1 u(R_\eta x) \quad \text{for } x \in \partial B_1, & \lim\limits_{|x|\to+\infty} u(|x|) = H_1[u,v], \\ v(R_1 x) = \delta_1 \dfrac{\partial v}{\partial r}(R_\xi x) \quad \text{for } x \in \partial B_1, & \lim\limits_{|x|\to+\infty} v(|x|) = H_2[u,v], \end{cases} \tag{1.2}$$

where $x \in \mathbb{R}^n$, $n \geq 3$, $\beta_1, \delta_1 \in \mathbb{R}$, $R_1 > 0$, $R_\eta, R_\xi \in (R_1, +\infty)$, $B_\rho = \{x \in \mathbb{R}^n : |x| < \rho\}$, $\partial/\partial r$ denotes (as in [18]) the differentiation in the radial direction $r = |x|$, and H_i are suitable compact functionals, not necessarily linear.

We stress that a variety of methods have been used to study the existence of solutions of elliptic equations subject to *homogeneous* BCs on exterior domains: for example, topological methods were employed by Lee [39], Stanczy [46], Han and Wang [22], do Ó *et al.* [12], Abebe and co-authors [1] and Orpel [42]; a priori estimates were utilized by Castro *et al.* [5]; sub and super solutions were used by Sankar *et al.* [45] and Djedali and Orpel [10]; and variational methods were used by Orpel [41].

In the context of non-homogeneous BCs, elliptic problems in exterior domains were studied by Aftalion and Busca [2] and do Ó *et al.* [13–16], and nonlinear BCs were investigated by Butler and others [4], Dhanya *et al.* [9], Ko and co-authors [33], and Lee and others [40].

In order to discuss the existence of non-zero solutions of the elliptic system (1.2), we study the associated system of perturbed Hammerstein integral equations

$$\begin{cases} u(t) = \left(1 + \dfrac{(\beta_1 - 1)t}{1 - \beta_1 \eta}\right) H_1[u,v] + \displaystyle\int_0^1 k_1(t,s) g_1(s) f_1(u(s), v(s))\, \mathrm{d}s, \\ v(t) = \left(1 - \dfrac{t}{1 - \beta_2}\right) H_2[u,v] + \displaystyle\int_0^1 k_2(t,s) g_2(s) f_2(u(s), v(s))\, \mathrm{d}s. \end{cases}$$

The existence of solutions of systems of (different kinds) of perturbed Hammerstein integral equations has been studied, for example, in [17, 19, 20, 23–25, 29, 31, 32, 49]. When using Krasnosel'skiĭ-type arguments for these kind of systems, one difficulty to be overcome is how to control the growth of the perturbations. In the recent manuscript [8] the authors used *local* estimates via *linear* functionals. Here we also use local estimates, but with *affine* functionals instead. This allows more flexibility when dealing with the existence results.

The paper is organized as follows: § 2 is devoted to the existence and non-existence results for the system (1.2); and in § 3 we briefly illustrate how our theory allows us to deal with more general conditions than those present in (1.1), giving a positive answer to Open Problem 3 of [15].

Our methodology relies on classical fixed point index theory (see, for example, [3, 21]) and also benefits from ideas from the papers [17, 24, 28–30, 37, 38, 47, 48].

2. A system of elliptic PDEs in exterior domains

We consider the system of boundary value problems

$$\begin{cases} \Delta u + h_1(|x|)f_1(u,v) = 0, & |x| \in [R_1, +\infty), \\ \Delta v + h_2(|x|)f_2(u,v) = 0, & |x| \in [R_1, +\infty), \\ u(R_1 x) = \beta_1 u(R_\eta x) \quad \text{for } x \in \partial B_1, & \lim\limits_{|x|\to+\infty} u(|x|) = H_1[u,v], \\ v(R_1 x) = \delta_1 \dfrac{\partial v}{\partial r}(R_\xi x) \quad \text{for } x \in \partial B_1, & \lim\limits_{|x|\to+\infty} v(|x|) = H_2[u,v], \end{cases} \tag{2.1}$$

where $x \in \mathbb{R}^n$, $\beta_1, \delta_1 \in \mathbb{R}$, $R_1 > 0$, and $R_\eta, R_\xi \in (R_1, +\infty)$. We assume that, for $i = 1, 2$,

- $f_i : \mathbb{R} \times \mathbb{R} \to [0, +\infty)$ is continuous;
- $h_i : [R_1, +\infty) \to [0, +\infty)$ is continuous and $h_i(|x|) \le ((1)/(|x|^{n+\mu_i}))$ for $|x| \to +\infty$ for some $\mu_i > 0$.

Consider in $\mathbb{R}^n$, $n \ge 3$, the equation

$$\triangle w + h(|x|)f(w) = 0, \quad |x| \in [R_1, +\infty). \tag{2.2}$$

In order to establish the existence of radial solutions $w = w(r)$, $r = |x|$, we proceed as in [4] and rewrite (2.2) in the form

$$w''(r) + \frac{n-1}{r} w'(r) + h(r)f(w(r)) = 0, \quad r \in [R_1, +\infty). \tag{2.3}$$

Set $w(t) = w(r(t))$, where

$$r(t) := R_1\, t^{1/2-n}, \quad t \in [0,1]$$

and take, for $t \in [0,1]$,

$$\phi(t) := r(t)\, \frac{R_1}{(n-2)^2}\, t^{((2n-3)/(2-n))};$$

then (2.3) becomes

$$w''(t) + \phi(t)h(r(t))f(w(t)) = 0, \quad t \in [0,1].$$

Set $u(t) = u(r(t))$ and $v(t) = v(r(t))$. Thus, with the system (2.1), we associate the system of ordinary differential equations (ODEs)

$$\begin{cases} u''(t) + g_1(t)f_1(u(t), v(t)) = 0, & t \in (0,1), \\ v''(t) + g_2(t)f_2(u(t), v(t)) = 0, & t \in (0,1), \\ u(0) = H_1[u,v], & u(1) = \beta_1 u(\eta), \\ v(0) = H_2[u,v], & v(1) = \beta_2 v'(\xi), \end{cases} \tag{2.4}$$

where

$$g_i(t) := \phi(t)h_i(r(t)), \quad \beta_2 = (2-n)\delta_1 \frac{\xi}{R_\xi}$$

and $\xi, \eta \in (0,1)$ are such that $r(\eta) = R_\eta$ and $r(\xi) = R_\xi$.

We study the existence of non-trivial solutions of the system (2.4), by means of the associated system of perturbed Hammerstein integral equations

$$\begin{cases} u(t) = \left(1 + \dfrac{(\beta_1 - 1)t}{1 - \beta_1 \eta}\right) H_1[u, v] + \displaystyle\int_0^1 k_1(t, s) g_1(s) f_1(u(s), v(s))\, \mathrm{d}s, \\ v(t) = \left(1 - \dfrac{t}{1 - \beta_2}\right) H_2[u, v] + \displaystyle\int_0^1 k_2(t, s) g_2(s) f_2(u(s), v(s))\, \mathrm{d}s, \end{cases} \tag{2.5}$$

where the Green's functions k_1 and k_2 are given by

$$k_1(t, s) := \frac{t}{1 - \beta_1 \eta}(1 - s) - \begin{cases} \dfrac{\beta_1 t}{1 - \beta_1 \eta}(\eta - s), & s \le \eta, \\ 0, & s > \eta, \end{cases} - \begin{cases} t - s, & s \le t, \\ 0, & s > t, \end{cases}$$

and

$$k_2(t, s) := \frac{t}{1 - \beta_2}(1 - s) - \begin{cases} \dfrac{\beta_2 t}{1 - \beta_2}, & s \le \xi, \\ 0, & s > \xi, \end{cases} - \begin{cases} t - s, & s \le t, \\ 0, & s > t. \end{cases} \tag{2.6}$$

Owing to the presence of the parameters $\beta_1, \beta_2, \eta, \xi$, several cases can occur; here we restrict our attention to the case

$$1 \le \beta_1 < \frac{1}{\eta} \quad \text{and} \quad 0 \le \beta_2 < 1 - \xi. \tag{2.7}$$

The choice (2.7) is owing to brevity and to the fact that it enables us to illustrate the different behaviour of the solution (u, v) in the two components: u is *non-negative* on $[0, 1]$ and v is positive on some sub-interval of $[0, 1]$ and is allowed to *change sign* elsewhere.

In the following Lemma we summarize some properties of the Green's functions k_1 and k_2 that can be found in [**27**, **47**].

Lemma 2.1. *The following hold.*

(1) *The kernel k_1 is non-negative and continuous in $[0, 1] \times [0, 1]$. Moreover, we have*

$$0 \le k_1(t, s) \le \Phi_1(s) \quad \textit{for } (t, s) \in [0, 1] \times [0, 1],$$
$$k_1(t, s) \ge c_{k_1} \Phi_1(s) \quad \textit{for } (t, s) \in [a_1, b_1] \times [0, 1],$$

where we take

$$\Phi_1(s) = \frac{\beta_1 s(1 - s)}{1 - \beta_1 \eta},$$

an arbitrary $[a_1, b_1] \subset (0, 1]$ and

$$c_{k_1} = \min\{a_1 \eta, 4 a_1 (1 - \beta_1 \eta) \eta, \eta(1 - \beta_1 \eta)\}.$$

(2) *The kernel* k_2 *in* $[0,1]\times[0,1]$ *is non-positive for*

$$1-\beta_2 \le t \le 1 \quad \text{and} \quad 0 \le s \le \xi$$

and is discontinuous on the segment $\{t\in[0,1], s=\eta\}$. *Moreover, we have*

$$|k_2(t,s)| \le \Phi_2(s) \quad \text{for } (t,s)\in[0,1]\times[0,1],$$
$$k_2(t,s) \ge c_{k_2}\Phi_2(s) \quad \text{for } (t,s)\in[a_2,b_2]\times[0,1],$$

where we take

$$\Phi_2(s) = \max\{1,\beta_2/\xi\}\,\frac{s(1-s)}{1-\beta_2},$$

an arbitrary $[a_2,b_2]\subset(0,1-\beta_2)$ *and*

$$c_{k_2} = \frac{\min\{4a_2(1-\beta_2-\xi),(1-b_2-\beta_2)\}}{\max\{1,\beta_2/\xi\}}.$$

We note that, for $i=1,2$,

$$g_i\,\Phi_i \in L^1[0,1], \quad g_i \ge 0$$

and assume that

$$\int_{a_i}^{b_i} \Phi_i(s)g_i(s)\,\mathrm{d}s > 0.$$

We denote by $C[0,1]$ the space of the continuous functions on $[0,1]$ equipped with the norm $\|w\| := \max\{|w(t)|,\ t\in[0,1]\}$ and set

$$\gamma_1(t) := 1+\frac{(\beta_1-1)t}{1-\beta_1\eta} \quad \text{and} \quad \gamma_2(t) := 1-\frac{t}{1-\beta_2}.$$

We have, for $i=1,2$, $\gamma_i\in C[0,1]$ and

$$\gamma_i(t) \ge c_{\gamma_i}\|\gamma_i\|, \quad t\in[a_i,b_i],$$

$$||\gamma_1|| = \frac{\beta_1(1-\eta)}{1-\beta_1\eta}, \quad c_{\gamma_1} = \frac{\beta_1-1}{\beta_1(1-\eta)}a_1 + \frac{1-\beta_1\eta}{\beta_1(1-\eta)},$$

$$||\gamma_2|| = \begin{cases} \dfrac{\beta_2}{1-\beta_2}, & \beta_2 \ge 1/2 \\ 1, & \beta_2 \le 1/2 \end{cases} \quad \text{and} \quad c_{\gamma_2} = \begin{cases} \dfrac{1-\beta_2}{\beta_2} - \dfrac{b_2}{\beta_2}, & \beta_2 \ge 1/2 \\ 1-\dfrac{b_2}{1-\beta_2}, & \beta_2 \le 1/2. \end{cases}$$

We consider the space $C[0,1]\times C[0,1]$ endowed (with abuse of notation) with the norm

$$\|(u,v)\| := \max\{\|u\|,\|v\|\}.$$

Recall that a *cone* K in a Banach space X is a closed convex set such that $\lambda x\in K$ for $x\in K$ and $\lambda\ge 0$ and $K\cap(-K)=\{0\}$. Owing to the properties above, we can work in

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the cone K in $C[0,1] \times C[0,1]$ defined by

$$K := \{(u,v) \in K_1 \times K_2\}, \tag{2.8}$$

where

$$K_1 := \left\{ w \in C[0,1] : \ w \geq 0, \ \min_{t \in [a_1,b_1]} w(t) \geq c_1 \|w\| \right\}$$

and

$$K_2 := \left\{ w \in C[0,1] : \ \min_{t \in [a_2,b_2]} w(t) \geq c_2 \|w\| \right\},$$

with $c_i = \min\{c_{k_i}, c_{\gamma_i}\}$. Note that the functions in K_1 are non-negative, while those in K_2 are allowed to change sign outside the interval $[a_2,b_2]$. K_1 is a kind of cone first used by Krasnosel'skiĭ (see [**34**]) and D. Guo (see e.g. [**21**]), while K_2 was introduced by Infante and Webb in [**28**].

By a *non-trivial* solution of the system (2.1) we mean a solution $(u,v) \in K$ of the system (2.5) such that $\|(u,v)\| > 0$.

We assume that for $i = 1,2$, $H_i : K \to [0,\infty)$ is a compact functional.

Under our assumptions, it is possible to show that the integral operator

$$T(u,v)(t) := \begin{pmatrix} \gamma_1(t)H_1[u,v] + F_1(u,v)(t) \\ \gamma_2(t)H_2[u,v] + F_2(u,v)(t) \end{pmatrix} := \begin{pmatrix} T_1(u,v)(t) \\ T_2(u,v)(t) \end{pmatrix}, \tag{2.9}$$

where

$$F_i(u,v)(t) := \int_0^1 k_i(t,s)g_i(s)f_i(u(s),v(s))\,\mathrm{d}s,$$

leaves the cone K invariant and is compact.

Lemma 2.2. *The operator T maps K into K and is compact.*

Proof. Take $(u,v) \in K$ such that $||(u,v)|| \leq r$. Then we have, for $t \in [0,1]$,

$$\begin{aligned} |T_2(u,v)(t)| &= \left| \gamma_2(t)H_2[u,v] + \int_0^1 k_2(t,s)g_2(s)f_2(u(s),v(s))\,\mathrm{d}s \right| \\ &\leq |\gamma_2(t)|H_2[u,v] + \int_0^1 |k_2(t,s)|g_2(s)f_2(u(s),v(s))\,\mathrm{d}s \end{aligned}$$

and, taking the supremum over $[0,1]$, we obtain

$$||T_2(u,v)|| \leq ||\gamma_2||H_2[u,v] + \int_0^1 \Phi_2(s)g_2(s)f_2(u(s),v(s))\,\mathrm{d}s.$$

Moreover, we have

$$\begin{aligned} \min_{t \in [a_2,b_2]} T_2(u,v)(t) &\geq c_{\gamma_2}||\gamma_2||H_2[u,v] + c_{k_2}\int_0^1 \Phi_2(s)g_2(s)f_2(u(s),v(s))\,\mathrm{d}s \\ &\geq c_2||T_2(u,v)||. \end{aligned}$$

Hence we have $T_2(u,v) \in K_2$. We proceed in a similar manner for $T_1(u,v)$.

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Moreover, the map T is compact since the components T_i are sums of two compact maps: by routine arguments, the compactness of F_i can be shown and, since γ_i is continuous, the perturbation $\gamma_i(t)H_i[u,v]$ maps bounded sets into bounded subsets of a finite-dimensional space. □

2.1. Index calculations

If U is an open bounded subset of a cone $\hat{K}$ (in the relative topology) in real Banach space X, we denote by $\overline{U}$ and ∂U the closure and the boundary relative to $\hat{K}$. When U is an open bounded subset of X we write $U_{\hat{K}} = U \cap \hat{K}$, an open subset of $\hat{K}$.

We summarize in the next lemma some classical results regarding the fixed point index; for more details see [**3**, **21**].

Lemma 2.3. *Let U be an open bounded set with $0 \in U_{\hat{K}}$ and $\overline{U}_{\hat{K}} \neq \hat{K}$. Assume that $S : \overline{U}_{\hat{K}} \to \hat{K}$ is a compact map such that $x \neq Sx$ for all $x \in \partial U_{\hat{K}}$. Then the fixed point index $i_{\hat{K}}(S, U_{\hat{K}})$ has the following properties.*

(1) *If there exists $e \in \hat{K} \setminus \{0\}$ such that $x \neq Sx + \lambda e$ for all $x \in \partial U_{\hat{K}}$ and all $\lambda > 0$, then $i_{\hat{K}}(S, U_{\hat{K}}) = 0$.*

(2) *If $\mu x \neq Sx$ for all $x \in \partial U_{\hat{K}}$ and for every $\mu \geq 1$, then $i_{\hat{K}}(S, U_{\hat{K}}) = 1$.*

(3) *If $i_{\hat{K}}(S, U_{\hat{K}}) \neq 0$, then S has a fixed point in $U_{\hat{K}}$.*

(4) *Let U^1 be open in X with $\overline{U^1_{\hat{K}}} \subset U_{\hat{K}}$. If $i_{\hat{K}}(S, U_{\hat{K}}) = 1$ and $i_{\hat{K}}(S, U^1_{\hat{K}}) = 0$, then S has a fixed point in $U_{\hat{K}} \setminus \overline{U^1_{\hat{K}}}$. The same result holds if $i_{\hat{K}}(S, U_{\hat{K}}) = 0$ and $i_{\hat{K}}(S, U^1_{\hat{K}}) = 1$.*

For our index calculations we use the following (relative) open bounded sets in the cone K defined in (2.8):

$$K_{\rho_1,\rho_2} = \{(u,v) \in K : \|u\| < \rho_1 \quad \text{and} \quad \|v\| < \rho_2\}$$

and

$$V_{\rho_1,\rho_2} = \left\{(u,v) \in K : \min_{t\in[a_1,b_1]} u(t) < \rho_1 \quad \text{and} \quad \min_{t\in[a_2,b_2]} v(t) < \rho_2\right\},$$

where $\rho_1, \rho_2 > 0$. The set $V_{\rho,\rho}$ was introduced in [**23**] and is equal to the set called $\Omega^{\rho/c}$ in [**17**], an extension to the case of systems of a set given by Lan [**36**]. The choice of different radii (used also in the papers [**6**, **8**, **26**]) allows more freedom in the growth of the nonlinearities.

We utilize the following Lemma, similar to [**17**, Lemma 5]. The proof follows the corresponding one in [**17**] and is omitted.

Lemma 2.4. *The sets K_{ρ_1,ρ_2} and V_{ρ_1,ρ_2} have the following properties.*

(1) $K_{\rho_1,\rho_2} \subset V_{\rho_1,\rho_2} \subset K_{\rho_1/c_1,\rho_2/c_2}$.

(2) *$(w_1, w_2) \in \partial V_{\rho_1,\rho_2}$ if and only if $(w_1, w_2) \in K$, $\min_{t\in[a_i,b_i]} w_i(t) = \rho_i$ for some $i \in \{1,2\}$ and $\min_{t\in[a_j,b_j]} w_j(t) \leq \rho_j$ for $j \neq i$.*

(3) *If* $(w_1, w_2) \in \partial V_{\rho_1,\rho_2}$, *then for some* $i \in \{1, 2\}$, $\rho_i \le w_i(t) \le \rho_i/c_i$ *for* $t \in [a_i, b_i]$, *and for* $j \neq i$, $0 \le w_j(t) \le \rho_j/c_j$ *for* $t \in [a_j, b_j]$.

(4) $(w_1, w_2) \in \partial K_{\rho_1,\rho_2}$ *if and only if* $(w_1, w_2) \in K$, $\|w_i\| = \rho_i$ *for some* $i \in \{1, 2\}$ *and* $\|w_j\| \le \rho_j$ *for* $j \neq i$.

Now we prove a result concerning the fixed point index of the operator T in (2.9) on the set K_{ρ_1,ρ_2}.

Lemma 2.5. *Assume that*

($\mathrm{I}^1_{\rho_1,\rho_2}$) *there exist* $\rho_1, \rho_2 > 0$, $A_1^{\rho_1,\rho_2}, A_2^{\rho_1,\rho_2} \ge 0$, *and linear functionals* $\alpha_{ij}^{\rho_1,\rho_2}[\cdot] : K_j \to [0, +\infty)$, *involving positive Stieltjes measures given by*

$$\alpha_{ij}^{\rho_1,\rho_2}[w] = \int_0^1 w(t)dC_{ij}(t),$$

where C_{ij} *is of bounded variation, such that, for* $i = 1, 2$,

- $\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i] < 1$,
- $H_i[u, v] \le A_i^{\rho_1,\rho_2} + \alpha_{i1}^{\rho_1,\rho_2}[u] + \alpha_{i2}^{\rho_1,\rho_2}[v]$ *for* $(u, v) \in \partial K_{\rho_1,\rho_2}$,
- *the following inequality holds with* $j = 1, 2$, $j \neq i$:

$$f_i^{\rho_1,\rho_2}\left(\frac{||\gamma_i||}{1-\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i]}\int_0^1 \mathcal{K}_i(s)g_i(s)\,\mathrm{d}s + \frac{1}{m_i}\right) + ||\gamma_i||\frac{A_i^{\rho_1,\rho_2} + \rho_j\alpha_{ij}^{\rho_1,\rho_2}[1]}{\rho_i(1-\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i])} < 1, \tag{2.10}$$

where

$$f_i^{\rho_1,\rho_2} := \sup\left\{\frac{f_i(u,v)}{\rho_i} : (u, v) \in [0, \rho_1] \times [-\rho_2, \rho_2]\right\},$$

$$\mathcal{K}_i(s) := \int_0^1 k_i(t, s)dC_{ii}(t) \quad \text{and} \quad \frac{1}{m_i} := \sup_{t\in[0,1]}\int_0^1 |k_i(t, s)|g_i(s)\,\mathrm{d}s.$$

Then $i_K(T, K_{\rho_1,\rho_2})$ *is equal to* 1.

Proof. We show that $\mu(u, v) \neq T(u, v)$ for every $(u, v) \in \partial K_{\rho_1,\rho_2}$ and for every $\mu \ge 1$. In fact, if this does not happen, there exist $\mu \ge 1$ and $(u, v) \in \partial K_{\rho_1,\rho_2}$ such that $\mu(u, v) = T(u, v)$. First, we assume that $\|u\| \le \rho_1$ and $\|v\| = \rho_2$. Then we have, for $t \in [0, 1]$,

$$\mu v(t) = \gamma_2(t)H_2[u, v] + F_2(u, v)(t). \tag{2.11}$$

Applying $\alpha_{22}^{\rho_1,\rho_2}$ to both sides of (2.11), we obtain

$$\begin{aligned}\mu\alpha_{22}^{\rho_1,\rho_2}[v] &= \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]H_2[u, v] + \alpha_{22}^{\rho_1,\rho_2}[F_2(u, v)]\\ &\le \alpha_{22}^{\rho_1,\rho_2}[\gamma_2](A_2^{\rho_1,\rho_2} + \alpha_{21}^{\rho_1,\rho_2}[u] + \alpha_{22}^{\rho_1,\rho_2}[v]) + \alpha_{22}^{\rho_1,\rho_2}[F_2(u, v)]\\ &\le \alpha_{22}^{\rho_1,\rho_2}[\gamma_2](A_2^{\rho_1,\rho_2} + \rho_1\alpha_{21}^{\rho_1,\rho_2}[1] + \alpha_{22}^{\rho_1,\rho_2}[v]) + \int_0^1 \mathcal{K}_2(s)g_2(s)f_2(u(s), v(s))\,\mathrm{d}s.\end{aligned}$$

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Therefore we obtain

$$(\mu - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2])\alpha_{22}^{\rho_1,\rho_2}[v] \le \rho_2\alpha_{22}^{\rho_1,\rho_2}[\gamma_2]\left(\frac{A_2^{\rho_1,\rho_2}}{\rho_2} + \frac{\rho_1}{\rho_2}\alpha_{21}^{\rho_1,\rho_2}[1]\right) + \rho_2\int_0^1 \mathcal{K}_2(s)g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s,$$

and, consequently, we get

$$\begin{aligned}\alpha_{22}^{\rho_1,\rho_2}[v] &\le \frac{\rho_2\,\alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}{\mu - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\left(\frac{A_2^{\rho_1,\rho_2}}{\rho_2} + \frac{\rho_1}{\rho_2}\alpha_{21}^{\rho_1,\rho_2}[1]\right)\\ &\quad + \frac{\rho_2}{\mu - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\int_0^1 \mathcal{K}_2(s)g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s\\ &\le \frac{\rho_2\,\alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}{1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\left(\frac{A_2^{\rho_1,\rho_2} + \rho_1\alpha_{21}^{\rho_1,\rho_2}[1]}{\rho_2}\right)\\ &\quad + \frac{\rho_2}{1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\int_0^1 \mathcal{K}_2(s)g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s.\end{aligned}$$

Moreover, we have

$$\begin{aligned}\mu|v(t)| &= |\gamma_2(t)H_2[u,v] + F_2(u,v)(t)|\\ &\le |\gamma_2(t)|H_2[u,v] + \int_0^1 |k_2(t,s)|g_2(s)f_2(u(s),v(s))\,\mathrm{d}s\\ &\le |\gamma_2(t)|(A_2^{\rho_1,\rho_2} + \alpha_{21}^{\rho_1,\rho_2}[u] + \alpha_{22}^{\rho_1,\rho_2}[v]) + \int_0^1 |k_2(t,s)|g_2(s)f_2(u(s),v(s))\,\mathrm{d}s\\ &\le \rho_2\Bigg(|\gamma_2(t)|\frac{A_2^{\rho_1,\rho_2} + \rho_1\alpha_{21}^{\rho_1,\rho_2}[1]}{\rho_2} + |\gamma_2(t)|\Bigg(\frac{\alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}{1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\left(\frac{A_2^{\rho_1,\rho_2} + \rho_1\alpha_{21}^{\rho_1,\rho_2}[1]}{\rho_2}\right)\\ &\quad + \frac{1}{1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\int_0^1 \mathcal{K}_2(s)g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s\Bigg)\\ &\quad + \int_0^1 |k_2(t,s)|g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s\Bigg)\\ &= \rho_2\Bigg(|\gamma_2(t)|\frac{A_2^{\rho_1,\rho_2} + \rho_1\alpha_{21}^{\rho_1,\rho_2}[1]}{\rho_2(1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2])} + \frac{|\gamma_2(t)|}{1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}\int_0^1 \mathcal{K}_2(s)g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s\\ &\quad + \int_0^1 |k_2(t,s)|g_2(s)\frac{f_2(u(s),v(s))}{\rho_2}\,\mathrm{d}s\Bigg).\end{aligned}$$

Taking the supremum over $[0,1]$ gives

$$\begin{aligned}\mu\rho_2 &\le \rho_2\Bigg(||\gamma_2||\frac{A_2^{\rho_1,\rho_2} + \rho_1\alpha_{21}^{\rho_1,\rho_2}[1]}{\rho_2(1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2])} + \frac{||\gamma_2||}{1 - \alpha_{22}^{\rho_1,\rho_2}[\gamma_2]}f_2^{\rho_1,\rho_2}\int_0^1 \mathcal{K}_2(s)g_2(s)\,\mathrm{d}s\\ &\quad + f_2^{\rho_1,\rho_2}\sup_{t\in[0,1]}\int_0^1 |k_2(t,s)|g_2(s)\,\mathrm{d}s\Bigg).\end{aligned}$$

Using the hypothesis (2.10) we obtain $\mu\rho_2 < \rho_2$. This contradicts the fact that $\mu \geq 1$ and proves the result.

The case $\|u\| = \rho_1$ and $\|v\| \leq \rho_2$ is simpler and therefore is omitted. □

Remark 2.6. Take $\omega \in L^1([0,1]\times[0,1])$ and denote

$$\omega^+(t,s) = \max\{\omega(t,s),0\}, \quad \omega^-(t,s) = \max\{-\omega(t,s),0\}.$$

Then we have

$$\left|\int_0^1 \omega(t,s)\,\mathrm{d}s\right| \leq \max\left\{\int_0^1 \omega^+(t,s)\,\mathrm{d}s, \int_0^1 \omega^-(t,s)\,\mathrm{d}s\right\} \leq \int_0^1 |\omega(t,s)|\,\mathrm{d}s, \tag{2.12}$$

since $\omega = \omega^+ - \omega^-$ and $|\omega| = \omega^+ + \omega^-$.

Note that, using the inequality (2.12) as in [**26**, **30**], it is possible to relax the growth assumptions on the nonlinearity f_2 in Lemma 2.5, by replacing the quantity

$$\sup_{t\in[0,1]} \int_0^1 |k_2(t,s)| g_2(s)\,\mathrm{d}s$$

with

$$\sup_{t\in[0,1]} \left\{\max\left\{\int_0^1 k_2^+(t,s) g_2(s)\,\mathrm{d}s, \int_0^1 k_2^-(t,s) g_2(s)\,\mathrm{d}s\right\}\right\}.$$

For example, if we fix $\beta_2 = 1/2$, $\xi = 1/3$ in (2.6) and $g_2 \equiv 1$, we obtain

$$\sup_{t\in[0,1]} \left\{\max\left\{\int_0^1 k_2^+(t,s)\,\mathrm{d}s, \int_0^1 k_2^-(t,s)\,\mathrm{d}s\right\}\right\} = \sup_{t\in[1/2,1]} \frac{1}{18}(-9t^2+14t-1) = 0.247$$

and

$$\sup_{t\in[0,1]} \int_0^1 |k_2(t,s)|\,\mathrm{d}s = \sup_{t\in[1/2,1]} \frac{1}{18}(-9t^2+16t-2) = 0.284.$$

We give a first Lemma that shows that the index of the operator T is 0 on a set V_{ρ_1,ρ_2}.

Lemma 2.7. *Assume that*

($\mathrm{I}^0_{\rho_1,\rho_2}$) *there exist* $\rho_1, \rho_2 > 0$, $A_1^{\rho_1,\rho_2}, A_2^{\rho_1,\rho_2} \geq 0$, *and linear functionals* $\alpha_{ij}^{\rho_1,\rho_2}[\cdot] : K_j \to [0,+\infty)$, *involving positive Stieltjes measures given by*

$$\alpha_{ij}^{\rho_1,\rho_2}[w] = \int_0^1 w(t)dC_{ij}(t),$$

where C_{ij} *is of bounded variation, such that, for* $i = 1, 2$,

- $\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i] < 1$,
- $H_i[u,v] \geq A_i^{\rho_1,\rho_2} + \alpha_{i1}^{\rho_1,\rho_2}[u] + \alpha_{i2}^{\rho_1,\rho_2}[v]$ *for* $(u,v) \in \partial V_{\rho_1,\rho_2}$,

- *the following inequality holds:*

$$f_{i,(\rho_1,\rho_2)}\left(\frac{c_{\gamma_i}\|\gamma_i\|}{1-\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i]}\int_{a_i}^{b_i}\mathcal{K}_i(s)g_i(s)\,\mathrm{d}s+\frac{1}{M_i}\right)+\frac{c_{\gamma_i}\|\gamma_i\|A_i^{\rho_1,\rho_2}}{\rho_i(1-\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i])}>1, \tag{2.13}$$

where

$$f_{1,(\rho_1,\rho_2)}=\inf\left\{\frac{f_1(u,v)}{\rho_1}:\ (u,v)\in[\rho_1,\rho_1/c_1]\times[-\rho_2/c_2,\rho_2/c_2]\right\},$$
$$f_{2,(\rho_1\rho_2)}=\inf\left\{\frac{f_2(u,v)}{\rho_2}:\ (u,v)\in[0,\rho_1/c_1]\times[\rho_2,\rho_2/c_2]\right\},$$
$$\frac{1}{M_i}:=\inf_{t\in[a_i,b_i]}\int_{a_i}^{b_i}k_i(t,s)g_i(s)\,\mathrm{d}s.$$

Then $i_K(T,V_{\rho_1,\rho_2})=0$.

Proof. Let $e(t)\equiv 1$ for $t\in[0,1]$. Then $(1,1)\in K$. We prove that

$$(u,v)\neq T(u,v)+\lambda(1,1)\quad\text{for }(u,v)\in\partial V_{\rho_1,\rho_2}\quad\text{and }\lambda\geq 0.$$

In fact, if this does not happen, there exist $(u,v)\in\partial V_{\rho_1,\rho_2}$ and $\lambda\geq 0$ such that $(u,v)=T(u,v)+\lambda(1,1)$. Without loss of generality, we can assume that for all $t\in[a_1,b_1]$ we have $\rho_1\leq u(t)\leq\rho_1/c_1$, $\min u(t)=\rho_1$ and $-\rho_2/c_2\leq v(t)\leq\rho_2/c_2$. For $t\in[0,1]$, we have

$$u(t)=\gamma_1(t)H_1[u,v]+F_1(u,v)(t)+\lambda;$$

thus, applying $\alpha_{11}^{\rho_1,\rho_2}$ to both sides of the equality, we obtain

$$\begin{aligned}\alpha_{11}^{\rho_1,\rho_2}[u]&=\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]H_1[u,v]+\alpha_{11}^{\rho_1,\rho_2}[F_1(u,v)]+\alpha_{11}^{\rho_1,\rho_2}[\lambda]\\&\geq\alpha_{11}^{\rho_1,\rho_2}[\gamma_1](A_1^{\rho_1,\rho_2}+\alpha_{11}^{\rho_1,\rho_2}[u]+\alpha_{12}^{\rho_1,\rho_2}[v])+\alpha_{11}^{\rho_1,\rho_2}[F_1(u,v)]\\&\geq\alpha_{11}^{\rho_1,\rho_2}[\gamma_1](A_1^{\rho_1,\rho_2}+\alpha_{11}^{\rho_1,\rho_2}[u])+\int_0^1\mathcal{K}_1(s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s.\end{aligned}$$

We get

$$(1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1])\alpha_{11}^{\rho_1,\rho_2}[u]\geq\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]A_1^{\rho_1,\rho_2}+\int_0^1\mathcal{K}_1(s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s$$

and

$$\alpha_{11}^{\rho_1,\rho_2}[u]\geq\frac{\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]A_1^{\rho_1,\rho_2}}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}+\frac{1}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}\int_0^1\mathcal{K}_1(s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s.$$

Consequently, for $t \in [a_1, b_1]$, we have

$$\begin{aligned}
u(t) &\geq \gamma_1(t)(A_1^{\rho_1,\rho_2} + \alpha_{11}^{\rho_1,\rho_2}[u] + \alpha_{12}^{\rho_1,\rho_2}[v]) + \int_0^1 k_1(t,s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s + \lambda \\
&\geq \gamma_1(t)A_1^{\rho_1,\rho_2} + \gamma_1(t)\alpha_{11}^{\rho_1,\rho_2}[u] + \int_0^1 k_1(t,s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s + \lambda \\
&\geq \gamma_1(t)A_1^{\rho_1,\rho_2} + \gamma_1(t)\frac{\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]A_1^{\rho_1,\rho_2}}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]} \\
&\quad + \frac{\gamma_1(t)}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}\int_0^1 \mathcal{K}_1(s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s \\
&\quad + \int_0^1 k_1(t,s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s + \lambda \\
&\geq \frac{\gamma_1(t)A_1^{\rho_1,\rho_2}}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]} + \frac{\gamma_1(t)}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}\int_{a_1}^{b_1} \mathcal{K}_1(s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s \\
&\quad + \int_{a_1}^{b_1} k_1(t,s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s + \lambda \\
&= \rho_1\frac{\gamma_1(t)A_1^{\rho_1,\rho_2}}{\rho_1(1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1])} + \rho_1\frac{\gamma_1(t)}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}\int_{a_1}^{b_1} \mathcal{K}_1(s)g_1(s)\frac{f_1(u(s),v(s))}{\rho_1}\,\mathrm{d}s \\
&\quad + \rho_1\int_{a_1}^{b_1} k_1(t,s)g_1(s)\frac{f_1(u(s),v(s))}{\rho_1}\,\mathrm{d}s + \lambda.
\end{aligned}$$

Therefore, taking the infimum over $[a_1, b_1]$, we obtain

$$\begin{aligned}
\rho_1 = \min_{t\in[a_1,b_1]} u(t) &\geq \rho_1\frac{c_{\gamma_1}\|\gamma_1\|A_1^{\rho_1,\rho_2}}{\rho_1(1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1])} \\
&\quad + \rho_1 f_{1,(\rho_1,\rho_2)}\left(\frac{c_{\gamma_1}\|\gamma_1\|}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}\int_{a_1}^{b_1}\mathcal{K}_1(s)g_1(s)\,\mathrm{d}s + \inf_{t\in[a_1,b_1]}\int_{a_1}^{b_1} k_1(t,s)g_1(s)\,\mathrm{d}s\right) + \lambda.
\end{aligned}$$

Using the hypothesis (2.13) we obtain $\rho_1 > \rho_1 + \lambda$, a contradiction since $\lambda \geq 0$. □

In the following Lemma, we provide a result of index 0 for the operator T on V_{ρ_1,ρ_2}, by controlling the growth of just one nonlinearity f_i in a larger domain. Nonlinearities with different growths were also studied in [**7**, **24**, **26**, **43**, **44**, **49**].

Lemma 2.8. *Assume that*

$(\mathrm{I}^0_{\rho_1,\rho_2})^\circ$ *there exist* $\rho_1, \rho_2 > 0$, $A_1^{\rho_1,\rho_2}, A_2^{\rho_1,\rho_2} \geq 0$, *and linear functionals* $\alpha_{ij}^{\rho_1,\rho_2}[\cdot] : K_j \to [0, +\infty)$, *involving positive Stieltjes measures given by*

$$\alpha_{ij}^{\rho_1,\rho_2}[w] = \int_0^1 w(t)dC_{ij}(t),$$

where C_{ij} *is of bounded variation, such that, for at least one* $i = 1, 2$,

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- $\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i] < 1$,
- $H_i[u,v] \geq A_i^{\rho_1,\rho_2} + \alpha_{i1}^{\rho_1,\rho_2}[u] + \alpha_{i2}^{\rho_1,\rho_2}[v]$ *for* $(u,v) \in \partial V_{\rho_1,\rho_2}$,
- *the following inequality holds:*

$$f^{\circ}_{i,(\rho_1,\rho_2)}\left(\frac{c_{\gamma_i}\|\gamma_i\|}{1-\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i]}\int_{a_i}^{b_i}\mathcal{K}_i(s)g_i(s)\,\mathrm{d}s + \frac{1}{M_i}\right) + \frac{c_{\gamma_i}\|\gamma_i\|A_i^{\rho_1,\rho_2}}{\rho_i(1-\alpha_{ii}^{\rho_1,\rho_2}[\gamma_i])} > 1, \tag{2.14}$$

where

$$f^{\circ}_{1,(\rho_1,\rho_2)} = \inf\left\{\frac{f_1(u,v)}{\rho_1} : (u,v) \in [0,\rho_1/c_1]\times[-\rho_2/c_2,\rho_2/c_2]\right\},$$
$$f^{\circ}_{2,(\rho_1,\rho_2)} = \inf\left\{\frac{f_2(u,v)}{\rho_2} : (u,v) \in [0,\rho_1/c_1]\times[0,\rho_2/c_2]\right\}.$$

Then $i_K(T, V_{\rho_1,\rho_2}) = 0$.

Proof. Suppose that condition (2.14) holds for $i = 1$. Let $(u,v) \in \partial V_{\rho_1,\rho_2}$ and $\lambda \geq 0$ such that $(u,v) = T(u,v) + \lambda(1,1)$. Therefore, for all $t \in [a_1,b_1]$, we have $\min u(t) \leq \rho_1$, $0 \leq u(t) \leq \rho_1/c_1$ and $-\rho_2/c_2 \leq v(t) \leq \rho_2/c_2$. For $t \in [0,1]$, we have

$$u(t) = \gamma_1(t)H_1[u,v] + \int_0^1 k_1(t,s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s + \lambda.$$

As in the proof of Lemma 2.7, taking the minimum over $[a_1,b_1]$ gives

$$\begin{aligned}\rho_1 \geq \min_{t\in[a_1,b_1]} u(t) \geq \rho_1\frac{c_{\gamma_1}\|\gamma_1\|A_1^{\rho_1,\rho_2}}{\rho_1(1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1])}\\ + \rho_1 f^{\circ}_{1,(\rho_1,\rho_2)}\left(\frac{c_{\gamma_1}\|\gamma_1\|}{1-\alpha_{11}^{\rho_1,\rho_2}[\gamma_1]}\int_{a_1}^{b_1}\mathcal{K}_1(s)g_1(s)\,\mathrm{d}s + \inf_{t\in[a_1,b_1]}\int_{a_1}^{b_1}k_1(t,s)g_1(s)\,\mathrm{d}s\right) + \lambda.\end{aligned}$$

Using the hypothesis (2.14) we obtain $\rho_1 > \rho_1 + \lambda$, a contradiction. □

Remark 2.9. In the case of $[a_1,b_1] = [a_2,b_2]$ the assumptions on the nonlinearities f_i can be relaxed; for example, in condition $(\mathrm{I}^0_{\rho_1,\rho_2})^{\circ}$ we have to control the growth of f_i in the smaller set $[0,\rho_1/c_1]\times[0,\rho_2/c_2]$. We refer to the paper [**26**] for a statement of similar results.

We now state a result regarding the existence of at least one, two or three non-trivial solutions. The proof, which follows by the properties of fixed point index, is omitted. We can state results for four or more non-trivial solutions by expanding the lists in conditions (S_5), (S_6); see, for example, the paper [**35**].

Theorem 2.10. *If one of the following conditions holds:*

(S_1) *for* $i = 1, 2$ *there exist* $\rho_i, r_i \in (0, \infty)$ *with* $\rho_i/c_i < r_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ [or $(\mathrm{I}^0_{\rho_1,\rho_2})^\circ$] *and* $(\mathrm{I}^1_{r_1,r_2})$ *hold;*

(S_2) *for* $i = 1, 2$ *there exist* $\rho_i, r_i \in (0, \infty)$ *with* $\rho_i < r_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$ *and* $(\mathrm{I}^0_{r_1,r_2})$ *hold*

then the system (2.5) has at least one non-trivial solution in K.

If one of the following conditions holds:

(S_3) *for* $i = 1, 2$ *there exist* $\rho_i, r_i, s_i \in (0, \infty)$ *with* $\rho_i/c_i < r_i < s_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ [or $(\mathrm{I}^0_{\rho_1,\rho_2})^\circ$], $(\mathrm{I}^1_{r_1,r_2})$ *and* $(\mathrm{I}^0_{s_1,s_2})$ *hold;*

(S_4) *for* $i = 1, 2$ *there exist* $\rho_i, r_i, s_i \in (0, \infty)$ *with* $\rho_i < r_i$ *and* $r_i/c_i < s_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$ *and* $(\mathrm{I}^1_{s_1,s_2})$ *hold*

then the system (2.5) has at least two non-trivial solutions in K.

If one of the following conditions holds:

(S_5) *for* $i = 1, 2$ *there exist* $\rho_i, r_i, s_i, \sigma_i \in (0, \infty)$ *with* $\rho_i/c_i < r_i < s_i$ *and* $s_i/c_i < \sigma_i$ *such that* $(\mathrm{I}^0_{\rho_1,\rho_2})$ [or $(\mathrm{I}^0_{\rho_1,\rho_2})^\circ$], $(\mathrm{I}^1_{r_1,r_2})$, $(\mathrm{I}^0_{s_1,s_2})$ *and* $(\mathrm{I}^1_{\sigma_1,\sigma_2})$ *hold;*

(S_6) *for* $i = 1, 2$ *there exist* $\rho_i, r_i, s_i, \sigma_i \in (0, \infty)$ *with* $\rho_i < r_i$ *and* $r_i/c_i < s_i < \sigma_i$ *such that* $(\mathrm{I}^1_{\rho_1,\rho_2})$, $(\mathrm{I}^0_{r_1,r_2})$, $(\mathrm{I}^1_{s_1,s_2})$ *and* $(\mathrm{I}^0_{\sigma_1,\sigma_2})$ *hold*

then the system (2.5) has at least three non-trivial solutions in K.

The results of this subsection—for example Theorem 2.10—can be applied to the integral system (2.5), yielding results for the elliptic system (2.1). Similar statements were given in [**37**, **38**] in the case of annular domains.

In the following example we illustrate the applicability of Theorem 2.10.

Example 2.11. In $\mathbb{R}^3$, consider the elliptic system

$$\begin{cases} \Delta u + |x|^{-4}\left[\dfrac{3}{10}(u^3 + |v|^3) + \dfrac{1}{2}\right] = 0, \ |x| \in [1, +\infty), \\ \Delta v + |x|^{-4}(u^{1/2} + v^2 + 1) = 0, \quad |x| \in [1, +\infty), \\ u(x) = 2u(4x) \quad \text{for } x \in \partial B_1, \ \lim\limits_{|x|\to+\infty} u(|x|) = \dfrac{1}{10}\sqrt{u(3x)} + \dfrac{1}{10}v^3\left(\dfrac{7}{2}x\right), \\ v(x) = -\dfrac{4}{3}\dfrac{\partial v}{\partial r}(2x) \quad \text{for } \ x \in \partial B_1, \ \lim\limits_{|x|\to+\infty} v(|x|) = \dfrac{1}{5}\sqrt{u\left(\dfrac{7}{3}x\right)} + \dfrac{1}{10}v^2\left(\dfrac{5}{2}x\right). \end{cases} \tag{2.15}$$

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With the system (2.15) we associate the system of second-order ODEs

$$\begin{cases} u''(t) + \frac{3}{10}(u^3(t) + |v|^3(t)) + \frac{1}{2} = 0, \ t \in [0,1], \\ v''(t) + u^{1/2}(t) + v^2(t) + 1 = 0, \ t \in [0,1], \\ u(0) = \frac{1}{10}\sqrt{u\left(\frac{1}{3}\right)} + \frac{1}{10}v^3\left(\frac{2}{7}\right), \quad u(1) = 2u\left(\frac{1}{4}\right), \\ v(0) = \frac{1}{5}\sqrt{u\left(\frac{3}{7}\right)} + \frac{1}{10}v^2\left(\frac{2}{5}\right), \quad v(1) = \frac{1}{3}v'\left(\frac{1}{2}\right). \end{cases}$$

By direct computation, we have

$$\frac{1}{m_1} = \sup_{t\in[0,1]} \int_0^1 k_1(t,s)\,\mathrm{d}s = \frac{49}{128} \quad \text{and} \quad \frac{1}{m_2} = \sup_{t\in[0,1]} \int_0^1 |k_2(t,s)|\,\mathrm{d}s = \frac{1}{8}.$$

We fix $[a_1, b_1] = [a_2, b_2] = [1/4, 1/2]$, obtaining $c_1 = 1/32, \ c_2 = 1/4$,

$$\frac{1}{M_1} = \inf_{t\in[1/4,1/2]} \int_{1/4}^{1/2} k_1(t,s)\,\mathrm{d}s = \frac{3}{16}$$

and

$$\frac{1}{M_2} = \inf_{t\in[1/4,1/2]} \int_{1/4}^{1/2} k_2(t,s)\,\mathrm{d}s = \frac{3}{32}.$$

With the choice of

$$\begin{aligned} \rho_1 &= \tfrac{1}{16}, \quad \rho_2 = \tfrac{1}{32}, \quad \alpha_{11}^{\rho_1,\rho_2}[u] = \alpha_{21}^{\rho_1,\rho_2}[u] = 0, \\ r_1 &= 2.01, \quad r_2 = 1, \quad A_1^{r_1,r_2} = \tfrac{1}{10}\sqrt{2.01}, \quad \alpha_{12}^{r_1,r_2}[v] = \tfrac{1}{10}\ v\left(\tfrac{2}{7}\right), \\ A_2^{r_1,r_2} &= \tfrac{1}{5}\sqrt{2.01}, \quad \alpha_{22}^{r_1,r_2}[v] = \tfrac{1}{10}\ v\left(\tfrac{2}{5}\right), \\ s_1 &= 5, \quad s_2 = 11, \quad \alpha_{11}^{s_1,s_2}[u] = \tfrac{1}{130}\,u\left(\tfrac{1}{3}\right), \\ \alpha_{21}^{s_1,s_2}[u] &= \tfrac{1}{100}\ u\left(\tfrac{3}{7}\right), \quad \alpha_{22}^{s_1,s_2}[v] = \tfrac{1}{100}v\left(\tfrac{2}{5}\right), \end{aligned}$$

we obtain

$$\begin{aligned} &\alpha_{22}^{r_1,r_2}[\gamma_2] < 1, \quad \alpha_{11}^{s_1,s_2}[\gamma_1] < 1, \quad \alpha_{22}^{s_1,s_2}[\gamma_2] < 1, \\ &H_1[u,v] \le A_1^{r_1,r_2} + \alpha_{12}^{r_1,r_2}[v], \end{aligned}$$

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$$H_2[u,v] \leq A_2^{r_1,r_2} + \alpha_{22}^{r_1,r_2}[v], \quad (u,v) \in [0,r_1] \times [0,r_2],$$

$$H_1[u,v] \geq \alpha_{11}^{s_1,s_2}[u], \quad (u,v) \in [s_1, s_1/c_1] \times [0, s_2/c_2],$$

$$H_2[u,v] \geq \alpha_{21}^{s_1,s_2}[u] + \alpha_{22}^{s_1,s_2}[v], \quad (u,v) \in [0, s_1/c_1] \times [s_2, s_2/c_2],$$

$$\inf\{f_2(u,v): \ (u,v) \in [0, 32\rho_1] \times [0, 4\rho_2]\} = f_2(0,0) = 1 > 0.33,$$

$$\sup\{f_1(u,v): \ (u,v) \in [0, r_1] \times [-r_2, r_2]\} = f_1(2.01, 1) = 3.236 < 3.878,$$

$$\sup\{f_2(u,v): \ (u,v) \in [0, r_1] \times [-r_2, r_2]\} = f_2(2.01, 1) = 3.417 < 5.125,$$

$$\inf\{f_1(u,v): \ (u,v) \in [s_1, 32s_1] \times [0, 4s_2]\} = f_1(5,0) = 38 > 26.557,$$

$$\inf\{f_2(u,v): \ (u,v) \in [0, 32s_1] \times [s_2, 4s_2]\} = f_2(0,11) = 122 > 117.075.$$

Thus the conditions $(\mathrm{I}^0_{\rho_1,\rho_2})^\circ$, $(\mathrm{I}^1_{r_1,r_2})$ and $(\mathrm{I}^0_{s_1,s_2})$ are satisfied and, from Theorem 2.10, the system (2.15) has at least two non-trivial solutions.

2.2. Non-existence results for perturbed integral systems

We now present some non-existence results for the integral system (2.5). We begin with a theorem wherein the nonlinearities and the functionals are sufficiently 'small'.

Theorem 2.12. *Assume that, for $i = 1, 2$, there exist $A_i > 0$ and $\lambda_i > 0$ such that for $f_i : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ one has*

- $H_i[u_1, u_2] \leq A_i \|u_i\|$ *for* $(u_1, u_2) \in K$,
- $f_i(z_1, z_2) \leq \lambda_i m_i |z_i|$ *for* $(z_1, z_2) \in [0, +\infty) \times \mathbb{R}$,
- $\|\gamma_i\| A_i + \lambda_i < 1$.

Then there is no non-trivial solution of system (2.5) in K.

Proof. Suppose that there exists $(u,v) \in K$ such that $(u,v) = T(u,v)$ and assume that $||v|| = \nu > 0$. Then we have, for $t \in [0,1]$,

$$\begin{aligned}
|v(t)| &= |\gamma_2(t) H_2[u,v] + F_2(u,v)(t)| \\
&\leq \|\gamma_2\| H_2[u,v] + \int_0^1 |k_2(t,s)|\, g_2(s)\, f_2(u(s), v(s))\, \mathrm{d}s \\
&\leq \|\gamma_2\| A_2 \|v\| + \lambda_2 m_2 \int_0^1 |k_2(t,s)|\, g_2(s) |v(s)|\, \mathrm{d}s \\
&\leq \|\gamma_2\| A_2 \nu + \lambda_2 m_2 \nu \int_0^1 |k_2(t,s)|\, g_2(s)\, \mathrm{d}s.
\end{aligned}$$

Taking the supremum over $[0,1]$ gives

$$\nu \leq \nu\left(\|\gamma_2\| A_2 + \lambda_2 m_2 \sup_{t\in[0,1]} \int_0^1 |k_2(t,s)|\, g_2(s)\,\mathrm{d}s \right).$$

We obtain $\nu < \nu$, a contradiction that proves the result.

The case $||u|| = \nu > 0$ is simpler and we omit the proof. □

In the next theorem, the nonlinearities and the functionals are sufficiently 'large'.

Theorem 2.13. *Assume that, for $i = 1, 2$, there exist $A_i > 0$ and $\lambda_i > 0$ such that for $f_i : [0, +\infty) \times \mathbb{R} \to [0, +\infty)$ one has*

- $H_i[u_1, u_2] \geq A_i \|u_i\|$ *for* $(u_1, u_2) \in K$,
- $f_i(z_1, z_2) \geq \lambda_i M_i\, z_i$ *for* $(z_1, z_2) \in [0, +\infty) \times \mathbb{R}$,
- $c_{\gamma_i} \|\gamma_i\| A_i + \lambda_i > 1$.

Then there is no non-trivial solution of system (2.5) in K.

Proof. Suppose that there exists $(u, v) \in K$ such that $(u, v) = T(u, v)$ with $(u, v) \neq 0$ and $\min_{t\in[a_1,b_1]} u(t) = \theta > 0$ (the other case is similar). Then we have, for $t \in [a_1, b_1]$,

$$\begin{aligned}
u(t) &= \gamma_1(t) H_1[u, v] + F_1(u, v)(t) \\
&\geq \gamma_1(t) H_1[u, v] + \int_{a_1}^{b_1} k_1(t,s) g_1(s) f_1(u(s), v(s))\,\mathrm{d}s \\
&\geq c_{\gamma_1} \|\gamma_1\| A_1 \|u\| + \lambda_1 M_1 \int_{a_1}^{b_1} k_1(t,s) g_1(s) u(s)\,\mathrm{d}s \\
&\geq c_{\gamma_1} \|\gamma_1\| A_1 \theta + \lambda_1 M_1 \theta \int_{a_1}^{b_1} k_1(t,s) g_1(s)\,\mathrm{d}s.
\end{aligned}$$

Taking the infimum for $t \in [a_1, b_1]$ gives

$$\theta \geq \theta\left(c_{\gamma_1} \|\gamma_1\| A_1 + \lambda_1 M_1 \inf_{t\in[a_1,b_1]} \int_{a_1}^{b_1} k_1(t,s) g_1(s)\,\mathrm{d}s \right).$$

We obtain $\theta > \theta$, a contradiction that proves the result. □

In the last non-existence result, the contribution of the nonlinearities and the functionals is of a 'mixed' type.

Theorem 2.14. *Assume that, for $i = 1, 2$, there exist $A_i, \lambda_i > 0$ such that the assumptions in Theorem 2.12 are verified for an $i \in \{1, 2\}$ and the assumptions in Theorem 2.13 are verified for the other index. Then there is no non-trivial solution of the system (2.5) in K.*

Proof. Assume, on the contrary, that there exists an element $(u,v) \in K$ such that $(u,v) = T(u,v)$ with, for example, $\|u\| \neq 0$. Then the functions γ_1, H_1 and f_1 satisfy either the assumptions in Theorem 2.12 or the assumptions in Theorem 2.13, and the proof follows. □

We present an example that illustrates the applicability of Theorem 2.14.

Example 2.15. Fix in the system (2.5) $g_1 \equiv 1$, $\beta_1 = 2$ and $\eta = 1/4$. By direct calculations, we obtain $m_1 = 128/49$. Thus, the functions

$$f_1(u,v) = \begin{cases} \frac{1}{4}u^2 \arctan v^2, & u \le 1, \\ \\ \frac{1}{4}u^{2/3} \arctan v^2, & u > 1, \end{cases} \quad \text{and} \quad H_1[u,v] = \tfrac{1}{6}u(1/2)\,|\sin(v(1/3))|$$

satisfy the conditions of Theorem 2.12 with $A_1 = 1/6$ and $\lambda_1 = 1/5$.

Now fix $g_2 \equiv 1$, $\beta_2 = 1/3$, $\xi = 1/2$ and $[a_2, b_2] = [1/4, 1/2]$. In this case we obtain $M_2 = 32/3$. Thus, the functions

$$f_2(u,v) = \begin{cases} \frac{32}{3}\,(2 + \cos u)\, v^{2/3}, & v \le 1, \\ \\ \frac{32}{3}\,(2 + \cos u)\, v^2, & v > 1, \end{cases} \quad \text{and} \quad H_2[u,v] = \mathrm{e}^{u(1/2)}\, v(1/3)$$

satisfy the conditions of Theorem 2.13 with $A_2 = 1/6$ and $\lambda_2 = 1$.

With these choices, by Theorem 2.14, the system (2.5) has no solutions in K.

3. Perturbing a system with Dirichlet BCs

In [**15**] the authors raised the question of the existence of multiple positive solutions of the elliptic system (1.1) under more general BCs.

Using our theory, we can deal with the system of elliptic equations

$$\begin{cases} \Delta u + h_1(|x|) f_1(u,v) = 0, & |x| \in [1, +\infty), \\ \Delta v + h_2(|x|) f_2(u,v) = 0, & |x| \in [1, +\infty) \end{cases} \tag{3.1}$$

subject to the BCs

$$\begin{cases} u(x) = H_1[u,v] \quad \text{for } x \in \partial B_1, & \lim\limits_{|x| \to +\infty} u(|x|) = 0, \\ \\ v(x) = H_2[u,v] \quad \text{for } x \in \partial B_1, & \lim\limits_{|x| \to +\infty} v(|x|) = 0 \end{cases} \tag{3.2}$$

or

$$\begin{cases} u(x) = 0 \quad \text{for } x \in \partial B_1, & \lim\limits_{|x| \to +\infty} u(|x|) = H_1[u,v], \\ \\ v(x) = 0 \quad \text{for } x \in \partial B_1, & \lim\limits_{|x| \to +\infty} v(|x|) = H_2[u,v] \end{cases}$$

or

$$\begin{cases} u(x) = H_1[u,v] \quad \text{for } x \in \partial B_1, \quad \lim\limits_{|x|\to+\infty} u(|x|) = 0, \\ v(x) = 0 \quad \text{for } x \in \partial B_1, \quad \lim\limits_{|x|\to+\infty} v(|x|) = H_2[u,v]. \end{cases}$$

We examine, for example, the perturbed integral system

$$\begin{cases} u(t) = t\,H_1[u,v] + \displaystyle\int_0^1 k(t,s)g_1(s)f_1(u(s),v(s))\,\mathrm{d}s, \\ v(t) = t\,H_2[u,v] + \displaystyle\int_0^1 k(t,s)g_2(s)f_2(u(s),v(s))\,\mathrm{d}s \end{cases}$$

associated with the system (3.1) with the BCs (3.2).

In this case, the Green's function k is given by

$$k(t,s) := \begin{cases} s(1-t), & s \le t, \\ t(1-s), & s > t. \end{cases}$$

In $[0,1]\times[0,1]$ the kernel k is continuous and non-negative and has been studied, for example, in [**48**], where it was shown that

$$\begin{aligned} k(t,s) &\le \Phi(s) \quad \text{for } t,\, s \in [0,1], \\ k(t,s) &\ge c_k\Phi(s) \quad \text{for } t \in [a,b] \text{ and } s \in [0,1], \end{aligned}$$

where $\Phi(s) = s(1-s)$, $[a,b]$ is an arbitrary subset of $(0,1)$ and $c_k = \min\{a, 1-b\}$. Set $\gamma_1(t) = \gamma_2(t) =: \gamma(t) = t$; we have $\gamma \in C[0,1]$ and

$$\gamma(t) \ge c_\gamma\|\gamma\| \quad \text{for } t \in [a,b],$$

where $||\gamma|| = 1$ and $c_\gamma = a$.

Owing to the properties above, we are able to work in the cone of non-negative functions $K := \tilde{K}\times\tilde{K}$ in $C[0,1]\times C[0,1]$, where

$$\tilde{K} = \{w \in C[0,1] :\ w \ge 0,\ \min_{t\in[a,b]} w(t) \ge c\|w\|\},$$

with $c = \min\{c_k, c_\gamma\}$. Our theory works in this case with suitable changes.

For example (with abuse of notation), in condition (2.10) of Lemma 2.5 we would consider

$$f_i^{\rho_1,\rho_2} := \sup\left\{\frac{f_i(u,v)}{\rho_i} :\ (u,v) \in [0,\rho_1]\times[0,\rho_2]\right\}, \quad \frac{1}{m_i} := \sup_{t\in[0,1]}\int_0^1 k(t,s)g_i(s)\,\mathrm{d}s,$$

and in condition (2.13) of Lemma 2.7 we would have

$$f_{1,(\rho_1,\rho_2)} = \inf\left\{\frac{f_1(u,v)}{\rho_1} :\ (u,v) \in [\rho_1,\rho_1/c]\times[0,\rho_2/c]\right\}.$$

In the next example we briefly illustrate the constants involved in our theory.

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Example 3.1. Consider in $\mathbb{R}^3$ the elliptic system

$$\begin{cases} \Delta u + |x|^{-4}\left(u^3 + v^2 + \dfrac{1}{2}\right) = 0, \ |x| \in [1, +\infty), \\ \Delta v + |x|^{-4}\left(\dfrac{\sqrt{u}}{2} + v^2\right) = 0, \ |x| \in [1, +\infty), \\ u(x) = \dfrac{1}{10} + \dfrac{\sqrt{v(2x)}}{2\sqrt{5}} \quad \text{for } x \in \partial B_1, \ \lim\limits_{|x|\to+\infty} u(|x|) = 0, \\ v(x) = \dfrac{1}{10} + \dfrac{u^2(3x)}{20} \quad \text{for } x \in \partial B_1, \ \lim\limits_{|x|\to+\infty} v(|x|) = 0. \end{cases} \tag{3.3}$$

With system (3.3) we associate the system of second-order ODEs

$$\begin{cases} u''(t) + u^3(t) + v^2(t) + \dfrac{1}{2} = 0, \ t \in [0,1], \\ v''(t) + \dfrac{\sqrt{u(t)}}{2} + v^2(t) = 0, \ t \in [0,1], \\ u(0) = 0, \quad u(1) = \dfrac{1}{10} + \dfrac{\sqrt{v(1/2)}}{2\sqrt{5}}, \\ v(0) = 0, \quad v(1) = \dfrac{1}{10} + \dfrac{u^2(1/3)}{20}, \end{cases}$$

and the perturbed integral system

$$\begin{cases} u(t) = t\,H_1[u,v] + \displaystyle\int_0^1 k(t,s) f_1(u(s), v(s))\,\mathrm{d}s, \\ v(t) = t\,H_2[u,v] + \displaystyle\int_0^1 k(t,s) f_2(u(s), v(s))\,\mathrm{d}s, \end{cases} \tag{3.4}$$

with $H_1[u,v] = 1/10 + \sqrt{v(1/2)}/2\sqrt{5}$ and $H_2[u,v] = 1/10 + ((u^2(1/3))/(20))$.

By direct computation, we have

$$\frac{1}{m} = \sup_{t\in[0,1]} \int_0^1 k(t,s)\,\mathrm{d}s = \frac{1}{8}.$$

We fix $[a,b] = [1/4, 3/4]$, obtaining $c = 1/4$,

$$\frac{1}{M} := \inf_{t\in[1/4,3/4]} \int_{1/4}^{3/4} k(t,s)\,\mathrm{d}s = \frac{1}{16}.$$

With the choice of

$$\rho_1 = \frac{1}{39}, \quad \rho_2 = \frac{1}{10}, \quad A_1^{\rho_1,\rho_2} = \frac{1}{10}, \quad \alpha_{12}^{\rho_1,\rho_2}[v] = \frac{1}{2\sqrt{5}}v(1/2),$$
$$r_1 = 2, \quad r_2 = 2, \quad A_1^{r_1,r_2} = \frac{1}{10} + \frac{1}{2\sqrt{5}}\sqrt{2}, \quad A_2^{r_1,r_2} = \frac{1}{10}, \quad \alpha_{21}^{r_1,r_2}[u] = \frac{1}{10}u(1/3),$$
$$s_1 = 5, \quad s_2 = 16, \quad A_1^{s_1,s_2} = \frac{1}{10}, \quad \alpha_{12}^{s_1,s_2}[v] = \frac{1}{16\sqrt{5}}v(1/2), \quad A_2^{s_1,s_2} = \frac{1}{10},$$

one can verify that

$$\begin{aligned}
&H_1[u,v] \geq A_1^{\rho_1,\rho_2} + \alpha_{12}^{\rho_1,\rho_2}[v], \quad (u,v) \in [0,4\rho_1] \times [0,4\rho_2],\\
&H_1[u,v] \leq A_1^{r_1,r_2}, \quad H_2[u,v] \leq A_2^{r_1,r_2} + \alpha_{21}^{r_1,r_2}[u], \quad (u,v) \in [0,r_1] \times [0,r_2],\\
&H_1[u,v] \geq A_1^{s_1,s_2} + \alpha_{12}^{s_1,s_2}[v], \quad (u,v) \in [s_1,4s_1] \times [0,4s_2],\\
&H_2[u,v] \geq A_2^{s_1,s_2}, \quad (u,v) \in [0,4s_1] \times [s_2,4s_2],\\
&\inf\{f_1(u,v) : (u,v) \in [0,4\rho_1] \times [0,4\rho_2]\} = f_1(0,0) = 0.5 > 0.01,\\
&\sup\{f_1(u,v) : (u,v) \in [0,r_1] \times [0,r_2]\} = f_1(2,2) = 12.5 < 12.67,\\
&\sup\{f_2(u,v) : (u,v) \in [0,r_1] \times [0,r_2]\} = f_2(2,2) = 4.70 < 13.6,\\
&\inf\{f_1(u,v) : (u,v) \in [s_1,4s_1] \times [0,4s_2]\} = f_1(5,0) = 125.5 > 78,\\
&\inf\{f_2(u,v) : (u,v) \in [0,4s_1] \times [s_2,4s_2]\} = f_2(0,16) = 256 > 171.6.
\end{aligned}$$

It follows that conditions $(\mathrm{I}^0_{\rho_1,\rho_2})^\circ$, $(\mathrm{I}^1_{r_1,r_2})$ and $(\mathrm{I}^0_{s_1,s_2})$ are satisfied and therefore the system (3.4) has at least two positive solutions.

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References

1. A. Abebe, M. Chhetri, L. Sankar and R. Shivaji, Positive solutions for a class of superlinear semipositone systems on exterior domains, *Bound. Value Probl.* **2014**(198) (2014), 9.
2. A. Aftalion and J. Busca, Symétrie radiale pour des problèmes elliptiques surdéterminés posés dans des domaines extérieurs, *C. R. Acad. Sci. Paris Sér. I Math.* **324** (1997), 633–638.
3. H. Amann, Fixed point equations and nonlinear eigenvalue problems in ordered Banach spaces, *SIAM. Rev.* **18** (1976), 620–709.
4. D. Butler, E. Ko, E. K. Lee, E. Kyoung and R. Shivaji, Positive radial solutions for elliptic equations on exterior domains with nonlinear boundary conditions, *Commun. Pure Appl. Anal.* **13** (2014), 2713–2731.
5. A. Castro, L. Sankar and R. Shivaji, Uniqueness of nonnegative solutions for semipositone problems on exterior domains, *J. Math. Anal. Appl.* **394** (2012), 432–437.
6. X. Cheng and C. Zhong, Existence of positive solutions for a second-order ordinary differential system, *J. Math. Anal Appl.* **312** (2005), 14–23.
7. F. Cianciaruso and P. Pietramala, Multiple positive solutions of a (p_1, p_2)-Laplacian system with nonlinear BCs, *Bound. Value Probl.* **2015**(163) (2015), 18.

8. F. Cianciaruso, G. Infante and P. Pietramala, Solutions of perturbed Hammerstein integral equations with applications, *Nonlinear Anal. Real World Appl.* **33** (2017), 317–347.
9. R. Dhanya, Q. Morris and R. Shivaji, Existence of positive radial solutions for superlinear, semipositone problems on the exterior of a ball, *J. Math. Anal. Appl.* **434** (2016), 1533–1548.
10. S. Djebali and A. Orpel, The continuous dependence on parameters of solutions for a class of elliptic problems on exterior domains, *Nonlinear Anal.* **73** (2010), 660–672.
11. J. M. do Ó, S. Lorca and P. Ubilla, Local superlinearity for elliptic systems involving parameters, *J. Differ. Equ.* **211** (2005), 1–19.
12. J. M. do Ó, S. Lorca, J. Sánchez and P. Ubilla, Positive radial solutions for some quasilinear elliptic systems in exterior domains, *Commun. Pure Appl. Anal.* **5** (2006), 571–581.
13. J. M. do Ó, S. Lorca, J. Sánchez and P. Ubilla, Non-homogeneous elliptic equations in exterior domains, *Proc. Roy. Soc. Edinb. Sect. A* **136** (2006), 139–147.
14. J. M. do Ó, S. Lorca, J. Sánchez and P. Ubilla, Positive solutions for a class of multiparameter ordinary elliptic systems, *J. Math. Anal. Appl.* **332** (2007), 1249–1266.
15. J. M. do Ó, S. Lorca, J. Sánchez and P. Ubilla, Superlinear ordinary elliptic systems involving parameters, *Mat. Contemp.* **32** (2007), 107–127.
16. J. M. do Ó, S. Lorca, J. Sánchez and P. Ubilla, Positive solutions for some nonlocal and nonvariational elliptic systems, *Complex Var. Elliptic Equ.* **61** (2016), 297–314.
17. D. Franco, G. Infante and D. O'Regan, Nontrivial solutions in abstract cones for Hammerstein integral systems, *Dyn. Contin. Discrete Impuls. Syst. Ser. A Math. Anal.* **14** (2007), 837–850.
18. B. Gidas, W.-M. Ni and L. Nirenberg, Symmetry and related properties via the maximum principle, *Comm. Math. Phys.* **68** (1979), 209–243.
19. C. S. Goodrich, Nonlocal systems of BVPs with asymptotically superlinear boundary conditions, *Comment. Math. Univ. Carolin.* **53** (2012), 79–97.
20. C. S. Goodrich, Nonlocal systems of BVPs with asymptotically sublinear boundary conditions, *Appl. Anal. Discrete Math.* **6** (2012), 174–193.
21. D. Guo and V. Lakshmikantham, *Nonlinear problems in abstract cones* (Academic Press, 1988).
22. G. Han and J. Wang, Multiple positive radial solutions of elliptic equations in an exterior domain, *Monatsh. Math.* **148** (2006), 217–228.
23. G. Infante and P. Pietramala, Eigenvalues and non-negative solutions of a system with nonlocal BCs, *Nonlinear Stud.* **16** (2009), 187–196.
24. G. Infante and P. Pietramala, Existence and multiplicity of non-negative solutions for systems of perturbed Hammerstein integral equations, *Nonlinear Anal.* **71** (2009), 1301–1310.
25. G. Infante and P. Pietramala, Multiple nonnegative solutions of systems with coupled nonlinear boundary conditions, *Math. Methods Appl. Sci.* **37** (2014), 2080–2090.
26. G. Infante and P. Pietramala, Nonzero radial solutions for a class of elliptic systems with nonlocal BCs on annular domains, *NoDEA Nonlinear Differ. Equ. Appl.* **22** (2015), 979–1003.
27. G. Infante and J. R. L. Webb, Nonzero solutions of Hammerstein integral equations with discontinuous kernels, *J. Math. Anal. Appl.* **272** (2002), 30–42.
28. G. Infante and J. R. L. Webb, Three point boundary value problems with solutions that change sign, *J. Integral Equations Appl.* **15** (2003), 37–57.
29. G. Infante, F. M. Minhós and P. Pietramala, Non-negative solutions of systems of ODEs with coupled boundary conditions, *Commun. Nonlinear Sci. Numer. Simul.* **17** (2012), 4952–4960.

30. G. Infante, P. Pietramala and F. A. F. Tojo, Nontrivial solutions of local and nonlocal Neumann boundary value problems, *Proc. Roy. Soc. Edinb. Sect. A* **146** (2016), 337–369.
31. P. Kang and Z. Wei, Three positive solutions of singular nonlocal boundary value problems for systems of nonlinear second-order ordinary differential equations, *Nonlinear Anal.* **70** (2009), 444–451.
32. G. L. Karakostas, Existence of solutions for an n-dimensional operator equation and applications to BVPs, *Electron. J. Differ. Equ.* **17** (2014), 1–17.
33. E. Ko, M. Ramaswamy and R. Shivaji, Uniqueness of positive radial solutions for a class of semipositone problems on the exterior of a ball, *J. Math. Anal. Appl.* **423** (2015), 399–409.
34. M. A. Krasnosel'skiĭ and P. P. Zabreĭko, *Geometrical methods of nonlinear analysis* (Springer-Verlag, Berlin, 1984).
35. K. Q. Lan, Multiple positive solutions of Hammerstein integral equations with singularities, *Diff. Eqns and Dynam. Syst.* **8** (2000), 175–195.
36. K. Q. Lan, Multiple positive solutions of semilinear differential equations with singularities, *J. Lond. Math. Soc.* **63** (2001), 690–704.
37. K. Q. Lan and W. Lin, Positive solutions of systems of singular Hammerstein integral equations with applications to semilinear elliptic equations in annuli, *Nonlinear Anal.* **74** (2011), 7184–7197.
38. K. Q. Lan and J. R. L. Webb, Positive solutions of semilinear differential equations with singularities, *J. Differ. Equ.* **148** (1998), 407–421.
39. Y. H. Lee, A multiplicity result of positive radial solutions for a multiparameter elliptic system on an exterior domain, *Nonlinear Anal.* **45** (2001), 597–611.
40. E. K. Lee, R. Shivaji and B. Son, Positive radial solutions to classes of singular problems on the exterior domain of a ball, *J. Math. Anal. Appl.* **434** (2016), 1597–1611.
41. A. Orpel, Nonlinear BVPS with functional parameters, *J. Differ. Equ.* **246** (2009), 1500–1522.
42. A. Orpel, Increasing sequences of positive evanescent solutions of nonlinear elliptic equations, *J. Differ. Equ.* **259** (2015), 1743–1756.
43. R. Precup, Componentwise compression-expansion conditions for systems of nonlinear operator equations and applications, *Mathematical models in engineering, biology and medicine*, in: AIP Conference Proceedings, Volume 1124 (American Institute of Physics, 2009), pp. 284–293.
44. R. Precup, Existence, localization and multiplicity results for positive radial solutions of semilinear elliptic systems, *J. Math. Anal. Appl.* **352** (2009), 48–56.
45. L. Sankar, S. Sasi and R. Shivaji, Semipositone problems with falling zeros on exterior domains, *J. Math. Anal. Appl.* **401** (2013), 146–153.
46. R. Stanczy, Positive solutions for superlinear elliptic equations, *J. Math. Anal. Appl.* **283** (2003), 159–166.
47. J. R. L. Webb, Positive solutions of some three point boundary value problems via fixed point index theory, *Nonlinear Anal.* **47** (2001), 4319–4332.
48. J. R. L. Webb and G. Infante, Positive solutions of nonlocal boundary value problems: a unified approach, *J. Lond. Math. Soc.* **74** (2006), 673–693.
49. Z. Yang, Positive solutions to a system of second-order nonlocal boundary value problems, *Nonlinear Anal.* **62** (2005), 1251–1265.