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MORE ON AUTOMORPHISM GROUPS OF LAMINATED NEAR-RINGS

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1. Introduction

We will assume throughout this paper that polynomials are nonconstant. Let P be any complex polynomial and let $\mathcal{N}_P$ denote the near-ring of all continuous selfmaps of the complex plane where addition of functions is pointwise and multiplication is defined by $fg=f\circ P\circ g$ for all $f,g\in\mathcal{N}_P$. The near-ring $\mathcal{N}_P$ is referred to as a laminated near-ring and P is referred to as the laminating element or laminator. In [1] the problem was posed of determining $\operatorname{Aut}\mathcal{N}_P$ the automorphism group of $\mathcal{N}_P$. It was shown that exactly three infinite groups occur as automorphism groups of the laminated near-rings $\mathcal{N}_P$ and for each of the three groups those polynomials P were characterized such that $\operatorname{Aut}\mathcal{N}_P$ is isomorphic to that particular group. The infinite groups turn out to be $GL(2)$, the full linear group of all 2×2 nonsingular real matrices and two of its subgroups.

In [2], as the title of that paper indicates, finite automorphism groups of the near-rings $\mathcal{N}_P$ were investigated and the results obtained there, combined with the results obtained in [1], yielded a description of $\operatorname{Aut}\mathcal{N}_P$ when P has real coefficients and $\operatorname{Deg}P\leqq 4$. In this paper, we complete the solution of the problem. That is, the main result of this paper, together with a result from [1] allows us to completely describe $\operatorname{Aut}\mathcal{N}_P$ with no restrictions whatsoever on the polynomial P. In Section 2 we recall some notation and state the main results. Proofs are given in Section 3.

2. Statements of main results

As we mentioned previously, $GL(2)$ denotes the full linear group of all real 2×2 nonsingular matrices. G_1 denotes the subgroup of $GL(2)$ consisting of all matrices of the form

$$\begin{bmatrix}1, & a\\ 0, & b\end{bmatrix} \quad \text{where } b\neq 0$$

and G_c denotes the subgroup of $GL(2)$ consisting of all matrices of the form

$$\begin{bmatrix}a, & -b\\ b, & a\end{bmatrix} \text{ and } \begin{bmatrix}a, & b\\ b, & -a\end{bmatrix}$$

where $a^2+b^2 \neq 0$. Let GR_m (m a positive integer) denote that subgroup of G_c where $a=\cos(2k\pi/m)$ and $b=\sin(2k\pi/m)$, $1 \leqq k \leqq m$. Finally, we denote by $\mathbb{Z}_m$ the cyclic group of order m. The groups $GL(2)$, G_1 and G_c are all infinite while GR_m is finite of order $2m$. Of course, GR_1 is isomorphic to $\mathbb{Z}_2$ and GR_2 is isomorphic to $\mathbb{K}_4$ the Klein group of order four. These are the only instances in which GR_m is abelian.

In all that follows, we let $P(z)=\sum_{j=0}^{n} a_j z^{n-j}$ where each a_j is a complex number and $a_0 \neq 0$. We now state two theorems which, together, completely describe $\operatorname{Aut}\mathcal{N}_P$ for all complex polynomials P. The first result appears in [1] as Theorem 4.4. We restate it here (without proof) for the sake of completeness.

Theorem 2.1. *Let P be any complex polynomial. Then:*

$\operatorname{Aut}\mathcal{N}_P$ *is isomorphic to* $GL(2)$ *if and only if* $\operatorname{Deg} P=1$ *or* $\operatorname{Deg} P=2$ *and* $a_1=0$, (2.1.1)

$\operatorname{Aut}\mathcal{N}_P$ *is isomorphic to* G_1 *if and only if* $\operatorname{Deg} P=2$ *and* $a_1 \neq 0$, (2.1.2)

$\operatorname{Aut}\mathcal{N}_P$ *is isomorphic to* G_c *if and only if* $\operatorname{Deg} P \geqq 3$ *and* $a_j=0$ *for* $1 \leqq j \leqq n-1$. (2.1.3)

It is evident from the previous result that it remains for us to consider the case where $\operatorname{Deg} P \geqq 3$ and $a_j \neq 0$ for some j such that $1 \leqq j \leqq n-1$. We do this in the next theorem which is the main result of the paper. In the statement we require $a_0=1$. At first glance this may appear to be a restriction but it really is not for Lemma 3.2 of [1] assures us that $\operatorname{Aut}\mathcal{N}_P$ and $\operatorname{Aut}\mathcal{N}_Q$ are isomorphic where $Q(z)=(1/a_0)P(z)$. In the statements of the following three results, we let $A=\{j: 1 \leqq j \leqq n-1 \text{ and } a_j \neq 0\}$.

Theorem 2.2. *Let $\operatorname{Deg} P \geqq 3$ and $a_0=1$. Suppose $A \neq \varnothing$ and let $m=\gcd A$. Then there exist integers b_j such that $m=\sum_{j\in A} j b_j$ and we define*

$$c=\prod_{j\in A}(a_j/\bar{a}_j)^{b_j}. \tag{2.2.1}$$

If there exists an mth *root σ of c such that*

$$a_j=\bar{a}_j\sigma^j \quad \text{for each } j\in A, \tag{2.2.2}$$

then $\operatorname{Aut}\mathcal{N}_P$ *is isomorphic to* GR_m. *If no* mth *root of c satisfies* (2.2.2), *then* $\operatorname{Aut}\mathcal{N}_P$ is isomorphic to $\mathbb{Z}_m$ *the cyclic group of order m.*

Corollary 2.3. *Let $\operatorname{Deg} P \geqq 3$ and let $a_0=1$. Suppose $A \neq \varnothing$ and a_j is real for each $j \in A$. Then $\operatorname{Aut}\mathcal{N}_P$ is isomorphic to GR_m where $m=\gcd A$.*

Corollary 2.4. *Let $\operatorname{Deg} P \geqq 3$, let $a_0=1$, suppose $A \neq \varnothing$ and suppose a_j is a pure imaginary number for each $j \in A$. Suppose further that the least element $m \in A$ divides every other element in A. Then:*

$\operatorname{Aut}\mathcal{N}_P$ *is isomorphic to GR_m if j/m is odd for each $j \in A$.* (2.4.1)

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and

$$\text{Aut}\,\mathcal{N}_P \textit{ is isomorphic to } \mathbb{Z}_m \textit{ if } j/m \textit{ is even for at least one } j\in A. \tag{2.4.2}$$

3. Supporting lemmas and proofs

Lemma 3.1. *Let $P(z)$ be any complex polynomial and let any real number $r>0$ be given. Then there exists an $R>0$ such that if $|z_0|>R$, then z_0 is the only zero of $P(z)-P(z_0)$ in the interior of the curve $C=\{z:|z-z_0|=r\}$.*

Proof. Let $Q(z)=P(z)-P(z_0)$. We then have

$$Q(z)=\sum_{j=0}^{n}\frac{Q^{(j)}(z_0)}{j!}(z-z_0)^j=\sum_{j=1}^{n}\frac{P^{(j)}(z_0)}{j!}(z-z_0)^j. \tag{3.1.1}$$

According to Lemma 3.4 of [1], we can choose R_1 large enough so that if $|z_0|>R_1$, then $Q(z)$ has no multiple roots. Thus, $Q'(z_0)=P'(z_0)\neq 0$ for $|z_0|>R_1$ and from this fact and (3.1.1) we get

$$Q(z)=P'(z_0)(z-z_0)[1+R(z)] \tag{3.1.2}$$

where

$$R(z)=\sum_{j=2}^{n}\frac{P^{(j)}(z_0)}{j!P'(z_0)}(z-z_0)^{j-1}. \tag{3.1.3}$$

Now let ε be any number such that $0<\varepsilon<1$ and choose $R_2\geqq R_1$ so that $|R(z)|<\varepsilon$ when $|z_0|>R_2$ and z is any point on the curve C. From this and (3.1.2) we get

$$|Q(z)|\geqq r|P'(z_0)|(1-\varepsilon) \tag{3.1.4}$$

for any z on C. Since $P'(z_0)\neq 0$, this means, among other things, that $Q(z)$ does not vanish on C.

In a similar manner,

$$Q'(z)=P'(z_0)[1+T(z)] \tag{3.1.5}$$

where

$$T(z)=\sum_{j=1}^{n-1}\frac{P^{(j+1)}(z_0)}{j!P'(z_0)}(z-z_0)^j. \tag{3.1.6}$$

Now choose $R\geqq R_2$ so that if $|z_0|>R$ and z is any point on C, then $|T(z)|<\varepsilon$. It follows from (3.1.5) that

$$|Q'(z)|\leqq|P'(z_0)|(1+\varepsilon). \tag{3.1.7}$$

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From (3.1.4) and (3.1.7) we then get

$$\left|\frac{1}{2\pi i}\int_C \frac{Q'(z)}{Q(z)}dz\right| \leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|Q'(z)|}{|Q(z)|} r d\theta$$
$$\leq \frac{1}{2\pi}\int_0^{2\pi} \frac{|P'(z_0)|(1+\varepsilon)}{r|P'(z_0)|(1-\varepsilon)} r d\theta = \frac{1+\varepsilon}{1-\varepsilon}. \tag{3.1.8}$$

The number ε can be chosen small enough so that $(1+\varepsilon)/(1-\varepsilon)<2$ and it then follows from the Principle of the Argument that z_0 is the only zero of $Q(z)=P(z)-P(z_0)$ within the curve C when $|z_0|>R$.

We next recall a result of J. L. Walsh [3, p. 21] which we will need in the proof of a subsequent lemma.

Theorem 3.2 (Walsh). *Let $P(z)=(z-\alpha_1)(z-\alpha_2)\ldots(z-\alpha_n)$ and let $\alpha_0=(\alpha_1+\alpha_2+\cdots+\alpha_n)/n$. For each $\varepsilon>0$ there exists an M_ε such that if $|A|>M_\varepsilon$ then every zero z_0 of the polynomial $P(z)-A$ satisfies an inequality*

$$|z_0-(\alpha_0+A^{1/n})|<\varepsilon \tag{3.2.1}$$

where $A^{1/n}$ is a suitably chosen nth *root of A.*

Any polynomial P decomposes the complex plane $\mathscr{C}$ into mutually disjoint subsets. Specifically, we define

$$\Pi(P)=\{P^{-1}(w):w\in\mathscr{C}\}.$$

Next, we regard $\mathscr{C}$ as a two-dimensional vector space over the reals and we denote by $LA(P)$ the group of all linear automorphisms t of $\mathscr{C}$ which satisfy the condition $t[A]\in\Pi(P)$ for each $A\in\Pi(P)$. Corollary (2.3) of [**1**] tells us that Aut $\mathscr{N}_P$ is isomorphic to $LA(P)$ so our efforts in this section will be directed toward determining $LA(P)$ for each complex polynomial P. There is another result we need to recall from [**1**]. It was stated there as Lemma 3.1.

Lemma 3.3. *A linear automorphism t of $\mathscr{C}$ belongs to $LA(P)$ if and only if for all $z_1, z_2\in\mathscr{C}$, the following two statements are equivalent:*

$$P(z_1)=P(z_2) \tag{3.3.1}$$

$$P(t(z_1))=P(t(z_2)). \tag{3.3.2}$$

And now we are in a position to prove

Lemma 3.4. *Let $P(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n$, let $t\in LA(P)$ and let θ be a primitive* nth *root of unity. Then the sets*

$$\{t(1), t(\theta), t(\theta^2), \ldots, t(\theta^{n-1})\} \tag{3.4.1}$$

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and

$$\{t(1), \theta t(1), \theta^2 t(1), \ldots, \theta^{n-1} t(1)\} \tag{3.4.2}$$

coincide.

Proof. According to Lemma 3.1 we can choose $R_0 > 0$ so that if $|z_0| > R_0$ then z_0 is the only zero of $P(z) - P(z_0)$ within the curve $|z - z_0| = 1$. Next let ε be given subject to the conditions $0 < \varepsilon < \frac{1}{2}$. According to Walsh's Theorem 3.2, there exists an M_ε such that if $|A| > M_\varepsilon$ then every zero z_0 of $P(z) - A$ satisfies (3.2.1). Next, let $M = \sup_{|z| \leq R_0} |P(z)|$ and then choose R_ε so that whenever $|z| > R_\varepsilon$, the following three conditions are satisfied.

$$|P(z_0)| > M + M_\varepsilon \tag{3.4.3}$$

$$|P(t(z_0))| > M_\varepsilon \tag{3.4.4}$$

$$P(z) - P(z_0) \text{ has } n \text{ distinct zeros.} \tag{3.4.5}$$

It is evident that the first two conditions can be satisfied and it follows from Lemma 3.4 of [1] that the third condition can be satisfied as well.

Now let $l > R_\varepsilon$, let $|P(l)|^{1/n}$ be the positive nth root of $|P(l)|$ and let z_1 be any zero of $P(z) - P(l)$. It follows from (3.4.3) and Walsh's Theorem that

$$|z_1 - (\alpha_0 + |P(l)|^{1/n} \theta^j)| < \varepsilon \tag{3.4.6}$$

where $1 \leqq j \leqq n$. Let z_2 be a zero of $P(z) - P(l)$ distinct from z_1. As in the case for z_1 we have

$$|z_2 - (\alpha_0 + |P(l)|^{1/n} \theta^i)| < \varepsilon \tag{3.4.7}$$

where $1 \leqq i \leqq n$ and we claim that $i \neq j$. Suppose, to the contrary, that $i = j$. It follows from (3.4.6) and (3.4.7) that

$$|z_1 - z_2| < 2\varepsilon < 1. \tag{3.4.8}$$

Thus, $P(z) - P(z_1) = P(z) - P(l)$ has at least two zeros within $|z - z_1| = 1$. But from (3.4.3) we see that $|P(z_1)| = |P(l)| > M$ which implies $|z_1| > R_0$. This, in turn, implies that $P(z) - P(z_1)$ has only one zero within $|z - z_1| = 1$ and we have reached a contradiction. Consequently, $i \neq j$ as we asserted. According to (3.4.5), $P(z) - P(l)$ has n distinct zeros and it follows from all this that for each integer j such that $1 \leqq j \leqq n$, there exists a zero z_{lj} of $P(z) - P(l)$ such that

$$|z_{lj} - (\alpha_0 + |P(l)|^{1/n} \theta^j)| < \varepsilon. \tag{3.4.9}$$

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Thus, we have

$$\lim_{l\to\infty} |z_{lj}-(\alpha_0+|P(l)^{1/n}\theta^j)|=0 \tag{3.4.10}$$

for each j. This implies

$$\lim_{l\to\infty}\left|\frac{z_{lj}}{l}-\frac{|P(l)|^{1/n}}{l}\theta^j\right|=0. \tag{3.4.11}$$

But

$$\lim_{l\to\infty}\frac{|P(l)|}{l^n}=1 \quad \text{so that} \quad \lim_{l\to\infty}\frac{|P(l)|^{1/n}}{l}=1.$$

This, together with (3.4.11) implies

$$\lim_{l\to\infty}\frac{z_{lj}}{l}=\theta^j \quad \text{for} \quad 1\leqq j\leqq n. \tag{3.4.12}$$

Since t is continuous, we also have

$$\lim_{l\to\infty}\frac{t(z_{lj})}{l}=t(\theta^j) \quad \text{for} \quad 1\leqq j\leqq n. \tag{3.4.13}$$

It follows from Lemma 3.3 that $\{t(z_{lj})\}_{j=1}^n$ is the collection of zeros for the polynomial $P(z)-P(t(l))$. Choose any $t(z_{lj})$. Since $l>R_\varepsilon$, $|P(t(z_{lj}))|=|P(t(l))|>M_\varepsilon$ and it follows from Walsh's Theorem that

$$|t(z_{lj})-(\alpha_0+P(t(l))^{1/n})|<\varepsilon \tag{3.4.14}$$

where $P(t(l))^{1/n}$ is a suitable nth root of $P(t(l))$. This implies

$$\lim_{l\to\infty}\left|\frac{t(z_{lj})}{lt(1)}-\frac{P(t(l))^{1/n}}{lt(1)}\right|=0. \tag{3.4.15}$$

Thus, (3.4.13) and (3.4.15) together imply that

$$\lim_{l\to\infty}\frac{P(t(l))^{1/n}}{lt(1)}=\frac{t(\theta^j)}{t(1)}. \tag{3.4.16}$$

But we have

$$\lim_{l\to\infty}\left[\frac{P(t(l))^{1/n}}{lt(1)}\right]^n=\lim_{l\to\infty}\frac{P(t(l))}{(t(l))^n}=1$$

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so that $[t(\theta^j)/t(1)]^n=1$. That is, $t(\theta^j)/t(1)=\theta^i$ for some i such that $1\leqq i\leqq n$. Thus, $t(\theta^j)=\theta^i t(1)\in\{t(1),\theta t(1),\ldots,\theta^{n-1}t(1)\}$ for $1\leqq j\leqq n$. Consequently, the sets (3.4.1) and (3.4.2) coincide as we asserted.

Lemma 3.5. *Let $P(z)=z^n+a_1z^{n-1}+a_2z^{n-2}+\cdots+a_n$ with $\operatorname{Deg} P\geqq 3$ and let $t\in LA(P)$. Then there exists a nonzero complex number v such that either $t(z)=vz$ for all $z\in\mathscr{C}$ or $t(z)=v\bar{z}$ for all $z\in\mathscr{C}$.*

Proof. We first consider the case where $\operatorname{Deg} P=4$. Then, by Lemma (3.4) we have

$$t(i)\in\{t(1),it(1),-t(1),-it(1)\}.$$

If $t(i)=t(1)$, then $0=t(i)-t(1)=t(i-1)$ which is a contradiction. Thus, $t(i)\neq t(1)$ and for similar reasons, $t(i)\neq -t(1)$. Consequently, either $t(i)=it(1)$ or $t(i)=-it(1)$. In the former case,

$$t(x+yi)=t(x)+t(yi)=xt(1)+yt(i)$$

$$=xt(1)+yit(1)=t(1)(x+yi)$$

and one shows in the same manner that $t(x+yi)=t(1)\overline{(x+yi)}$ in the latter case.

Now suppose $\operatorname{Deg} P\neq 4$ and let $\theta=x+yi$ be a primitive nth root of unity where $n=\operatorname{Deg} P$. Then $x\neq 0\neq y$ (since $\operatorname{Deg} P\geqq 3$ and $\operatorname{Deg} P\neq 4$) and $x^2\neq 1$. The vectors 1, θ and $\bar{\theta}$ all have absolute value 1 and by Lemma 3.4, $t(1)$, $t(\theta)$ and $t(\bar{\theta})$ all have absolute value $|t(1)|$. It now follows from Lemma 4.1 of [**1**] that there exists a nonzero complex number v such that either $t(z)=vz$ for each $z\in\mathscr{C}$ or $t(z)=v\bar{z}$ for each $z\in\mathscr{C}$.

Notation. Let v be a nonzero complex number. In all that follows t_v is the linear automorphism of $\mathscr{C}$ which is defined by $t_v(z)=vz$ and $\hat{t}_v$ is defined by $\hat{t}_v(z)=v\bar{z}$. As before, we let $A=\{j:1\leqq j\leqq n-1 \text{ and } a_j\neq 0\}$ and we assume $A\neq\varnothing$. Furthermore, we assume without further mention that $\operatorname{Deg} P\geqq 3$ and $a_0=1$.

Lemma 3.6. *$t_v\in LA(P)$ if and only if $v^j=1$ for each $j\in A$.*

Proof. Suppose $t_v\in LA(P)$ and choose z_1 so that $P^{-1}(P(z_1))$ consists of n distinct points $\{z_j\}_{j=1}^n$. Then we have

$$P(z)-P(z_1)=(z-z_1)(z-z_2)\ldots(z-z_n). \tag{3.6.1}$$

Now $\{z_j\}_{j=1}^n\in\Pi(P)$ so $\{vz_j\}_{j=1}^n\in\Pi(P)$ since $t_v\in LA(P)$. It follows that $P^{-1}(P(vz_1))=\{vz_j\}_{j=1}^n$ and this implies

$$P(z)-P(vz_1)=(z-vz_1)(z-vz_2)\ldots(z-vz_n). \tag{3.6.2}$$

From (3.6.1) and (3.6.2), we get

$$P(vz)-P(vz_1)=v^n[P(z)-P(z_1)]. \tag{3.6.3}$$

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Choose $j\in A$. The coefficient of z^{n-j} in $P(vz)-P(vz_1)$ is $v^{n-j}a_j$ and the coefficient of z^{n-j} in $v^n[P(z)-P(z_1)]$ is $v^n a_j$. It follows from (3.6.3) that $v^{n-j}a_j=v^n a_j$ for each $j\in A$ and this implies $v^j=1$ for each $j\in A$.

Conversely, suppose that $v^j=1$ for each $j\in A$. Then we have

$$\begin{aligned} P(vz) &= a_n+v^n z^n+\sum_{j\in A} a_j v^{n-j} z^{n-j} \\ &= a_n+v^n z^n+v^n\sum_{j\in A} a_j z^{n-j} \\ &= a_n+v^n[P(z)-a_n]. \end{aligned} \tag{3.6.4}$$

It readily follows from (3.6.4) that for any $z_1, z_2\in\mathscr{C}$, we have $P(z_1)=P(z_2)$ if and only if $P(vz_1)=P(vz_2)$. Thus, $t_v\in LA(P)$ by Lemma 3.3.

Lemma 3.7. *$\hat{t}_v\in LA(P)$ if and only if $a_j/\bar{a}_j=v^j$ for each $j\in A$.*

Proof. Suppose $\hat{t}_v\in LA(P)$. Again choose z_1 so that $P^{-1}(P(z_1))$ consists of n distinct elements $\{z_i\}_{i=1}^n$. As before, we have

$$P(z)-P(z_1)=(z-z_1)(z-z_2)\dots(z-z_n) \tag{3.7.1}$$

and this time $\{v\bar{z}_i\}_{i=1}^n\in\Pi(P)$ which implies

$$P(z)-P(v\bar{z}_1)=(z-v\bar{z}_1)(z-v\bar{z}_2)\dots(z-v\bar{z}_n). \tag{3.7.2}$$

From (3.7.1) and (3.7.2), we get

$$P(v\bar{z})-P(v\bar{z}_1)=v^n\overline{[P(z)-P(z_1)]}. \tag{3.7.3}$$

For each $j\in A$, the coefficient of $\bar{z}^{n-j}$ in $P(v\bar{z})-P(v\bar{z}_1)$ is $a_j v^{n-j}$ while the coefficient of $\bar{z}^{n-j}$ in $v^n\overline{[P(z)-P(z_1)]}$ is $v^n\bar{a}_j$. Thus, $a_j v^{n-j}=v^n\bar{a}_j$ by (3.7.3) and it follows that $a_j/\bar{a}_j=v^j$ for each $j\in A$.

Suppose, conversely, that $(a_j/\bar{a}_j)=v^j$ for each $j\in A$. We then have

$$\begin{aligned} P(v\bar{z}) &= a_n+v^n\bar{z}^n+\sum_{j\in A} a_j v^{n-j}\bar{z}^{n-j} \\ &= a_n+v^n\bar{z}^n+v^n\sum_{j\in A}\bar{a}_j\bar{z}^{n-j} \\ &= a_n+v^n[P(\bar{z})-a_n]. \end{aligned} \tag{3.7.4}$$

It readily follows from (3.7.4) that for any $z_1, z_2\in\mathscr{C}$ we have $P(z_1)=P(z_2)$ if and only if $P(v\bar{z}_1)=P(v\bar{z}_2)$ and we appeal to Lemma 3.3 once again to conclude that $\hat{t}_v\in LA(P)$.

Notation. We let $B(P)=\{v\in\mathscr{C}: t_v\in LA(P)\}$ and $\hat{B}(P)=\{v\in\mathscr{C}:\hat{t}_v\in LA(P)\}$.

Lemma 3.8. *Let $m=\gcd A$ and let θ be a primitive mth root of unity. Then $B(P)=\{\theta^i\}_{i=1}^m$.*

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Proof. Suppose $v \in B(P)$. Then $t_v \in LA(P)$ and $v^j = 1$ for each $j \in A$ by Lemma 3.6. Furthermore, there exist integers $\{b_j\}_{j \in A}$ so that $m = \sum_{j \in A} jb_j$ and we have

$$v^m = \prod_{j \in A} (v^j)^{b_j} = 1.$$

That is, v is an mth root of unity. Since θ is a primitive mth root, we have $\theta^k = v$ for some k where $1 \leqq k \leqq m$. On the other hand, it is immediate from Lemma 3.6 that for each such k, we have $\theta^k \in B(P)$.

Lemma 3.9. *Let $m = \gcd A$. Then there exist integers b_j such that $m = \sum_{j \in A} jb_j$ and we define*

$$c = \prod_{j \in A} (a_j/\bar{a}_j)^{b_j}. \tag{3.9.1}$$

If there exists an m*th root σ of c such that*

$$a_j = \bar{a}_j \sigma^j \quad \textit{for all } j \in A, \tag{3.9.2}$$

then

$$\hat{B}(P) = \{\sigma\theta^i\}_{i=1}^{m} \tag{3.9.3}$$

where θ is a primitive m*th root of unity. If no* m*th root of c satisfies* (3.9.2), *then*

$$\hat{B}(P) = \varnothing. \tag{3.9.4}$$

Proof. Suppose σ satisfies (3.9.2). Then $\theta^j = 1$ for each $j \in A$ and we get $(\sigma\theta^i) = a_j/\bar{a}_j$ for each $j \in A$ which, by Lemma 3.7, means $\sigma\theta^i \in \hat{B}(P)$. On the other hand, suppose $v \in \hat{B}(P)$. Then $a_j/\bar{a}_j = v^j$ for each $j \in A$. We use (3.9.1) and get

$$v^m = \prod_{j \in A} (v^j)^{b_j} = \prod_{j \in A} (a_j/\bar{a}_j)^{b_j} = c.$$

That is, v is an mth root of c and it follows that $v = \sigma\theta^i$ for some i such that $1 \leqq i \leqq m$.

To prove the last assertion of the lemma, deny it and suppose $v \in \hat{B}(P)$. Then $a_j/\bar{a}_j = v^j$ for each $j \in A$ by Lemma 3.7. From (3.9.1) we get $v^m = c$ just as before. But this is a contradiction since we now have an mth root of c which satisfies (3.9.2). Therefore, we conclude that $\hat{B}(P) = \varnothing$ when no mth roots of c satisfy (3.9.2).

Our next result shows that the two sets $B(P)$ and $\hat{B}(P)$ do not intersect except under very special circumstances.

Corollary 3.10. *The following statements are equivalent:*

$$B(P) \cap \hat{B}(P) \neq \varnothing \tag{3.10.1}$$

$$B(P) = \hat{B}(P). \tag{3.10.2}$$

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All the coefficients of P are real with the possible exception of a_n. (3.10.3)

Proof. We show (3.10.1) implies (3.10.3). Suppose $v \in B(P) \cap \hat{B}(P)$. Then $a_j/\bar{a}_j = 1$ for each $j \in A$ by Lemmas 3.6 and 3.7. Thus, a_j is real for each $j \in A$ and, of course, $a_0 = 1$. It is only a_n that may possibly not be a real number.

Next, we show that (3.10.3) implies (3.10.2). We appeal to Lemma 3.9. In that lemma, $c = 1$ and we take $\sigma = 1$. Then (3.9.2) is satisfied and $\hat{B}(P) = \{\theta^i\}_{i=1}^m$ by (3.9.3). It now follows from Lemma 3.8 that $B(P) = \hat{B}(P)$. It is evident (since $B(P) \neq \emptyset$) that (3.10.2) implies (3.10.1) and the proof is complete.

Notation. Let θ be a primitive mth root of unity. We let $G_m(\theta) = \{t_v, \hat{t}_v : v = \theta^i, 1 \leqq i \leqq m\}$. $G_m(\theta)$ is, of course, a finite subgroup of the group of linear automorphisms of $\mathscr{C}$.

Lemma 3.11. *Let* $m = \gcd A$. *Then there exist integers* b_j *such that* $m = \sum_{j \in A} j b_j$ *and we define*

$$c = \prod_{j \in A} (a_j/\bar{a}_j)^{b_j}. \tag{3.11.1}$$

Suppose there exists an mth *root* σ *of* c *such that*

$$a_j = \bar{a}_j \sigma^j \quad \text{for all } j \in A. \tag{3.11.2}$$

Then $LA(P)$ *is isomorphic to* $G_m(\theta)$.

Proof. Since $a_j/\bar{a}_j = \sigma^j$, we have $\bar{a}_j/a_j = \bar{\sigma}^j$ which implies $(\sigma\bar{\sigma})^j = 1$. Now $\sigma\bar{\sigma}$ is a positive real number so we must have $\sigma\bar{\sigma} = 1$. According to Lemmas 3.5, 3.8 and 3.9,

$$LA(P) = \{t_v : v = \theta^i, 1 \leqq i \leqq m\} \cup \{\hat{t}_w : W = \sigma\theta^i, 1 \leqq i \leqq m\}.$$

Since $\sigma\bar{\sigma} = 1$, one easily verifies that the mapping ϕ from$LA(P)$ to $G_m(\theta)$ defined by $\phi(t_v) = t_v$ and $\phi(\hat{t}_w) = \hat{t}_u$ where $u = \theta^i$ whenever $w = \sigma\theta^i$, is an isomorphism.

It is now an easy matter to complete the proof of Theorem 2.2 and to derive its corollaries. Suppose first that there exists an mth root of c satisfying condition (2.2.2) of Theorem 2.2. According to Lemma 3.11, $LA(P)$ is isomorphic to $G_m(\theta)$ and one easily verifies that if $v = \theta^k$, $1 \leqq k \leqq m$, the map which sends t_v to

$$\begin{bmatrix} \cos(2k\pi/m), & -\sin(2k\pi/m) \\ \sin(2k\pi/m), & \cos(2k\pi/m) \end{bmatrix} \text{ and } \hat{t}_v \text{ to } \begin{bmatrix} \cos(2k\pi/m), & \sin(2k\pi/m) \\ \sin(2k\pi/m), & -\cos(2k\pi/m) \end{bmatrix}$$

is an isomorphism from $G_m(\theta)$ onto GR_m. It now follows from Corollary 2.3 of [1] that in this particular case, $\operatorname{Aut} \mathscr{N}_P$ is isomorphic to GR_m.

Now consider the remaining case where no mth root of c satisfies (2.2.2). It follows from Lemmas 3.5, 3.8 and 3.9 that $LA(P) = \{t_v : v = \theta^i, 1 \leqq i \leqq m\}$ which is cyclic of order m. Consequently, in this case $\operatorname{Aut} \mathscr{N}_P$ is isomorphic to $\mathbb{Z}_m$.

Corollary 2.3 follows easily from Theorem 2.2. One has only to observe that if a_j is real for every $j \in A$, then $c=1$ and one can then choose $\sigma=1$ and (2.2.2) is satisfied. As for Corollary 2.4, we take $b_m=1$ and $b_j=0$ for all $j \in A-\{m\}$. Then c, as defined by (2.2.1), is -1. Choose any mth root σ of -1. Condition (2.2.2) will be satisfied if and only if j/m is odd for each $j \in A$. Consequently, it follows from Theorem 2.2 that Aut $\mathcal{N}_P$ is isomorphic to GR_m if j/m is odd for each $j \in A$ and Aut $\mathcal{N}_P$ is isomorphic to $\mathbb{Z}_m$ if j/m is even for at least one $j \in A$.

REFERENCES

1. K. D. Magill, P. R. Misra and U. B. Tewari, Automorphism groups of laminated near-rings determined by complex polynomials, *Proc. Edinburgh Math. Soc.* **26** (1983), 73–84.

2. K. D. Magill, P. R. Misra and U. B. Tewari, Finite automorphism groups of laminated near-rings, *Proc. Edinburgh Math. Soc.* **26** (1983), 297–306.

3. J. L. Walsh, *The Location of Critical Points of Analytic and Harmonic Functions* (Colloquium Pub. Vol. **34**, Amer. Math. Soc., New York, 1950).

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