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A CLASS OF FOURTH ORDER DAMPED WAVE EQUATIONS WITH ARBITRARY POSITIVE INITIAL ENERGY

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Abstract In this paper, we study the initial boundary value problem for a class of fourth order damped wave equations with arbitrary positive initial energy. In the framework of the energy method, we further exploit the properties of the Nehari functional. Finally, the global existence and finite time blow-up of solutions are obtained.

Keywords: fourth order damped wave equations; global existence; finite time blow-up; arbitrary positive initial energy

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1. Introduction

In this paper, we study the following fourth order weakly damped wave equations

$$u_{tt} + \Delta^2 u + \mu u_t + au = |u|^{p-2}u, \quad (x, y, t) \in \Omega \times [0, T], \tag{1.1}$$

with the initial condition

$$u(x, y, 0) = u_0(x, y), \qquad u_t(x, y, 0) = u_1(x, y), \quad (x, y) \in \Omega, \tag{1.2}$$

and the boundary condition

$$\begin{cases} u(0, y, t) = u_{xx}(0, y, t) = u(\pi, y, t) = u_{xx}(\pi, y, t) = 0, \ (y, t) \in (-l, l) \times [0, T], \\ u_{yy}(x, \pm l, t) + \sigma u_{xx}(x, \pm l, t) = 0, \ (x, t) \in (0, \pi) \times [0, T], \\ u_{yyy}(x, \pm l, t) + (2 - \sigma)u_{xxy}(x, \pm l, t) = 0, \ (x, t) \in (0, \pi) \times [0, T], \end{cases} \tag{1.3}$$

where $\Omega = (0, \pi) \times (-l, l) \subset \mathbb{R}^2$, $T > 0$, $\mu > 0$, $2 < p < \infty$, $\sigma \in (0, (1/2))$ and $a = a(x, y, t)$ is a sign-changing and bounded measurable function.

Problem (1.1)–(1.3) arises from the physical model for the nonlinear dynamic suspension bridge (see [**5**, **22**]). The weak damping term μu_t represents internal friction. The term au describes the restoring force due to the hangers of the suspension bridge and the source term $|u|^{p-2}u$ represents the other external forces acting on the bridge. The open rectangular plate $\Omega = (0, \pi) \times (-l, l)$ represents the roadway of a suspension bridge, and the edges $x = 0, \pi$ connect with ground while the edges $y = \pm l$ are free.

As is well known, a reliable model for suspension bridge should be nonlinear and it should have enough degrees of freedom to display torsional oscillations. There have been some studies on the nonlinear behaviour of a suspension bridge, we refer the readers to [**1**, **6**, **9**, **18**] and the references therein. Recently, Ferrero and Gazzola [**5**] suggested the following fourth order damped wave equations from a plate model describing the dynamics of a suspension bridge

$$u_{tt} + \Delta^2 u + \mu u_t + h(x, y, u) = f(x, y, t), \tag{1.4}$$

with the initial condition (1.2) and the boundary condition (1.3), where $h(x, y, u)$ is restoring force due to the hangers of the suspension bridge, and $f(x, y, t)$ is the external force including the gravity. Moreover, the kinetic energy was added to the total energy of the nonlinear dynamic suspension bridge. They investigated existence, uniqueness and qualitative behaviour of solutions for problem (1.2)–(1.4). Subsequently, Wang [**22**] studied local existence, global existence and finite time blow-up of solutions for problem (1.1)–(1.3) by employing the potential well theory. However, the results of [**22**] are restricted to the case of low initial energy $E(0) < d$, that is, the initial energy $E(0)$ is less than the depth of the potential well d.

The potential well is also called stable set and started with Sattinger [**19**] (also see Payne and Sattinger [**17**]). In general, by the energy functional $J(u)$ and the Nehari functional $I(u)$, the classical potential well and its outside set (namely unstable set) can be usually defined respectively as follows

$$W = \{u | J(u) < d, I(u) > 0\} \cup \{0\},$$

$$V = \{u | J(u) < d, I(u) < 0\}.$$

The critical points of $J(u)$ are stationary solutions of the problem concerned. Under certain assumptions on the parameters such as the growth power of source term, $J(u)$ satisfies the Palais–Smale condition and the problem concerned admits at least a positive stationary solution whose energy d (that is generally called the depth of the potential well) can be defined by

$$d = \inf_{u \in \mathcal{N}} J(u),$$

where the Nehari manifold

$$\mathcal{N} = \{u | I(u) = 0\} \setminus \{0\}.$$

Thus, under the situation where $E(0)$ is controlled by d, the well-posedness for solutions to the problem concerned can be investigated by employing the potential well theory (see e.g. [**2**–**4**, **8**, **12**–**16**, **20**–**23**] and the references therein). Here it is worth mentioning that Gazzola and Squassina [**7**] modified the potential well theory and the concavity method developed by Levine [**10**, **11**] to derive the finite time blow-up of solutions for a class of second order damped wave equations with high initial energy.

In this paper, we are mainly interested in the global existence and finite time blow-up of solutions for problem (1.1)–(1.3) with $E(0) > 0$, namely, arbitrary positive initial energy or high initial energy without restrictions of upper bound. In other words, the goal of this work is to complement the results in [**22**]. From the physical point of view, this work is of a great importance for the high energy case of a model for a suspension bridge in some sense. To the best of our knowledge, much less effort has been devoted to problem (1.1)–(1.3) with arbitrary positive initial energy. In addition, inspired by Gazzola and Squassina [**7**], our main tools are the modified potential well theory and the modified concavity method. On the one hand, we would like to mention that our techniques are not equal to those in [**7**] since the complexities of equation (1.1) itself and the boundary condition (1.3) as well as additional difficulties in the proof of the global existence of solutions. On the other hand, in order to study the global well-posedness for solutions to problem (1.1)–(1.3) with $E(0) > 0$, we have to break through the classical potential well theory so that our results are not restricted to $E(0) < d$, which will cause additional technical obstacles to our energy estimates. More precisely, unlike in the previous studies [**2**–**4**, **8**, **12**–**17**, **19**–**23**], the innovation of this paper is that we use directly the relationship between the energy functionals associated with problem (1.1)–(1.3) to discuss the global well-posedness for solutions without the aid of d. In order to obtain the global existence and finite time blow-up of solutions with $E(0) > 0$, we further exploit the properties of the Nehari functional, which will play an essential role in the proofs of our main results. Finally, in the case $E(0) > 0$, we shall provide appropriate and relaxed sufficient conditions on the global existence and finite time blow-up of solutions for problem (1.1)–(1.3), respectively.

This paper is organized as follows. In § 2 we recall some definitions, lemmas and theorems related to problem (1.1)–(1.3). Moreover, we state our main results with arbitrary positive initial energy. In § 3 we establish the global existence of solutions by introducing an evolution property of solutions. In § 4 we are devoted to the proof of finite time blow-up of solutions by a combination of an unstable set and the modified concavity method.

2. Preliminaries and main results

Throughout this paper, the following notations are used for precise statements: $\|\cdot\|_p = \|\cdot\|_{L^p(\Omega)}$, $\|\cdot\| = \|\cdot\|_{L^2(\Omega)}$, $(u, v) = \int_\Omega uv \,\mathrm{d}x\,\mathrm{d}y$, $\|\cdot\|_{H^2} = (\|\cdot\|^2 + \|D^2\cdot\|^2)^{1/2}$ and $\|\cdot\|_* = \|\cdot\|_{H^2_*(\Omega)}$, where

$$H^2_*(\Omega) = \{u \in H^2(\Omega) | u = 0 \text{ on } \{0, \pi\} \times (-l, l)\}.$$

Clearly, $H^2_0(\Omega) \subset H^2_*(\Omega) \subset H^2(\Omega)$. According to [**5**], we see that $H^2_*(\Omega)$ is a Hilbert space with the norm

$$\|u\|_* = \left(\int_\Omega |\Delta u|^2 \,\mathrm{d}x\,\mathrm{d}y + 2(1-\sigma)\int_\Omega (u_{xy}^2 - u_{xx}u_{yy}) \,\mathrm{d}x\,\mathrm{d}y\right)^{1/2},$$

which is equivalent to $\|\cdot\|_{H^2}$ for $\sigma \in (0, (1/2))$. Moreover, in view of [**22**] we have a Sobolev embedding inequality for this case.

Lemma 2.1. *Assume that* $1 \leq q < \infty$. *Then for any* $u \in H_*^2(\Omega)$, *there holds the inequality*

$$\|u\|_q \leq S_q \|u\|_*,$$

where

$$S_q = \left(\frac{\pi}{2l} + \frac{\sqrt{2}}{2}\right)(2\pi l)^{(q+2)/2q}\left(\frac{1}{1-\sigma}\right)^{1/2}.$$

As in [22], we now define the total energy associated with problem (1.1)–(1.3)

$$E(t) = \frac{1}{2}\|u_t\|^2 + \frac{1}{2}\|u\|_*^2 + \frac{1}{2}(au, u) - \frac{1}{p}\|u\|_p^p, \tag{2.1}$$

which satisfies the identity

$$E(t) + \mu \int_0^t \|u_t(\tau)\|^2 \, d\tau = E(0), \tag{2.2}$$

for all $t \in [0, T_{\max})$, where $T_{\max}$ is the maximum existence time of $u(x, y, t)$. In addition, we also define the energy functional J on $H_*^2 \to \mathbb{R}$

$$J(u) = \frac{1}{2}\|u\|_*^2 + \frac{1}{2}(au, u) - \frac{1}{p}\|u\|_p^p,$$

and the Nehari functional

$$I(u) = \|u\|_*^2 + (au, u) - \|u\|_p^p. \tag{2.3}$$

Thus, the Nehari manifold can be defined by

$$\mathcal{N} = \{u \in H_*^2(\Omega) \setminus \{0\} | I(u) = 0\},$$

which separates two sets

$$\mathcal{N}_+ = \{u \in H_*^2(\Omega) | I(u) > 0\} \cup \{0\},$$

and

$$\mathcal{N}_- = \{u \in H_*^2(\Omega) | I(u) < 0\}.$$

Next, we recall the following preliminary lemma, definition of weak solutions and local existence theorem in [22], see [22] for the proofs.

Lemma 2.2. *Assume that* $-\Lambda_1 < a_1 \le a \le a_2$, *where* $\{\Lambda_i\}_{i=1}^{\infty}$ *is the eigenvalue sequence of the eigenvalue problem*

$$\begin{cases} \Delta^2 u = \Lambda u, \ (x,y) \in \Omega, \\ u(0,y) = u_{xx}(0,y) = u(\pi,y) = u_{xx}(\pi,y) = 0, \ y \in (-l,l), \\ u_{yy}(x,\pm l) + \sigma u_{xx}(x,\pm l) = u_{yyy}(x,\pm l) + (2-\sigma) u_{xxy}(x,\pm l) = 0, \ x \in (0,\pi), \end{cases}$$

and $\Lambda_1 < 1$. *Then for any* $u \in H_*^2(\Omega)$, *there holds*

$$A_1 \|u\|_*^2 \le \|u\|_*^2 + (au,u) \le A_2 \|u\|_*^2,$$

where

$$A_1 = \begin{cases} 1 + \dfrac{a_1}{\Lambda_1}, & a_1 < 0, \\ 1, & a_1 \ge 0, \end{cases} \quad \text{and} \quad A_2 = \begin{cases} 1, & a_2 < 0, \\ 1 + \dfrac{a_2}{\Lambda_1}, & a_2 \ge 0. \end{cases}$$

Definition 2.3. A function $u \in C([0,T]; H_*^2(\Omega)) \cap C^1([0,T]; L^2(\Omega)) \cap C^2([0,T]; \mathcal{H}(\Omega))$ with $u_t \in L^2(0,T; L^2(\Omega))$ is called a weak solution of problem (1.1)–(1.3), if $u(0) = u_0$, $u_t(0) = u_1$ and

$$\langle u_{tt}, \eta \rangle + (u,\eta)_* + \mu (u_t, \eta) + (au, \eta) = (|u|^{p-2} u, \eta)$$

for all $\eta \in H_*^2(\Omega)$ and a.e. $t \in [0,T]$, where $\mathcal{H}(\Omega)$ denotes the dual space of $H_*^2(\Omega)$ and corresponding duality between them is denoted by $\langle \cdot, \cdot \rangle$.

Theorem 2.4. *Let* $-\Lambda_1 < a_1 \le a \le a_2$. *Then for any* $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$, *there exists* $T > 0$ *such that problem (1.1)–(1.3) has a unique local weak solution* u *on* $[0,T]$. *Moreover, if*

$$T_{\max} = \sup\{T > 0 : u = u(t) \text{ exists on } [0,T]\} < \infty,$$

then

$$\lim_{t \to T_{\max}} \|u(t)\|_q = \infty,$$

for $q \ge 1$ *such that* $q > ((p-2)/2)$.

Theorem 2.4 shows that the weak solution of problem (1.1)–(1.3) exists globally if $T_{\max} = \infty$, while the weak solution blows up if $T_{\max} < \infty$.

Now we are in a position to state the main results of this paper.

Theorem 2.5. *Let* u *be the unique local solution of problem (1.1)–(1.3),* $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. *Assume that* $-\wedge_1 < a_1 \le a \le a_2$, $E(0) > 0$ *and the initial data satisfy the following assumptions*

$$2(u_0, u_1) + (\mu + 1)\|u_0\|^2 + \frac{2p}{p+2} E(0) \le 0, \tag{A$_1$}$$

$$\|u_0\|_*^2 + (a_0 u_0, u_0) > \|u_1\|^2 + \|u_0\|_p^p, \tag{A$_2$}$$

where $a_0 := a(x,y,0)$. *Then the solution* $u(t)$ *of problem (1.1)–(1.3) exists globally.*

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Theorem 2.6. *Let u be the unique local solution of problem (1.1)–(1.3), $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $-\wedge_1 < a_1 \leq a \leq a_2$, $a_1 \geq 0$, $E(0) > 0$, $u_0 \in \mathcal{N}_-$ and the initial data satisfy the following assumption*

$$(u_0, u_1) \geq 0, \qquad \|u_0\|^2 \geq \frac{2pS_2^2}{p-2} E(0). \tag{A$_3$}$$

Then the solution $u(t)$ of problem (1.1)–(1.3) blows up in finite time.

3. Proof of Theorem 2.5

In this section, we first introduce a property for u in order to prove the global existence. For given $T \in (0, \infty)$, we say that $u \in C([0,T]; H_*^2(\Omega)) \cap C^1([0,T]; L^2(\Omega))$ satisfies property (P_{t_0}) for some $t_0 \in (0, T]$ if

$$I(u(t)) > \|u_t(t)\|^2 + \mu \int_0^t \|u_t(\tau)\|^2 \, d\tau \quad \text{for any } t \in (0, t_0). \tag{P$_{t_0}$}$$

We now consider the function $M : [0, T_{\max}) \to \mathbb{R}^+$ defined by

$$M(t) = \|u(t)\|^2 + \mu \int_0^t \|u(\tau)\|^2 \, d\tau. \tag{3.1}$$

Lemma 3.1. *Let u be the unique local solution of problem (1.1)–(1.3), $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $E(0) > 0$ and the initial data satisfy (A_1). Then $M(t)$ is strictly decreasing, provided u satisfies* ($P_{T_{\max}}$).

Proof. By (3.1) we get

$$M'(t) = 2(u, u_t) + \mu \|u\|^2. \tag{3.2}$$

Further,

$$\begin{aligned} M''(t) &= 2\|u_t\|^2 + 2\langle u, u_{tt} \rangle + 2\mu(u, u_t) \\ &= 2\|u_t\|^2 + 2(u, -\Delta^2 u - \mu u_t - au + |u|^{p-2}u) + 2\mu(u, u_t) \\ &= 2\|u_t\|^2 - 2\|u\|_*^2 - 2(au, u) + 2\|u\|_p^p, \end{aligned}$$

for $t \in [0, T_{\max})$. Consequently, from (2.3) we obtain

$$M''(t) = 2\|u_t\|^2 - 2I(u),$$

which together with ($P_{T_{\max}}$) gives

$$M''(t) < 0. \tag{3.3}$$

Further, $M'(t) < M'(0)$ for $t \in (0, T_{\max})$. From $E(0) > 0$ and (A_1) it follows that

$$2(u_0, u_1) + \mu \|u_0\|^2 < 0,$$

i.e.,

$$M'(0) < 0.$$

Therefore, $M'(t) < 0$ for $t \in [0, T_{\max})$. This implies that $M(t)$ is strictly decreasing. □

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Lemma 3.2. *Let u be the unique local solution of problem (1.1)–(1.3), $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $-\wedge_1 < a_1 \le a \le a_2$, $E(0) > 0$ and the initial data satisfy (A_1)–(A_2). Then u satisfies* ($\mathrm{P}_{T_{\max}}$).

Proof. Suppose that u does not satisfy ($\mathrm{P}_{T_{\max}}$). Then from (A_2) and the continuity of u in time we see that there exists the first time $0 < t_0 < T_{\max}$ such that u satisfies (P_{t_0}) and

$$I(u(t_0)) = \|u_t(t_0)\|^2 + \mu \int_0^{t_0} \|u_t(\tau)\|^2 \,\mathrm{d}\tau. \tag{3.4}$$

From (2.1)–(2.3) it follows that

$$\begin{aligned} E(0) &= \frac{1}{2}\left(\|u_t(t_0)\|^2 + 2\mu \int_0^{t_0} \|u_t(\tau)\|^2 \,\mathrm{d}\tau\right) + \frac{1}{p} I(u(t_0)) \\ &\quad + \left(\frac{1}{2} - \frac{1}{p}\right)(\|u(t_0)\|_*^2 + (au(t_0), u(t_0))). \end{aligned}$$

By Lemma 2.2 we get

$$\|u(t_0)\|_*^2 + (au(t_0), u(t_0)) \ge A_1 \|u(t_0)\|_*^2 \ge 0.$$

We further have

$$E(0) \ge \frac{1}{2}\left(\|u_t(t_0)\|^2 + \mu \int_0^{t_0} \|u_t(\tau)\|^2 \,\mathrm{d}\tau\right) + \frac{1}{p} I(u(t_0)),$$

which together with (3.4) yields

$$E(0) \ge \frac{p+2}{2p}\left(\|u_t(t_0)\|^2 + \mu \int_0^{t_0} \|u_t(\tau)\|^2 \,\mathrm{d}\tau\right). \tag{3.5}$$

By virtue of

$$\|u_t(t_0)\|^2 = \|u_t(t_0) + u(t_0)\|^2 - \|u(t_0)\|^2 - 2(u(t_0), u_t(t_0))$$

and

$$\int_0^{t_0} \|u_t(\tau)\|^2 \,\mathrm{d}\tau = \int_0^{t_0} \|u_t(\tau) + u(\tau)\|^2 \,\mathrm{d}\tau - \int_0^{t_0} \|u(\tau)\|^2 \,\mathrm{d}\tau - 2\int_0^{t_0} (u(\tau), u_t(\tau)) \,\mathrm{d}\tau,$$

(3.5) becomes

$$\begin{aligned} E(0) &\ge \frac{p+2}{2p}\|u_t(t_0) + u(t_0)\|^2 + \frac{p+2}{2p}\mu \int_0^{t_0} \|u_t(\tau) + u(\tau)\|^2 \,\mathrm{d}\tau \\ &\quad - \frac{p+2}{2p}\|u(t_0)\|^2 - \frac{p+2}{2p}\mu \int_0^{t_0} \|u(\tau)\|^2 \,\mathrm{d}\tau \\ &\quad - \frac{p+2}{p}(u(t_0), u_t(t_0)) - \frac{p+2}{p}\mu \int_0^{t_0} (u(\tau), u_t(\tau)) \,\mathrm{d}\tau. \end{aligned} \tag{3.6}$$

By applying (3.1)–(3.2) to the last term in (3.6), we obtain

$$\begin{aligned}\int_0^{t_0}(u(\tau),u_t(\tau))\,\mathrm{d}\tau &= \frac{1}{2}\int_0^{t_0}(M'(\tau)-\mu\|u(\tau)\|^2)\,\mathrm{d}\tau\\ &= \frac{1}{2}(M(t_0)-M(0))-\frac{1}{2}\mu\int_0^{t_0}\|u(\tau)\|^2\,\mathrm{d}\tau\\ &= \frac{1}{2}(\|u(t_0)\|^2-\|u_0\|^2). \end{aligned}\tag{3.7}$$

Substituting (3.7) into (3.6), we deduce that

$$\begin{aligned}E(0) \geq &-\frac{p+2}{2p}\left(\|u(t_0)\|^2+\mu\int_0^{t_0}\|u(\tau)\|^2\,\mathrm{d}\tau\right)\\ &-\frac{p+2}{p}\left((u(t_0),u_t(t_0))+\frac{1}{2}\mu(\|u(t_0)\|^2-\|u_0\|^2)\right).\end{aligned}\tag{3.8}$$

From (3.3) it is easy to see that $M'(t_0)<M'(0)$, i.e.,

$$(u(t_0),u_t(t_0))+\tfrac{1}{2}\mu(\|u(t_0)\|^2-\|u_0\|^2)<(u_0,u_1).$$

Substituting this inequality into (3.8), we obtain

$$E(0)>-\frac{p+2}{2p}\left(\|u(t_0)\|^2+\mu\int_0^{t_0}\|u(\tau)\|^2\,\mathrm{d}\tau\right)-\frac{p+2}{p}(u_0,u_1).$$

As a consequence, we may write

$$-\frac{2p}{p+2}E(0)<\|u(t_0)\|^2+\mu\int_0^{t_0}\|u(\tau)\|^2\,\mathrm{d}\tau+2(u_0,u_1).$$

By recalling (3.1), we conclude that

$$M(t_0)>-2(u_0,u_1)-\frac{2p}{p+2}E(0).$$

According to Lemma 3.1 and (A_1), this contradicts

$$M(t_0)<M(0)=\|u_0\|^2\leq -2(u_0,u_1)-\mu\|u_0\|^2-\frac{2p}{p+2}E(0).$$

Thus the proof of Lemma 3.2 is complete. □

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Proof of Theorem 2.5. By Lemma 3.2 we infer that u satisfies ($\mathrm{P}_{T_{\max}}$), i.e.,

$$I(u) > \|u_t\|^2 + \mu \int_0^t \|u_t(\tau)\|^2 \,\mathrm{d}\tau, \tag{3.9}$$

for any $t \in (0, T_{\max})$. From (2.1)–(2.3) it follows that

$$\begin{aligned} E(0) &= \frac{1}{2}\left(\|u_t\|^2 + 2\mu \int_0^t \|u_t(\tau)\|^2 \,\mathrm{d}\tau\right) + \frac{1}{p} I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)(\|u\|_*^2 + (au, u)) \\ &\geq \frac{1}{2}\left(\|u_t\|^2 + \mu \int_0^t \|u_t(\tau)\|^2 \,\mathrm{d}\tau\right) + \frac{1}{p} I(u) + \left(\frac{1}{2} - \frac{1}{p}\right)(\|u\|_*^2 + (au, u)). \end{aligned}$$

Combining this with (3.9), we get

$$E(0) > \frac{p+2}{2p}\left(\|u_t\|^2 + \mu \int_0^t \|u_t(\tau)\|^2 \,\mathrm{d}\tau\right) + \frac{p-2}{2p}(\|u\|_*^2 + (au, u)).$$

This implies that u is bounded in $C([0, T_{\max}); H_*^2(\Omega)) \cap C^1([0, T_{\max}); L^2(\Omega))$ and u_t is bounded in $L^2(0, T_{\max}; L^2(\Omega))$. Therefore, we conclude from Theorem 2.4 that $T_{\max} = \infty$ and the solution of problem (1.1)–(1.3) exists globally. □

4. Proof of Theorem 2.6

We start this section by the following Lemma.

Lemma 4.1. *Let u be the unique local solution of problem (1.1)–(1.3), $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $(u_0, u_1) \geq 0$. Then $\theta(t) = \|u(t)\|^2$ is strictly increasing on $[0, T_{\max})$, provided $u(t) \in \mathcal{N}_-$.*

Proof. From the expression of $\theta(t)$ it is easy to see that

$$\theta'(t) = 2(u, u_t).$$

A simple calculation yields

$$\begin{aligned} \theta''(t) &= 2\|u_t\|^2 + 2\langle u, u_{tt}\rangle \\ &= 2\|u_t\|^2 + 2(u, -\Delta^2 u - \mu u_t - au + |u|^{p-2}u) \\ &= 2\|u_t\|^2 - 2\|u\|_*^2 - 2\mu(u, u_t) - 2(au, u) + 2\|u\|_p^p \\ &= 2(\|u_t\|^2 - I(u) - \mu(u, u_t)). \end{aligned}$$

Consequently,

$$\theta''(t) + \mu\theta'(t) = 2(\|u_t\|^2 - I(u)).$$

Noticing that $u(t) \in \mathcal{N}_-$, i.e., $I(u) < 0$ for all $t \in [0, T_{\max})$, we conclude from $\theta'(0) = (u_0, u_1) \geq 0$ and [7, Lemma 8.1] that $\theta'(t) > 0$ for all $t \in [0, T_{\max})$. Hence $\theta(t)$ is strictly increasing on $[0, T_{\max})$. □

Lemma 4.2. *Let u be the unique local solution of problem (1.1)–(1.3), $u_0 \in H_*^2(\Omega)$, $u_1 \in L^2(\Omega)$. Assume that $-\wedge_1 < a_1 \leq a \leq a_2$, $a_1 \geq 0$ and there holds (A_3). Then the solution $u(t)$ of problem (1.1)–(1.3) with $E(0) > 0$ belongs to $\mathcal{N}_-$, provided $u_0 \in \mathcal{N}_-$.*

Proof. If conditions hold, we have $u(t) \in \mathcal{N}_-$ for all $t \in (0, T_{\max})$. Indeed, if it was not the case, there would exist the first time $0 < t_0 < T_{\max}$ such that $u(t_0) \in \mathcal{N}$ and $u(t) \in \mathcal{N}_-$ for all $t \in (0, t_0)$, i.e., $I(u(t_0)) = 0$ and $I(u(t)) < 0$ for all $t \in (0, t_0)$.

According to Lemma 4.1 and (A_3), we have

$$\theta(t) > \theta(0) = \|u_0\|^2 \geq \frac{2pS_2^2}{p-2}E(0),$$

for all $t \in (0, t_0)$. We then conclude that

$$\theta(t_0) > \frac{2pS_2^2}{p-2}E(0). \tag{4.1}$$

On the other hand, it follows from (2.1) and (2.3) that

$$E(t_0) = \frac{1}{2}\|u_t(t_0)\|^2 + \frac{p-2}{2p}\|u(t_0)\|_*^2 + \frac{p-2}{2p}(au(t_0), u(t_0)) + \frac{1}{p}I(u(t_0)).$$

Noticing that $I(u(t_0)) = 0$, we get

$$\begin{aligned} E(t_0) &= \frac{1}{2}\|u_t(t_0)\|^2 + \frac{p-2}{2p}\|u(t_0)\|_*^2 + \frac{p-2}{2p}(au(t_0), u(t_0)) \\ &\geq \frac{p-2}{2p}\|u(t_0)\|_*^2 + \frac{p-2}{2p}(au(t_0), u(t_0)), \end{aligned}$$

and so

$$\|u(t_0)\|_*^2 + (au(t_0), u(t_0)) \leq \frac{2p}{p-2}E(t_0).$$

This together with (2.2) gives

$$\|u(t_0)\|_*^2 + (au(t_0), u(t_0)) \leq \frac{2p}{p-2}E(0).$$

Combining this with Lemmas 2.1 and 2.2, we deduce that

$$\theta(t_0) = \|u(t_0)\|^2 \leq S_2^2\|u(t_0)\|_*^2 \leq S_2^2(\|u(t_0)\|_*^2 + (au(t_0), u(t_0))) \leq \frac{2pS_2^2}{p-2}E(0),$$

which contradicts (4.1). Thus the proof of Lemma 4.2 is complete. □

Clearly, Lemma 4.2 shows that the unstable set $\mathcal{N}_-$ is invariant under the flow of problem (1.1)–(1.3) with arbitrary positive initial energy.

In the end, we finish the proof of Theorem 2.6.

Proof of Theorem 2.6. Suppose that $T_{\max} = \infty$. For any $0 < T < \infty$ we now consider the auxiliary function $\phi : [0, T] \to \mathbb{R}^+$ defined by

$$\phi(t) = \|u\|^2 + \mu \int_0^t \|u(\tau)\|^2 \,\mathrm{d}\tau + \mu(T - t)\|u_0\|^2. \tag{4.2}$$

A direct calculation yields

$$\begin{aligned} \phi'(t) &= 2(u, u_t) + \mu\|u\|^2 - \mu\|u_0\|^2 \\ &= 2(u, u_t) + 2\mu \int_0^t (u, u_t) \,\mathrm{d}\tau, \end{aligned} \tag{4.3}$$

and so

$$\begin{aligned} \phi''(t) &= 2\|u_t\|^2 + 2\langle u, u_{tt}\rangle + 2\mu(u, u_t) \\ &= 2(\|u_t\|^2 - \|u\|_*^2 - (au, u) + \|u\|_p^p) \qquad (4.4) \\ &= 2(\|u_t\|^2 - I(u)), \qquad (4.5) \end{aligned}$$

for $t \in [0, T]$. From (4.2)–(4.4) we get

$$\begin{aligned} \phi(t)\phi''(t) - \frac{p+2}{4}\phi'(t)^2 =& 2\phi(t)(\|u_t\|^2 - \|u\|_*^2 - (au, u) + \|u\|_p^p) \\ &+ (p+2)\Big[\varphi(t) - (\phi(t) - \mu(T - t)\|u_0\|^2) \\ &\times \Big(\|u_t\|^2 + \mu \int_0^t \|u_t\|^2 \,\mathrm{d}\tau\Big)\Big], \end{aligned}$$

where

$$\varphi(t) = \Big(\|u\|^2 + \mu \int_0^t \|u\|^2 \,\mathrm{d}\tau\Big)\Big(\|u_t\|^2 + \mu \int_0^t \|u_t\|^2 \,\mathrm{d}\tau\Big) - \Big((u, u_t) + \mu \int_0^t (u, u_t) \,\mathrm{d}\tau\Big)^2.$$

By Schwarz's inequality, we obtain

$$\|u\|^2\|u_t\|^2 \geq (u, u_t)^2,$$

$$\mu \int_0^t \|u\|^2 \,\mathrm{d}\tau \cdot \mu \int_0^t \|u_t\|^2 \,\mathrm{d}\tau \geq \Big(\mu \int_0^t (u, u_t) \,\mathrm{d}\tau\Big)^2,$$

and

$$\begin{aligned} (u, u_t) \cdot \mu \int_0^t (u, u_t) \,\mathrm{d}\tau &\leq \|u\|\|u_t\|\Big(\mu \int_0^t \|u\|^2 \,\mathrm{d}\tau\Big)^{1/2}\Big(\mu \int_0^t \|u_t\|^2 \,\mathrm{d}\tau\Big)^{1/2} \\ &\leq \frac{1}{2}\Big(\|u\|^2\mu \int_0^t \|u_t\|^2 \,\mathrm{d}\tau + \|u_t\|^2\mu \int_0^t \|u\|^2 \,\mathrm{d}\tau\Big). \end{aligned}$$

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These three inequalities entail $\varphi(t) \geq 0$ for all $t \in [0, T]$. We further get

$$\begin{aligned}\phi(t)\phi''(t) - \frac{p+2}{4}\phi'(t)^2 &\geq -p\phi(t)\|u_t\|^2 - 2\phi(t)\|u\|_*^2 - 2\phi(t)(au, u) \\ &\quad + 2\phi(t)\|u\|_p^p - (p+2)\phi(t)\mu\int_0^t \|u_t\|^2\,\mathrm{d}\tau \\ &= -2p\phi(t)\left(E(t) - \frac{1}{2}\|u\|_*^2 - \frac{1}{2}(au, u)\right) - 2\phi(t)\|u\|_*^2 \\ &\quad - 2\phi(t)(au, u) - (p+2)\phi(t)\mu\int_0^t \|u_t\|^2\,\mathrm{d}\tau,\end{aligned}$$

which together with (2.2) gives

$$\phi(t)\phi''(t) - \frac{p+2}{4}\phi'(t)^2 \geq \phi(t)\psi(t), \tag{4.6}$$

for $t \in [0, T]$, where

$$\psi(t) = -2pE(0) + (p-2)\|u\|_*^2 + (p-2)(au, u) + (p-2)\mu\int_0^t \|u_t\|^2\,\mathrm{d}\tau. \tag{4.7}$$

According to $u_0 \in \mathcal{N}_-$ and Lemma 4.2, we have $u \in \mathcal{N}_-$. Moreover, by recalling Lemmas 2.1, 4.1 and (A_3), it is easy to see that

$$\|u\|_*^2 \geq \frac{1}{S_2^2}\|u\|^2 > \frac{2p}{p-2}E(0),$$

i.e.,

$$-2pE(0) + (p-2)\|u\|_*^2 > 0.$$

Combining this with Lemma 2.2, we infer from (4.7) that there exists a constant $\rho_1 > 0$ that is independent of the choice of T such that $\psi(t) \geq \rho_1$. Moreover, notice that $\phi(t)$ is continuous on $[0, T]$. Then there exists a $\rho_2 > 0$ such that $\phi(t) \geq \rho_2$, where ρ_2 is independent of the choice of T. Hence we conclude from (4.6) that

$$\phi(t)\phi''(t) - \frac{p+2}{4}\phi'(t)^2 \geq \rho_1\rho_2.$$

From $u \in \mathcal{N}_-$ and (4.5) it follows that $\phi''(t) > 0$, and so $\phi'(t) > 0$. Thus

$$(\phi^{-\alpha}(t))' = -\frac{\alpha\phi'(t)}{\phi^{\alpha+1}(t)} < 0,$$

and

$$(\phi^{-\alpha}(t))'' = \frac{-\alpha}{\phi^{\alpha+2}(t)}(\phi(t)\phi''(t) - (\alpha+1)\phi'(t)^2) < 0,$$

where $\alpha = (p-2)/4$. Hence there exists a finite time T_0 that is independent of the choice of T such that

$$\lim_{t\to T_0} \phi^{-\alpha}(t) = 0.$$

Therefore,

$$\lim_{t\to T_0} \phi(t) = \infty,$$

which contradicts $T_{\max} = \infty$. This completes the proof of Theorem 2.6. □

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References

1. J. M. W. Brownjohn, Observations on non-linear dynamic characteristics of suspension bridges, *Earthq. Eng. Struct. D* **23**(12) (1994), 1351–1367.
2. M. M. Cavalcanti and V. N. Domingos Cavalcanti, Existence and asymptotic stability for evolution problem on manifolds with damping and source terms, *J. Math. Anal. Appl.* **291**(1) (2004), 109–127.
3. M. M. Cavalcanti, V. N. Domingos Cavalcanti and P. Martinez, Existence and decay rate estimates for the wave equation with nonlinear boundary damping and source term, *J. Differential Equations* **203**(1) (2004), 119–158.
4. J. A. Esquivel-Avila, A characterization of global and nonglobal solutions of nonlinear wave and Kirchhoff equations, *Nonlinear Anal.* **52**(4) (2003), 1111–1127.
5. A. Ferrero and F. Gazzola, A partially hinged rectangular plate as a model for suspension bridges, *Discrete Contin. Dyn. Syst. Ser. A* **35**(12) (2015), 5879–5908.
6. F. Gazzola, Nonlinearity in oscillating bridges, *Electron. J. Differential Equations* **211**(3) (2013), 1642–1654.
7. F. Gazzola and M. Squassina, Global solutions and finite time blow up for damped semilinear wave equations, *Ann. Inst. Henri Poincaré Probab. Stat.* **23** (2006), 185–207.
8. R. Ikehata, Some remarks on the wave equations with nonlinear damping and source terms, *Nonlinear Anal.* **27**(10) (1996), 1165–1175.
9. W. Lacarbonara, *Nonlinear structural mechanics* (Springer, 2013).
10. H. A. Levine, Instability and nonexistence of global solutions to nonlinear wave equations of the form $Pu_{tt} = -Au + \mathcal{F}(u)$, *Trans. Amer. Math. Soc.* **192** (1974), 1–21.
11. H. A. Levine, Some additional remarks on the nonexistence of global solutions to nonlinear wave equations, *SIAM J. Math. Anal.* **5** (1974), 138–146.
12. J. L. Lions, *Quelques méthodes de résolution des problèmes aux limites non linéaires* (Dunod, 1969).
13. Y. Liu and J. Zhao, On potential wells and applications to semilinear hyperbolic equations and parabolic equations, *Nonlinear Anal.* **64**(12) (2006), 2665–2687.
14. Y. Liu, R. Xu and T. Yu, Global existence, nonexistence and asymptotic behavior of solutions for the Cauchy problem of semilinear heat equations, *Nonlinear Anal.* **68**(11) (2008), 3332–3348.
15. M. Nakao and K. Ono, Existence of global solutions to the Cauchy problem for the semilinear dissipative wave equations, *Math. Z.* **214** (1993), 325–342.

16. K. Ono, Global existence, asymptotic behaviour, and global non-existence of solutions for damped non-linear wave equations of Kirchhoff type in the whole space, *Math. Methods Appl. Sci.* **23**(6) (2000), 535–560.
17. L. E. Payne and D. H. Sattinger, Sadle points and instability of nonlinear hyperbolic equations, *Israel J. Math.* **22**(3–4) (1975), 273–303.
18. R. H. Plaut and F. M. Davis, Sudden lateral asymmetry and torsional oscillations of section models of suspension bridges, *J. Sound Vib.* **307**(3) (2007), 894–905.
19. D. H. Sattinger, On global solution of nonlinear hyperbolic equations, *Arch. Ration. Mech. Anal.* **30**(2) (1968), 148–172.
20. M. Tsutsumi, On solutions of semilinear differential equations in a Hilbert space, *Math. Japan.* **17** (1972), 173–193.
21. E. Vitillaro, Global nonexistence theorems for a class of evolution equations with dissipation, *Arch. Ration. Mech. Anal.* **149**(2) (1999), 155–182.
22. Y. Wang, Finite time blow-up and global solutions for fourth order damped wave equations, *J. Math. Anal. Appl.* **418**(2) (2014), 713–733.
23. R. Xu, Asymptotic behavior and blow up of solutions for semilinear parabolic equations at critical energy level, *Math. Comput. Simulation* **80**(4) (2009), 808–813.