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# ASYMPTOTICS FOR THE DISCRETE-TIME AVERAGE OF THE GEOMETRIC BROWNIAN MOTION AND ASIAN OPTIONS

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**Abstract**

The time average of geometric Brownian motion plays a crucial role in the pricing of Asian options in mathematical finance. In this paper we consider the asymptotics of the discrete-time average of a geometric Brownian motion sampled on uniformly spaced times in the limit of a very large number of averaging time steps. We derive almost sure limit, fluctuations, large deviations, and also the asymptotics of the moment generating function of the average. Based on these results, we derive the asymptotics for the price of Asian options with discrete-time averaging in the Black–Scholes model, with both fixed and floating strike.

*Keywords:* Asian option; central limit theorem; Berry–Esseen bound; large deviations

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## 1. Introduction

Asian (or average) options are widely traded instruments in the financial markets, which involve the time average of the price of an asset $S_t$. Most commonly, $S_t$ is a stock price or a commodity futures contract price, for example, oil or natural gas futures. An Asian call option has payoff of the form

$$\text{payoff} = \max\left\{\frac{1}{n}\sum_{i=1}^{n} S_{t_i} - K, 0\right\},$$

where $0 \le t_1 < t_2 < \cdots < t_n$ is a sequence of strictly increasing times, called sampling or averaging dates. Under risk-free neutral pricing, the price of such an option is given by the expectation of the payoff in the risk-neutral measure. Assuming the Black–Scholes (BS) model, we study the distributional properties of the discrete-time average of the asset price

$$A_n = \frac{1}{n}\sum_{i=1}^{n} S_{t_i} \tag{1}$$

under the assumption that $S_n$ follows a geometric Brownian motion (GBM)

$$\mathrm{d}S_t = (r-q)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}Z_t,$$

where $Z_t$ is a standard Brownian motion, $r$ is the risk-free rate, $q$ is the dividend yield, and $\sigma$ is the volatility.

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The main technical difficulty for pricing Asian options is that the probability distribution of the discrete-time average (1) does not have a simple expression. If the averaging times are uniformly distributed, the time average can be well approximated, for sufficiently small time steps, by a continuous average

$$A_n = \frac{1}{t_n}\int_0^{t_n} S_t \,\mathrm{d}t.$$

When $S_t$ follows a GBM, the problem is reduced to the study of the distributional properties of the time integral of the GBM, which has been extensively studied in the literature. See [17] for a review of the main results and their applications to the Asian options pricing.

A wide variety of methods have been proposed for pricing Asian options, and a brief survey is given below.

*Partial differential equation (PDE) methods (see [15], [37], [44], [51], [52], and [54]).* The pricing of an Asian option can be reduced to the solution of a $1 + 1$ PDE, which is solved numerically. This method can be applied both to continuous-time and discrete-time averaging Asian options [1]. See also [2].

*The Laplace transform method (see [7] and [26]).* The Asian option price with random exponentially distributed maturity can be found in closed form for the case when the asset price $S_t$ follows a GBM. This reduces the problem of the Asian option pricing in the BS model to the inversion of a Laplace transform.

*Spectral method (see [36]).* The probability distribution of the time integral of the GBM can be related to that of a Bessel process [13], [14]. The transition density of this Bessel process can be expanded in an eigenfunction series [55], and Asian option prices can be evaluated using the eigenfunction expansion, truncated to a sufficiently high order [34], [36]. A method based on expansion in Laguerre polynomials was proposed in [12].

*Bounds and control variates methods.* There is a large literature on deriving bounds on Asian option prices. Both lower and upper bounds have been given; see [37] for an overview. They can be used also in conjunction with Monte Carlo (MC) methods as control variates. One precise method of this type which is popular in practice was given by Curran [9]. Other methods which take into account the discrete-time averaging have been proposed in [22], [23], [24].

*MC simulation.* See [21], [30], and [31].

*Analytical approximations*. Various numerical methods have been proposed which approximate the distribution of the arithmetic average $A_n$ using parametric forms, such as log-normal [33] or inverse gamma distributions [38].

We note also the more general approach of [53] which can be applied for a wide class of models.

Most of the theoretical results in the literature concerning the distribution of the time average of the GBM refer to the continuous-time average. The discrete sum of the GBM is a particular case of the sum of correlated log-normals which has been studied extensively in the literature; see [3] for an overview. In [16] Dufresne obtained a limit distribution for the discrete-time average in the limit of very small volatility $\sigma \to 0$. A recent work by the present authors [41] studied the properties of the discrete-time sum of the GBM at fixed $\sigma$ in the limit $n \to \infty$, and its convergence to the continuous-time integral as the time-step $\tau \to 0$.

In this paper we concentrate on the discrete-time average of the GBM, $A_n = (1/n)\sum_{i=1}^n S_{t_i}$. We assume the BS model, that is, the asset price follows a GBM

$$S_t = S_0 \mathrm{e}^{\sigma Z_t + (r-q-\sigma^2/2)t},$$

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where $Z_t$ is a standard Brownian motion. We would like to study the distributional properties of the average of the discretely sampled asset price (1) defined on the discrete times uniformly spaced $t_i = i\tau$ with time-step $\tau$.

In this paper we derive asymptotic results about $A_n$ in the limit $n \to \infty$ by keeping fixed the following combinations of model parameters:

$$\beta = \tfrac{1}{2}\sigma^2 t_n n = \tfrac{1}{2}\sigma^2 \tau n^2, \tag{2}$$

$$(r-q)\tau n = \rho. \tag{3}$$

Note that $\beta$ is always positive but $\rho$ can be both positive and negative. We also note that conditions (2) and (3) can be replaced by $\lim_{n\to\infty} \frac{1}{2}\sigma^2\tau n^2 = \beta$ and $\lim_{n\to\infty}(r-q)\tau n = \rho$ and all the results in this paper will still hold.

Constraints (2) and (3) include two interesting regimes.

- When the maturity $t_n = \tau n$ is constant, and so are the interest rates $r$ and dividend yield $q$, then (2) assumes that the volatility $\sigma$ is of the order $O(1/\sqrt{n})$. Therefore, conditions (2) and (3) include the *small volatility* regime.
- When the maturity $t_n = \tau n$ is small, that is, $t_n \to 0$ as $n \to \infty$ and, in particular, is of the order $1/n$, then by (3), the volatility $\sigma$ is a constant. If the interest rates $r$ and dividend yield $q$ are constant then (3) is replaced by $\lim_{n\to\infty}(r-q)\tau n = 0$, that is, $\rho = 0$. Therefore, conditions (2) and (3) include the *short maturity* regime.

We emphasize that we do not make any assumptions about the values of $\rho, \beta$, and they can be arbitrary. The validity of our asymptotic results require only that $n \gg 1$, such that these regimes cover most cases of practical interest, provided that the number of averaging times is sufficiently large.

In this paper we present three asymptotic results for the distributional properties of the discrete-time average of a GBM in the limit of a large number of averaging time steps $n$:

(i) almost sure limit and fluctuation results for $A_n$,

(ii) an asymptotic result for the moment generating function of the partial sums $nA_n$ for $n \to \infty$, and

(iii) large deviations results for $\mathbb{P}(A_n \in \cdot)$.

Using these asymptotic results, we derive rigorously asymptotics for the prices of out-of-the-money (OTM), in-the-money (ITM), and at-the-money (ATM) Asian options.

In Section 2 we present the almost sure and fluctuations results for $A_n$ in the $n \to \infty$ limit. In Section 3 we present an asymptotic result for the Laplace transform of the finite sum of the GBM sampled on $n$ discrete-times $nA_n$, in the limit $n \to \infty$. In Section 4 we consider the asymptotics of fixed-strike Asian options following from the large deviations result (iii), and in Section 5 we treat the case of the floating-strike Asian options. These asymptotic results can be used to obtain approximative pricing formulae for Asian options, and in Section 6 we compare the numerical performance of the asymptotic result against alternative methods for pricing Asian options under the BS model. Some of the proposed methods are known to be less efficient numerically in the small maturity and/or small volatility limit [26], [36]. The asymptotic results derived in this paper are of practical interest as they complement these approaches in a region where their numerical performance is not very good. We demonstrate good agreement of our asymptotic results with alternative pricing methods for Asian options with realistic values of the model parameters.

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## 2. Asymptotics for the discrete-time average of a GBM

**Proposition 1.** *We have the almost sure (a.s.) limit*

$$\lim_{n\to\infty} A_n = A_\infty \equiv S_0 \frac{1}{\rho}(\mathrm{e}^{\rho} - 1).$$

*Proof.* Note that $\max_{1\le i\le n} \sigma|Z_{t_i}| = \max_{1\le i\le n} \sqrt{(2\beta/\tau)}|Z_{i\tau}|/n$ and from the property of Brownian motion, $1/n \max_{1\le i\le n}|Z_{i\tau}| \to 0$ a.s. as $n\to\infty$. Moreover, $\frac{1}{2}\sigma^2 t_i = \beta i/n^2 \le \beta/n \to 0$ as $n\to\infty$ uniformly in $1\le i\le n$. Therefore, $S_{t_i}$ can be approximated by $S_0\mathrm{e}^{(r-q)t_i}$ uniformly in $1\le i\le n$, that is, $\max_{1\le i\le n}|S_{t_i} - S_0\mathrm{e}^{(r-q)t_i}| \to 0$ a.s. as $n\to\infty$. Finally, note that

$$\frac{1}{n}\sum_{i=1}^{n} \mathrm{e}^{(r-q)t_i} = \frac{1}{n}\sum_{i=1}^{n}\mathrm{e}^{\rho i/n} = \frac{1}{n}\frac{\mathrm{e}^{\rho}-1}{1-\mathrm{e}^{-\rho/n}} \to \frac{1}{\rho}(\mathrm{e}^{\rho}-1), \qquad n\to\infty.$$

Hence, we proved the desired result. □

We have also the following fluctuation result.

**Proposition 2.** *The time average $A_n$ converges in distribution to a normal distribution in the $n\to\infty$ limit*

$$\lim_{n\to\infty}\sqrt{n}\frac{A_n - A_\infty}{S_0} = N(0, 2\beta v(\rho)).$$

*with*

$$v(a) := \frac{1}{a^3}\bigg[a\mathrm{e}^{2a} - \frac{3}{2}\mathrm{e}^{2a} + 2\mathrm{e}^{a} - \frac{1}{2}\bigg].$$

*Proof.* We have

$$\begin{aligned}
\sqrt{n}\frac{A_n - A_\infty}{S_0} &= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\bigg(\exp\bigg(\sigma Z_i + \bigg(r-q-\frac{1}{2}\sigma^2\bigg)t_i\bigg) - \exp\bigg(\rho\frac{i}{n}\bigg)\bigg) \\
&\quad + \bigg[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\bigg(\rho\frac{i}{n}\bigg) - \sqrt{n}\frac{\exp(\rho)-1}{\rho}\bigg] \\
&= \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\bigg(\rho\frac{i}{n}\bigg)\bigg(\exp\bigg(\frac{\sqrt{2\beta}}{n}B_i - \beta\frac{i}{n^2}\bigg) - 1\bigg) \\
&\quad + \bigg[\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\bigg(\rho\frac{i}{n}\bigg) - \sqrt{n}\,\frac{\exp(\rho)-1}{\rho}\bigg],
\end{aligned} \tag{4}$$

where $Z_i = \sqrt{\tau}B_i$ with $B_i$ a standard Brownian motion. We can rewrite the second term in (4) as

$$\begin{aligned}
\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\mathrm{e}^{\rho i/n} - \sqrt{n}\,\frac{\mathrm{e}^{\rho}-1}{\rho} &= \frac{1}{\sqrt{n}}\frac{\mathrm{e}^{\rho}-1}{1-\mathrm{e}^{-\rho/n}} - \sqrt{n}\,\frac{\mathrm{e}^{\rho}-1}{\rho} \\
&= (\mathrm{e}^{\rho}-1)\frac{1}{\sqrt{n}}\bigg[\frac{1}{\rho/n - \rho^2/2n^2 + O(n^{-3})} - \frac{n}{\rho}\bigg] \\
&= (\mathrm{e}^{\rho}-1)\frac{1}{\sqrt{n}}\frac{\rho^2/2n + O(n^{-2})}{\rho(\rho/n - \rho^2/2n^2 + O(n^{-3}))} \\
&\to 0 \quad \text{as } n\to\infty.
\end{aligned}$$

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The first term in (4) can be written further as

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\biggl(\exp\biggl(\frac{\sqrt{2\beta}}{n}B_i-\beta\frac{i}{n^2}\biggr)-1\biggr)=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\frac{\sqrt{2\beta}}{n}B_i+\xi_n, \quad (5)$$

where we define

$$\xi_n \equiv \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\biggl(\exp\biggl(\frac{\sqrt{2\beta}}{n}B_i-\beta\frac{i}{n^2}\biggr)-\frac{\sqrt{2\beta}}{n}B_i-1\biggr).$$

We claim that $\xi_n \to 0$ in probability as $n\to\infty$.

We have the following upper bound on $\xi_n$:

$$\xi_n \le \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\biggl(\exp\biggl(\frac{\sqrt{2\beta}}{n}B_i\biggr)-\frac{\sqrt{2\beta}}{n}B_i-1\biggr)\equiv \xi_n^{(\mathrm{up})}.$$

The upper bound $\xi_n^{(\mathrm{up})}$ is a nonnegative random variable since $\mathrm{e}^x-1-x\ge 0$ for any real $x$. The expectation of $\xi_n^{(\mathrm{up})}$ can be computed exactly as

$$\begin{aligned}\mathbb{E}[\xi_n^{(\mathrm{up})}]&=\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\biggl(\exp\biggl(\frac{\beta}{n^2}i\biggr)-1\biggr)\\&=\frac{1}{\sqrt{n}}\biggl(\frac{\exp(\rho+\beta/n)-1}{1-\exp(-\rho/n-\beta/n^2)}-\frac{\exp(\rho)-1}{1-\exp(-\rho/n)}\biggr)\\&=\frac{1}{\sqrt{n}}\biggl(\frac{\beta}{\rho}+o\biggl(\frac{1}{n}\biggr)\biggr).\end{aligned}$$

This goes to 0 as $n\to\infty$. The Markov inequality implies that $\xi_n^{(\mathrm{up})}\to 0$ in probability as $n\to\infty$.

Next, let us estimate the lower bound on $\xi_n$. We have

$$\begin{aligned}\xi_n &\ge \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\biggl(\exp\biggl(\frac{\sqrt{2\beta}}{n}B_i-\frac{\beta}{n}\biggr)-\biggl(\frac{\sqrt{2\beta}}{n}B_i-\frac{\beta}{n}\biggr)-1-\frac{\beta}{n}\biggr)\\&\ge \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\biggl(-\frac{\beta}{n}\biggr)\\&=-\frac{\beta}{\sqrt{n}}\frac{\exp(\rho)-1}{n(1-\exp(-\rho/n))}\\&\to 0,\end{aligned}$$

where we used again in the second step the inequality $\mathrm{e}^x\ge 1+x$.

The first term in (5) is a normal random variable and converges in distribution to a normal distribution with mean 0 and variance to be determined. We have

$$\frac{1}{\sqrt{n}}\sum_{i=1}^{n}\exp\biggl(\rho\frac{i}{n}\biggr)\frac{\sqrt{2\beta}}{n}B_i\to N(0,2\beta v(\rho)).$$

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This can be computed by writing $B_i = \sum_{j=0}^{i-1} V_j$ with $V_j \sim N(0,1)$ independent and identically distributed (i.i.d.) normally distributed random variables with mean 0 and unit variance. The sum can be written as

$$\begin{aligned}
&\frac{1}{\sqrt{n}} \sum_{i=1}^{n} \exp\left(\rho \frac{i}{n}\right) \frac{\sqrt{2\beta}}{n} B_i \\
&\quad = \frac{\sqrt{2\beta}}{n^{3/2}} \sum_{j=0}^{n-1} V_j \sum_{i=j+1}^{n} \exp\left(\rho \frac{i}{n}\right) \\
&\quad = \frac{\sqrt{2\beta}}{n^{3/2}} \sum_{j=0}^{n-1} V_j \frac{1}{\exp(\rho/n) - 1} \left\{ \exp\left(\rho \frac{n+1}{n}\right) - \exp\left(\rho \frac{j+1}{n}\right) \right\}.
\end{aligned}$$

We can compute the variance of this random variable as

$$\begin{aligned}
&\operatorname{var}\left( \frac{1}{\sqrt{n}} \sum_{i=1}^{n} \exp\left(\rho \frac{i}{n}\right) \frac{\sqrt{2\beta}}{n} B_i \right) \\
&\quad = \frac{2\beta}{n^3} \sum_{j=0}^{n-1} \frac{1}{(\exp(\rho/n) - 1)^2} \left( \exp\left(\rho \frac{n+1}{n}\right) - \exp\left(\rho \frac{j+1}{n}\right) \right)^2 \\
&\quad = 2\beta \frac{1}{n^2 (\exp(\rho/n) - 1)^2} \sum_{j=0}^{n-1} \left( \exp\left(\rho \frac{n+1}{n}\right) - \exp\left(\rho \frac{j+1}{n}\right) \right)^2 \frac{1}{n} \\
&\quad \to \frac{2\beta}{\rho^2} \int_0^1 (\exp(\rho) - \exp(\rho x))^2 \, \mathrm{d}x \quad \text{as } n \to \infty,
\end{aligned}$$

where we can compute

$$\frac{2\beta}{\rho^2} \int_0^1 (\mathrm{e}^{\rho} - \mathrm{e}^{\rho x})^2 \, \mathrm{d}x = \frac{2\beta}{\rho^3} \left[ \rho \mathrm{e}^{2\rho} - \frac{3}{2} \mathrm{e}^{2\rho} + 2\mathrm{e}^{\rho} - \frac{1}{2} \right]. \qquad \square$$

## 3. Moment generating function

Define the moment generating function of $nA_n$ as

$$F_n(\theta) := \mathbb{E}[\mathrm{e}^{\theta n A_n}].$$

For $\theta < 0$, this is the Laplace transform of the distribution function of $nA_n$.

We are interested in the limit $\lim_{n\to\infty} (1/n) \log F_n(\theta)$. We will compute this limit using the theory of large deviations. Before we proceed, recall that a sequence $(P_n)_{n\in\mathbb{N}}$ of probability measures on a topological space $X$ satisfies the large deviation principle with rate function $I\colon X \to \mathbb{R}$ if $I$ is nonnegative, lower semicontinuous, and, for any measurable set $A$, we have

$$-\inf_{x\in A^o} I(x) \le \liminf_{n\to\infty} \frac{1}{n} \log P_n(A) \le \limsup_{n\to\infty} \frac{1}{n} \log P_n(A) \le -\inf_{x\in\overline{A}} I(x).$$

Here, $A^o$ is the interior of $A$ and $\overline{A}$ is its closure. The rate function $I$ is said to be good if, for any $m$, the level set $\{x\colon I(x) \le m\}$ is compact. We refer the reader to [10] or [50] for general background information on large deviations and the applications.

We have the following limit theorem for the moment generating function in the limit $n \to \infty$ at fixed $\beta$.

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**Theorem 1.** *For any $\theta > 0$, $F_n(\theta) = \infty$, and, for any $\theta \le 0$,*

$$\lim_{n\to\infty} \frac{1}{n} \log F_n(\theta) = \sup_{g\in\mathcal{AC}_0[0,1]} \biggl\{ \theta S_0 \int_0^1 \mathrm{e}^{\sqrt{2\beta} g(x)}\,\mathrm{d}x - \frac{1}{2}\int_0^1 \biggl( g'(x) - \frac{\rho}{\sqrt{2\beta}} \biggr)^2 \mathrm{d}x \biggr\}.$$

*Proof.* Since $\mathbb{E}[\exp(\theta X)] = \infty$ for any $\theta > 0$ and for any log-normal random variable $X$, it is clear that $\mathbb{E}[\exp(\theta n A_n)] = \infty$ for any $\theta > 0$. Next, for any $\theta \le 0$,

$$\begin{aligned}
\mathbb{E}[\exp(\theta n A_n)] &= \mathbb{E}\biggl[\exp\biggl(\theta \sum_{k=0}^{n-1} S_0 \exp\biggl(\sigma Z_{t_k} + \biggl(r - q - \frac{1}{2}\sigma^2\biggr)t_k\biggr)\biggr)\biggr] \\
&= \mathbb{E}\biggl[\exp\biggl(\theta S_0 \sum_{k=0}^{n-1} \exp\biggl(\sigma\sqrt{\tau}\sum_{j=1}^{k} V_j + \biggl(r - q - \frac{1}{2}\sigma^2\biggr)k\tau\biggr)\biggr)\biggr] \\
&= \mathbb{E}\biggl[\exp\biggl(\theta S_0 \sum_{k=0}^{n-1} \exp\biggl(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{k} V_j + \frac{\rho k}{n} - \frac{\beta}{n^2}k\biggr)\biggr)\biggr] \\
&= \mathbb{E}\biggl[\exp\biggl(\theta S_0 \sum_{k=0}^{n-1} \exp\biggl(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{k} \biggl(V_j + \frac{\rho}{\sqrt{2\beta}}\biggr) - \frac{\beta}{n^2}k\biggr)\biggr)\biggr],
\end{aligned}$$

where $V_j := (Z_j - Z_{j-1})/\sqrt{\tau}$, $1 \le j \le k$, are i.i.d. $N(0, 1)$ random variables. Note that $\sum_{j=1}^{0} V_j$ is defined as 0. By Mogulskii's theorem, see, e.g. [10], $\mathbb{P}((1/n)\sum_{j=1}^{\lfloor \cdot n\rfloor}(V_j + \rho/\sqrt{2\beta}) \in \cdot)$ satisfies a large deviation principle on $L^\infty[0, 1]$ with the good rate function

$$I(g) = \int_0^1 \Lambda(g'(x))\,\mathrm{d}x,$$

if $g \in \mathcal{AC}_0[0, 1]$, that is, absolutely continuous, $g(0) = 0$, and $I(g) = +\infty$ otherwise, where

$$\Lambda(x) := \sup_{\theta\in\mathbb{R}} \biggl\{\theta x - \log \mathbb{E}\biggl[\exp\biggl(\theta\biggl(V_1 + \frac{\rho}{\sqrt{2\beta}}\biggr)\biggr)\biggr]\biggr\} = \frac{1}{2}\biggl(x - \frac{\rho}{\sqrt{2\beta}}\biggr)^2.$$

Let $g(x) := (1/n)\sum_{j=1}^{\lfloor xn\rfloor}(V_j + \rho/\sqrt{2\beta})$. Then

$$\begin{aligned}
\int_0^1 \exp(\sqrt{2\beta} g(x))\,\mathrm{d}x &= \sum_{k=0}^{n-1} \int_{k/n}^{(k+1)/n} \exp(\sqrt{2\beta} g(x))\,\mathrm{d}x \\
&= \frac{1}{n}\sum_{k=0}^{n-1} \exp\biggl(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{k}\biggl(V_j + \frac{\rho}{\sqrt{2\beta}}\biggr)\biggr).
\end{aligned}$$

Moreover, we claim that

$$g \mapsto \int_0^1 \exp(\sqrt{2\beta} g(x))\,\mathrm{d}x$$

is a continuous map. Let $g_n$ be any sequence in $L^\infty[0, 1]$ so that $g_n \to g$ in $L^\infty[0, 1]$. Observe that, for any $|x| \le \frac{1}{2}$,

$$|\mathrm{e}^x - 1| = \biggl| x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \biggr| \le |x|(1 + |x| + |x|^2 + \cdots) \le 2|x|.$$

Let $n$ be sufficiently large so that $\sqrt{2\beta}\|g_n - g\|_{L^\infty[0,1]} \le \frac{1}{2}$. Therefore, we have

$$\begin{aligned}
&\left|\int_0^1 \exp(\sqrt{2\beta} g_n(x))\,\mathrm{d}x - \int_0^1 \exp(\sqrt{2\beta} g(x))\,\mathrm{d}x\right| \\
&\quad = \left|\int_0^1 \exp(\sqrt{2\beta} g(x))(\exp(\sqrt{2\beta}(g_n(x) - g(x))) - 1)\,\mathrm{d}x\right| \\
&\quad \le \exp(\sqrt{2\beta}\|g\|_{L^\infty[0,1]}) \int_0^1 |\exp(\sqrt{2\beta}(g_n(x) - g(x))) - 1|\,\mathrm{d}x \\
&\quad \le 2\sqrt{2\beta}\exp(\sqrt{2\beta}\|g\|_{L^\infty[0,1]})\|g_n - g\|_{L^\infty[0,1]},
\end{aligned}$$

which converges to 0 as $n \to \infty$. Hence, the map is continuous. Let us recall the celebrated Varadhan's lemma from large deviations theory; see, e.g. [10]. If $\mathbb{P}(Z_n \in \cdot)$ satisfies a large deviation principle with good rate function $I\colon \mathcal{X} \to [0, +\infty]$, and if $\phi$ is a continuous map and

$$\lim_{M\to+\infty} \limsup_{n\to\infty} \frac{1}{n} \log \mathbb{E}[\mathrm{e}^{n\phi(Z_n)} \mathbf{1}_{\{\phi(Z_n)\ge M\}}] = -\infty, \tag{6}$$

then

$$\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[\mathrm{e}^{n\phi(Z_n)}] = \sup_{x\in\mathcal{X}} \{\phi(x) - I(x)\}.$$

In our case,

$$\phi(g) = \theta S_0 \int_0^1 \mathrm{e}^{\sqrt{2\beta} g(x)}\,\mathrm{d}x$$

is a continuous map. Moreover, for $\theta \le 0$, $\phi(g) \le 0$ and, thus, condition (6) is trivially satisfied. Hence, we can apply Varadhan's lemma to obtain

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}\biggl[\exp\biggl(\theta S_0 \sum_{k=0}^{n-1} \exp\biggl(\frac{\sqrt{2\beta}}{n} \sum_{j=1}^{k} \biggl(V_j + \frac{\rho}{\sqrt{2\beta}}\biggr)\biggr)\biggr)\biggr] \\
&\quad = \sup_{g\in\mathcal{AC}_0[0,1]} \biggl\{\theta S_0 \int_0^1 \exp(\sqrt{2\beta} g(x))\,\mathrm{d}x - \frac{1}{2}\int_0^1 \biggl(g'(x) - \frac{\rho}{\sqrt{2\beta}}\biggr)^2 \mathrm{d}x\biggr\}.
\end{aligned}$$

Finally, note that

$$\begin{aligned}
&\mathbb{E}\biggl[\exp\biggl(\theta S_0 \sum_{k=0}^{n-1} \exp\biggl(\frac{\sqrt{2\beta}}{n} \sum_{j=1}^{k} \biggl(V_j + \frac{\rho}{\sqrt{2\beta}}\biggr)\biggr)\biggr)\biggr] \\
&\quad \le \mathbb{E}[\exp(\theta n A_n)] \\
&\quad \le \mathbb{E}\biggl[\exp\biggl(\theta S_0 \exp\biggl(-\frac{\beta}{n}\biggr) \sum_{k=0}^{n-1} \exp\biggl(\frac{\sqrt{2\beta}}{n} \sum_{j=1}^{k} \biggl(V_j + \frac{\rho}{\sqrt{2\beta}}\biggr)\biggr)\biggr)\biggr].
\end{aligned}$$

Hence, for any $\theta \le 0$,

$$\begin{aligned}
&\lim_{n\to\infty} \frac{1}{n} \log \mathbb{E}[\exp(\theta n A_n)] \\
&\quad = \sup_{g\in\mathcal{AC}_0[0,1]} \biggl\{\theta S_0 \int_0^1 \exp(\sqrt{2\beta} g(x))\,\mathrm{d}x - \frac{1}{2}\int_0^1 \biggl(g'(x) - \frac{\rho}{\sqrt{2\beta}}\biggr)^2 \mathrm{d}x\biggr\}. \qquad \square
\end{aligned}$$

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### 3.1. Solution of the variational problem

The variational problem in Theorem 1 can be restated as

$$\lim_{n\to\infty}\frac{1}{n}\log F_n(\theta) = \lambda(-\theta S_0, \sqrt{2\beta}; \rho),$$

where $\lambda(a, b; \rho)$ is the solution of the variational problem

$$\lambda(a, b; \rho) = \sup_{g\in\mathcal{AC}_0[0,1]}\left\{-a\int_0^1 \mathrm{e}^{bg(x)}\,\mathrm{d}x - \frac{1}{2}\int_0^1\left(g'(x) - \frac{\rho}{b}\right)^2\mathrm{d}x\right\}. \tag{7}$$

Here we have $a, b > 0$.

This variational problem can be solved explicitly, and the solution is given by the following result.

**Proposition 3.** *The function $\lambda(a, b; \rho)$ is given by one of the two expressions*

$$\begin{aligned}\lambda_1(a, b; \rho) = a\left\{1 + \sinh^2\left(\frac{\delta}{2}\right)\left(1 - \frac{4\rho}{\delta^2} + \frac{\rho^2}{\delta^2}\right) - \frac{2-\rho}{\delta}\sinh\delta\right\}\\ + \frac{2}{b^2}\rho\log\left[\cosh\left(\frac{\delta}{2}\right) + \frac{\rho}{\delta}\sinh\left(\frac{\delta}{2}\right)\right] - \frac{\rho^2}{b^2}\end{aligned} \tag{8}$$

*or*

$$\begin{aligned}\lambda_2(a, b; \rho) = a\left\{1 - \sin^2\xi\left(1 + \frac{\rho}{\xi^2} - \frac{\rho^2}{4\xi^2}\right) + \frac{\rho-2}{2\xi}\sin(2\xi)\right\}\\ + \frac{2\rho}{b^2}\log\left[\cos\xi + \frac{\rho}{2\xi}\sin\xi\right] - \frac{\rho^2}{b^2}.\end{aligned} \tag{9}$$

*In (8), $\delta$ is the solution of*

$$\rho^2 - \delta^2 = 2ab^2\left(\cosh\left(\frac{1}{2}\delta\right) + \frac{\rho}{\delta}\sinh\left(\frac{1}{2}\delta\right)\right)^2, \tag{10}$$

*and in (9), $\xi$ is the unique solution $\xi \in (0, \xi_{\max})$ of*

$$2\xi^2(4\xi^2 + \rho^2) = ab^2(2\xi\cos\xi + \rho\sin\xi)^2. \tag{11}$$

*Note that $\xi_{\max}$ is the smallest solution of* $\tan\xi_{\max} = -2\xi_{\max}/\rho$.

*For given $(a > 0, b, \rho)$ only one of the two equations (10) and (11) has a solution, such that the solution of the variational problem is unique.*

*Proof.* The proof will be given in Appendix A. □

Recall that $\lim_{n\to\infty}(r - q)\tau n = \rho$, and in the short maturity limit $t_n \to 0$ at constant $r, q$, we have $\rho = 0$. Therefore, the special case $\rho = 0$ is of practical interest when considering the short maturity limit. For this case it is clear that only (11) has a solution for $a > 0$ so we obtain the simpler result.

**Corollary 1.** *The function $\lambda(a, b; 0)$ in the $\rho = 0$ limit is given by*

$$\lambda(a, b; 0) = a\left(\cos^2\xi - \frac{1}{\xi}\sin(2\xi)\right),$$

*where $\xi$ is the solution of $2\xi^2 = ab^2\cos^2\xi$.*

In conclusion, the result of Theorem 1 and Proposition 3 gives an asymptotic expression for the Laplace transform of the discrete sum of the GBM $nA_n$ in the limit $n \to \infty$, of the form $\mathbb{E}[\exp(-\theta n A_n)] = \exp(n\lambda(\theta S_0, \sqrt{2\beta}; \rho) + o(n))$. This result could be used for numerical simulations of $nA_n$, similar to the approach presented in [32] using an asymptotic result for the Laplace transform of the sum of correlated log-normals. Another possible application would be to obtain a first-order approximation of Asian options prices in the asymptotic $n \geq 1$ limit using the Carr–Madan formula [6].

In the next section we present the leading asymptotics for the Asian option prices using the theory of large deviations.

## 4. Asymptotics for Asian options prices

Asymptotics for the option pricing is a well-studied subject in mathematical finance. There is a vast literature on the asymptotics for option pricing, especially the asymptotics for the vanilla option pricing and the corresponding implied volatility for various continuous-time models; see, e.g. [4], [18], [19], [25], and [48]. We are interested in the asymptotics for the pricing of the Asian options in the discrete-time setting, under assumptions (2) and (3).

Let us consider an Asian option with strike price $K$, in the BS model with volatility $\sigma$, risk free rate $r$, and dividend yield $q$. The prices of the put and call options at time 0 are given by

$$P(n) := \mathrm{e}^{-rt_n}\mathbb{E}[(K - A_n)^+], \qquad C(n) := \mathrm{e}^{-rt_n}\mathbb{E}[(A_n - K)^+],$$

respectively, where $A_n = (1/n)\sum_{i=1}^n S_{t_i}$ and the expectations are taken under the risk-neutral probability measure under which the asset price satisfies the stochastic differential equation $\mathrm{d}S_t = (r-q)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t$. Also note that $\exp(-rt_n) = \exp(-(r/(r-q))(r-q)\tau n) = \exp(-(r/(r-q))\rho)$. Recall that we have proved that $A_n \to A_\infty = S_0(\mathrm{e}^\rho - 1)/\rho$ a.s. as $n \to \infty$. Since $(K - A_n)^+ \leq K$, by the bounded convergence theorem from real analysis, we have

$$\begin{aligned}
\lim_{n\to\infty} P(n) &= \exp\Big(-\frac{r}{r-q}\rho\Big)\lim_{n\to\infty}\mathbb{E}[(K - A_n)^+] \\
&= \exp\Big(-\frac{r}{r-q}\rho\Big)\Big(K - \frac{S_0}{\rho}(\exp(\rho) - 1)\Big)^+.
\end{aligned}$$

From put-call parity,

$$\begin{aligned}
C(n) - P(n) &= \exp(-rt_n)\mathbb{E}[A_n - K] \\
&= \exp\Big(-\frac{r}{r-q}\rho\Big)\Big[\frac{1}{n}\sum_{i=1}^n \mathbb{E}[S_{t_i}] - K\Big] \\
&= \exp\Big(-\frac{r}{r-q}\rho\Big)\Big[\frac{1}{n}S_0\sum_{i=1}^n \exp\Big(\rho\frac{i}{n}\Big) - K\Big] \\
&\to \exp\Big(-\frac{r}{r-q}\rho\Big)\Big(\frac{S_0}{\rho}(\exp(\rho) - 1) - K\Big) \quad \text{as } n \to \infty.
\end{aligned}$$

Therefore,

$$\lim_{n\to\infty} C(n) = \exp\Big(-\frac{r}{r-q}\rho\Big)\Big(\frac{S_0}{\rho}(\exp(\rho) - 1) - K\Big)^+.$$

### 4.1. The OTM case

When $K < S_0(e^{\rho} - 1)/\rho$, $\lim_{n\to\infty} P(n) = 0$ and the put option is OTM and the decaying rate of $P(n)$ to 0 as $n \to \infty$ is governed by the left tail of the large deviations of $A_n$. When $K > S_0(e^{\rho} - 1)/\rho$, $\lim_{n\to\infty} C(n) = 0$ and the call option is OTM and the decaying rate of $C(n)$ to 0 as $n \to \infty$ is governed by the right tail of the large deviations of $A_n$. Before we proceed, let us first derive the large deviation principle for $\mathbb{P}(A_n \in \cdot)$.

**Proposition 4.** *It holds that $\mathbb{P}(A_n \in \cdot)$ satisfies a large deviation principle with rate function*

$$\mathcal{I}(x) = \inf_{g\in\mathcal{AC}_0[0,1],\ \int_0^1 \exp(\sqrt{2\beta}g(y))\,\mathrm{d}y = x/S_0} \frac{1}{2}\int_0^1 \bigg(g'(x) - \frac{\rho}{\sqrt{2\beta}}\bigg)^2 \mathrm{d}x \tag{12}$$

*for $x \geq 0$ and $\mathcal{I}(x) = +\infty$ otherwise.*

*Proof.* We proved already that

$$\frac{1}{n}\sum_{k=0}^{n-1}\exp\bigg(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{k}\bigg(V_j + \frac{\rho}{\sqrt{2\beta}}\bigg)\bigg) = \int_0^1 \exp(\sqrt{2\beta}g(x))\,\mathrm{d}x,$$

where $g(x) = (1/n)\sum_{j=1}^{\lfloor xn\rfloor}(V_j + \rho/\sqrt{2\beta})$ and the map $g \mapsto \int_0^1 \exp(\sqrt{2\beta}g(x))\,\mathrm{d}x$ is continuous in the supremum norm. Since $\mathbb{P}((1/n)\sum_{j=1}^{\lfloor\cdot n\rfloor}(V_j + \rho/\sqrt{2\beta}) \in \cdot)$ satisfies a large deviation principle on $L^{\infty}[0, 1]$ with rate function $\frac{1}{2}\int_0^1(g'(x) - \rho/\sqrt{2\beta})^2\,\mathrm{d}x$ if $g \in \mathcal{AC}_0[0, 1]$ and $+\infty$ otherwise. From the contraction principle, and the fact that $e^{-\beta/n} \leq e^{-\beta k/n^2} \leq 1$ uniformly in $0 \leq k \leq n - 1$, we conclude that $\mathbb{P}(A_n \in \cdot)$ satisfies a large deviation principle with rate function defined in (12). Finally, note that $A_n$ is positive and, thus, $\mathcal{I}(x) = +\infty$ for any $x < 0$. This completes the proof. □

**Remark 1.** Note that $\mathcal{I}(x) = 0$ in (12) if and only if the optimal $g$ satisfies $g'(x) = \rho/\sqrt{2\beta}$, which is equivalent to $g(x) = \rho x/\sqrt{2\beta}$ since $g(0) = 0$. This gives us $\int_0^1 e^{\sqrt{2\beta}g(y)}\,\mathrm{d}y = \int_0^1 e^{\rho y}\,\mathrm{d}y = (e^{\rho} - 1)/\rho$. Thus, $\mathcal{I}(x) = 0$ if and only if $x = S_0((e^{\rho} - 1)/\rho) = A_{\infty}$, which is consistent with the a.s. limit of $A_n$ as $n \to \infty$.

**Remark 2.** We have proved that $\Gamma(\theta) := \lim_{n\to\infty}(1/n)\log\mathbb{E}[e^{\theta n A_n}]$ exists for any $\theta \leq 0$ and is differentiable and $\Gamma(\theta) = +\infty$ for any $\theta > 0$. Since $\Gamma(\theta) = +\infty$ for any $\theta > 0$, we cannot use the Gärtner–Ellis theorem to obtain large deviations for $\mathbb{P}(A_n \in \cdot)$. One may speculate that we might have subexponential tails. But the intriguing fact is that we still have large deviations as stated in Proposition 4.

We can further analyze and solve the variational problem (12). For $\rho \neq 0$, the solution is given by the following result.

**Proposition 5.** *The rate function of the discrete-time average of the GBM is given by*

$$\mathcal{I}(x) = \frac{1}{2\beta}\mathcal{J}\bigg(\frac{x}{S_0}, \rho\bigg), \tag{13}$$

*with*

$$\mathcal{J}\bigg(\frac{x}{S_0}, \rho\bigg) = \begin{cases} \mathcal{J}_1\bigg(\dfrac{x}{S_0}, \rho\bigg), & x/S_0 \geq 1 + \frac{1}{2}\rho, \\ \mathcal{J}_2\bigg(\dfrac{x}{S_0}, \rho\bigg), & x/S_0 \leq 1 + \frac{1}{2}\rho, \end{cases}$$

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*where*

$$\mathcal{J}_1\left(\frac{x}{S_0},\rho\right) = \frac{1}{2}(\delta^2-\rho^2)\left(1-\frac{2\tanh(\delta/2)}{\delta+\rho\tanh(\delta/2)}\right) - 2\rho\log\left[\cosh\left(\frac{\delta}{2}\right)+\frac{\rho}{\delta}\sinh\left(\frac{\delta}{2}\right)\right]+\rho^2,$$

$$\mathcal{J}_2\left(\frac{x}{S_0},\rho\right) = 2\left(\xi^2+\frac{\rho^2}{4}\right)\left\{\frac{\tan\xi}{\xi+(\rho/2)\tan\xi}-1\right\} - 2\rho\log\left(\cos\xi+\frac{\rho}{2\xi}\sin\xi\right)+\rho^2,$$

*and* $\delta, \xi$ *are the solutions of*

$$\frac{1}{\delta}\sinh\delta+\frac{2\rho}{\delta^2}\sinh^2\left(\frac{\delta}{2}\right)=\frac{x}{S_0} \tag{14}$$

*and*

$$\frac{1}{2\xi}\sin(2\xi)\left(1+\frac{\rho}{2}\frac{\tan\xi}{\xi}\right)=\frac{x}{S_0}.$$

*Proof.* The proof is given in Appendix A. □

**Remark 3.** We note that the equations for $\mathcal{J}_{1,2}(K/S_0,\rho)$ can be put into a unique form by denoting $z = 2\xi = i\delta$. Expressed in terms of this variable, we have

$$\mathcal{J}\left(\frac{x}{S_0},\rho\right) = \frac{1}{2}(z^2+\rho^2)\left(\frac{2\tan(z/2)}{z+\rho\tan(z/2)}-1\right) - 2\rho\log\left[\cos\left(\frac{z}{2}\right)+\frac{\rho}{z}\sin\left(\frac{z}{2}\right)\right]+\rho^2,$$

where $z$ is the solution of

$$\frac{1}{z}\sin z+\frac{2\rho}{z^2}\sin^2\left(\frac{z}{2}\right)=\frac{x}{S_0}.$$

**Remark 4.** The rate function $\mathcal{J}(K/S_0,\rho)$ vanishes for $K = A_\infty = S_0(\mathrm{e}^\rho-1)/\rho$, as expected from the general properties of the rate function. Since we have $(\mathrm{e}^\rho-1)/\rho \geq 1+\frac{1}{2}\rho$ for any $\rho\in\mathbb{R}$, this 0 occurs for $\mathcal{J}_1(K/S_0,\rho)$. We note that the rate function $\mathcal{J}_1(K/S_0,\rho)$ vanishes at $\delta=\pm\rho$. Both these values of $\delta$ satisfy (14) with $K/S_0 = (\mathrm{e}^\rho-1)/\rho$, which corresponds to $K = A_\infty$. However, the true solution of the variational problem (12) corresponds to $\delta=-\rho$, which gives the optimal function $g(x)=\rho x/\sqrt{2\beta}$; see (44).

For $\rho = 0$, the solution to the variational problem (12) simplifies and is given as follows.

**Corollary 2.** *For the special case* $\rho = 0$*,*

$$\mathcal{J}(x) = \begin{cases} \dfrac{\delta^2}{2}-\delta\tanh\left(\dfrac{\delta}{2}\right), & x/S_0 \geq 1, \\ 2\xi(\tan\xi-\xi), & 0 < x/S_0 \leq 1, \end{cases}$$

*and* $\mathcal{J}(x)=\infty$ *otherwise, where* $\delta$ *is the unique solution of*

$$\frac{1}{\delta}\sinh\delta=\frac{x}{S_0},$$

*and* $\xi$ *is the unique solution in* $(0,\frac{1}{2}\pi)$ *of*

$$\frac{1}{2\xi}\sin(2\xi)=\frac{x}{S_0}.$$

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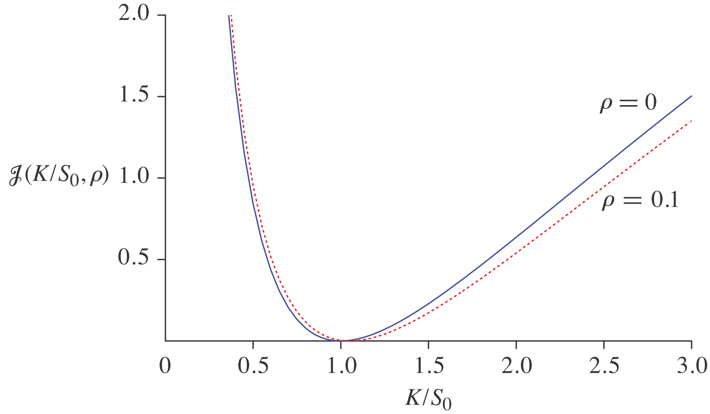


FIGURE 1: Plot of the rate function $\mathcal{J}(K/S_0, \rho)$ versus $K/S_0$ for two values of the $\rho$ parameter $\rho = 0, 0.1$. This is related to the rate function $\mathcal{I}(x)$ for the large deviations of the average of the GBM $A_n$ as in (13).

It can be shown that this is identical to the rate function for the short maturity asymptotics of Asian options with continuous-time averaging in the BS model [42]. The rate function $\mathcal{J}(K/S_0, \rho)$ can be evaluated numerically using the result of Proposition 5. The plot of $\mathcal{J}(x/S_0, \rho)$ is shown in Figure 1 for $\rho = 0, 0.1$.

Using the large deviations results for $\mathbb{P}(A_n \in \cdot)$, we can obtain the asymptotics of the OTM Asian options prices. This is given by the following result.

**Proposition 6.** *When* $K < (S_0/\rho)(\mathrm{e}^{\rho} - 1)$,

$$P(n) = \mathrm{e}^{-n\mathcal{I}(K)+o(n)} \quad \text{as } n \to \infty, \tag{15}$$

*and when* $K > (S_0/\rho)(\mathrm{e}^{\rho} - 1)$,

$$C(n) = \mathrm{e}^{-n\mathcal{I}(K)+o(n)} \quad \text{as } n \to \infty, \tag{16}$$

*where* $\mathcal{I}(\cdot)$ *was defined in (12).*

*Proof.* For any $0 < \epsilon < K$,

$$P(n) \geq \exp\biggl(-\frac{r}{r-q}\rho\biggr)\mathbb{E}[(K - A_n)\,\mathbf{1}_{\{A_n \leq K-\epsilon\}}] \geq \exp\biggl(-\frac{r}{r-q}\rho\biggr)\epsilon\mathbb{P}(A_n \leq K - \epsilon).$$

Therefore, $\liminf_{n\to\infty}(1/n)\log P(n) \geq -\mathcal{I}(K - \epsilon)$. Since it holds for any $\epsilon \in (0, K)$, we conclude that

$$\liminf_{n\to\infty} \frac{1}{n} \log P(n) \geq -\mathcal{I}(K).$$

On the other hand,

$$P(n) = \exp\biggl(-\frac{r}{r-q}\rho\biggr)\mathbb{E}[(K - A_n)\,\mathbf{1}_{\{A_n \leq K\}}] \leq \exp\biggl(-\frac{r}{r-q}\rho\biggr)K\mathbb{P}(A_n \leq K),$$

which implies that $\limsup_{n\to\infty}(1/n)\log P(n) \leq -\mathcal{I}(K)$. Hence, we have proved (15).

For any $\epsilon > 0$,

$$C(n) \geq \exp\biggl(-\frac{r}{r-q}\rho\biggr)\mathbb{E}[(A_n - K)\,\mathbf{1}_{\{A_n \geq K+\epsilon\}}] \geq \exp\biggl(-\frac{r}{r-q}\rho\biggr)\epsilon\mathbb{P}(A_n \geq K + \epsilon).$$

Therefore, $\liminf_{n\to\infty}(1/n)\log C(n) \geq -\mathcal{I}(K+\epsilon)$. Since it holds for any $\epsilon > 0$, we have

$$\liminf_{n\to\infty}\frac{1}{n}\log C(n) \geq -\mathcal{I}(K).$$

For any $1/p + 1/q = 1$, $p, q > 1$, by Hölder's inequality,

$$\begin{aligned} C(n) &= \exp\biggl(-\frac{r}{r-q}\rho\biggr)\mathbb{E}[(A_n - K)^{+}\mathbf{1}_{\{A_n\geq K\}}] \\ &\leq \exp\biggl(-\frac{r}{r-q}\rho\biggr)(\mathbb{E}[[(A_n - K)^{+}]^p])^{1/p}(\mathbb{E}[(\mathbf{1}_{\{A_n\geq K\}})^q])^{1/q} \\ &\leq \exp\biggl(-\frac{r}{r-q}\rho\biggr)(\mathbb{E}[(A_n + K)^p])^{1/p}\mathbb{P}(A_n \geq K)^{1/q}. \end{aligned} \tag{17}$$

By Jensen's inequality, for any $x, y > 0$, it is clear that, for any $p \geq 2$, $(\frac{1}{2}(x+y))^p \leq \frac{1}{2}(x^p + y^p)$. Therefore, for any $p \geq 2$,

$$\mathbb{E}[(A_n + K)^p] \leq 2^{p-1}(\mathbb{E}[A_n^p] + K^p). \tag{18}$$

We can compute that

$$\begin{aligned} \mathbb{E}[A_n^p] &= n^{-p}\mathbb{E}\biggl[\biggl(\sum_{i=1}^{n} S_0 \exp\biggl(\sigma Z_{t_i} + \biggl(r - q - \frac{\sigma^2}{2}\biggr)t_i\biggr)\biggr)^p\biggr] \\ &= n^{-p}\mathbb{E}\biggl[\biggl(\sum_{i=1}^{n} S_0 \exp\biggl(\sigma\sqrt{\tau}Z_i + \biggl(r - q - \frac{\sigma^2}{2}\biggr)\tau i\biggr)\biggr)^p\biggr] \\ &\leq n^{-p}\mathbb{E}\biggl[\biggl(\sum_{i=1}^{n} S_0 \exp\Bigl(\sigma\sqrt{\tau}\max_{1\leq i\leq n} Z_i + |\rho|\Bigr)\biggr)^p\biggr] \\ &\leq S_0^p \exp(|\rho|p)\mathbb{E}\biggl[\exp\biggl(\frac{\sqrt{2\beta}}{n}p\max_{1\leq i\leq n} Z_i\biggr)\biggr] \\ &= S_0^p \exp(|\rho|p)\mathbb{E}\biggl[\exp\biggl(\frac{\sqrt{2\beta}}{n}p|Z_n|\biggr)\biggr] \\ &= S_0^p \exp(|\rho|p)\mathbb{E}\biggl[\exp\biggl(\frac{\sqrt{2\beta}}{\sqrt{n}}p|Z_1|\biggr)\biggr], \end{aligned} \tag{19}$$

where we used the reflection principle for the Brownian motion and the Brownian scaling property. Note that $\mathbb{E}[\mathrm{e}^{\theta|Z_1|}]$ is finite for any $\theta > 0$. Hence, from (17)–(19), we conclude that, for any $1 < q < 2$ (and, thus, $p > 2$, where $1/p + 1/q = 1$),

$$\limsup_{n\to\infty}\frac{1}{n}\log C(n) \leq -\frac{1}{q}\lim_{n\to\infty}\frac{1}{n}\log\mathbb{P}(A_n \geq K) = -\frac{1}{q}\mathcal{I}(K).$$

Since it holds for any $1 < q < 2$, by letting $q \downarrow 1$, we prove (16). □

### 4.2. The ITM case

We consider the case of ITM Asian options, that is, $K > (S_0/\rho)(\mathrm{e}^{\rho} - 1)$ for the put option (and $K < S_0(\mathrm{e}^{\rho} - 1)/\rho$ for the call option). Since $A_n \to A_\infty$ a.s., from the bounded convergence theorem and put-call parity, it follows that $P(n) \to K - S_0(\mathrm{e}^{\rho} - 1)/\rho$ and $C(n) \to S_0(\mathrm{e}^{\rho} - 1)/\rho - K$. The next results concern the speed of the convergence.

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**Proposition 7.** *When $K < S_0(\mathrm{e}^\rho - 1)/\rho$ and $\rho \neq 0$,*

$$C(n) = \exp\Big(-\frac{r}{r-q}\rho\Big)\Big(\frac{S_0}{\rho}(\exp(\rho)-1)-K\Big)+\frac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n}+O(n^{-2}), \tag{20}$$

*and when $K > S_0(\mathrm{e}^\rho - 1)/\rho$ and $\rho \neq 0$,*

$$P(n) = \exp\Big(-\frac{r}{r-q}\rho\Big)\Big(K-\frac{S_0}{\rho}(\exp(\rho)-1)\Big)-\frac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n}+O(n^{-2}). \tag{21}$$

*The $\rho = 0$ case is similar. When $K < S_0$,*

$$C(n) = (S_0 - K) + \mathrm{e}^{-n\mathcal{I}(K)+o(n)},$$

*and when $K > S_0$,*

$$P(n) = (K - S_0) + \mathrm{e}^{-n\mathcal{I}(K)+o(n)}.$$

*Proof.* When $K < S_0(\mathrm{e}^\rho - 1)/\rho$, we proved that $P(n) = \mathrm{e}^{-n\mathcal{I}(k)+o(n)}$. From put-call parity,

$$C(n) - P(n) = \exp(-rt_n)\mathbb{E}[A_n - K] = \exp\Big(-\frac{r}{r-q}\rho\Big)\Big[S_0\frac{\exp(\rho)-1}{n(1-\exp(-\rho/n))} - K\Big].$$

Therefore,

$$\begin{aligned}
C(n) - P(n) - \exp\Big(-\frac{r}{r-q}\rho\Big)\Big(\frac{S_0}{\rho}(\exp(\rho)-1)-K\Big)
&= \exp\Big(-\frac{r\rho}{r-q}\Big)S_0(\exp(\rho)-1)\Big[\frac{1}{n(1-\exp(-\rho/n))}-\frac{1}{\rho}\Big]\\
&= \exp\Big(-\frac{r\rho}{r-q}\Big)S_0(\exp(\rho)-1)\Big[\frac{1}{\rho-\rho^2/2n+O(n^{-2})}-\frac{1}{\rho}\Big]\\
&= \frac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n}+O(n^{-2}).
\end{aligned}$$

Since $P(n) = \mathrm{e}^{-n\mathcal{I}(k)+o(n)}$, we prove (20). Similarly, we have (21). □

### 4.3. The ATM case

Consider next the case of ATM Asian options, that is, $K = S_0(\mathrm{e}^\rho - 1)/\rho = A_\infty$. Since $A_n \to A_\infty$ a.s., using the bounded convergence theorem, we have $P(n) \to 0$ as $n \to \infty$. Put-call parity implies that $C(n) \to 0$ as $n \to \infty$ as well. Note that in the case of OTM, we have already seen that both $P(n)$ and $C(n)$ decay to 0 exponentially fast in $n$, where the exponent is given by $\mathcal{I}(K)$. The next result concerns the speed that $P(n)$ and $C(n)$ decay to 0 as $n \to \infty$ for ATM Asian options. We will see that, unlike the OTM Asian options, whose asymptotics are governed by the large deviations results, the asymptotics for the ATM case are governed by the normal fluctuations from the central limit theorem and nonuniform Berry–Esseen bound.

**Proposition 8.** *When the Asian option is ATM, that is, $K = S_0(\mathrm{e}^\rho - 1)/\rho = A_\infty$,*

$$\left.\begin{aligned}
P(n) &= \exp\Big(-\frac{r\rho}{r-q}\Big)S_0\sqrt{\frac{\beta v(\rho)}{\pi}}\frac{1}{\sqrt{n}}(1+o(1)),\\
C(n) &= \exp\Big(-\frac{r\rho}{r-q}\Big)S_0\sqrt{\frac{\beta v(\rho)}{\pi}}\frac{1}{\sqrt{n}}(1+o(1)),
\end{aligned}\right\}\quad \text{as } n \to \infty.$$

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*Proof.* We have

$$\begin{aligned} C(n) &= \exp(-rt_n)\mathbb{E}[(K-A_n)^+] \\ &= \exp\biggl(-\frac{r\rho}{r-q}\biggr)\mathbb{E}[(A_n-A_\infty)\,\mathbf{1}_{\{A_n\ge A_\infty\}}] \\ &= \exp\biggl(-\frac{r\rho}{r-q}\biggr)S_0\frac{1}{\sqrt{n}}\mathbb{E}\biggl[\sqrt{n}\frac{(A_n-A_\infty)}{S_0}\,\mathbf{1}_{\{\sqrt{n}((A_n-A_\infty)/S_0)\ge 0\}}\biggr]. \end{aligned}$$

In Proposition 2 we proved that $\sqrt{n}((A_n - A_\infty)/S_0) \to N(0, 2\beta v(\rho))$ as $n \to \infty$. Intuitively, it is clear that

$$\mathbb{E}\biggl[\sqrt{n}\frac{(A_n-A_\infty)}{S_0}\,\mathbf{1}_{\{\sqrt{n}((A_n-A_\infty)/S_0)\ge 0\}}\biggr] \to \mathbb{E}[Z\,\mathbf{1}_{\{Z\ge 0\}}],$$

where $Z \sim N(0, 2\beta v(\rho))$. But in order to prove this, the central limit theorem is not sufficient. We need a nonuniform Berry–Esseen bound [5], [39], which we recall next. See, e.g. [40] for a survey on this subject.

**Theorem 2.** (Nonuniform Berry–Esseen bound.) *For any independent and not necessarily identically distributed random variables $X_1, X_2, \dots, X_n$ with zero means and finite variances and* $\operatorname{var}(W_n) = 1$, *where* $W_n = \sum_{i=1}^n X_i$, *let $F_n$ be the cumulative distribution function of $W_n$ and $\Phi$ the standard normal cumulative distribution function, that is,* $\Phi(x) := (1/\sqrt{2\pi})\int_{-\infty}^{x} \mathrm{e}^{-y^2/2}\,\mathrm{d}y$.

*The difference between the two distributions is bounded as (see [5] and [39])*

$$|F_n(x) - \Phi(x)| \le \frac{C\sum_{i=1}^n \mathbb{E}|X_i|^3}{1+|x|^3} \quad \textit{for any } -\infty < x < \infty,$$

*where $C$ is a constant. The best known bound on this constant in the general (nonidentical $X_i$) case is $C < 31.935$ [40].*

We have proved that

$$\sqrt{n}\frac{(A_n-A_\infty)}{S_0} = \sum_{i=1}^n X_i + \xi_n + \varepsilon_n, \tag{22}$$

where

$$X_i := \frac{\sqrt{2\beta}}{n^{3/2}} V_i \frac{\exp(\rho(n+1)/n) - \exp(\rho i/n)}{\exp(\rho/n) - 1}, \qquad 1 \le i \le n, \tag{23}$$

where $V_i$ are i.i.d. $N(0, 1)$ random variables and

$$\varepsilon_n := \frac{1}{\sqrt{n}}\sum_{i=1}^n \exp\biggl(\rho\frac{i}{n}\biggr) - \sqrt{n}\frac{\exp(\rho)-1}{\rho}.$$

The plan of the proof will be to show that the contributions from the second and third terms in (22) are negligible, and to apply the nonuniform Berry–Esseen bound to the first term in (22).

From (22), we have

$$\mathbb{E}\biggl[\sqrt{n}\frac{(A_n-A_\infty)}{S_0}\,\mathbf{1}_{\{\sqrt{n}((A_n-A_\infty)/S_0)\ge 0\}}\biggr] = \mathbb{E}\biggl[\biggl(\sum_{i=1}^n X_i + \xi_n + \varepsilon_n\biggr)\mathbf{1}_{\{\sum_{i=1}^n X_i+\xi_n+\varepsilon_n\ge 0\}}\biggr],$$

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which implies that

$$\left|\mathbb{E}\left[\sqrt{n}\frac{(A_n - A_\infty)}{S_0} \mathbf{1}_{\{\sqrt{n}((A_n-A_\infty)/S_0)\geq 0\}}\right] - \mathbb{E}\left[\left(\sum_{i=1}^n X_i\right) \mathbf{1}_{\{\sum_{i=1}^n X_i+\xi_n+\varepsilon_n\geq 0\}}\right]\right| \leq \mathbb{E}|\xi_n| + |\varepsilon_n|.$$

We have already proved that $\mathbb{E}|\xi_n|$ and $|\varepsilon_n| \to 0$ as $n \to \infty$. Next, note that

$$\mathbb{E}\left[\left(\sum_{i=1}^n X_i\right) \mathbf{1}_{\{\sum_{i=1}^n X_i+\xi_n+\varepsilon_n\geq 0\}}\right] = \sqrt{\sum_{i=1}^n \operatorname{var}(X_i)}\,\mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0\}}\right], \tag{24}$$

where

$$Y_i = \left(\sum_{i=1}^n \operatorname{var}(X_i)\right)^{-1/2} X_i, \qquad \bar{\xi}_n = \left(\sum_{i=1}^n \operatorname{var}(X_i)\right)^{-1/2} \xi_n,$$
$$\bar{\varepsilon}_n = \left(\sum_{i=1}^n \operatorname{var}(X_i)\right)^{-1/2} \varepsilon_n,$$

so that $\operatorname{var}(\sum_{i=1}^n Y_i) = 1$. Recall that we have already proved

$$\lim_{n\to\infty} \sum_{i=1}^n \operatorname{var}(X_i) = 2\beta v(\rho).$$

The expectation on the right-hand side of (24) can be written as

$$\begin{aligned}
&\mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0\}}\right] \\
&\quad = \mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\leq\delta\}}\right] \\
&\qquad + \mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|>\delta\}}\right] \quad \text{for any } \delta > 0.
\end{aligned} \tag{25}$$

The second term is bounded from above by the Cauchy–Schwarz inequality

$$\begin{aligned}
&\left|\mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|>\delta\}}\right]\right| \\
&\quad \leq \mathbb{E}\left[\left(\left(\sum_{i=1}^n Y_i\right) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0\}}\right)^2\right]^{1/2} \mathbb{E}[(\mathbf{1}_{\{|\bar{\xi}_n+\bar{\varepsilon}_n|>\delta\}})^2]^{1/2} \\
&\quad \leq \mathbb{E}\left[\left(\sum_{i=1}^n Y_i\right)^2\right]^{1/2} \mathbb{P}(|\bar{\xi}_n + \bar{\varepsilon}_n| > \delta)^{1/2} \\
&\quad = \mathbb{P}(|\bar{\xi}_n + \bar{\varepsilon}_n| > \delta)^{1/2} \\
&\quad \to 0 \quad \text{as } n \to \infty.
\end{aligned} \tag{26}$$

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The first term in (25) can be written, furthermore, as

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta\}}\bigg] \\
&\qquad = \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta,\ \sum_{i=1}^n Y_i\ge 0\}}\bigg] \\
&\qquad\quad + \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta,\ \sum_{i=1}^n Y_i\le 0\}}\bigg].
\end{aligned}\tag{27}$$

The second term in (27) is negative and is bounded in absolute value as

$$\begin{aligned}
0 &< \bigg|\mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta,\ \sum_{i=1}^n Y_i\le 0\}}\bigg]\bigg| \\
&\le \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)^2\bigg]^{1/2}\mathbb{P}\bigg(\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta,\ \sum_{i=1}^n Y_i\le 0\bigg)^{1/2} \\
&\le \mathbb{P}\bigg(-\delta\le\sum_{i=1}^n Y_i\le 0\bigg)^{1/2} \\
&\to [\Phi(0)-\Phi(-\delta)]^{1/2} \quad \text{as } n\to\infty
\end{aligned}\tag{28}$$

by the central limit theorem.

Next, we need to estimate the first term in (27). We first give an upper bound,

$$\mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta,\ \sum_{i=1}^n Y_i\ge 0\}}\bigg] \le \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i\ge 0\}}\bigg].\tag{29}$$

Next, we give a lower bound,

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\ge 0,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta,\ \sum_{i=1}^n Y_i\ge 0\}}\bigg] \\
&\qquad \ge \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i\ge\delta,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta\}}\bigg].
\end{aligned}\tag{30}$$

This can be written, furthermore, as

$$\begin{aligned}
&\mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i\ge\delta,\ |\bar{\xi}_n+\bar{\varepsilon}_n|\le\delta\}}\bigg] \\
&\qquad = \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i\ge\delta\}}\bigg] - \mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i\ge\delta,\ |\bar{\xi}_n+\bar{\varepsilon}_n|>\delta\}}\bigg].
\end{aligned}\tag{31}$$

By following the same argument as in (26), we have

$$\lim_{n\to\infty}\mathbb{E}\bigg[\bigg(\sum_{i=1}^n Y_i\bigg)\mathbf{1}_{\{\sum_{i=1}^n Y_i\ge\delta,\ |\bar{\xi}_n+\bar{\varepsilon}_n|>\delta\}}\bigg] = 0.\tag{32}$$

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The bounds (29) and (30) can be combined with the bounds (28) to obtain simpler bounds on the expectation in (27) in the $n \to \infty$ limit. By (25)–(32), these bounds translate into corresponding bounds for the expectation (28). We have, for any $\delta \geq 0$,

$$\liminf_{n\to\infty} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0\}}\bigg] \geq \liminf_{n\to\infty} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i\geq \delta\}}\bigg] - [\Phi(0)-\Phi(-\delta)]^{1/2},$$

and

$$\limsup_{n\to\infty} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0\}}\bigg] \leq \limsup_{n\to\infty} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i\geq 0\}}\bigg].$$

Finally, take the $\delta \to 0$ limit, which gives

$$\lim_{n\to\infty} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i+\bar{\xi}_n+\bar{\varepsilon}_n\geq 0\}}\bigg] = \lim_{n\to\infty} \mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i\geq 0\}}\bigg].$$

The nonuniform Berry–Esseen bound can be applied to compute the expectation on the right-hand side.

The sums of third moments appearing in the nonuniform Berry–Esseen bound are estimated as follows. Recalling that $Y_i = (\sum_{i=1}^n \operatorname{var}(X_i))^{-1/2} X_i$, where $X_i$ are defined in terms of $N(0, 1)$ i.i.d. random variables $V_i$ as given in (23), we find that

$$\begin{aligned}
\sum_{i=1}^{n} \mathbb{E}|Y_i|^3 &= \frac{1}{(\sum_{i=1}^n \operatorname{var}(X_i))^{3/2}} \sum_{i=1}^{n} \mathbb{E}|X_i|^3 \\
&= \frac{1}{(\sum_{i=1}^n \operatorname{var}(X_i))^{3/2}} \frac{(2\beta)^{3/2}}{n^{1/2}} \mathbb{E}|V_1|^3 \sum_{i=1}^{n} \bigg(\frac{\exp(\rho(n+1)/n) - \exp(\rho i/n)}{n(\exp(\rho/n)-1)}\bigg)^3 \frac{1}{n} \\
&\leq \frac{C_0(\rho)}{n^{1/2}},
\end{aligned}$$

where $C_0(\rho) > 0$ depends only on $\rho$. Therefore, by the nonuniform Berry–Esseen bound, we have

$$|F_n(x) - \Phi(x)| \leq \frac{C_1(\rho)}{n^{1/2}} \frac{1}{1+|x|^3} \quad \text{for any } -\infty < x < \infty,$$

where $F_n$ is the cumulative distribution function of $\sum_{i=1}^n Y_n$ and $C_1(\rho) > 0$ is another constant. Hence, we have, with $Z \sim N(0, 2\beta v(\rho))$,

$$\begin{aligned}
\bigg|\mathbb{E}\bigg[\bigg(\sum_{i=1}^{n} Y_i\bigg) \mathbf{1}_{\{\sum_{i=1}^n Y_i\geq 0\}}\bigg] - \mathbb{E}[Z\,\mathbf{1}_{\{Z\geq 0\}}]\bigg| &= \bigg|\int_0^\infty x\,\mathrm{d}F_n(x) - \int_0^\infty x\,\mathrm{d}\Phi(x)\bigg| \\
&= \bigg|\int_0^\infty F_n(x)\,\mathrm{d}x - \int_0^\infty \Phi(x)\,\mathrm{d}x\bigg| \\
&\leq \int_0^\infty \frac{C_1(\rho)}{n^{1/2}} \frac{1}{1+|x|^3}\,\mathrm{d}x,
\end{aligned}$$

which goes to 0 as $n \to \infty$. We conclude that

$$C(n) = \exp\left(-\frac{r\rho}{r-q}\right) S_0 \mathbb{E}[Z\,\mathbf{1}_{\{Z\geq 0\}}] \frac{1}{\sqrt{n}}(1+o(1)) \quad \text{as } n \to \infty,$$

where $Z \sim N(0, 2\beta v(\rho))$. The expectation is given explicitly by

$$\mathbb{E}[Z\,\mathbf{1}_{\{Z\geq 0\}}] = \sqrt{2\beta v(\rho)} - \frac{1}{\sqrt{2\pi}} \int_0^\infty x \exp\left(-\frac{x^2}{2}\right) \mathrm{d}x = \sqrt{\frac{\beta v(\rho)}{\pi}}.$$

This completes the proof of the asymptotics for the ATM call option $C(n)$. The asymptotics for the price of the ATM put option $P(n)$ can be obtained by using put-call parity. The proof is complete. □

## 5. Asymptotics for floating-strike Asian options

We consider in this section the floating-strike Asian options, which are a variation of the standard Asian option. The floating-strike Asian call option with strike $K$ and weight $\kappa$ has payoff $(\kappa S_T - A_T)^+$ at maturity $T$ and the floating-strike put option has payoff $(A_T - \kappa S_T)^+$ at maturity $T$.

The floating-strike Asian option is more difficult to price than the fixed-strike case because the joint law of $S_T$ and $A_T$ is needed. Also, the one-dimensional PDE that the floating-strike Asian price satisfies after a change of numéraire is difficult to solve numerically as the Dirac delta function appears as a coefficient; see, e.g. [2], [29], and [44]. See [8], [28], and [43] for alternative methods which have been proposed to deal with this problem.

It has been shown by Henderson and Wojakowski [28] that the floating-strike Asian options with continuous-time averaging can be related to fixed strike ones. These equivalence relations have been extended to discrete-time averaging Asian options in [49]. According to these relations, we have

$$\mathrm{e}^{-rt_n}\mathbb{E}[(\kappa S_{t_n} - A_n)^+] = \mathrm{e}^{-qt_n}\mathbb{E}_*[(\kappa S_0 - A_n)^+], \tag{33}$$

$$\mathrm{e}^{-rt_n}\mathbb{E}[(A_n - \kappa S_{t_n})^+] = \mathrm{e}^{-qt_n}\mathbb{E}_*[(A_n - \kappa S_0)^+]. \tag{34}$$

The expectations on the right-hand side are taken with respect to a different measure $\mathbb{Q}_*$, where the asset price $S_t$ follows the process

$$\mathrm{d}S_t = (q-r)S_t\,\mathrm{d}t + \sigma S_t\,\mathrm{d}W_t^*,$$

with $W_t^*$ a standard Brownian motion in the $\mathbb{Q}_*$ measure.

We are interested in the asymptotics of the price of the Asian call/put options with payoffs $(\kappa S_{t_n} - A_n)^+$ and $(A_n - \kappa S_{t_n})^+$,

$$\begin{aligned} C(n) &:= \mathrm{e}^{-rt_n}\mathbb{E}[(\kappa S_{t_n} - A_n)^+], \\ P(n) &:= \mathrm{e}^{-rt_n}\mathbb{E}[(A_n - \kappa S_{t_n})^+]. \end{aligned} \tag{35}$$

As $n \to \infty$, $\kappa S_{t_n} - A_n \to \kappa S_0 \mathrm{e}^\rho - S_0((\mathrm{e}^\rho - 1)/\rho)$ a.s. When $\kappa < (1-\mathrm{e}^{-\rho})/\rho$ the call option is OTM and the put option is ITM. When $\kappa > (1-\mathrm{e}^{-\rho})/\rho$, the call option is ITM and the put option is OTM. When $\kappa = (1-\mathrm{e}^{-\rho})/\rho$, the call and put options are ATM.

For the expectations on the right-hand side of the equivalence relations (33) and (34), it follows, as $n \to \infty$, $\kappa S_0 - A_n \to \kappa S_0 - S_0((\mathrm{e}^{-\rho} - 1)/(-\rho))$ a.s. We conclude that for

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$\kappa < (1 - \mathrm{e}^{-\rho})/\rho$ these equivalence relations map an OTM floating-strike call (put) Asian option onto an OTM fixed-strike put (call) Asian option. For $\kappa > (1 - \mathrm{e}^{-\rho})/\rho$ a similar relation holds between the respective ITM Asian options.

Let us derive the asymptotics of the price of the floating-strike Asian options. This could be expressed in terms of the asymptotics of the fixed-strike Asian options obtained in the previous sections, with the help of the equivalence relations. An alternative way is to derive directly the large deviation result for the floating-strike Asian options. Then we will relate the rate function to that for the fixed-strike Asian options and show that this is consistent with the equivalence relations.

We have the following result for the asymptotics of floating-strike Asian options.

**Proposition 9.** (i) *When $\kappa < (1 - \mathrm{e}^{-\rho})/\rho$, the call option is OTM,*

$$C(n) = \mathrm{e}^{-n\mathcal{H}(0)+o(n)} \quad \text{as } n \to \infty,$$

*and the put option is ITM,*

$$P(n) = \begin{cases} -\kappa S_0 \exp\Big(-\dfrac{r}{r-q}\rho\Big) + S_0 \exp\Big(-\dfrac{r}{r-q}\rho\Big)\dfrac{\exp(\rho)-1}{\rho} \\ \quad + \dfrac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n} + O(n^{-2}), & \rho \neq 0, \\ (1-\kappa)S_0 + \exp(-n\mathcal{H}(0) + o(n)), & \rho = 0. \end{cases}$$

(ii) *When $\kappa > (1 - \mathrm{e}^{-\rho})/\rho$, the put option is OTM*

$$P(n) = \mathrm{e}^{-n\mathcal{H}(0)+o(n)} \quad \text{as } n \to \infty,$$

*and the call option is ITM,*

$$C(n) = \begin{cases} \kappa S_0 \exp\Big(-\dfrac{r}{r-q}\rho\Big) - S_0 \exp\Big(-\dfrac{r}{r-q}\rho\Big)\dfrac{\exp(\rho)-1}{\rho} \\ \quad - \dfrac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n} + O(n^{-2}), & \rho \neq 0, \\ S_0(\kappa - 1) + \exp(-n\mathcal{H}(0) + o(n)), & \rho = 0. \end{cases}$$

*The rate function in (i) and (ii) is given by*

$$\mathcal{H}(0) := \inf_{g \in \mathcal{AC}_0[0,1],\ \kappa \exp(\sqrt{2\beta} g(1)) - \int_0^1 \exp(\sqrt{2\beta} g(y))\,\mathrm{d}y = 0} \frac{1}{2}\int_0^1 \Big(g'(x) - \frac{\rho}{\sqrt{2\beta}}\Big)^2 \mathrm{d}x.$$

(iii) *When $\kappa = (1 - \mathrm{e}^{-\rho})/\rho$, the call and put options are ITM,*

$$\lim_{n\to\infty} \sqrt{n} C(n) = \lim_{n\to\infty} \sqrt{n} P(n) = S_0 \exp\Big(-\frac{r\rho}{r-q}\Big)\mathbb{E}[Z\, \mathbf{1}_{\{Z \geq 0\}}],$$

*where $Z = N(0, s)$ is a normal random variable with mean* 0 *and variance*

$$s^2 = \frac{2\beta}{\rho^2}\Big[1 - \frac{2}{\rho}(\mathrm{e}^{\rho} - 1) + \frac{\mathrm{e}^{2\rho} - 1}{2\rho}\Big].$$

*Proof.* The proof is similar to the fixed-strike case. The sketch of the proof will be given in Appendix A. □

We show next that the rate function $\mathcal{H}(0)$ can be simply related to $\mathcal{I}(x)$ defined in (12). Recall that we showed explicitly the dependence of $\rho$ of the respective rate functions $H(\cdot)$ and $I(\cdot)$. With a slight abuse of notation to emphasize the dependence on $\rho$, let $H(\cdot;\rho) := H(\cdot)$ and $I(\cdot;\rho) := I(\cdot)$. We have the following result, which is clearly consistent with the equivalence relations (33) and (34).

**Proposition 10.** *The rate functions for the fixed-strike and floating-strike Asian options are related as*

$$\mathcal{H}(0;\rho) = \mathcal{I}(\kappa S_0; -\rho). \tag{36}$$

*Proof.* The functionals in the variational problems for $\mathcal{H}(0)$ and $\mathcal{I}(x)$ are identical, and the only difference is in the constraints on $g(x)$. The constraints can be related as follows.

Let us express $g(x)$ in the variational problem for $\mathcal{H}(0)$ in terms of a new function $h(x)$ defined as $g(x) = g(1) + h(1-x)$. This function satisfies the constraint $h(0) = 0$. The rate function is now given by

$$\mathcal{H}(0) := \inf_{h\in\mathcal{AC}_0[0,1],\ \kappa-\int_0^1 \exp(\sqrt{2\beta}h(y))\,\mathrm{d}y=0} \frac{1}{2}\int_0^1 \bigg(h'(x) + \frac{\rho}{\sqrt{2\beta}}\bigg)^2 \,\mathrm{d}x.$$

It is easy to see that this variational problem is identical to that for the rate function $\mathcal{I}(x)$, identifying $K/S_0 = \kappa$ and $\rho \to -\rho$. This concludes the proof of (36). □

## 6. Implied volatility and numerical tests

It has become accepted market practice to quote European option prices in terms of their implied volatility. This is defined as the value of the log-normal volatility which, upon substitution into the BS formula, reproduces the market option prices. A similar normal implied volatility can be defined in terms of the Bachelier formula.

Although Asian options are quoted in practice by price, and not by implied volatility, it is convenient to define an equivalent implied volatility also for these options. We will define the equivalent log-normal implied volatility of an Asian option with strike $K$ and maturity $T$ as that value of the volatility $\Sigma_{\mathrm{LN}}(K,T)$ which reproduces the Asian option price when substituted into the BS formula for a European option with the same parameters $(K,T)$,

$$\begin{aligned} C(K,S_0,T) &= \mathrm{e}^{-rT}(A_\infty\Phi(d_1) - K\Phi(d_2)), \\ P(K,S_0,T) &= \mathrm{e}^{-rT}(K\Phi(-d_2) - A_\infty\Phi(-d_1)), \end{aligned} \tag{37}$$

where

$$A_\infty = S_0\frac{1}{\rho}(\mathrm{e}^{\rho} - 1) = \frac{1}{(r-q)T}(\mathrm{e}^{(r-q)T} - 1),$$

and

$$d_{1,2} = \frac{1}{\Sigma_{\mathrm{LN}}(K,T)\sqrt{T}}\bigg(\log\frac{A_\infty}{K} \pm \frac{1}{2}\Sigma^2_{\mathrm{LN}}(K,T)T\bigg).$$

The equivalent log-normal volatility $\Sigma_{\mathrm{LN}}$ defined in this way exists for any Asian option call price $C(K,S_0,T)$ satisfying the Merton bounds $(A_\infty - K)^+ \le \mathrm{e}^{rT}C(K,S_0,T) \le A_\infty$ [45].

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For finite $n$ the price of the Asian option is bounded as $(\mathbb{E}[A_n] - K)^+ \leq \mathrm{e}^{rT} C(K, S_0, T) \leq \mathbb{E}[A_n]$ with

$$\mathbb{E}[A_n] = \frac{1}{n} \frac{\mathrm{e}^{\rho} - 1}{1 - \mathrm{e}^{-\rho/n}},$$

so the required bounds are satisfied for $n \to \infty$. (The lower bound follows from the convexity of the payoff $(x - K)^+$ and the upper bound follows from $(x - K)^+ \leq x$.)

One can define also a normal equivalent volatility $\Sigma_{\mathrm{N}}(K, T)$ of an Asian option as that volatility which reproduces the Asian option price when substituted into the Bachelier option pricing formula.

We would like to study the implications of the asymptotic results for Asian option prices derived in Section 4 for the equivalent log-normal volatility $\Sigma_{\mathrm{LN}}$, and for the equivalent normal volatility $\Sigma_{\mathrm{N}}$. This is given by the following result.

**Proposition 11.** (i) *The asymptotic normal and log-normal equivalent implied volatilities of an OTM Asian option in the $n \to \infty$ limit at constant $\beta = \frac{1}{2}\sigma^2 t_n n$ are given by*

$$\lim_{n\to\infty} \frac{\Sigma^2_{\mathrm{LN}}(K, n)}{\sigma^2} = \frac{1}{2} \frac{\log^2(K/A_\infty)}{\mathcal{J}(K/S_0, \rho)}, \tag{38}$$

$$\lim_{n\to\infty} \frac{\Sigma^2_{N}(K, n)}{\sigma^2} = \frac{1}{2} \frac{(K - A_\infty)^2}{\mathcal{J}(K/S_0, \rho)}, \tag{39}$$

*where $\mathcal{J}(K/S_0, \rho)$ is related to the rate function $\mathcal{I}(x)$ as in (13), and is given by Proposition 5.*

(ii) *The equivalent log-normal implied volatility for $n \to \infty$ of an ATM Asian option is*

$$\lim_{n\to\infty} \frac{\Sigma_{\mathrm{LN}}(A_\infty, n)}{\sigma} = \frac{S_0}{A_\infty}\sqrt{v(\rho)}, \tag{40}$$

*and the corresponding result for the equivalent normal implied volatility is*

$$\lim_{n\to\infty} \frac{\Sigma_{\mathrm{N}}(A_\infty, n)}{\sigma} = S_0\sqrt{v(\rho)}.$$

*Proof.* The proof is given in Appendix A. □

We note that in (38), $\sigma$ depends implicitly on $n$ as the limit is taken at fixed $\beta$. In particular, in the fixed maturity regime $\tau n = T$ fixed, we have $\sigma \sim n^{-1/2}$ such that both $\sigma$ and $\Sigma_{\mathrm{LN}}(K)$ approach 0 as $n \to \infty$, in such a way that their ratio approaches a finite nonzero value. We will use this relation for finite $n$ to approximate the equivalent log-normal implied volatility $\Sigma_{\mathrm{LN}}(K)$ as

$$\Sigma^2_{\mathrm{LN}}(K, n) = \sigma^2 \frac{1}{2} \frac{\log^2(K/A_\infty)}{\mathcal{J}(K/S_0, \rho)},$$

and analogously for $\Sigma_{\mathrm{N}}(K)$. These volatilities can be used together with (37) to obtain approximations for Asian option prices.

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TABLE 1: Numerical results for Asian call options under the seven scenarios considered in, e.g. [21] and [36]. *FPP3:* the third-order approximations in [20]. *Vecer:* the PDE method from [52]. *MAE3:* the matched asymptotic expansions from [11]. *Mellin500:* the Mellin transform based method in [46]. *Linetsky:* the results from the spectral expansion in [36]. *LN:* the result of the log-normal approximation [33]. *PZ:* the results of the asymptotic result of this paper using (37).

| Scenario | FPP3 | MAE3 | Mellin500 | Vecer | PZ | LN | Linetsky |
|---|---|---|---|---|---|---|---|
| 1 | 0.055 986 | 0.055 986 | 0.056 036 | 0.055 986 | 0.055 998 | 0.056 054 | 0.055 986 |
| 2 | 0.218 387 | 0.218 369 | 0.218 360 | 0.218 388 | 0.218 480 | 0.219 829 | 0.218 387 |
| 3 | 0.172 267 | 0.172 263 | 0.172 369 | 0.172 269 | 0.172 460 | 0.173 490 | 0.172 269 |
| 4 | 0.193 164 | 0.193 188 | 0.192 972 | 0.193 174 | 0.193 692 | 0.195 379 | 0.193 174 |
| 5 | 0.246 406 | 0.246 382 | 0.246 519 | 0.246 416 | 0.246 944 | 0.249 791 | 0.246 416 |
| 6 | 0.306210 | 0.306 139 | 0.306 497 | 0.306 220 | 0.306 744 | 0.310 646 | 0.306 220 |
| 7 | 0.350 040 | 0.349 909 | 0.348 926 | 0.350 095 | 0.351 517 | 0.359 204 | 0.350 095 |

TABLE 2: The seven benchmark scenarios considered for pricing Asian options in, e.g. [21] and [36]. Here, $q = 0$.

| Scenario | $r$ | $T$ | $S_0$ | $K$ | $\sigma$ |
|---|---|---|---|---|---|
| 1 | 0.0200 | 1 | 2.0 | 2 | 0.10 |
| 2 | 0.1800 | 1 | 2.0 | 2 | 0.30 |
| 3 | 0.0125 | 2 | 2.0 | 2 | 0.25 |
| 4 | 0.0500 | 1 | 1.9 | 2 | 0.50 |
| 5 | 0.0500 | 1 | 2.0 | 2 | 0.50 |
| 6 | 0.0500 | 1 | 2.1 | 2 | 0.50 |
| 7 | 0.0500 | 2 | 2.0 | 2 | 0.50 |

We show in Table 1 numerical results for the asymptotic approximation for the Asian options obtained from (37) for a few scenarios proposed in [21], and defined as in Table 2. They are compared against a few alternative methods considered in the literature: the method of Linetsky [36], PDE methods [20], [52], inversion of Laplace transform [11], [46], and the log-normal approximation [33] corresponding to continuous-time averaging.

The numerical agreement of the asymptotic result with the precise results of the spectral expansion [36] is very good, and the difference is always below 0.5% in relative value. A more appropriate test compares the difference to the option vega $\mathcal{V}$: the approximation error of the asymptotic result is always below $0.24\mathcal{V}$ (compared with the log-normal approximation which has an error as large as $1.54\mathcal{V}$ (for scenario 7; see Table 2)). This is smaller than the typical precision on $\sigma$ around the ATM point, and compares well with typical bid-ask spreads for Asian options which can be $\sim 1\mathcal{V}$ for maturities up to 1–2Y.

**Remark 5.** We comment on the relation of the asymptotic implied volatility (38) to the log-normal approximation [33]. The log-normal approximation [33] corresponds to a flat equivalent log-normal volatility $\Sigma_{\mathrm{LN}}^{(\text{Lévy})}(T)$. In contrast, the asymptotic equivalent log-normal implied volatility $\Sigma_{\mathrm{LN}}(K)$ given by (38) has a nontrivial dependence on strike. It can be easily shown that the log-normal implied volatility reproduces the asymptotic equivalent implied volatility at the ATM point in the limit $\lim_{\sigma^2 T\to 0,\, rT=\rho} \Sigma_{\mathrm{LN}}^{(\text{Lévy})}(T) = \Sigma_{\mathrm{LN}}(K = A_\infty)$.

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The results of Table 1 show that the asymptotic result is an improvement over the log-normal approximation.

**Remark 6.** The results of [21] and [36] were obtained using continuous-time averaging, while our result (38) was derived for discrete-time Asian options. However, we note that (38) does not depend on the size of the time-step $\tau$, so it should hold for an arbitrarily small time-step. It is shown elsewhere [42] that a result similar to (38) holds for the small maturity limit of continuous-time Asian options at fixed $\sigma, r, q$, with the substitution $\rho = 0$. The limiting procedure adopted in this paper, of taking $n \to \infty$ at fixed $\beta, \rho$, allows one to take into account the dependence on $r, q$ in the short maturity expansion.

In order to address the performance of the asymptotic results in the small volatility and maturity regime, we compare our results against those in Table 4 of [20]. As pointed out in [21] and [46], some of the methods proposed in the literature have numerical issues in these regimes in the model parameters. The scenarios considered for this test correspond to $\sigma = 0.01$, $S_0 = 100$, $r = 0.05$, $q = 0$, and three choices of maturity and strike as shown in Table 3. For reasons of space economy, we present only a subset of the test results in Table 4 of [20], which show the best agreement with a MC calculation. The asymptotic results are in very good agreement with the alternative methods shown. We note that the computing time required by the asymptotic method is very good, as it requires only the solution of a simple nonlinear algebraic equation, and the evaluation of a function.

In Table 4 we present a comparison with the test results for discretely sampled Asian options corresponding to the scenarios considered in Table B of [52]. These scenarios have parameters $r = 0.1$, $q = 0$, $\sigma = 0.4$, $T = 1$, and $K = 100$. The results are compared against those obtained in [9], [47], and [52]. The asymptotic results agree with the alternative methods up to about 1%–1.5% in relative error.

Finally, in order to test the asymptotic relation (38) for the equivalent log-normal implied volatility, in Figure 2 we show the equivalent log-normal implied volatility of several Asian options obtained by numerical simulation (circles). These results are obtained by MC pricing of Asian options with parameters $\sigma = 0.2$, $r = q = 0$, $\tau = 0.01$, and $n = 50, 100, 200$ averaging dates. The MC calculation used $N_{\text{MC}} = 10^6$ samples. The strikes considered cover

Table 3: Test results for Asian call options under small volatility $\sigma = 0.01$, $S_0 = 100$, $r = 0.05$, and $q = 0$. *FPP3:* the third-order approximations in [20]. *MAE3:* the results using the matched asymptotic expansions from [11]. *Mellin500:* the results of the Mellin transform based method in [46]. *PZ:* the asymptotic results of this paper.

| $T$ | $K$ | PZ | FPP3 | MAE3 | Mellin500 |
|---|---|---|---|---|---|
| 0.25 | 99 | 1.607 390 0 | $1.607\,39 \times 10^{0}$ | $1.607\,39 \times 10^{0}$ | $1.517\,18 \times 10^{0}$ |
| 0.25 | 100 | 0.621 359 0 | $6.213\,59 \times 10^{-1}$ | $6.213\,59 \times 10^{-1}$ | $6.968\,55 \times 10^{-1}$ |
| 0.25 | 101 | 0.013 761 5 | $1.376\,18 \times 10^{-2}$ | $1.376\,15 \times 10^{-2}$ | $1.603\,61 \times 10^{-2}$ |
| 1.00 | 97 | 5.271 900 0 | $5.271\,90 \times 10^{0}$ | $5.271\,90 \times 10^{0}$ | $5.274\,74 \times 10^{0}$ |
| 1.00 | 100 | 2.418 210 0 | $2.418\,21 \times 10^{0}$ | $2.418\,21 \times 10^{0}$ | $2.433\,03 \times 10^{0}$ |
| 1.00 | 103 | 0.072 433 9 | $7.269\,10 \times 10^{-2}$ | $7.243\,37 \times 10^{-2}$ | $8.508\,16 \times 10^{-2}$ |
| 5.00 | 80 | 26.1756 | $2.617\,56 \times 10^{1}$ | $2.617\,56 \times 10^{1}$ | $2.617\,56 \times 10^{1}$ |
| 5.00 | 100 | 10.5996 | $1.059\,96 \times 10^{1}$ | $1.059\,96 \times 10^{1}$ | $1.059\,93 \times 10^{1}$ |
| 5.00 | 120 | $5.8331 \times 10^{-6}$ | $2.066\,99 \times 10^{-5}$ | $5.733\,17 \times 10^{-6}$ | $1.422\,35 \times 10^{-3}$ |

TABLE 4: Asymptotic results for discretely sampled Asian call options under the scenarios considered in Table B of [52], comparing with the results of [9], [47], and [52].

| | $S_0 = 95$ | $S_0 = 100$ | $S_0 = 105$ |
|---|---|---|---|
| | Vecer | | |
| $n = 250$ | 8.4001 | 11.1600 | 14.3073 |
| $n = 500$ | 8.3826 | 11.1416 | 14.2881 |
| $n = 1000$ | 8.3741 | 11.1322 | 14.2786 |
| $\infty$ | 8.3661 | 11.1233 | 14.2696 |
| | Tavella–Randall | | |
| $n = 250$ | 8.3972 | 11.1573 | 14.3054 |
| $n = 500$ | 8.3804 | 11.1392 | 14.2866 |
| $n = 1000$ | 8.3719 | 11.1300 | 14.2771 |
| $\infty$ | 8.3640 | 11.1215 | 14.2681 |
| | Curran | | |
| $n = 250$ | 8.3972 | 11.1572 | 14.3048 |
| $n = 500$ | 8.3801 | 11.1388 | 14.2857 |
| $n = 1000$ | 8.3715 | 11.1296 | 14.2762 |
| $\infty$ | – | – | – |
| PZ | 8.3789 | 11.1362 | 14.2818 |

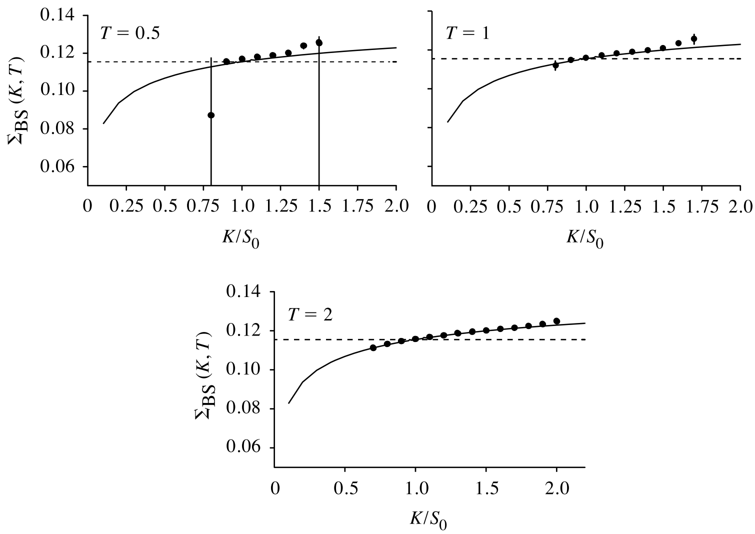


FIGURE 2: The equivalent log-normal volatility $\Sigma_{LN}(K, S_0)$ of Asian options in the BS model given by (38) (*solid line*). The dashed line is at $\sigma/\sqrt{3}$ and corresponds to the ATM equivalent volatility. The circles indicate the log-normal equivalent volatility obtained by MC pricing of the Asian options with maturity $T = 0.5, 1, 2$. The BS model parameters are $r = q = 0, \sigma = 0.2$. The time step of the MC simulation is $\tau = 0.01$ and the number of paths $N_{MC} = 1m$.

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a region around the ATM point $K = S_0$; the numerical precision of the simulation decreases rapidly outside of this region. We note very good agreement with the asymptotic result of Proposition 11, even for $n$ as low as 50.

## Appendix A

*Proof of Proposition 3.* The variational problem appearing in (7) can be written equivalently by introducing the function $f(x) = bg(x)$ as

$$\lambda(a, b; \rho) = \frac{1}{b^2} \sup_{f \in \mathcal{AC}_0[0,1]} \left\{ -ab^2 \int_0^1 \mathrm{e}^{f(x)}\, \mathrm{d}x - \frac{1}{2} \int_0^1 (f'(x) - \rho)^2\, \mathrm{d}x \right\}.$$

The functional $\Lambda[f]$ appearing in this variational problem can be written as

$$\begin{aligned} \Lambda[f] &= -ab^2 \int_0^1 \mathrm{d}x \mathrm{e}^{f(x)} - \frac{1}{2} \int_0^1 (f'(x) - \rho)^2\, \mathrm{d}x \\ &= -ab^2 \int_0^1 \mathrm{e}^{f(x)}\, \mathrm{d}x - \frac{1}{2} \int_0^1 [f'(x)]^2\, \mathrm{d}x + f(1)\rho - \frac{\rho^2}{2}. \end{aligned}$$

In the second line we integrated by parts and wrote $\int_0^1 f'(x)\, \mathrm{d}x = f(1)$, where we took into account the constraint $f(0) = 0$. Although in Proposition 3 we have $a > 0$, the variational problems in Section 4 require also the case of negative $a$. For this reason we will treat here both cases of positive and negative $a$.

The optimal function $f(x)$ satisfies the Euler–Lagrange equation

$$f''(x) = ab^2 \mathrm{e}^{f(x)} \tag{41}$$

with the boundary conditions

$$f(0) = 0, \qquad f'(1) = \rho. \tag{42}$$

The second boundary condition (at $x = 1$) is a transversality condition.

We observe that the quantity

$$E = -ab^2 \mathrm{e}^{f(x)} + \tfrac{1}{2}[f'(x)]^2 = -ab^2 \mathrm{e}^{f(1)} + \tfrac{1}{2}\rho^2$$

is a constant of motion of the differential equation (41). Its value was expressed in terms of $f(1)$ by taking $x = 1$ and using the boundary condition (42). Taking the integral of this relation over $x\colon (0, 1)$ can be used to eliminate the integral of $\frac{1}{2}[f'(x)]^2$ in the functional $\Lambda[f]$. This can be put into the equivalent form

$$\Lambda[f] = -2ab^2 \int_0^1 \mathrm{e}^{f(x)}\, \mathrm{d}x + ab^2 \mathrm{e}^{f(1)} + f(1)\rho - \rho^2. \tag{43}$$

The Euler–Lagrange equation (41) can be solved exactly. Two independent solutions of this equation are

$$f_1(x) = \delta x - 2 \log\left(\frac{\mathrm{e}^{\delta x} + \gamma}{1 + \gamma}\right), \tag{44}$$

$$f_2(x) = -2 \log|\cos(\xi x + \eta)| + 2 \log|\cos \eta|. \tag{45}$$

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The first solution was given in [27], where a related differential equation appears in the context of optimal sampling for MC pricing of Asian options. It is easy to see by direct substitution into (41) that these functions satisfy this equation, with the appropriate boundary condition at $x = 0$. Requiring that the coefficient in this equation and the boundary condition $f'(1) = \rho$ are satisfied yielding two conditions.

For $f_1(x)$ we have the conditions

$$2\gamma\delta^2 = -ab^2(1+\gamma)^2, \qquad \delta\frac{\gamma - e^{\delta}}{\gamma + e^{\delta}} = \rho. \tag{46}$$

Eliminating $\gamma$ between these two equations as $\gamma = ((\delta+\rho)/(\delta-\rho))e^{\delta}$ gives an equation for $\delta$:

$$\delta^2 - \rho^2 = -2ab^2\bigg(\cosh\bigg(\frac{\delta}{2}\bigg) + \frac{\rho}{\delta}\sinh\bigg(\frac{\delta}{2}\bigg)\bigg)^2. \tag{47}$$

For $f_2(x)$ we obtain the conditions

$$2\xi^2 = ab^2\cos^2\eta, \qquad 2\xi\tan(\xi+\eta) = \rho. \tag{48}$$

The second relation allows us to eliminate $\eta$ as

$$\tan\eta = \frac{\rho/2 - \xi\tan\xi}{\xi + (\rho/2)\tan\xi}.$$

We obtain the equation for $\xi$:

$$2\xi^2(4\xi^2+\rho^2) = ab^2(2\xi\cos\xi + \rho\sin\xi)^2. \tag{49}$$

Finally, the integral appearing in $\Lambda[f]$ can be computed in closed form for each solution, and we have

$$T_1(\delta,\rho) = \int_0^1 dx\, e^{f_1(x)} = \frac{1}{\delta}\sinh\delta + \frac{2\rho}{\delta^2}\sinh^2\bigg(\frac{\delta}{2}\bigg),$$
$$T_2(\xi,\rho) = \int_0^1 dx\, e^{f_2(x)} = \frac{1}{2\xi}\sin(2\xi)\bigg(1 + \frac{\rho}{2\xi}\tan\xi\bigg).$$

Substituting these results into (43), we find the following results for the function $\lambda(a, b; \rho)$:

$$\begin{aligned}
\lambda_1(a,b;\rho) &= -2aT_1(\delta) + ae^{\delta}\bigg(\frac{1+\gamma}{e^{\delta}+\gamma}\bigg)^2 + \frac{1}{b^2}\rho\bigg(\delta - 2\log\bigg(\frac{e^{\delta}+\gamma}{1+\gamma}\bigg)\bigg) - \frac{\rho^2}{b^2}\\
&= a\bigg\{1 + \sinh^2\frac{\delta}{2}\bigg(1 - \frac{4\rho}{\delta^2} + \frac{\rho^2}{\delta^2}\bigg) - \frac{2-\rho}{\delta}\sinh\delta\bigg\}\\
&\quad + \frac{2}{b^2}\rho\log\bigg[\cosh\frac{\delta}{2} + \frac{\rho}{\delta}\sinh\frac{\delta}{2}\bigg] - \frac{\rho^2}{b^2},\\
\lambda_2(a,b;\rho) &= -2aT_2(\xi) + a\frac{\cos^2\eta}{\cos^2(\xi+\eta)} + \frac{1}{b^2}\rho\log\frac{\cos^2\eta}{\cos^2(\xi+\eta)} - \frac{\rho^2}{b^2}\\
&= a\bigg\{1 - \sin^2\xi\bigg(1 + \frac{\rho}{\xi^2} - \frac{\rho^2}{4\xi^2}\bigg) + \frac{\rho-2}{2\xi}\sin(2\xi)\bigg\}\\
&\quad + \frac{2\rho}{b^2}\log\bigg[\cos\xi + \frac{\rho}{2\xi}\sin\xi\bigg] - \frac{\rho^2}{b^2},
\end{aligned}$$

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where $\delta$ and $\xi$ are given by the solutions of (47) and (49), respectively. For given $(a > 0, b, \rho)$, only one of these two equations has a solution, which determines the optimal function $f(x)$ uniquely, and the function $\lambda(a, b; \rho)$. This completes the proof of Proposition 3. $\square$

*Proof of Proposition 5.* The variational problem (12) can be written equivalently in terms of $\mathcal{J}(x, \rho)$ defined as in (13), by introducing the function $f(y) = \sqrt{2\beta}g(y)$ as

$$\mathcal{J}(x, \rho) = \inf_{f \in \mathcal{AC}_0[0,1],\ \int_0^1 \exp(f(y))\,\mathrm{d}y = x/S_0} \frac{1}{2}\int_0^1 (f'(y) - \rho)^2\,\mathrm{d}y.$$

The integral constraint on $f(y)$ is taken into account by introducing a Lagrange multiplier $a$ and defining an auxiliary functional

$$\begin{aligned}\Lambda[f] &= \frac{1}{2}\int_0^1 \mathrm{d}y(f'(y) - \rho)^2 + a\biggl(\int_0^1 \mathrm{d}y\mathrm{e}^{f(y)} - \frac{x}{S_0}\biggr)\\ &= \frac{1}{2}\int_0^1 \mathrm{d}y[f'(y)]^2 + a\int_0^1 \mathrm{d}y\mathrm{e}^{f(y)} - \rho f(1) + \frac{\rho^2}{2} - a\frac{x}{S_0}.\end{aligned}$$

The solution of this variational problem $f(y)$ satisfies the Euler–Lagrange equation

$$f''(y) = a\mathrm{e}^{f(y)} \tag{50}$$

with boundary conditions (the condition at $y = 1$ is a transversality condition)

$$f(0) = 0, \qquad f'(1) = \rho.$$

This differential equation and the associated boundary conditions are identical to the equation appearing in the proof of Proposition 3. As shown, this can be solved exactly, and the solutions are given in (44) and (45). The details of the proof will be slightly different, as in the present case the coefficient $a$ (the Lagrange multiplier) is not known, but is one of the unknowns of the variational problem. However, we will show that it can be determined using the integral constraint

$$\int_0^1 \mathrm{e}^{f(y)}\,\mathrm{d}y = \frac{K}{S_0}. \tag{51}$$

Before proceeding with the solution of the variational problem, we give a preliminary result which expresses the rate function only in terms of $a$, $f(1)$.

**Lemma 1.** *The rate function $\mathcal{J}(x, \rho)$ is given by*

$$\mathcal{J}\biggl(\frac{K}{S_0}, \rho\biggr) = a\biggl(\frac{K}{S_0} - \mathrm{e}^{f(1)}\biggr) - \rho f(1) + \rho^2. \tag{52}$$

*Proof.* The Euler–Lagrange equation (50) conserves the quantity

$$E = \tfrac{1}{2}(f'(y))^2 - a\mathrm{e}^{f(y)},$$

which yields

$$\tfrac{1}{2}[f'(y)]^2 - a\mathrm{e}^{f(y)} = \tfrac{1}{2}\rho^2 - a\mathrm{e}^{f(1)}.$$

Taking the integral of this relation over $x\colon (0, 1)$, and using the constraint (51) yields the result (52). $\square$

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The only remaining part of the proof is determining $a$, $f(1)$. This can be done from the constraint (51). Substituting (44) into this constraint gives

$$\int_0^1 \mathrm{d}x \mathrm{e}^{f_1(x)} = \frac{1}{\delta}\sinh\delta + \frac{2\rho}{\delta^2}\sinh^2\left(\frac{\delta}{2}\right) = \frac{K}{S_0},$$

which is an equation for $\delta$. This equation has solutions only for $K/S_0 \geq 1+\rho/2$. Once $\delta$, $\gamma$ are known, the Lagrange multiplier $a$ is determined using the relation (46). Substituting into (52), we find the rate function

$$\mathcal{J}\left(\frac{K}{S_0}, \rho\right) = \frac{1}{2}(\beta^2-\rho^2)\left(1 - \frac{2\tanh(\beta/2)}{\beta+\rho\tanh(\beta/2)}\right) \\ - 2\rho\log\left[\cosh\left(\frac{\beta}{2}\right) + \frac{\rho}{\beta}\sinh\left(\frac{\beta}{2}\right)\right] + \rho^2.$$

A similar calculation using $f_2(x)$ yields

$$\int_0^1 \mathrm{d}x \mathrm{e}^{f_2(x)} = \frac{1}{2\xi}\sin(2\xi)\left(1 + \frac{\rho}{2}\frac{\tan\xi}{\xi}\right) = \frac{K}{S_0}. \tag{53}$$

Both $\eta$ and $\xi+\eta$ must be in the $(-\pi/2, \pi/2)$ range. Equation (53) has solutions only for $K/S_0 \leq 1+\rho/2$.

Using the solution for $\xi$, the Lagrange multiplier $a$ is found from (48). Substituting into (52), we find the rate function

$$\mathcal{J}\left(\frac{K}{S_0}, \rho\right) = 2\left(\xi^2 + \frac{\rho^2}{4}\right)\left\{\frac{\tan\xi}{\xi+(\rho/2)\tan\xi} - 1\right\} - 2\rho\log\left(\cos\xi + \frac{\rho}{2\xi}\sin\xi\right) + \rho^2.$$

This completes the proof of Proposition 5. □

*Proof of Proposition 9.* Start by noting that $S_{t_n} = S_0\exp(\sigma Z_{t_n} + (r-q-\frac{1}{2}\sigma^2)t_n)$ can be written equivalently as $S_0\exp((\sqrt{2\beta}/n)\sum_{j=1}^n (V_j + \rho/\sqrt{2\beta}) - \beta/n)$ in distribution, where $V_j$ are i.i.d. $N(0,1)$ random variables. Let us also recall that

$$A_n = \frac{1}{n}S_0\sum_{k=0}^{n-1}\exp\left(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{k}\left(V_j + \frac{\rho}{\sqrt{2\beta}}\right) - \frac{\beta k}{n^2}\right).$$

The terms $\beta/n$, $\beta k/n^2$ are uniformly bounded and negligible and if we let

$$g(x) = \left(\frac{1}{n}\right)\sum_{j=1}^{\lfloor xn\rfloor}\left(V_j + \frac{\rho}{\sqrt{2\beta}}\right),$$

then

$$\kappa\exp\left(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{n}\left(V_j + \frac{\rho}{\sqrt{2\beta}}\right)\right) - \frac{1}{n}\sum_{k=0}^{n-1}\exp\left(\frac{\sqrt{2\beta}}{n}\sum_{j=1}^{k}\left(V_j + \frac{\rho}{\sqrt{2\beta}}\right)\right) \\ = \kappa\exp(\sqrt{2\beta}g(1)) - \int_0^1 \exp(\sqrt{2\beta}g(x))\,\mathrm{d}x.$$

The map $g \mapsto \kappa\exp(\sqrt{2\beta}g(1)) - \int_0^1 \exp(\sqrt{2\beta}g(x))\,\mathrm{d}x$ is continuous in the supremum norm and by the contraction principle, $\mathbb{P}(\kappa S_{t_n} - A_n \in \cdot)$ satisfies a large deviation principle with the rate function

$$\mathcal{H}(x) = \inf_{g\in\mathcal{AC}_0[0,1],\,\kappa\exp(\sqrt{2\beta}g(1))-\int_0^1\exp(\sqrt{2\beta}g(y))\,\mathrm{d}y=x/S_0} \frac{1}{2}\int_0^1\bigg(g'(y) - \frac{\rho}{\sqrt{2\beta}}\bigg)^2\,\mathrm{d}y.$$

As $n\to\infty$, $\kappa S_{t_n} - A_n \to \kappa S_0\mathrm{e}^{\rho} - S_0((\mathrm{e}^{\rho}-1)/\rho)$ a.s. When $\kappa < (1-\mathrm{e}^{-\rho})/\rho$, the call option is OTM and

$$C(n) = \mathrm{e}^{-n\mathcal{H}(0)+o(n)} \quad \text{as } n\to\infty,$$

and by put-call parity, when $\rho\neq 0$,

$$\begin{aligned}
C(n) - P(n) &= \exp(-rt_n)\mathbb{E}[\kappa S_{t_n} - A_n] \\
&= \kappa S_0\exp\bigg(-\frac{r}{r-q}\rho\bigg) - \exp\bigg(-\frac{r}{r-q}\rho\bigg)S_0\frac{\exp(\rho)-1}{n(1-\exp(-\rho/n))} \\
&= \kappa S_0\exp\bigg(-\frac{r}{r-q}\rho\bigg) - S_0\exp\bigg(-\frac{r}{r-q}\rho\bigg)\frac{\exp(\rho)-1}{\rho} \\
&\quad - \frac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n} + O(n^{-2}).
\end{aligned}$$

Therefore, as $n\to\infty$, the asymptotics for the ITM put option are

$$\begin{aligned}
P(n) &= -\kappa S_0\exp\bigg(-\frac{r}{r-q}\rho\bigg) + S_0\exp\bigg(-\frac{r}{r-q}\rho\bigg)\frac{\exp(\rho)-1}{\rho} \\
&\quad + \frac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n} + O(n^{-2}).
\end{aligned}$$

When $\rho = 0$,

$$P(n) = (1-\kappa)S_0 + \mathrm{e}^{-n\mathcal{H}(0)+o(n)} \quad \text{as } n\to\infty.$$

When $\kappa = (1-\mathrm{e}^{-\rho})/\rho$, that is, ATM, the asymptotics for $C(n)$ and $P(n)$ are governed by the central limit theorem. We can approximate $(\kappa S_{t_n} - A_n)/\sqrt{n}S_0$ by

$$\kappa\exp(\rho)\frac{\sqrt{2\beta}}{\sqrt{n}}\sum_{j=0}^{n-1}V_j - \frac{\sqrt{2\beta}}{n^{3/2}}\sum_{j=0}^{n-1}V_j\sum_{i=j+1}^{n}\exp\bigg(\rho\frac{i}{n}\bigg)$$

with $V_j = N(0,1)$ i.i.d. random variables. The variance of this expression converges to

$$\begin{aligned}
\int_0^1\bigg[\kappa\mathrm{e}^{\rho}\sqrt{2\beta} - \frac{\sqrt{2\beta}}{\rho}(\mathrm{e}^{\rho}-\mathrm{e}^{\rho x})\bigg]^2\,\mathrm{d}x &= \frac{2\beta}{\rho^2}\int_0^1(1-\mathrm{e}^{\rho x})^2\,\mathrm{d}x \\
&= \frac{2\beta}{\rho^2}\bigg[1 - \frac{2}{\rho}(\mathrm{e}^{\rho}-1) + \frac{\mathrm{e}^{2\rho}-1}{2\rho}\bigg].
\end{aligned}$$

We can further use the nonuniform Berry–Esseen bound for the central limit theorem to obtain the following asymptotics:

$$\lim_{n\to\infty}\sqrt{n}C(n) = \lim_{n\to\infty}\sqrt{n}P(n) = S_0\exp\bigg(-\frac{r\rho}{r-q}\bigg)\mathbb{E}[Z\,\mathbf{1}_{\{Z\geq 0\}}],$$

where $Z$ is a normal random variable with mean 0 and variance

$$\frac{2\beta}{\rho^2}\bigg[1-\frac{2}{\rho}(\mathrm{e}^{\rho}-1)+\frac{\mathrm{e}^{2\rho}-1}{2\rho}\bigg].$$

When $\kappa > (1-\mathrm{e}^{-\rho})/\rho$, the put option is OTM and

$$P(n)=\mathrm{e}^{-n\mathcal{H}(0)+o(n)} \quad \text{as } n\to\infty,$$

and when $\rho\neq 0$, we have, for the ITM call option,

$$\begin{aligned}C(n)&=\kappa S_0\exp\Big(-\frac{r}{r-q}\rho\Big)-S_0\exp\Big(-\frac{r}{r-q}\rho\Big)\frac{\exp(\rho)-1}{\rho}\\&\quad-\frac{\exp(-r\rho/(r-q))S_0(\exp(\rho)-1)}{2n}+O(n^{-2}),\end{aligned}$$

and when $\rho=0$,

$$C(n)=S_0(\kappa-1)+\mathrm{e}^{-n\mathcal{H}(0)+o(n)} \quad \text{as } n\to\infty. \qquad \square$$

*Proof of Proposition 11.* (i) The price of an undiscounted European option in the BS model depends only on $\sigma^2 T$ and $K/F$ with $F$ the forward asset price. In our case given by (37), we have $F=A_\infty$, and we denote this dependence as $\mathrm{e}^{-rT}A_\infty\bar{C}_{\mathrm{BS}}(K/A_\infty,\sigma^2T)$, with

$$\bar{C}_{\mathrm{BS}}(k,v):=\Phi\bigg(\frac{1}{\sqrt{v}}\Big(-\log k+\frac{v}{2}\Big)\bigg)-k\Phi\bigg(\frac{1}{\sqrt{v}}\Big(-\log k-\frac{v}{2}\Big)\bigg).$$

By definition of the equivalent log-normal implied volatility, we have

$$C(n)=\mathrm{e}^{-rT}A_\infty\bar{C}_{\mathrm{BS}}\Big(\frac{K}{A_\infty},\Sigma_{\mathrm{LN}}^2T\Big).$$

Consider an OTM Asian call option $K>A_\infty$. We have, from Proposition 6,

$$\lim_{n\to\infty}\frac{1}{n}\log C(n)=-\frac{1}{2\beta}\mathcal{J}\Big(\frac{K}{S_0},\rho\Big).$$

Also, we have

$$\lim_{T\to 0}(\Sigma_{\mathrm{LN}}^2T)\log\Big(A_\infty\bar{C}_{\mathrm{BS}}\Big(\frac{K}{A_\infty},\Sigma_{\mathrm{LN}}^2T\Big)\Big)=-\frac{1}{2}\log^2\Big(\frac{K}{A_\infty}\Big).$$

We thus obtain, setting $T=t_n$,

$$\lim_{n\to\infty}\Sigma_{\mathrm{LN}}^2(K,n)n^2\tau=\lim_{n\to\infty}\frac{\Sigma_{\mathrm{LN}}^2(K,n)n\tau\log[A_\infty\bar{C}_{\mathrm{BS}}(K/A_\infty,\Sigma_{\mathrm{LN}}^2T)]}{(1/n)\log C(n)}=\beta\frac{\log^2(K/A_\infty)}{\mathcal{J}(K/S_0,\rho)}.$$

Recalling that $\beta=\frac{1}{2}\sigma^2n^2\tau$, this is written equivalently as

$$\lim_{n\to\infty}\frac{1}{\sigma^2}\Sigma_{\mathrm{LN}}^2(K,n)=\frac{1}{2}\frac{\log^2(K/A_\infty)}{\mathcal{J}(K/S_0,\rho)},$$

which reproduces the result (38).

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(ii) ATM Asian option. For this case, the BS formula yields

$$\bar{C}_{\mathrm{BS}}(1, \Sigma_{\mathrm{LN}}^2 T) = \Phi\left(\frac{\Sigma_{\mathrm{LN}}\sqrt{T}}{2}\right) - \Phi\left(-\frac{\Sigma_{\mathrm{LN}}\sqrt{T}}{2}\right) = \frac{1}{\sqrt{2\pi}}\Sigma_{\mathrm{LN}}\sqrt{T}(1 + O(\Sigma_{\mathrm{LN}}^2 T)).$$

The large $n$ asymptotics of the ATM Asian option given in Proposition 8 reads as

$$C(A_\infty, n) = \frac{1}{\sqrt{\pi}} S_0 \exp\left(-\frac{r\rho}{r-q}\right)\sqrt{\beta v(\rho)}\frac{1}{\sqrt{n}}.$$

The two results are related as $C(A_\infty, n) = \mathrm{e}^{-rT} A_\infty \bar{C}_{\mathrm{BS}}(1, \Sigma_{\mathrm{LN}}^2 T)$. Recalling that we have $\sigma\sqrt{t_n} = \sqrt{2\beta}/\sqrt{n}$, we obtain the asymptotics of the equivalent implied volatility of the ATM Asian option

$$\lim_{n\to\infty} \frac{\Sigma_{\mathrm{LN}}(A_\infty, n)}{\sigma} = \frac{S_0}{A_\infty}\sqrt{v(\rho)}.$$

This reproduces (40).

The proof of (39) proceeds in a similar way, starting with the Bachelier formula for the call option prices. □

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