The Journal of Symbolic Logic
Volume 82, Number 3, September 2017

# DENSITY-1-BOUNDING AND QUASIMINIMALITY IN THE GENERIC DEGREES

PETER CHOLAK AND GREGORY IGUSA

**Abstract.** We consider the question "Is every nonzero generic degree a density-1-bounding generic degree?" By previous results [8] either resolution of this question would answer an open question concerning the structure of the generic degrees: A positive result would prove that there are no minimal generic degrees, and a negative result would prove that there exist minimal pairs in the generic degrees.

We consider several techniques for showing that the answer might be positive, and use those techniques to prove that a wide class of assumptions is sufficient to prove density-1-bounding.

We also consider a historic difficulty in constructing a potential counterexample: By previous results [7] any generic degree that is not density-1-bounding must be quasiminimal, so in particular, any construction of a non-density-1-bounding generic degree must use a method that is able to construct a quasiminimal generic degree. However, all previously known examples of quasiminimal sets are also density-1, and so trivially density-1-bounding. We provide several examples of non-density-1 sets that are quasiminimal.

Using cofinite and mod-finite reducibility, we extend our results to the uniform coarse degrees, and to the nonuniform generic degrees. We define all of the above terms, and we provide independent motivation for the study of each of them.

Combined with a concurrently written paper of Hirschfeldt, Jockusch, Kuyper, and Schupp [4], this paper provides a characterization of the level of randomness required to ensure quasiminimality in the uniform and nonuniform coarse and generic degrees.

**§1. Introduction.** Generic computability was introduced by Kapovich, Miasnikov, Schupp, and Shpilrain [10] as a computability-theoretic analogue of the real-world phenomenon in which a problem is apparently much easier to solve than would be suggested by complexity theory. The idea of generic-case complexity is to measure the complexity of the majority of instances of a problem, while disregarding "difficult" instances if they are sufficiently uncommon. Generic computability as well as coarse computability, a similarly defined notion, were later studied by Jockusch and Schupp [9] in the framework of the theory of computability theory.

In generic and coarse computability, we think of a real $A$ that we are trying to compute as the problem, and the bits of $A$ as the instances of the problem. The goal, then, is to compute the majority of the bits of $A$. In generic computability, we are not allowed to make any mistakes, but we are allowed to not always give answers. In coarse computability, we must give answers everywhere, but we are

Received January 19, 2016.
2010 *Mathematics Subject Classification.* Primary 03D30,25,28,32.
*Key words and phrases*. generic computability, coarse computability, generic reducibility, coarse reducibility, density 1 bounding, quasiminimal.


0022-4812/17/8203-0006
DOI:10.1017/jsl.2016.50

allowed to make some mistakes. Coarse computability can be thought of as the analogue of algorithms that take shortcuts, sacrificing accuracy for speed, while generic computability is the analogue of completely accurate algorithms that run very quickly in most cases, but more slowly or perhaps not at all in others. Following the notation of Jockusch and Schupp [9], we make this precise as follows.

Definition 1.1. Let $A \subseteq \omega$. Then $A$ is *density-1* if the limit of the densities of its initial segments is 1, or in other words, if $\lim_{n\to\infty} \frac{|A\cap n|}{n} = 1$.

In this paper a real $A$ is thought of either as a subset of $\omega$, or as a function $A : \omega \to \{0, 1\}$. In situations where it will not cause confusion, the two notations will be used interchangably, so $n \in A$ means the same thing as $A(n) = 1$, and $n \notin A$ means the same thing as $A(n) = 0$.

Definition 1.2. A *partial computation of a real $A$* is a partial computation $\phi$ (potentially with an oracle) such that for any $n$, if $\phi(n)\downarrow$, then $\phi(n) = A(n)$.

Definition 1.3. A real $A$ is *generically computable* if there exists a partial computable function $\phi$ such that $\mathrm{dom}(\phi)$ is density-1, and $\phi$ is a partial computation of $A$.

Definition 1.4. A real $A$ is *coarsely computable* if there exists a total computable function $\phi$, whose range is contained in $\{0, 1\}$ such that $\{n \mid \phi(n) = A(n)\}$ is density-1.

In order to obtain degree structures for these, we need to make sure that our notion of relative computability is transitive, or in other words, that if $X \leq Y \leq Z$, then $X \leq Z$. The outputs of our "computations" are generic and coarse descriptions of our reals, and so the inputs of our computations should also be generic and coarse descriptions. This is fairly straightforward to define for coarse reducibility.

Definition 1.5. Let $A$ and $B$ be reals. Then *$B$ is (uniformly) coarsely reducible to $A$* if there exists a Turing functional $\phi$ such that for any $C$ for which $\{n : A(n) = C(n)\}$ is density-1, $\phi^C$ is a coarse computation of $B$. In this case, we write $B \leq_c A$.

In nonuniform coarse reducibility, the functional $\phi$ is allowed to depend on $C$. There is no major reason to prefer the uniform or nonuniform reducibilities, although it appears that for coarse reducibility, the nonuniform version is slightly easier to work with, while for generic reducibility, the uniform version is slightly easier to work with. This paper is primarily focused on generic reducibility, but will occasionally derive conclusions about (uniform) coarse reducibility from arguments concerning (uniform) generic reducibility, and so in this paper, unless specifically specified otherwise, all reducibilities are assumed to be uniform.

Generic reducibility is somewhat more difficult to define than coarse reducibility, because to define generic reducibility, we are forced to discuss what it means to use a partial computation as an oracle in a computation. Our generic computations are not even required to tell us whether or not they will give an output on a given value, and so our generic computations must be able to work with oracles that also do not tell them which outputs they will give. For this reason, we first define partial oracles, and discuss what it means for a Turing functional to work with a partial oracle.

**§2. Partial oracles.** We wish to define computations with partial oracles so that the following happens: We must be able to ask questions of our oracles and make decisions based on the outputs. Second, we must be able to avoid being paralyzed when an oracle does not give outputs, and we must also not be able to know whether or not the oracle will give an output in the future, if it has not yet given an output. We may formalize this either with time delays built in to our oracles, or with enumeration operators, which are designed to be blind to exactly the sorts of information that we do not wish to be able to use.

For uniform reducibilities, these are equivalent in terms of what is reducible to what, although not necessarily in terms of the actual procedures [7]. For nonuniform reducibilities, it is not known whether or not the two ways to approach partial oracles are equivalent. In this paper, we present the reducibility in terms of time-dependent oracles.

DEFINITION 2.1. Let $A$ be a real. Then a (time-dependent) *partial oracle,* $(A)$, for $A$ is a set of ordered triples $\langle n, x, l\rangle$ such that:
$n \in \omega,\ \ x \in 2,\ \ l \in \omega.$
$\exists l\left(\langle n, 0, l\rangle \in (A)\right) \implies n \notin A.$
$\exists l\left(\langle n, 1, l\rangle \in (A)\right) \implies n \in A.$
For every $n, x$, there exists at most one $l$ such that $\langle n, x, l\rangle \in (A)$.

When using such an oracle, querying whether or not $n \in A$ consists of initiating a search for some value of $l$ such that either $\langle n, 0, l\rangle \in (A)$, or $\langle n, 1, l\rangle \in (A)$. If there exists such an $l$, we say that $(A)(n)$ converges, or that $(A)(n)\downarrow$. The domain of $(A)$, written $\mathrm{dom}((A))$, is the set of $n$ for which there exists such an $l$. If $\langle n, x, l\rangle \in (A)$, then we write that $(A)(n) = x$. Other computations, processes, and queries may be carried out while searching for such an $l$. When working with reducibilities that use partial oracles, we will frequently abuse notation and refer to $A$ as the partial oracle for $A$ that converges immediately on all inputs.

In previous work [3, 7, 8], the second author omitted the uniqueness requirement on $l$. Including this convention does not change any reducibilities that we will define, and it will be convenient to us in Section 6, as it will ensure that for any $n$, $(A) \upharpoonright n$ must always be computable.

DEFINITION 2.2. Let $A$ be a real. Then a *generic oracle,* $(A)$, for $A$ is a partial oracle for $A$ such that $\mathrm{dom}((A))$ is density-1.

Note that a generic computation of $A$ is the same thing as a computation of a generic oracle for $A$.

DEFINITION 2.3. Let $A$ and $B$ be reals. Then *$B$ is (uniformly) generically reducible to $A$* if there exists a Turing functional $\phi$ such that for every generic oracle, $(A)$, for $A$, $\phi^{(A)}$ is a generic computation of $B$. In this case, we write $B \leq_{\mathrm{g}} A$.

When working with partial oracles in a context where (1) mistakes are not allowed to be made, and (2) one must act uniformly, it can be shown that no advantage can be gained by actively using the time dependence of the partial oracles (see Observation 2.9). It is frequently much more convenient to work only with Turing functionals which ignore the time dependence of their partial oracles, and in this paper, we will later be assuming that all Turing functionals are of that form.

Those familiar with enumeration reduction will see that what we define below is an enumeration operator on the graph of the partial function given by a partial oracle.

DEFINITION 2.4. Let $\phi$ be a Turing functional. Then the *time-independent version of* $\phi$ is the (potentially multi-valued) functional $\psi$ such that if $X$ is any partial oracle, then $\psi^X(n)$ is defined by considering all partial oracles $Y$ whose domains are finite subsets of $\text{dom}(X)$, and which agree with $X$ on their domains, and giving every output that $\phi^Y(n)$ would give on any of those partial oracles $Y$.

We refer to a $\psi$ defined in this way as a *time-independent Turing functional.*

Note that this process can be carried out effectively. There are countably many finite partial oracles agreeing with $X$, and they can be enumerated effectively in $X$. The computations of $\phi^Y(n)$ can be carried out in parallel. Thus if $\psi$ is a time-independent Turing functional, then for a partial oracle $X$, the outputs of $\psi^X$ are $\Sigma_1^{0,X}$ in much the same way that the outputs of $\phi^X$ are $\Sigma_1^{0,X}$ for an ordinary Turing functional $\phi$ and oracle and $X$.

In the remark after Theorem 3.6 we will see why multi-valued functionals are necessary for our purposes.

We now prove some basic facts concerning partial oracles and time-independent functionals. Our first observation is a justification of why these are referred to as time-independent.

OBSERVATION 2.5. *Assume $\phi$ is a time-independent functional. Let $X$ and $Y$ be partial oracles that have the same domains and agree on their domains. (So they agree as partial functions, although perhaps with different $l$ values at the locations where they converge.) Then $\phi^X = \phi^Y$ as a (potentially partial, potentially multi-valued) function.*

Note that this justifies the abuse of notation in which $A$ is the partial oracle that halts everywhere. Every oracle for $A$ that halts everywhere gives the same outputs when given to a time-independent functional, and so it does not matter which specific one we use.

PROOF. The definition of a time-independent functional specifically ensures that $\phi^X$ depends only on $X$ as a partial function. ⊣

This next observation is the primary reason that we will use time-independent functionals. One important use of the following observation is that if we are checking whether $A \leq_{\text{g}} B$ via $\phi$, if we check that $\phi^B$ never makes any mistakes about $A$, then it also ensures that for every partial oracle $(B)$ for $B$, $\phi^{(B)}$ also never makes any mistakes about $A$.

OBSERVATION 2.6. *Assume $\phi$ is a time-independent functional. Let $X$ and $Y$ be partial oracles such that $dom(Y) \subseteq dom(X)$, and such that $Y$ agrees with $X$ on its domain. Then if $\phi^Y(n) = x$, then $\phi^X(n) = x$.*

Note that time-independent functionals are sometimes multivalued, and so it is possible that $\phi^X(n)$ has more than one value and that $\phi^Y$ does not. This observation simply notes that a "larger" oracle must cause the functional to give more outputs.

PROOF. Assume that $\phi$ is the time-independent version of $\theta$. Assume that $\phi^Y(n) = x$. This is because, at some finite stage, we see a finite partial oracle $Z$ agreeing with $Y$ on its domain, such that $\theta^Z(n) = x$. We have that $\text{dom}(Y) \subseteq \text{dom}(X)$,

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

and that $Y$ agrees with $X$ on its domain. Therefore, at some stage, we will see that $Z$ is a partial oracle that agrees with $X$ on its domain, and at that stage, we will begin computing $\theta^Z(n)$. ⊣

The following two observations will justify us in only using time-independent functionals, so that the aforementioned conveniences will be available to us.

OBSERVATION 2.7. *Let $A$ and $B$ be reals. Assume that $B \leq_g A$ via $\phi$. Then for any $n, x$, and for any partial oracle $(A)$ for $A$, if $\phi^{(A)}(n) = x$, then $B(n) = x$.*

*In particular, given any $X$ and $Y$, if $\phi^Y$ is multi-valued, then it cannot be the case that $X \leq_g Y$ via $\phi$.*

REMARK 2.8. The second part of the observation is particularly useful in that it allows us to make conclusions about whether or not $X \leq_g Y$ via $\phi$ without knowing $X$. This is similar to the fact that in ordinary Turing computation, if $\phi^Y$ is not total, then (without knowing $X$, one knows that) it is not the case that $X \leq_T Y$ via $\phi$. It is also similar to the fact that if one sees that for every $Z$, there is some $n$ such that $\phi^Z(n) \neq X(n)$, then (without knowing $Y$, one knows that) it is not the case that $X \leq_T Y$ via $\phi$. These sorts of observations are helpful in, for example, the classical construction of a real of minimal Turing degree.

PROOF. By definition of generic reduction, we have that, for any generic oracle $(A)$ for $A$, if $\phi^{(A)}(n) = x$, then $B(n) = x$. Furthermore, given any partial oracle $(A)$ for $A$, any initial segment of $(A)$ can be extended to a generic oracle for $A$ by adding more outputs at values not yet queried (or with $l$ values larger than have yet been checked). That generic oracle would not be able to have given any incorrect outputs about $B$, and so there can be no finite stage at which $\phi^{(A)}$ gives any incorrect outputs about $B$.

It cannot be the case that $X(n)$ has more than one value, and so if $\phi^{(Y)}(n)$ has more than one value, then the conclusion of the first part of the observation does not hold. ⊣

OBSERVATION 2.9. *Let $A$ and $B$ be reals. Assume that $B \leq_g A$ via $\phi$. Then $B \leq_g A$ via the time-independent version of $\phi$.*

PROOF. Assume that $B \leq_g A$ via $\phi$. Let $\psi$ be the time-independent version of $\phi$. Let $(A)$ be any generic oracle for $A$. If $\phi^{(A)}(n)\downarrow$, then it halts having queried only finitely much of $(A)$. Let $Y$ be the finite partial oracle that agrees with that portion of $(A)$, and that does not halt anywhere else. Then $\phi^Y(n)\downarrow$, and so as a result, we have that $\psi^{(A)}(n)\downarrow$. Thus $\mathrm{dom}(\phi^{(A)}) \subseteq \mathrm{dom}(\psi^{(A)})$ and because $\mathrm{dom}(\phi^{(A)})$ is density-1, we also have that $\mathrm{dom}(\psi^{(A)})$ is density-1.

Furthermore, every partial oracle that agrees with $(A)$ is a partial oracle for $A$, so by Observation 2.7, we have that if $\psi^{(A)}(n) = x$, then $B(n) = x$. Therefore $\psi^{(A)}$ never gives any incorrect outputs about $B$, and so $\psi^{(A)}$ is a generic computation of $B$. ⊣

§**3. Relationships between degree structures.** The Turing degrees embed both into the coarse and into the generic degrees [3,4,9]. These embeddings factor through the mod-finite and cofinite degrees, respectively [3]. These additional degree structures will be useful in terms of analyzing the embedded Turing degrees, as there are a number of lemmas making them convenient and relevant.

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

DEFINITION 3.1. Let $A$ and $B$ be reals. Then $B$ *is mod-finitely reducible to* $A$ if there exists a Turing functional $\phi$ such that for any $C \equiv A$ (mod finite), $\phi^C$ is total, and computes a set that is $\equiv B$ (mod finite). In this case, we write $B \leq_{\mathrm{mf}} A$.

DEFINITION 3.2. Let $A$ be a real. Then a *cofinite oracle,* $(A)$, for $A$ is a partial oracle for $A$ such that $\mathrm{dom}((A))$ is cofinite.

DEFINITION 3.3. Let $A$ and $B$ be reals. Then $B$ *is cofinitely reducible to* $A$ if there exists a Turing functional $\phi$ such that for every cofinite oracle, $(A)$, for $A$, $\phi^{(A)}$ is a partial computation of $B$ with cofinite domain. In this case, we write $B \leq_{\mathrm{cf}} A$.

Note that all the results from the previous section concerning time-independent functionals in generic reduction apply equally well when working with cofinite reduction.

Note also that cofinitely or mod-finitely computing a real is traditionally equivalent to computing the real, since the finite error can be directly coded into the machine. The difference here comes from the demand that the reduction is uniform, a single reduction that works over all potential oracles. The implications between these two reducibilities and Turing reducibility are as follows.

THEOREM 3.4 (Dzhafarov, Igusa). *$B \leq_{\mathrm{mf}} A \Rightarrow B \leq_{\mathrm{cf}} A \Rightarrow B \leq_T A$, and all of the implications are strict*.

The embeddings between the Turing, generic, coarse, cofinite, and mod-finite degrees can be induced by the following maps on reals.

DEFINITION 3.5. If $X \subseteq \omega$, then $\mathcal{R}(X) = \{(2m+1)2^n : m \in \omega, n \in X\}$.
If $X \subseteq \omega$, then $\widetilde{\mathcal{R}}(X) = \{m : (\exists n \in X)(2^n \leq m < 2^{n+1})\}$.

The idea behind $\mathcal{R}$ is that each bit of $X$ is coded redundantly over infinitely many bits of $\mathcal{R}(X)$ (in fact, positive density many bits). On the other hand, for $\widetilde{\mathcal{R}}$, each bit of $X$ is coded into progressively larger and larger (finite) initial segments of $\widetilde{\mathcal{R}}(X)$. These maps induce embeddings as follows.

THEOREM 3.6 (Dzhafarov, Igusa [3]). *The map $X \mapsto \mathcal{R}(X)$ induces an embedding of the Turing degrees into either the mod-finite or cofinite degrees.*
*The map $X \mapsto \widetilde{\mathcal{R}}(X)$ induces an embedding of the mod-finite degrees into the coarse degrees or of the cofinite degrees into the generic degrees.*
*Symbolically, we have that for any reals $A$ and $B$*:
$(B \leq_T A) \Leftrightarrow (\mathcal{R}(B) \leq_{\mathrm{mf}} \mathcal{R}(A)) \Leftrightarrow (\mathcal{R}(B) \leq_{\mathrm{cf}} \mathcal{R}(A))$.
$(B \leq_{\mathrm{mf}} A) \Leftrightarrow (\widetilde{\mathcal{R}}(B) \leq_{\mathrm{c}} \widetilde{\mathcal{R}}(A))$.
$(B \leq_{\mathrm{cf}} A) \Leftrightarrow (\widetilde{\mathcal{R}}(B) \leq_{\mathrm{g}} \widetilde{\mathcal{R}}(A))$.

We describe the idea behind the embedding of the cofinite degrees into the generic degrees. The embeddings of the mod-finite degrees into the coarse degrees and of the Turing degrees into the cofinite degrees are similar, although the embedding of the Turing degrees into the mod-finite degrees is somewhat more subtle. See Proposition 3.3 and Lemma 3.4 from [3] for a more thorough explanation.

In essence, a generic oracle for $\widetilde{\mathcal{R}}(X)$ has precisely the same information in it as a cofinite oracle for $X$. This is because for cofinitely many $n$, there must be some $m$ between $2^n$ and $2^n + 1$ in the domain of a generic oracle, or else its domain cannot be density-1. Conversely, a cofinite oracle for $X$ can compute a cofinite (and hence generic) oracle for $\widetilde{\mathcal{R}}(X)$. The embedding of the mod-finite degrees into the coarse

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

degrees is by a voting algorithm that must eventually give correct answers, and the embedding of the Turing degrees into the cofinite degrees is by an unbounded search that must eventually halt for every $n$.

REMARK 3.7. The algorithm described above illustrates the reason that time-independent functionals are by their nature potentially multivalued. If we were to use the functional described above for computing $X$ from a generic oracle for $\widetilde{\mathcal{R}}(X)$, but if we were to give, as input, a real $Y$ that was not in the range of $\widetilde{\mathcal{R}}$, then there would be intervals $[2^n, 2^n+1)$ on which the oracle gave more than one different output, and so our algorithm would also give more than one different output on those values of $n$.

If we did not demand that our algorithms were time-independent, then we could give the *first* output that we saw from our oracle, but then our output would depend on the order in which our oracle gave its outputs.

Using Theorem 3.6, we may embed the Turing degrees into any of the other four degree structures discussed in this section. In any of these degree structures, we define a real to be "quasiminimal" if it is nonzero, but is not an upper bound for any embedded Turing degrees.

DEFINITION 3.8. Let $A$ be a real, and let $\leq$ be any of: $\leq_c, \leq_g, \leq_{mf}, \leq_{cf}$.

Then $A$ is *quasiminimal* in the $\leq$ degrees if $A \nleq 0$, and if, for every $B$, if $B \nleq_T 0$, then,

- (if $\leq$ is either $\leq_c$ or $\leq_g$) $\widetilde{\mathcal{R}}(\mathcal{R}(B)) \nleq A$,
- (if $\leq$ is either $\leq_{mf}$ or $\leq_{cf}$) $\mathcal{R}(B) \nleq A$.

In these degree structures, a degree is quasiminimal if any (equivalently all) of its elements are quasiminimal.

In the next section, we will see additional motivation as to what makes quasiminimality interesting, but the basic idea is that a quasiminimal degree is a degree that, on one hand is not computable, but on the other hand, does not contain any actual information, in that there are no noncomputable reals that it can compute.

The implication in Theorem 3.4 allows quasiminimality to propagate in a surprisingly robust manner.

PROPOSITION 3.9. *Assume $A$ is quasiminimal in the cofinite degrees. Then $A$ is quasiminimal in the mod-finite, generic, and coarse degrees.*

PROOF. Let $A$ be quasiminimal in the cofinite degrees, and let $B$ be noncomputable in the Turing degrees.

By definition of quasiminimality, $\mathcal{R}(B) \nleq_{cf} A$. By Theorem 3.4, we therefore have that $\mathcal{R}(B) \nleq_{mf} A$. Thus for any noncomputable $B$, we have that $\mathcal{R}(B) \nleq_{mf} A$, and so $A$ is quasiminimal in the mod-finite degrees.

Using Theorem 3.6 and the above two statements, we have that $\widetilde{\mathcal{R}}(\mathcal{R}(B)) \nleq \widetilde{\mathcal{R}}(A)$ in either the generic or coarse degrees, so it remains to show that $A \leq \widetilde{\mathcal{R}}(A)$ in both the coarse and generic degrees, because then the fact that $\widetilde{\mathcal{R}}(\mathcal{R}(B)) \nleq \widetilde{\mathcal{R}}(A)$ will also show that $\widetilde{\mathcal{R}}(\mathcal{R}(B)) \nleq A$.

This last fact follows from the idea behind the proof of Theorem 3.6, mentioned in this paper under the statement of the theorem. A generic oracle for $\widetilde{\mathcal{R}}(A)$ contains enough information to cofinitely (and hence generically) compute $A$ in a uniform

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

manner. Likewise a coarse oracle for $\widetilde{\mathcal{R}}(A)$ contains enough information to mod-finitely (and hence coarsely) compute $A$, again uniformly. ⊣

**§4. Quasiminimality and density-1 bounding.** The results in this paper are motivated in large part by the following question, posed by the second author as Question 3 in [8].

QUESTION 4.1. *Is it true that for every nonzero generic degree* a *there exists a nonzero generic degree* b *such that* b $\leq_g$ a *and such that* b *is the generic degree of a density*-1 *real*?

The resolution of this question would provide insight into the structure of the generic degrees: If the answer is "yes," then there are no minimal generic degrees, and if the answer is "no," then there are minimal pairs in the generic degrees [8]. Also, the question can be rephrased as a question about the relationship between generic computability and coarse computability: We see below that a generic degree is the degree of a density-1 real if and only if it is the generic degree of a coarsely computable real, so the question is about how ubiquitous the coarsely computable reals are at the bottom of the generic degrees.

OBSERVATION 4.2. *Let $B$ be a real, then $B$ is coarsely computable if and only if the generic degree of $B$ has a density*-1 *set*.

PROOF. If $B$ is density-1, then $B$ is coarsely computable because it agrees with $\mathbb{N}$ on a set of density 1.

Conversely, if $B$ is coarsely computable, then fix a computable $X$ such that $B$ agrees with $X$ on a set of density 1. Let $Y = \{n : X(n) = B(n)\}$. Then $Y \leq_g B$ because $X$ is computable, and any generic oracle for $B$ can enumerate a density-1 set of locations where it agrees with $X$. Likewise, $B \leq_g Y$ because again, $X$ is computable, and any generic oracle for $Y$ can enumerate a density-1 set of locations where $X$ is correct about $B$, and then output the values of $X$ on those locations. ⊣

To help us study Question 4.1, we introduce the following terminology.

DEFINITION 4.3. A generic degree $a$ is *density-1-bounding* if there is a nonzero generic degree $b$ such that $b \leq_g a$ and such that $b$ is the generic degree of a density-1 real.

A real $A$ is density-1-bounding if it is of density-1-bounding generic degree.

In prior work, the second author showed that every noncomputable real can generically compute a density-1 real that is not generically computable [7]. In our context, this can be rephrased as saying that every nonzero embedded Turing degree is density-1-bounding in the generic degrees. In particular, this implies the following.

PROPOSITION 4.4 ([7]). *If* b *is not quasiminimal, then* b *is density*-1-*bounding*.

We will reprove this proposition in this paper, as the proof can be modified to prove slightly more. One of the important consequences of this proposition, however, is that any attempt to construct a $b$ that is *not* density-1-bounding must be a construction that is capable of producing a quasiminimal $a$. Unfortunately, currently, every example of a construction of a quasiminimal real $A$ constructs $A$ to be both quasiminimal, and also density-1, and therefore trivially density-1-bounding.

To help us understand quasiminimality, and therefore what sorts of constructions might potentially be able to produce sets that are not density-1 bounding, we provide

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

several examples of quasiminimal sets that are not density-1. We also prove that all of the sets we construct are density-1-bounding, and prove that a few additional sorts of sets are density-1-bounding. This can be taken as evidence toward the answer to Question 4.1 being "yes."

In the remainder of this section, we prove Proposition 4.4, and we modify the proof to show that if $a$ is an embedded cofinite degree, then it is density-1-bounding, and also to show a rather curious result linking non-density-1-bounding with all $\Pi^0_1$-basis theorems. In the next section, we show that 1-generics and 1-randoms are both quasiminimal in the generic degrees, and also that they are density-1-bounding.

For the constructions in the rest of this section, we will use the following notation.

Definition 4.5. Let $P_i = [2^i, 2^{i+1})$. Let $X \subseteq \omega$.
If $e \leq i$, say that $X$ *has a gap of size* $2^{-e}$ *at* $P_i$ if $|X \cap P_i| \leq 2^i - 2^{i-e}$.

The following lemma illustrates the control that these $P_i$ give over a construction that uses them.

Lemma 4.6. *Let* $X \subseteq \omega$.
*Then* $X$ *is density*-1 *if and only if for every* $e$, *there are at most finitely many* $i \geq e$ *such that* $X$ *has a gap of size* $2^{-e}$ *at* $P_i$.

This is a slight strengthening of Lemma 2.3 from [7], which required that the gaps be at the ends of the $P_i$. We will require this lemma in the generality just stated later in this section.

Proof. If $X$ has a gap of size $2^{-e}$ at $P_i$ then, because each $P_i$ contains half the elements of $\omega$ up to the end of $P_i$, $\frac{|X \upharpoonright 2^{i+1}|}{2^{i+1}} \leq 1 - 2^{-e-1}$. If, for a single value of $e$, this happens infinitely often, then $X$ is not density-1.

Conversely, assume that for every $e$, there are finitely many $i$ such that $X$ has a gap of size $2^{-e}$ at $P_i$. Let $\epsilon > 0$. We must show that there is an $m$ such that $\forall n \geq m \left(\frac{|X \upharpoonright n|}{n} > 1 - \epsilon\right)$. Choose $e \in \omega$ such that $(1 - 3 \cdot 2^{-e}) > 1 - \epsilon$. Fix $j$ such that for $\forall i > j$, $X$ does not have a gap of size $2^{-e}$ at $P_i$. Let $m = 2^{j+e+1}$. Then we claim that for $n \geq m$, $\frac{|X \upharpoonright n|}{n} > 1 - \epsilon$.

The reason for this is that for $n \geq m$, the smallest value that $\frac{|X \upharpoonright n|}{n}$ could possibly take is at the beginning of some $P_k$, after omitting the largest number of elements that can be omitted from $P_k$ without causing $X$ to have a gap of size $2^{-e}$ at $P_k$. If this happens, then the elements missing from $X \upharpoonright n$ can, at most, consist of: these elements from the beginning of $P_k$, all of the elements less than $2^{j+1}$, and the elements missing from the $P_i$ for $j < i < k$. The first elements are at most $n2^{-e}$ many elements because the number of elements of $P_k$ is at most $n$. The second elements are at most $n2^{-e}$ many elements because they are at most $2^{j+1}$ many elements, and $n \geq 2^{j+e+1}$. The last elements are at most $n2^{-e}$ many elements because, for each $i$ between $j$ and $k$, $X$ does not have a gap of size $2^{-e}$ at $P_i$.

Thus, there are at most $3n2^{-e}$ elements missing from $X \upharpoonright n$, and so $\frac{|X \upharpoonright n|}{n} > (1 - 3 \cdot 2^{-e}) > 1 - \epsilon$. ⊣

Proof (Proposition 4.4). We define a pair of functionals, $\phi$, $\psi$ such that for any real $X$, if $X$ is not left c.e. (not the leftmost path of any computable tree) then $\phi^X$ is an enumeration of a density-1 set with no density-1 c.e. subset, and if $X$ is not right c.e. (not the rightmost path of any computable tree) then $\psi^X$ is an enumeration of a

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

density-1 set with no density-1 c.e. subset. This will suffice to prove Proposition 4.4 as follows.

Let $a$ be a generic degree that is not quasiminimal, and let $A$ have generic degree $a$. By definition of quasiminimality, fix $X_0$ noncomputable such that $\widetilde{\mathcal{R}}(\mathcal{R}(X_0)) \leq_{\mathrm{g}} A$. Because $X_0$ is noncomputable, $X_0$ must be either not left c.e. or not right c.e. By symmetry, assume $X_0$ is not left c.e. Let $B$ the set enumerated by $\phi^{X_0}$, and let $b$ be the generic degree of $B$. By construction of $\phi$, $B$ is density-1. Furthermore $b \leq_{\mathrm{g}} a$ because any generic oracle for $A$ can generically compute $\widetilde{\mathcal{R}}(\mathcal{R}(X_0))$, and this generic computation can uniformly be used to compute $X_0$, and hence to enumerate $B$. This enumeration is a generic computation of $B$ because it halts on density-1, and it is correct about $B$ wherever it halts. Finally, $B$ is not generically computable because it has no density-1 c.e. subset, and if there were a generic computation of $B$, then the set of $n$ where that computation halted and outputted a 1 would be a density-1 c.e. set. (Density-1 because it is the intersection of two density-1 sets, c.e. because halting and outputting a 1 is a $\Sigma_1$ condition.)

We construct $\phi$ as follows. $\psi$ will be constructed symmetrically. For each $e$, we will have an $e$th strategy, which will act to ensure that if the $e$th c.e. set $W_e$ has density 1, then $W_e$ is not a subset of $\phi^X$ for any $X \in 2^\omega$. In doing so, there will be at most one $X = X_e \in 2^\omega$ such that the strategy prevents $\phi^{X_e}$ from enumerating a density-1 set, and that $X_e$ will be left c.e. Lemma 4.6 will then be used to ensure that the only reals $X$ such that $\phi^X$ is not density-1 are the $X_e$ from these strategies.

We define $\phi$ using the subintervals $P_i$ from Definition 4.5. At stage $s$, we simultaneously define $\phi^X$ on $P_s$ for every $X \in 2^\omega$. Note that because $\phi^X$ enumerates its elements in increasing order, the set enumerated by $\phi^X$ is actually uniformly computable from $X$, not just uniformly generically computable from $X$, although this is not relevant for our purposes.

At stage $s$, for each $e < s$, the $e$th strategy acts as follows. Consider the tree $T_{e,s}$ whose paths are the reals $X$ such that the numbers less than $2^s$ enumerated by $\phi^X$ are a superset of the numbers less than $2^e$ enumerated by $W_e$ by stage $s$. The reals $X$ not on $T_{e,s}$ are the reals that have already "beaten" $W_e$, in that $W_e$ has enumerated an element that they will never enumerate, and so the $e$th strategy will never need to work with those $X$ again. If $T_{e,s}$ has no paths, then the $e$th strategy has accomplished its task, and so it does nothing.

If $T_{e,s}$ has a path, then the $e$th strategy places a "marker" $p_{e,s}$ on the shortest unmarked node of the leftmost path of $T_{e,s}$. Here, an unmarked node is a node such that the $e$th strategy has not yet placed a marker on that node, regardless of whether or not other strategies have marked that node. When it places the marker $p_{e,s}$ on the node $\sigma$, it declares that for every $X \succ \sigma$, $\phi^X$ must have a gap of size $2^{-e}$ at $P_s$. Specifically, the strategy requires that the last $2^{s-e}$ elements of $P_s$ are not in $\phi^X$. The strategy makes no other requirements for $\phi^X$ if $X \not\succ \sigma$.

To define $\phi^X \restriction P_s$ for any given $X$, we simply enumerate every element of $P_s$ that no strategy requires us to not enumerate.

We now prove that this construction has the desired properties.

First we show that for every $e$, and for every $X \in 2^\omega$, $W_e$ is not a density-1 subset of $\phi^X$. Let $e$ be given. Either there is some $s$ such that $T_{e,s}$ has no paths, or for every $s$, $T_{e,s}$ has at least one path.

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

In the first case we have that for every $X$, $W_e$ has enumerated a number not enumerated by $\phi^X$, and so for every $X$, $W_e$ cannot possibly be a density-1 subset of the set enumerated by $\phi^X$.

In the second case, let $T_e = \bigcap_s T_{e,s}$, and let $X_e$ be the leftmost path of $T_e$. Note that for every $\sigma \prec X_e$, there is some $s$ such that $\sigma$ has the marker $p_{e,s}$. This is because every $\tau$ to the left of that $\sigma$ eventually is removed from some $T_{e,s}$, and once all of those $\tau$ are removed, after at most $|\sigma|$ many more steps, $\sigma$ must have a marker placed on it by the $e$th strategy.

All of these $\sigma$ are in $T_e$, and so, in particular, there are infinitely many nodes in $T_e$ with markers placed by the $e$th strategy. Therefore there are infinitely many $s$ for which $W_e$ has a gap of size $2^{-e}$ at $P_s$, because if $W_e$ ever enumerates an element from the last $2^{s-e}$ many elements of $P_s$, then at that stage $s'$, every $X$ extending the node marked with $p_{e,s}$ will be removed from $T_{e,s'}$. Thus, each marker on a $\sigma \in T$ corresponds to a gap of size $2^{-e}$ in $W_e$. Therefore $W_e$ is not an enumeration of a density-1 set, and in particular, not an enumeration of a density-1 subset of the set enumerated by $\phi^X$ for any $X$.

Next we show that if $X$ is not left c.e., then $\phi^X$ is an enumeration of a density-1 set.

Each $X_e$, if it is defined, is left c.e. because it is the leftmost path of a $\Pi^0_1$ tree, and because for every $\Pi^0_1$ tree, there is a computable tree having the same infinite paths. It remains to show that if $X$ is not equal to any of the $X_e$, then $\phi^X$ enumerates a density-1 set. To show this, we show that for every $e$, if $X_e$ is not defined, or if $X$ is not equal to $X_e$, then only finitely many markers are placed on initial segments of $X$ by strategy $e$. This will show that for every $e$, $\phi^X$ has finitely many gaps of size $2^{-e}$, and so by Lemma 4.6, $\phi^X$ is density-1.

So, fix $e$, assume $X_e$ is defined, and assume $X \neq X_e$. If $X$ is to the left of $X_e$, then at some stage $s$, $X$ was removed from $T_{e,s}$, and after that stage, $X$ stopped receiving new markers from strategy $e$. No marker ever gets placed to the right of $X_e$ by strategy $e$, so if $X$ is to the right of $X_e$, then the only markers from strategy $e$ on $X$ are those placed on $\sigma$ that are initial segments of both $X$ and $X_e$, of which there are at most finitely many. If $X_e$ is not defined, it is because there is some stage at which strategy $e$ stops acting, and so strategy $e$ uses at most finitely many markers, so in particular, each $X$ can receive at most finitely many markers from strategy $e$.

Thus, if $X$ is not equal to any of the $X_e$, then for every $e$ only finitely many markers are placed on initial segments of $X$ by strategy $e$, so $\phi^X$ is density-1.

To construct $\psi$, we do the same construction but using rightmost paths instead of leftmost paths. ⊣

We now modify the proof to prove a stronger result.

Proposition 4.7. *Assume $A$ is noncomputable. Then $\widetilde{\mathcal{R}}(A)$ is density-1-bounding in the generic degrees.*

Proposition 4.4 says that embedded noncomputable Turing degrees are density-1-bounding, while Proposition 4.7 says that embedded noncomputable cofinite degrees are density-1-bounding. In the next section, we will demonstrate examples of quasiminimal cofinite degrees, which, in particular, embed into the generic degrees as quasiminimal, and hence this proposition is strictly stronger than Proposition 4.4.

PROOF. Assume $A$ is noncomputable, and further assume that $A$ is not left c.e. Let $\phi$ be the functional $\phi$ from the proof of Proposition 4.4. Let $\tilde{\phi}$ be the time-independent functional defined from $\phi$ as follows.

Given a partial oracle $(X)$, $\tilde{\phi}^{(X)}(n) \downarrow= 1$ if and only if for every $Y$ such that $(X)$ is a partial oracle for $Y$, $\phi^Y(n) \downarrow= 1$.

(Note that this definition is very similar to the definition of $\psi$ as in Definition 2.4. The primary difference is that in this case, $\phi$ was created as a functional that uses ordinary Turing oracles, whereas in Definition 2.4, $\phi$ is a functional that uses partial oracles.)

By compactness, this is a $\Sigma_1$ operation: if every $Y$ agreeing with $(X)$ enumerates some $n$, then there is a finite stage at which we see this happen. Note also that the quantifier over $Y$ agreeing with $(X)$ does not interfere with the fact that $(X)$ enumerates its domain: As $(X)$ produces more answers, the set of $Y$ agreeing with $(X)$ becomes smaller, and so the set of $n$ for which $\tilde{\phi}^{(X)}(n) \downarrow$ becomes larger, so we never need to "take back" any $n$ for which $\tilde{\phi}^{(X)}(n)$ has halted.

We now claim that if $(A)$ is a cofinite oracle for $A$ (recall that any generic oracle for $\widetilde{\mathcal{R}}(A)$ can uniformly produce a cofinite oracle for $A$), then $\tilde{\phi}^{(A)}$ is a generic computation of the set enumerated by $\phi^A$, which we have proved is density-1 and not generically computable. Because $A$ is one of the $Y$'s agreeing with $(A)$, we have that the domain of $\tilde{\phi}^{(A)}$ is a subset of the domain of $\phi^A$, so in particular, $\tilde{\phi}^{(A)}$ never makes any mistakes about the set it is computing. It remains to show that $\mathrm{dom}(\tilde{\phi}^{(A)})$ is density-1.

To prove this, let $S$ be the set of initial segments of $Y$'s such that $(A)$ is a partial oracle for $Y$. We show that strategy $e$ from the proof of Proposition 4.4 places at most finitely many markers on elements of $S$. To show this, define $T_e$ as in the proof of Proposition 4.4. If $T_e$ is finite, then strategy $e$ places at most finitely many markers, and so it places at most finitely many markers on elements of $S$. If $T_e$ is infinite, then let $X_e$ be the leftmost path of $T_e$, and we claim that for every $\sigma \prec X_e$, strategy $e$ places at most finitely many markers on $\tau$ that are not extensions of $\sigma$. This is because everything to the left of $\sigma$ eventually gets removed from $T_e$, and everything to the right of $\sigma$ never gets a marker, and there are only finitely many things below $\sigma$.

So it remains to show that there is some $\sigma \prec X_e$ such that $\sigma$ has no extensions in $S$. This follows from the fact that $A$ is not left c.e., $(A)$ is a cofinite oracle for $A$, and every $Y$ agreeing with $(A)$ is mod-finitely equal to $A$, and so also not left c.e. Therefore there must be some $n$ such that $(A)(n) \downarrow\neq X_e(n)$, because $X_e$ is left c.e., and so $(A)$ is not a partial oracle for $X_e$.

We therefore conclude that for every $e$, $S$ has only finitely many markers on it placed by strategy $e$, and thus that the domain of $\tilde{\phi}^{(A)}$ has at most finitely many gaps of size $2^{-e}$, because if the $e$th strategy places a marker $p_{e,s}$ on some $\sigma \notin S$, then for every $Y$ agreeing with $(A)$, $\sigma \not\prec Y$, and so $\phi^Y$ does not have a gap of size $2^{-e}$ at $P_s$. (Or rather, the $e$th strategy does not cause it to have such a gap. The gaps of size $2^{-e}$ can be created by strategy $e'$ for any $e' < e$, but this proof shows that each one of those strategies causes at most finitely many gaps to appear in the domain of $\tilde{\phi}^{(A)}$.) Thus, by Lemma 4.6, the domain of $\tilde{\phi}^{(A)}$ is density-1. ⊣

As a corollary to the proof of Proposition 4.7, we make an observation that will have a number of consequences.

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

Observation 4.8. *Assume there is no left c.e. set $X$ such that $\{n : X(n) = A(n)\}$ is density*-1. *Then $A$ is density*-1-*bounding.*

Proof. Define $\tilde{\phi}$ as above. If $A$ is a real, there is no left c.e. set $X$ such that $\{n : X(n) = A(n)\}$ is density-1, and $(A)$ is a generic oracle for $A$, then, in particular, for every $e$, $X_e$ is not one of the reals $Y$ that agree with $(A)$. In particular, we therefore again have that there is some $\sigma \prec X_e$ such that no $Y \succ \sigma$ agrees with $(A)$. After some finite stage, all markers placed by strategy $e$ are placed on extensions of $\sigma$, and so therefore, there are at most finitely many markers placed on initial segments of $Y$'s that agree with $(A)$. This again proves that for every $e$, there are at most finitely many gaps of size $2^{-e}$ in the set enumerated by $\tilde{\phi}^{(A)}$. ⊣

This observation can be modified by a technique from a previous paper of the second author (Lemma 2.6 from [7]) which says that "left c.e." can be replaced by any property that can by realized uniformly in $0'$ effective basis theorem.

Observation 4.9. *Let $\mathcal{F}$ be any function from reals to reals such that for any $\Pi^0_1$ tree $T$, if $T$ is infinite, then $\mathcal{F}(T)$ is an infinite path through $T$ that is uniformly computable in $0'$ together with a $\Pi^0_1$ index for $T$.*

*Assume there is no $X$ in the range of $\mathcal{F}$ such that $\{n : X(n) = A(n)\}$ is density*-1. *Then $A$ is density*-1-*bounding.*

In particular, using the Low Basis Theorem, this says that if $A$ is not density-1-bounding, then $A$ agrees with a low set on a set of density 1. Likewise, using the cone avoidance basis theorem, this says that if $A$ is not density-1-bounding, then for any noncomputable $\Delta^0_2$ $B$, there is a $\Delta^0_2$ set $X$ such that $X \not\geq_T B$ and $A$ agrees with $X$ on a set of density-1, etc. In essence, Observation 4.9 is an observation schema across $0'$ computable basis theorems.

Proof (Sketch). In the proof of Proposition 4.4, wherever strategy $e$ would normally place a marker on the shortest unmarked node of $T_{e,s}$, have it instead place a marker on the shortest unmarked node of the stage-$s$ $\Delta^0_2$ approximation to $\mathcal{F}(T_e)$. (A $\Pi^0_1$ index for $T_e$ can be obtained using the recursion theorem.) Let $X_e = \mathcal{F}(T_e)$. Because $\Delta^0_2$ approximations eventually converge, for every $\sigma \prec X_e$, we have that eventually all markers are placed on extensions of $\sigma$, which is sufficient for the verification for Proposition 4.4 and also for Observation 4.8. ⊣

Remark 4.10. If the proof of Proposition 4.4 could be modified somehow to make each $X_e$ computable, then Observation 4.8 would probably be able to be modified to say that if there is no computable set $X$ such that $\{n : X(n) = A(n)\}$ is density-1 then $A$ is density-1-bounding. Note that Observation 4.2 implies that if $A$ is coarsely computable, then $A$ is in the same generic degree as a density-1 set, and so in particular is density-1-bounding. Combining these two would prove that if $A$ is not generically computable, then $A$ is density-1-bounding, solving Question 4.1.

Simplifying the analysis above leads to a question that has been open since the writing of [7], but that now appears to be sufficiently motivated to be worth stating as an open problem.

Question 4.11. *Is there a uniform proof of Proposition* 4.4?

This question asks whether there is a single $\phi$ such that for every $A$, if $A$ is not computable, then $\phi^A$ is an enumeration of a density-1 set with no density-1 c.e. subset. A positive solution would not necessarily answer Question 4.1 as well, because the solution might not admit the modification required for Observation 4.8, but the question is elegant and simple enough that it might merit study in its own right.

We conclude the section with a proof of a result that effectively says that the observations stated here are not already sufficient to prove that every nonzero degree is density-1-bounding in the generic degrees.

PROPOSITION 4.12 (Dzhafarov, Igusa). *There exists a real $A$ such that for every $\mathcal{F}$ as in Observation* 4.9, *there is an $X$ in the range of $\mathcal{F}$ such that $\{n : X(n) = A(n)\}$ is density*-1, *but such that $A$ is neither coarsely computable nor generically computable.*

Note that by Observation 4.9 and Lemma 4.2, if there existed a $A$ that was not density-1-bounding, then such a $A$ would necessarily need to be a witness to Proposition 4.12.

This construction is a slight modification of a construction by Dzhafarov, Igusa, and Westrick [personal communication] in which they proved Proposition 4.12 without the additional requirement that $A$ not be generically computable. The previous construction could only build generically computable $A$ although it contained most of the ideas necessary to write this proof.

PROOF. We build an infinite computable tree $T \subseteq 2^{<\omega}$ such that given any two paths in $[T]$, the two paths agree on density 1, and such that no path in $[T]$ is either coarsely computable or generically computable. Any path in $[T]$, can then be used as our $A$. This is because, given any $\mathcal{F}$, there must be a path in $[T]$ that is in the range of $\mathcal{F}$, and by construction of $T$, that path must agree with $A$ on density 1. Furthermore, by construction of $T$, $A$ is neither coarsely computable nor generically computable.

The construction is as follows. We have requirements:

$\mathcal{C}_i$: $\phi_i$ does not coarsely compute a path through $T$.

$\mathcal{G}_i$: $\phi_i$ does not generically compute a path through $T$.

We first remark that if each individual requirement can be met uniformly, then we can combine those trees to produce $T$. More formally:

CLAIM. *Assume there is a computable function $f$ such that $\phi_{f(2i)}$ computes an infinite tree $T_{2i} \subseteq 2^{<\omega}$ all of whose paths agree on density* 1 *such that $\phi_i$ does not coarsely compute a path through $T_{2i}$, and such that $\phi_{f(2i+1)}$ computes an infinite tree $T_{2i+1} \subseteq 2^{<\omega}$ all of whose paths agree on density* 1 *such that $\phi_i$ does not generically compute a path through $T_{2i+1}$. Then there is an infinite computable tree $T \subseteq 2^{<\omega}$ all of whose paths agree on density* 1 *such that no path is either generically computable or coarsely computable.*

PROOF OF CLAIM. Given $X \in 2^{\omega}$, let $X_k = \{n : (2n+1)2^k \in X\}$. Note that if $X$ is generically computable, then each $X_k$ is generically computable, and if $X$ is coarsely computable, then each $X_k$ is coarsely computable.

For each $k$, let $T_k$ be as in the statement of the claim. Define $T$ by $X \in [T]$ iff for each $k$, $X_k \in [T_k]$. (Note that because the $T_k$ are uniformly computable, this is a $\Pi^0_1$ class, and so in particular, there is a $T$ whose paths are the reals such that $\forall k X_k \in T_k$.) We then claim that if $X \in [T]$, then $X$ is neither coarsely nor generically computable.

To see this, assume otherwise. Assume $X \in [T]$ and $\psi$ is a Turing functional that coarsely computes $X$ (the proof for generic computation will be nearly identical).

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

Define $g$ so that $\phi_{g(i)}(n) = \psi((2n+1)2^{2i})$. Note then that if $\psi$ is a coarse computation of $X$, then $\phi_{g(i)}$ is a coarse computation of $X_{2i}$. By the recursion theorem, there exists an $i$ such that $\phi_{g(i)} = \phi_i$, providing a contradiction, because $X_i$ was specifically constructed to not be coarsely computable via $\phi_i$.

It remains to show that any two paths in $[T]$ must agree on density-1. Let $X, Y \in [T]$, let $\epsilon > 0$. Let $l$ be such that $2^{-l} < \frac{\epsilon}{2}$. For each $k < l$, let $n_k$ be such that for $m > n_k$, $X_k \upharpoonright m$ and $Y_k \upharpoonright m$ agree on $m(1-\frac{\epsilon}{2})$ many bits. Let $n = \max_{k<l}(n_k 2^k)$. It is straightforward to check that for $m > n$, $X \upharpoonright m$ and $Y \upharpoonright m$ agree on $m(1-\epsilon)$ many bits.

This concludes the proof of the claim. To complete the proof of the theorem, it now remains to construct the $T_k$ uniformly.

*Meeting requirement $\mathcal{C}_i$*: We meet all of these requirements uniformly with a single tree. Let $X_0$ be any real that is generically computable but not coarsely computable (exists by [9]).

Let $\phi$ be the Turing functional that generically computes $X_0$. For every $i$, let $T_{2i}$ be the tree of all $X$ such that $\phi$ is a generic computation of $X$. Note that all such $X$ agree on density 1 (because they agree on $\mathrm{dom}(\phi)$), and are not coarsely computable (because a coarse computation of one of them would be a coarse computation of every one of them, and $X_0$ is not coarsely computable).

*Meeting requirement $\mathcal{G}_i$*: We construct a tree $T_{2i+1} \subseteq 2^{<\omega}$ all of whose paths agree on density 1 such that $\phi_i$ does not generically compute a path through $T_{2i+1}$.

We construct a computable tree $\widetilde{T} \subseteq 2^{<\omega}$ such that any two paths in $[\widetilde{T}]$ agree on density 1 but such that, for any $n \in \omega$, there is at least one $X \in [\widetilde{T}]$ with $X(n) = 0$ and at least one $X \in [\widetilde{T}]$ with $X(n) = 1$. If we ever see $\phi_i(n)$ halt for any value of $n$, then we let $n_i$ be the first value of $n$ for which $\phi_i(n)\downarrow$, and $s_i > n_i$ be the number of stages required to see that $\phi_i(n_i)\downarrow$.

We then define:

$$T_{2i+1} = \{\sigma \in \widetilde{T} : |\sigma| < s_i\} \cup \{\sigma \in \widetilde{T} : |\sigma| \geq s_i \wedge \sigma(n_i) \neq \phi_i(n_i)\}.$$

Note, in particular, that $T_{2i+1}$ will be defined uniformly in $i$, and that if $\phi_i$ never halts (or if $\phi_i(n_i) \notin \{0,1\}$) then $T_{2i+1} = \widetilde{T}$.

To ensure that all paths in $[\widetilde{T}]$ agree on density 1, we ensure that every path in $[\widetilde{T}]$ is density-1 (as a subset of $\omega$), and so any two paths agree on a set containing their intersection, which is a density-1 set.

We put the empty string into $\widetilde{T}$.

When we define $\widetilde{T} \upharpoonright \{\sigma : 2^n \leq |\sigma| < 2^{n+1}\}$ we will have that there exist exactly $2^n$-many $\sigma \in \widetilde{T}$ with $|\sigma| = 2^n - 1$, and we will ensure that there exist exactly $2^{n+1}$-many $\sigma \in \widetilde{T}$ with $|\sigma| = 2^{n+1} - 1$.

To define $\widetilde{T} \upharpoonright \{\sigma : 2^n \leq |\sigma| < 2^{n+1}\}$, each $\sigma \in \widetilde{T}$ with $|\sigma| = 2^n - 1$ selects one element $m_\sigma$ of $[2^n, 2^{n+1})$, and extends itself so that the extensions of $\sigma$ of length $2^{n+1} - 1$ are precisely the two $\tau$ of that length such that $\tau \succ \sigma$ and $\tau(m) = 1$ if $(m \in [2^n, 2^{n+1})$ and $m \neq m_\sigma)$.

If each $\sigma$ selects a different $m_\sigma$, then every $m \in [2^n, 2^{n+1})$ will be selected, because $|[2^n, 2^{n+1})| = 2^n$. This is easy to arrange.

At the end of the construction, for each $X \in \widetilde{T}$, for each $n \in \omega$, there will be at most one $m \in [2^n, 2^{n+1})$ such that $m \notin X$, and so $X$ will be density-1. Also, every $\sigma$

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

in $\widetilde{T}$ extends to an $X \in [\widetilde{T}]$, and so, in particular, for every $m$, there is an $X \in [\widetilde{T}]$ such that $X(m) = 0$ and an $X \in [\widetilde{T}]$ such that $X(m) = 1$. ⊣

Given the results of this section, the paths through this tree $T$ are potential candidates for sets that are not density-1-bounding in the generic degrees. However, we now show that all of the paths through $T$ are density-1-bounding.

Proposition 4.13. *Let $T$ be the tree constructed in Proposition 4.12. Every $X \in [T]$ is density-1-bounding in the generic degrees.*

Proof. Let $T$ be the tree constructed in Proposition 4.12, and for each $k \in \omega$, let $T_k$ be constructed as in the proof of Proposition 4.12.

Let $X \in [T]$. Let $X_k = \{n : (2n+1)2^k \in X\}$, and note that by definition of $T$, we have that $X_k \in [T_k]$.

Define $Y$ so that $Y_i = X_{2i+1}$. More formally, so that $(2n+1)2^i \in Y \leftrightarrow n \in X_{2i+1}$. Note that $Y$ is not generically computable by the same argument as why $X$ is neither generically nor coarsely computable. Note also that $Y$ is density-1, because for every $i$, every path through $T_{2i+1}$ is density-1. Finally, $Y \leq_g X$, because a generic oracle for $X$ must contain density-1 many of the bits of $X_k$ for every $k$. ⊣

The reason we are able to prove Proposition 4.13 is that the non-coarse computability requirements and non-generic computability requirements are addressed independently in the proof of Proposition 4.12. If there were a way of meeting both sorts of requirements simultaneously, perhaps by meeting general "non-dense-computability" requirements, then this might shed additional light on Question 4.1.

Question 4.14. *If we define a real $A$ to be "densely computable" if there is a partial computable $\phi$ such that $\{n : \phi(n) = A(n)\}$ is density 1, then does there exist an infinite computable tree $T \subseteq 2^{<\omega}$ such that given any two paths in $[T]$, the two paths agree on density 1, and such that no path in $[T]$ is densely computable?*

**§5. Randoms and generics.** In this section, we investigate the generic degrees of random reals, and of generic reals. We show that both randomness and genericity imply quasiminimality in the cofinite degrees, and therefore (by Lemma 3.9) in the mod-finite, generic, and coarse degrees. In particular, this provides examples of quasiminimal sets that are not density-1, potentially helping along the way to a construction of a set that is not density-1-bounding. We also show that both randomness and genericity imply density-1-bounding in the generic degrees, potentially helping along the way to a proof that all non-generically-computable sets are density-1-bounding.

Proposition 5.1. *If $A$ is a weakly 1-random real, then $A$ is density-1-bounding in the generic degrees.*

Remark 5.2. Weak 1-randomness is implied by 1-randomness, and also by weak 1-genericity and therefore 1-genericity. Therefore, in particular, this shows that 1-randoms and 1-generics are density-1-bounding in the generic degrees.

Proof. The proof we present is very similar to that of Theorem 2.2 of Hirschfeldt, Jockusch, McNicholl, and Schupp [5].

Define $B = \{\sigma \in 2^{<\omega} : \sigma \not\prec A\}$. For purposes of density, it is important to determine which coding of $2^{<\omega}$ as a subset of $\omega$ is being used. We use the standard order on $2^{<\omega}$: first by length of $\sigma$, and then lexicographically among $\sigma$ of the same length, with one small adjustment. We start with the empty string being coded by $n = 1$, not $n = 0$, so that $\{\sigma : |\sigma| = i\}$ will be coded in to $P_i$, as in Definition 4.5. (There is no string coded by $n = 0$.)

Note that for any $A$ at all, $B$ as defined above is density-1, because there is exactly one $\sigma$ of each length not in $B$. Likewise, for any $A$ at all, $A \geq_g B$, as follows.

If $(A)$ is any partial oracle for $A$, consider the partial computation of $B$ given by enumerating all $\sigma$ that $(A)$ is able to rule out as potential initial segments of $A$:

$$\phi^{(A)}(\sigma) = 1 \leftrightarrow \exists n\big(n \in \text{dom}((A))\ \&\ n < |\sigma|\ \&\ (A)(n) \neq \sigma(n)\big).$$

Note that for any partial oracle $(A)$, for $A$, $\phi^{(A)}$ is a partial computation of $B$.

If $\text{dom}((A))$ is infinite, then we claim that $\phi^{(A)}$ is a generic computation of $B$. To show this, we show that for each $e$, there are at most finitely many $i$ such that $\text{dom}(\phi^{(A)})$ has a gap of size $2^{-e}$ at $P_i$, and then appeal to Lemma 4.6. To see this, let $e$ be given, and fix $i_0$ such that $|\text{dom}((A)) \upharpoonright i_0| \geq e + 1$. For $i > i_0$, there are at most $2^{i-(e+1)}$ many $\sigma$ of length $i$ agreeing with $(A)$, and so $\phi^{(A)}$ enumerates at least $2^i - 2^{i-(e+1)}$ many $\sigma$ of length $i$, and so $\text{dom}(\phi^{(A)})$ does not have a gap of size $2^{-e}$ at $P_i$.

It now remains to show that if $A$ is weakly 1-random, then the $B$ that we built is not generically computable.

Because $B$ is density-1, it is generically computable if and only if it has a density-1 c.e. subset. To prove that this cannot be the case, let $W_e$ be a c.e. set, thought of as coding a subset of $2^{<\omega}$. Consider $\mathcal{V} = \{X \in 2^\omega : \forall \sigma \in W_e, \sigma \not\prec X\}$. This is a $\Pi^0_1$ class, and we claim that if $W_e$ is density-1, then $\mathcal{V}$ is null, and we also claim that if $W_e$ is a subset of $B$, then $A$ is an element of $\mathcal{V}$.

Both of these claims are straightforward from the definitions, and we leave the verification to the reader. If $A$ is weakly 1-random, then $A$ cannot be a member of any null $\Pi^0_1$ class, and so $W_e$ cannot be a generic computation of $B$. ⊣

We now go on to prove that 1-generics and 1-randoms are quasiminimal in the cofinite, (and hence mod-finite, generic and coarse) degrees.

Both of these proofs will be by the following lemma, which summarizes and compiles the results of Sections 2 and 3 that we will use in this section. Note that the results of Section 2 concerning time-independent functionals apply equally well to cofinite reduction as they do to generic reduction.

Lemma 5.3. *Assume $A$ is not quasiminimal in the cofinite degrees. Then there exists a time-independent Turing functional $\phi$ such that for any cofinite oracle $(A)$, for $A$, $\phi^{(A)}$ is total, and furthermore such that $\phi^A$ is not multivalued, and is a computation of a noncomputable real $B$.*

Proof. If $A$ is not quasiminimal in the cofinite degrees, then by definition of quasiminimality, there is a noncomputable $B$ such that $\mathcal{R}(B) \leq_{\text{cf}} A$. By Observation 2.9, there is a time-independent Turing functional $\psi$ such that $\mathcal{R}(B) \leq_{\text{cf}} A$ via $\phi$. Any cofinite oracle for $\mathcal{R}(B)$ can be uniformly used to compute $B$, and if we use the time-independent version of the Turing functional that computes $B$ from a cofinite oracle for $\mathcal{R}(B)$, and compose that functional with $\psi$, then we obtain a time-independent Turing functional $\psi$ such that for any cofinite oracle $(A)$, for $A$, $\phi^{(A)}$ is a total computation of $B$.

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

In particular, because cofinite computations are never allowed to make mistakes, $\phi^A$ only produces correct outputs concerning $B$, and so is not multivalued. ⊣

PROPOSITION 5.4. *If $A$ is weakly 1-generic, then $A$ is quasiminimal in the cofinite degrees, and hence in the mod-finite, generic, and coarse degrees.*

PROOF. Let $A$ be weakly 1-generic, and assume $A$ is not quasiminimal. By Lemma 5.3, there is a noncomputable $B$ and a time-independent Turing functional $\phi$ such that for any cofinite oracle $(A)$, for $A$, $\phi^{(A)}$ is total, and is a computation of $B$. We prove then that $B$ is computable as follows.

Consider $S = \{\sigma \in 2^{<\omega} : \phi^\sigma \text{ is multivalued}\}$. Note that $S$ is $\Sigma^0_1$ because $\sigma \in S \leftrightarrow \exists n\,(\phi^\sigma(n)\downarrow= 1\ \&\ \phi^\sigma(n)\downarrow= 0)$. If $S$ is dense, then because $A$ is weakly 1-generic, $A$ meets $S$, and so by Observation 2.7 it cannot be the case that $\phi^A$ is a computation of $B$.

If $S$ is not dense, then choose any $\tau$ that has no extensions in $S$. Let $m = |\tau|$. For any $X \in 2^\omega$, let $X_m$ be the partial oracle for $X$ that does not halt on any of its first $m$ bits. Then we claim that $B$ is computable by the functional $\psi$ such that $\psi(n)$ searches for an $X$ such that $\phi^{X_m}(n)\downarrow$ and then outputs the same value as the found output.

This computation is total because for any cofinite oracle $(A)$, for $A$, $\phi^{(A)}$ is total, and so in particular, for any $n$, $\phi^{(A)_m}(n)\downarrow$. Furthermore, when this computation halts, it gives a correct output for the following reason. Assume not, and fix $X$ such that $\phi^{X_m}(n)\downarrow\neq B(n)$. Let $k$ be the use of this computation (the smallest number such that only $X \upharpoonright k$ was required for the computation of $\phi^{X_m}(n)$). Note then that by hypothesis $\phi^{A_k}(n)\downarrow= B(n)$. Let $l$ be the use of this computation.

Let $\sigma \in 2^{<\omega}$ be given by $\sigma \upharpoonright m = \tau$, $\sigma \upharpoonright [m,k) = X \upharpoonright [m,k)$, and $\sigma \upharpoonright [k,l) = A \upharpoonright [k,l)$. Note then that $\sigma \succeq \tau$, and furthermore, because $\phi$ is a time-independent functional, $\phi^\sigma(n) = \phi^{X_m}(n) \neq B$, and also $\phi^\sigma(n) = \phi^{A_k}(n) = B$, so $\phi^\sigma$ is multivalued, contradicting our choice of $\tau$. ⊣

PROPOSITION 5.5. *If $A$ is 1-random, then $A$ is quasiminimal in the cofinite degrees, and hence in the mod-finite, generic, and coarse degrees.*

The proof of Proposition 5.5 is somewhat more involved than the proof of Proposition 5.4. It will use Lemma 5.3 as well as a fairly subtle control of the halting measure of $\phi$. Throughout the remainder of the section, we will always assume that $\phi$ is a time-independent Turing functional.

DEFINITION 5.6. Fix $\phi$, and an integer $n$. Let $k \leq l$ be integers.

For any real $X$, let $X_k$ be the partial oracle for $X$ that does not halt on inputs less than $k$, and let $X_{k,l}$ be the partial oracle for $X$ that does not halt on inputs less than $k$ or greater than or equal to $l$.

Define $\mu_k = \mu(\{X : \phi^{X_k}(n)\downarrow\})$, the halting measure of $\phi$-computations that do not use the first $k$ bits of their oracles.

Similarly, define $\mu_{k,l} = \mu(\{X : \phi^{X_{k,l}}(n)\downarrow\})$.

Note that $\phi$ and $n$ are suppressed in the notation for brevity.

OBSERVATION 5.7.
*If $k_0 < k_1$, then for any $l$, $\mu_{k_0,l} \geq \mu_{k_1,l}$, and also $\mu_{k_0} \geq \mu_{k_1}$.*
*If $l_0 < l_1$, then for any $k$, $\mu_{k,l_0} \leq \mu_{k,l_1}$.*
*For any $k$, $\mu_k = \lim_{l\to\infty} \mu_{k,l}$.*

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

PROOF. The first two facts follow from Observation 2.6, which says that larger oracles never have smaller halting sets. The third fact is because any computation that halts does so using only finitely much from its oracle. ⊣

LEMMA 5.8. *Let $A$ be 1-random, $\phi$ a time-independent Turing functional, and $n$ an integer. If for every cofinite oracle $(A)$ for $A$, $\phi^{(A)}(n)\downarrow$, then, for that $\phi$ and $n$, for every $k$, $\mu_k = 1$.*

PROOF. Assume not. Fix $k$ such that $\mu_k \neq 1$, and fix some computable $\epsilon > 0$ such that $\mu_k < 1 - \epsilon$.

Consider the open sets $U_i$ defined by $X \in U_i$ if $\phi^{X_k}$ converges for at least $i$ different independent reasons. We make this precise by an inductive definition as follows.

Let $U_0 = 2^\omega$, and for every $X$, $k_{X,0} = k$.

Having defined $U_i$, and $k_{X,i}$ for every $X$ in $U_i$, we let $U_{i+1} = \{X \in U_i : \phi^{X_{k_{X,i}}}\downarrow\}$, and for every $X$ in $U_{i+1}$, we let $k_{X,i+1}$ be the minimum number $s$ such that $\phi^{X_{k_{X,i}}}\downarrow$ in less than $s$ steps, and using less than the first $s$ bits of the oracle $X$. Such an $s$ can be found computably in $X$ for any $X$ in $U_{i+1}$, and so each $U_i$ is defined $\Sigma_1$ in the previous one. Thus, the $U_i$ are uniformly $\Sigma_1$ sets.

Furthermore, $\mu(U_i) \leq (1-\epsilon)^i$. This is because to determine whether or not $X \in U_i$, we must see that $\phi^{X_k}$ halts (a measure $< 1-\epsilon$ event), then we ignore everything that caused $\phi^{X_k}$ to halt, and require that $\phi^{X_{k_{X,0}}}$ halts (another measure $< 1-\epsilon$ event) and so on. Each event is independent, and so the probability of meeting all of them is equal to their product.

If for every $l$, $\phi^{A_l}\downarrow$, then $A$ must be in every $U_i$. This contradicts the assumption that $A$ is 1-random. ⊣

This lemma will be what we require in order to construct a collection of "towers" that will prove our contradiction. We will use the ideas of a "90%-halting tower," an "80%-agreement tower," and a "60%-disagreement tower." The numbers 90%, 80%, and 60% are not special: the only important facts about them are that 100% > 90% > 80% > 50%, and also that 80% > 60% > 0%. For our proof, it will be convenient that $0.6 < 0.8^2$.

DEFINITION 5.9. Fix $\phi$ and $n$, and $k$. Then for that $\phi$, $n$ and $k$, a *90%-halting tower starting at $k$* is a sequence of numbers $\langle k_i : i \in \omega\rangle$ such that $k_0 = k$, for every $i$, $k_{i+1} > k_i$, and $\mu_{k_i,\mu_{k_{i+1}}} > 0.9$.

OBSERVATION 5.10. *Fix $\phi, n$.*

*If there exists some $k$ such that there is a 90%-halting tower starting at $k$, then for every $k$, there is a 90%-halting tower starting at $k$. Moreover, that 90% halting tower can be found uniformly computably in $n, k$.*

PROOF. The obvious greedy algorithm works for this. Let $k_0 = k$, and search for a $k_1$ such that $\mu_{k_0,\mu_{k_1}} > 0.9$. Eventually such a $k_1$ will be found because there is a 90%-halting tower, $\langle l_i : i \in \omega\rangle$ starting somewhere, and that other halting tower must have some $i$ such that $l_i \geq k_0$. But then $\mu_{k_0,l_{i+1}} \geq \mu_{l_i,l_{i+1}} \geq 0.9$ (using Observation 5.7). Thus, $l_{i+1}$ would work as $k_1$, although some other smaller or larger $k$ might be found first. We then proceed inductively to define each $k_i$. ⊣

LEMMA 5.11. *Assume that $\phi$ is such that for every $n$, and for every $k$, $\mu_k = 1$. Then, for that $\phi$ and for every $n$ and $k$, there exists a 90%-halting tower starting at $k$.*

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

PROOF. Once again, the obvious greedy algorithm works. We let $k_0 = k$, and for each $i$, let $k_{i+1}$ be the first $l$ found such that $\mu_{k_i,l} \geq 0.9$. Such an $l$ exists because $\lim_{l\to\infty} \mu_{k_i} = 1$. ⊣

DEFINITION 5.12. Fix $\phi, n, k$, and $\langle k_i : i \in \omega\rangle$, a 90%-halting tower starting at $k$. Then $\langle k_i : i \in \omega\rangle$ is an *80%-agreement tower* if there exists some $v \in \{0, 1\}$ such that:

(1) There exists an $i$ such that $\mu\{X : \phi^{X_{k_i,k_{i+1}}}(n) = v\} > 0.8$ and,
(2) There does not exist an $i$ such that $\mu\{X : \phi^{X_{k_i,k_{i+1}}}(n) \neq v\} > 0.8$.

In this case, we will sometimes say that $\langle k_i : i \in \omega\rangle$ is an 80%-agreement tower with value $v$.

The purpose of these 80% agreement towers is that they will allow us to compute $B$ without needing $A$ as an oracle. We will later prove that they must exist, but first we show how they can be used to compute $B$.

LEMMA 5.13. *Assume that $A$ is 1-random, and that $\phi^A$ is a computation of $B$. Fix $n, k, v$. Assume that for that $\phi, n$, there exists a computable 80%-agreement tower, $\langle k_i : i \in \omega\rangle$, with value $v$ starting at $k$. Then $B(n) = v$.*

PROOF. Assume the hypotheses are true and the conclusion is false.

Consider the open sets $U_i = \{X : \phi^{X_{k_i,k_{i+1}}}(n) = v\}$. We have that $A$ is not in any $U_i$ because $\phi^A(n) = B(n) \neq v$. Also, for every $i$, $\mu\{X : \phi^{X_{k_i,k_{i+1}}}(n)\downarrow\} > 0.9$, and there does not exist an $i$ such that $\mu\{X : \phi^{X_{k_i,k_{i+1}}}(n) \neq v\} > 0.8$. Therefore $\mu(U_i > 10\%)$.

Let $C_i$ be the complement of $U_i$, and let $C = \bigcap_i C_i$. The measure of the intersection of the $C_i$ is equal to the product of the measures of the $C_i$ because each $C_i$ is defined in terms of only the bits of $X$ between $k_i$ and $k_{i+1}$. Therefore $C$ is a null $\Pi^0_1$ set, and $A \in C$. This contradicts the assumption that $A$ is 1-random. ⊣

To prove that they must exist, we will use 60%-disagreement towers, which will derandomize $X$ if $\phi$ does not produce enough 80% agreement towers. These 60%-disagreement towers, unlike most of our other work in this proof, will not be fixed to a specific $n$.

DEFINITION 5.14. Let $\phi$ be given. Then a 60%-*disagreement tower* for $\phi$ is a sequence $\langle m_i : i \in \omega\rangle$ such that for every $i$,
$\mu\left\{X : (\exists n)\left(\phi^{X_{m_i,m_{i+1}}} \text{ is a multivalued function on } n\right)\right\} > 0.6$.

OBSERVATION 5.15. *If there is a 60%-disagreement tower for $\phi$, then there is a computable 60%-disagreement tower for $\phi$.*

PROOF. Again, a greedy algorithm produces a computable 60%-disagreement tower. ⊣

LEMMA 5.16. *Let $\phi$ be given. Then, either there exists a $k$ such that every 90%-halting tower (for any $n$) starting at $k$ is an 80%-agreement tower, or there exists a 60%-disagreement tower for $\phi$.*

PROOF. Assume the first clause is false. We construct a 60%-disagreement tower as follows.

Let $m_0 = 0$. Choose some $n$ and some $\langle k_i : i \in \omega\rangle$ that is a 90%-halting tower for that $n$ starting at $k = m_0$ that is not an 80%-agreement tower.

CASE 1. If it is not an 80%-agreement because it never reaches a consensus (i.e., it fails the first clause in the definition of 80%-agreement towers), then choose $j$ to be large enough that $(0.9)(1-(0.9)^{j-1}) > (0.6)$, and let $m_1 = k_j$. We claim then that

$$\mu\left\{X : \left(\phi^{X_{m_0,m_1}} \text{ is a multivalued function on } n\right)\right\} > 0.6.$$

This is because of the following calculation.

Let $S$ be the set of $X$ such that $\phi^{X_{k_0,k_1}}(n)$ halts. Note that $\mu(S) \geq 0.9$, by definition of a 90% halting tower.

We show that given any $i > 0$, and any $\sigma$ with $|\sigma| = k_i$, if $\sigma \prec X$ for some $X \in S$, and $\phi^\sigma(n)$ is not multivalued, then $\phi^\tau(n)$ is multivalued for at least 10% of all $\tau \succ \sigma$ with $|\tau| = k_{i+1}$. This will show that, $\phi^\tau(n)$ is multivalued for at least measure $(0.9)(1-(0.9)^{j-1})$ many $\tau$ of length $k_j$ because each time we increase $i$, 10% of all strings that have not yet become multivalued, and we started with measure 0.9 many strings.

Let $\sigma$ be as above, and note that because $\sigma \prec X$ for some $X \in S$, we have that $\phi^\sigma(n)$ is defined. By hypothesis of Case 1, at least 10% many $X$ give the opposite output from the rest of the $X$ when computing $\phi^{X_{i,i+1}}$. In particular, this implies that at least 10% many $X$ halt and give the opposite output from $\phi^\sigma(n)$ when computing $\phi^{X_{i,i+1}}$. This computation does not use any bits of $X$ less than $|\sigma|$, or greater than $k_{i+1}$, and so 10% many of all $\tau \succ \sigma$ with $|\tau| = k_{i+1}$ cause $\phi^\tau(n)$ to halt and give the opposite output from $\phi^\sigma(n)$. But because $\tau \succ \sigma$, $\phi^\tau(n)$ also halts and gives the same output as $\phi^\sigma(n)$, and so $\phi^\tau(n)$ is multivalued.

CASE 2. If the 90%-halting tower is not an 80%-agreement tower because it does reach a consensus, but it also reaches the opposite consensus at some point, then let $j$ be large enough that both kinds of consensus get reached before $k_j$, and let $m_1 = k_j$. Then at least 80% of the oracles from the first consensus arrive at the opposite conclusion with their later information, and so, in particular, they give a multivalued function if enough of the oracle is taken to witness both of those computations. We have that $0.8^2 = 0.64 > 0.6$, and so at that stage, at least measure 0.6-many oracles produce multivalued functions.

We then repeat the construction, choosing a potentially new $n$, and a new 90%-halting tower beginning at $k = m_1$ to find $m_2$, and then we repeat with $m_2$, $m_3$ etc. At each stage, this ensures that for 60% of all $X$, $\phi^{X_{m_i,m_{i+1}}}(n)$ is multivalued for some value of $n$. ⊣

LEMMA 5.17. *Assume that $A$ is* 1*-random, and that there is some $B$ such that for every $k$, $\phi^{A_k}$ is a computation of $B$. Then there is no* 60%*-disagreement tower for $\phi$.*

PROOF. If it is true that for every $k$, $\phi^{A_k}$ is a computation of $B$, then in particular, there is no partial oracle for $A$ that causes $\phi$ to be a multi-valued function. Thus, in particular, there can be no $m_i, m_{i+1}$ such that $A_{m_i,m_{i+1}}$ is in the "60% disagreement part" of the 60%-disagreement tower.

Thus, if there existed a 60%-disagreement tower for $\phi$, then $A$ would be in the intersection, over all $i$, of the reals $X$ such that $\phi^{X_{m_i,m_{i+1}}}$ is not a multivalued function on $n$. This is an intersection of uniformly $\Pi^0_1$ sets, and so it is a $\Pi^0_1$ set. Furthermore, it is a measure 0 set because each one of the sets was at most measure 0.4, and the sets each use disjoint parts of the oracle, so the measure of their intersection is the

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

product of their measures. A 1-random real $A$ cannot be in a $\Pi^0_1$ null set, and so there cannot be a 60%-disagreement tower for $\phi$. ⊣

We are now ready to prove Proposition 5.5, which we restate here for clarity.

PROPOSITION 5.5. *If $A$ is 1-random, then $\widetilde{\mathcal{R}}(A)$ (and hence $A$) is quasiminimal in the generic degrees.*

PROOF. Assume $A$ is 1-random. We show that $A$ is quasiminimal in the cofinite degrees, and hence that $\widetilde{\mathcal{R}}(A)$ is quasiminimal in the generic degrees. To prove this, assume that for every cofinite oracle $(A)$ for $A$, $\phi^{(A)}$ is a computation of $B$. We must prove that $B$ is therefore computable.

By Lemma 5.8, for that $\phi$ and for every $n$, and every $k$, we must have that $\mu_k = 1$. By Lemma 5.11, we therefore have that for every $n$ and $k$, there is a 90%-halting tower starting at $k$. By Lemmas 5.16 and 5.17, we have that there exists an $l$ such that every 90%-halting tower (for any $n$) starting at $l$ is an 80%-agreement tower.

Fix such an $l$. We now compute $B$ through a "majority vote" trick. To compute $B(n)$, wait until 80% of all $X$ have the property that $\phi^{X_l}(n)\downarrow$, giving the same output, then halt and give that output. We must now verify that this will happen at some point, and that when it happens, it gives the correct output.

We know this will eventually happen, because there is a 90%-halting tower starting at $l$, and it is an 80% agreement tower. Thus there exist some $v, k_i, k_{i+1}$ such that $l \le k_i \le k_{i+1}$, and $\mu\{X : \phi^{X_{k_i,k_{i+1}}}(n) = v\} > 0.8$. However, for any $X$, if $\phi^{X_{k_i,k_{i+1}}}(n) = v$, then $\phi^{X_l}(n) = v$, because $X_l$ is a partial oracle extension of $X_{k_i,k_{i+1}}$, and so we have that $\mu\{X : \phi^{X_l}(n) = v\} > 0.8$.

Furthermore, the $v$ that we find must be the value $v$ of some 80% agreement tower. This is because we may build a 90% halting tower for which $k_0 = l$, and $k_1$ is large enough to witness that measure 80% many $X$ give output $v$. By assumption, this tower is an 80% agreement tower, and by definition of 80% agreement tower, no "floor" of the tower can have 80% many $X$ give an output other than the value $v$ of that tower, and so the $v$ that we found is the $v$ of that tower.

Therefore, by Lemma 5.13, that $v$ must be $B(n)$, and so we have correctly computed $B(n)$ without using $A$ as an oracle. ⊣

**§6. Nonuniform generic reducibility.** In this section, we consider nonuniform generic reducibility, which has the property that the functional $\phi$ is allowed to change depending on the generic oracle for $A$. This section is joint work with Denis Hirschfeldt.

DEFINITION 6.1. Let $A$ and $B$ be reals. Then $B$ *is non-uniformly generically reducible to* $A$ if for every generic oracle, $(A)$, for $A$, there exists a Turing functional $\phi$ such that $\phi^{(A)}$ is a generic computation of $B$. In this case, we write $B \le_{\mathrm{ng}} A$.

Note that the Turing degrees embed into the nonuniform generic degrees by the same map as the one used to embed them into the uniform generic degrees, and so we may again define a degree to be quasiminimal if it is not above any nonzero embedded Turing degrees. We show weakly 2-randoms are quasiminimal in the nonuniform generic degrees, and also that 1-generics are quasiminimal in the nonuniform generic degrees. The inspiration for this section comes from a preprint of [4], and the realization that much of their work was complementary to the work in this paper.

In a preprint of [4], Hirschfeldt, Jockusch, Kuyper, and Schupp proved that $\Delta^0_2$ 1-randoms are not quasiminimal in the nonuniform coarse degrees (Corollary 3.11 from [4]), but that weakly 2-randoms are quasiminimal in the nonuniform coarse degrees (Corollary 3.3 from [4]). They also asked whether 1-randoms are quasiminimal in the uniform coarse degrees.

Proposition 5.5 answers this, showing that 1-randoms are quasiminimal in both the uniform coarse and uniform generic degrees. In the published version of [4], the authors also modify their proof of Corollary 3.11 to show that $\Delta^0_2$ 1-randoms are not quasiminimal in the nonuniform generic degrees. In this section, we combine our proof of Proposition 5.5 with the proof of Corollary 3.3 from [4] to prove that weakly 2-randoms are quasiminimal in the nonuniform generic degrees.

We use the following result, implicitly proved in the proof of Theorem 3.2 of [4], but stated here in the form that we will use.

Lemma 6.2 (Hirschfeldt, Jockusch, Kuyper, Schupp [4]). *Assume $A$ is weakly 2-random, $B$ is noncomputable, and $k > 1$. For each $i < k$, let $A_{=i} = \{n : kn + i \in A\}$, and let $A_{\neq i} = \bigoplus_{j \neq i} A_{=j}$.*

*Then $(\exists i < k)(B \not\leq_T A_{\neq i})$.*

*Furthermore, for every $i$, $A_{=i}$ is 1-random relative to $A_{\neq i}$.*

Proof (Sketch). Assume that for every $i$, $B \leq_T A_{\neq i}$. By a generalized form of Van Lambalgen's Theorem [11], we have that for every $i$, $A_{=i}$ is 1-random relative to $A_{\neq i}$, and so therefore relative to $B \oplus A_{\neq i} \equiv_T A_{\neq i}$. By the same generalized form of Van Lambalgen's Theorem relativized to $B$, we therefore have that $A$ is 1-random relative to $B$. We also have that $B \leq_T A$ (because $B \leq_T A_{\neq i}$), and so we can conclude that $B$ is a base for 1-randomness, and hence is $K$-trivial [6].

A weakly 2-random cannot compute any noncomputable $\Delta^0_2$ sets [2], and so cannot compute any noncomputable $K$-trivials. ⊣

Theorem 6.3 (Cholak, Hirschfeldt, Igusa). *Assume $A$ is weakly 2-random. Then $A$ is quasiminimal in the nonuniform generic degrees.*

The proof will use a relativized version of Proposition 5.5. The proof of Proposition 5.5 made ample use of uniformity, so we begin our argument with a forcing argument that will allow us to reduce the question to a uniform question. The uniform question will then be answered using Lemma 6.2 and Proposition 5.5.

Note also that in Section 2 we were working with uniform generic reducibility, and so we do not have access to Observation 2.9, which said that we may use time-independent functionals, and hence ignore the time dependence in our partial oracles. Because of this we will work directly with time-dependent partial oracles. In subsequent work Astor, Hirshfeldt, and Jockusch generalize the proof of Theorem 6.3 to prove a full analogue of Observation 2.9, which would simplify many of the steps of our proof.

We remind the reader that a partial oracle is coded as a set of ordered triples $\langle n, x, l\rangle$, with $n$ as the input, $x$ as the output, and $l$ as the number of steps required for the oracle to halt. As such, we will use the following nonuniform version of Lemma 5.3.

Lemma 6.4. *Assume $A$ is not quasiminimal in the nonuniform generic degrees. Then there exists a noncomputable real $B$ such that for any generic oracle $(A)$, for $A$, $B \leq_T (A)$.*

https://doi.org/10.1017/jsl.2016.50 Published online by Cambridge University Press

Proof (Lemma 6.4). By definition of quasiminimality, if $A$ is not quasiminimal, then there is a noncomputable $B$ such that $\widetilde{\mathcal{R}}(\mathcal{R}(B)) \leq_{\text{ng}} A$. So $\widetilde{\mathcal{R}}(\mathcal{R}(B))$ is generically computable from every generic oracle for $A$. But every generic oracle for $\widetilde{\mathcal{R}}(\mathcal{R}(B))$ can compute $B$, and so $B$ is computable from every generic oracle for $A$. ⊣

Proof (Theorem 6.3). First, we construct a forcing poset that will allow us to define a generic generic oracle, $G$ for $A$. The poset $\mathcal{P}$ will consist of finite approximations to partial oracles for $A$, together with a restriction saying that in the future, the partial oracles will need to have domain at least a certain size. In order to ensure that generic generic oracles are not total, $p$ will determine the entire behavior of $G \upharpoonright m$ for some $m$, so in particular, extensions of $p$ will not be allowed to halt at locations smaller than $m$.

A *finite partial oracle* $\sigma$ for $A$ is given by a number $m = |\sigma|$ and a subset of $m \times 2 \times \omega$ which, thought of as a subset of $\omega \times 2 \times \omega$ would be a partial oracle for $A$. For $n < m$, $\sigma \upharpoonright n$ is a shorthand for the partial oracle $\tau$ such that $|\tau| = n$ and for $k < n$, $\langle k, x, l\rangle \in \tau \leftrightarrow \langle k, x, l\rangle \in \sigma$. As with other partial oracles, $\text{dom}(\sigma) = \{n < |\sigma| : \exists x \exists l \langle n, x, l\rangle \in \sigma\}$.

We define the poset $\mathcal{P}$ to be the set of ordered pairs $\langle\sigma, \epsilon\rangle$ such that $\sigma$ is a partial oracle for $A$, $\epsilon > 0$, and $\frac{\text{dom}(\sigma)}{|\sigma|} > 1 - \epsilon$. Given conditions $p = \langle\sigma, \epsilon\rangle$, and $q = \langle\tau, \delta\rangle$, we say that $q \leq p$ if $|\tau| \geq |\sigma|$, $\tau \upharpoonright |\sigma| = \sigma$ $\delta \leq \epsilon$, and for all $n$, if $|\sigma| \leq n \leq |\tau|$, then $\frac{\text{dom}(\tau \upharpoonright n)}{n} > 1 - \epsilon$.

A generic generic oracle $G$ for $A$ is given by taking a sufficiently generic filter $\widetilde{G}$ for $\mathcal{P}$ and letting $G = \bigcup_{\langle\sigma,\epsilon\rangle \in \widetilde{G}} \sigma$. Note that it is dense to decrease $\epsilon$ below any positive number, and so a generic generic oracle for $A$ is a generic oracle for $A$.

In this proof, we only use genericity of $\widetilde{G}$ for two purposes: ensuring that $G$ is a generic oracle, and ensuring that, given an arbitrary $\phi$, if $\phi^G$ is a computation of $B$, then there is a condition $p \in \widetilde{G}$ that forces that $\phi^G$ is a computation of $B$. We do not wish to explicitly count quantifiers, but $\mathcal{P}$ is $A$-computable, so some small level of genericity relative to $A \oplus B$ is sufficient.

So, let $A$ be weakly 2-random, and assume that $A$ is not quasiminimal. By Lemma 6.4, fix $B$ noncomputable such that $B \leq_T (A)$ for every generic oracle $(A)$ for $A$. Let $G$ be a generic generic oracle for $A$, and fix $\phi$ such that $\phi^G$ is a computation of $B$. Fix a condition $p = \langle\sigma, \epsilon\rangle$ that forces that $\phi^G$ is a computation of $B$. Fix $k$ such that $\frac{1}{k} < \epsilon$. By Lemma 6.2, fix $i < k$ such that $B \nleq_T A_{\neq i}$.

We then claim that relativized to $A_{\neq i}$, $B$ is uniformly computable from an arbitrary cofinite oracle for $A_{=i}$, and also that, relative to $A_{\neq i}$, $A_{=i}$ is 1-random, contradicting Proposition 5.5 relativized to $A_{\neq i}$.

Proof of Claim. Let $X$ be an arbitrary cofinite oracle for $A_{=i}$. Let $\mathcal{F}(X)$ be the cofinite oracle for $A$ defined as follows.

For $m \geq |\sigma|$, let $S_m = \{n : (n \in \text{dom}(\sigma)) \vee (|\sigma| \leq n < m) \vee (n \geq m \;\&\; n \not\equiv i \mod k)\}$. Choose $m_0$ sufficiently large that for all $m \geq \text{dom}(\sigma)$, $\frac{|S_{m_0} \upharpoonright m|}{m} > 1 - \epsilon$. (Such an $m_0$ exists because $k$ was chosen so that $\frac{1}{k} < \epsilon$.)

Let $\mathcal{F}(X)$ be the cofinite oracle for $A$ that agrees with $\sigma$ on $|\sigma|$, that halts immediately on all $m$ between $|\sigma|$ and $m_0$, that halts immediately on all $m \geq m_0$ if $m \not\equiv i \mod k$, and so that if $m > m_0$ and $m \equiv i \mod k$, then $\mathcal{F}(X)(m) = X(\frac{m-i}{k})$

(halting if and only if $X$ halts, giving the same output if it does halt, and halting with the same $l$ value).

Finally, we define $\psi$ so that $\psi^{X \oplus A_{\neq i}}(n)$ searches for a partial oracle $Y$ such that:

- $Y \upharpoonright |\sigma| = \sigma$,
- $\text{dom}(Y) \subseteq \text{dom}(\mathcal{F}(X))$,
- for each $m \in \text{dom}(Y)$, $Y(m) = \mathcal{F}(X)(m)$,
- $\phi^Y(n)\downarrow$

and when it finds such a $Y$, then $\psi^{X \oplus A_{\neq i}}(n)\downarrow= \phi^Y(n)$.

(In essence, $\psi^{X \oplus A_{\neq i}}$ is almost a time-independent version of $\phi^{\mathcal{F}(X)}$, except that it restricts its attention only to $Y$'s that look potentially like extensions of $p$.)

It remains to show that $\psi^{X \oplus A_{\neq i}}$ is total, and that it is correct about $B$ wherever it halts. (Note that showing that $\psi^{X \oplus A_{\neq i}}$ is correct whenever it halts will also show that it is not multivalued.)

To show that if $\psi^{X \oplus A_{\neq i}}(n)\downarrow$ then $\psi^{X \oplus A_{\neq i}}(n) = B(n)$, let $Y$ be as above. Define $q = \langle \tau, \epsilon \rangle$ where $\tau$ is defined as the finite partial oracle for $A$ that agrees with $\sigma$ on $|\sigma|$, that agrees with the portion of $Y$ that is queried in the computation of $\phi^Y(n)$, and that "halts late" at all locations larger than $|\sigma|$ if $Y$ was queried at that location, but $Y$ was not seen to halt at that location.

Here, "halting late" means that $\tau$ halts at those locations, but with an $l$ value larger than any $l$ value queried in the computation of $\phi^Y(n)$. This ensures that $\tau$ agrees with the portion of $Y$ that was queried while also having a large enough domain to not violate the $\epsilon$ condition imposed by $p$.

Then $q \Vdash \phi^G(n) = \phi^Y(n)$, because $q$ agrees with the portion of $Y$ used in the computation. But also that $q \leq p$, and so $q \Vdash \phi^G(n) = B(n)$. Therefore $\phi^Y(n) = B(n)$. This proof was shown for an arbitrary $Y$ as above, and so we have that if $\psi^{X \oplus A_{\neq i}}(n)\downarrow$ then $\psi^{X \oplus A_{\neq i}}(n) = B(n)$.

To show that if $\psi^{X \oplus A_{\neq i}}$ is total, we show that there is a generic generic oracle $G_0$ for $A$, extending $p$, whose domain is contained in $\text{dom}(\mathcal{F}(X))$, and so, for every $n$, $G_0$ will be found as one of the $Y$ as above. From this, because $p \Vdash \phi^G$ is total, we will have that for every $n$, $\phi^{G_0}(n)\downarrow$, and so $\psi^{X \oplus A_{\neq i}}(n)\downarrow$.

To show that there exists such a $G_0$, let $m_1$ be the largest number such that $m_1 \notin \text{dom}(\mathcal{F}(X))$. Let $\tau$ be defined as the finite partial oracle $\mathcal{F}(X) \upharpoonright m_1 + 1$, and let $q = \langle \tau, \epsilon \rangle$. By construction of $\mathcal{F}$, we have that $q \leq p$. Let $G_0$ be any generic generic oracle for $A$ extending $q$. Then $G_0$ extends $p$, and its domain is contained in $\text{dom}(\mathcal{F}(X))$ because its domain restricted to $m_1 + 1$ is equal to the domain of $\mathcal{F}(X)$ restricted to $m_1 + 1$, and $\mathcal{F}(X)$ is a total oracle past $m_1 + 1$. ⊣

Remark 6.5. Combining Proposition 5.5, and Theorem 6.3, with Corollaries 3.3, 3.11, and 3.14 from [4], provides the following characterization of the level of randomness required for to ensure quasiminimality in the uniform or nonuniform cofinite, mod-finite, coarse, or generic degrees:

Theorem 6.6. *In the uniform coarse and generic degrees, and also in the cofinite, and mod-finite degrees, every* 1*-random is quasiminimal.*

*In the nonuniform coarse or generic degrees, every weakly* 2*-random is quasiminimal, but there exist* 1*-randoms (any* 1*-random that is also* $\Delta^0_2$*) which are not quasiminimal.*

In light of this, one might ask whether this provides a characterization of the weakly 2-randoms, but it does not for a fairly trivial reason:

OBSERVATION 6.7. *There exists a* 1*-random* $A$ *that is not weakly* 2*-random that is quasiminimal in both the nonuniform coarse and generic degrees.*

PROOF (Sketch). Let $B$ be 1-random but not weakly 2-random, and let $C$ be weakly 2-random relative to $B$. Let $A$ be the asymmetric join of $B$ and $C$ defined by $A = \{2^n : n \in B\} \cup \left(C \setminus \{2^n : n \in \omega\}\right)$.

Note then that $A$ is 1-random but not weakly 2-random, but that $A$ is coarsely (and generically) equivalent to $C$, and so quasiminimal in the nonuniform coarse (and generic) degrees. ⊣

We observe now that our proof of Theorem 6.3 allows us to also prove that 1-generics are quasiminimal in the nonuniform generic degrees. The following analogue of Lemma 6.2 is proved implicitly in the proof of Theorem 4.2 of [4].

LEMMA 6.8 (Hirschfeldt, Jockusch, Kuyper, Schupp [4]). *Assume* $A$ *is* 1*-generic,* $B$ *is noncomputable, and* $k > 1$. *For each* $i < k$, *let* $A_{=i} = \{n : kn + i \in A\}$, *and let* $A_{\neq i} = \bigoplus_{j \neq i} A_{=j}$.

*Then* $(\exists i < k)(B \not\leq_T A_{\neq i})$.

*Furthermore, for every* $i$, $A_{=i}$ *is* 1*-generic relative to* $A_{\neq i}$.

PROOF (Sketch). A theorem of Yu [12] replaces the generalized form of Van Lambalgen's Theorem that is used in the proof of Lemma 6.2, and $K$-triviality is not needed because if $A$ is 1-generic relative to a noncomputable $B$, then $B \not\leq_T A$. ⊣

PROPOSITION 6.9 (Cholak, Hirschfeldt, Igusa). *Assume* $A$ *is* 1*-generic. Then* $A$ *is quasiminimal in the nonuniform generic degrees.*

PROOF (Sketch). The proof is identical to the proof of Theorem 6.3, using Lemma 6.8 in place of Lemma 6.2, and using Proposition 5.4 in place of Proposition 5.5 ⊣

§7. **Acknowledgements.** This work was partially supported by a grant from the Simons Foundation (#315283 to Peter Cholak). Igusa was partially supported by EMSW21-RTG-0838506.

## REFERENCES

[1] E. ASTOR, D. HIRSCHFELDT, and C. JOCKUSCH, *Dense computability, upper cones, and minimal pairs* (tentative title), in preparation.

[2] R. G. DOWNEY, A. NIES, R. WEBER, and L. YU, *Lowness and* $\Pi^0_2$ *nullsets*, this JOURNAL, vol. 71 (2006), pp. 1044–1052.

[3] D. DZHAFAROV and G. IGUSA, *Notions of robust information coding*, ***Computability***, to appear.

[4] D. HIRSCHFELDT, C. JOCKUSCH, R. KUYPER, and P. SCHUPP, *Coarse reducibility and algorithmic randomness*, this JOURNAL, to appear.

[5] D. HIRSCHFELDT, C. JOCKUSCH, T. H. MCNICHOLL, and P. SCHUPP, *Asymptotic density and the coarse computability bound*. ***Computability***, to appear.

[6] D. R. HIRSCHFELDT, A. NIES, and F. STEPHAN, *Using random sets as oracles*. ***Journal of the London Mathematical Society***, vol. 75 (2007), pp. 610–622.

[7] G. IGUSA, *Nonexistence of minimal pairs for generic computability*, this JOURNAL, vol. 78 (2013), no. 2, pp. 511–522.

[8] ———, *The generic degrees of density-1 sets, and a characterization of the hyperarithmetic reals*, this JOURNAL, vol. 80 (2015), no. 4, pp. 1290–1314.

[9] C. Jockusch and P. Schupp, *Generic computability, Turing degrees, and asymptotic density*. ***Journal of the London Mathematical Society***, vol. 85 (2012), no. 2, pp. 472–490.

[10] I. Kapovich, A. Miasnikov, P. Schupp, and V. Shpilrain, *Generic-case complexity, decision problems in group theory, and random walks*. ***Journal of Algebra***, vol. 264 (2003), no. 2, pp. 665–694.

[11] M. van Lambalgen, *The axiomatization of randomness*, this Journal, vol. 55 (1990), pp. 1143–1167.

[12] L. Yu, *Lowness for genericity*. ***Archive for Mathematical Logic***, vol. 45 (2006), pp. 233–238.

DEPARTMENT OF MATHEMATICS
UNIVERSITY OF NOTRE DAME
NOTRE DAME, IN 46556-5683, USA
*E-mail*: peter.cholak.1@nd.edu
*URL*: http://www.nd.edu/~cholak
*E-mail*: gigusa@nd.edu
*URL*: http://www3.nd.edu/~gigusa