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ON O. BONNET III-ISOMETRY OF SURFACES IN THREE DIMENSIONAL EUCLIDEAN SPACE

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Abstract

In this paper we consider O. Bonnet III-isometry (or BIII-isometry) of surfaces in 3-dimensional Euclidean space E^3. Suppose a map $F: M \to M^*$ is a diffeomorphism, and $F^*(\mathrm{III}^*) = \mathrm{III}$, $\kappa_i(m) = \kappa_i^*(m^*)$, $i = 1, 2$, where $m \in M$, $m^* \in M^*$, $m^* = F(m)$, κ_i and κ_i^* are the principal curvatures of surfaces M and M^* at the points m and m^*, respectively, III and III^* are the third fundamental forms of M and M^*, respectively. In this case, we call F an O. Bonnet III-isometry from M to M^*. O. Bonnet I-isometries were considered in references [1]–[5].

We distinguish three cases about BIII-surfaces, which admits a non-trivial BIII-isometry. We obtain some geometric properties of BIII-surfaces and BIII-isometries in these three cases; see Theorems 1, 2, 3 (in Section 2). We study some special BIII-surfaces: the minimal BIII-surfaces; BIII-surfaces of revolution; and BIII-surfaces with constant Gaussian curvature; see Theorems 4, 5, 6 (in Section 3).

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0. Introduction

O. Bonnet [1] was the first to study the isometric deformations of surfaces in 3-dimensional Euclidean space E^3 which preserve mean curvature. Also W. C. Graustein [4] and E. Cartan [2] did some work in this area. Recently, S. S. Chern [3] obtained an interesting result about the surfaces with mean curvature $H \neq$ constant. After that, I. M. Roussos [5] got some detailed results. In this paper, a more general definition of O. Bonnet deformations is given.

Let M and M^* be two surfaces in the Euclidean space E^3. Suppose I, II, III are the first, second, third fundamental forms of the surface M, respectively. We shall denote the quantities pertaining to M^* by the same symbols with asterisks " $*$ ".

DEFINITION. Suppose $F: M \to M^*$ is a diffeomorphism, and $F^*(\mathrm{I}^*) = \mathrm{I}$ or $F^*(\mathrm{II}^*) = \mathrm{II}$ or $F^*(\mathrm{III}^*) = \mathrm{III}$, where F^* represents F's cotangent map. Then we call F a I- or II- or III-*isometry* of M and M^*, respectively. Moreover, suppose F preserves the principal curvatures at the corresponding points:

$$\kappa_i(m) = \kappa_i^*(m^*), \qquad m^* = F(m), \ i = 1, 2,$$

where $m \in M$, $m^* \in M^*$, κ_i and κ_i^* are the principal curvatures of M and M^*. In this case, we call F an *O. Bonnet* I or II or III-*isometry*, denoted by BI or BII or BIII-isometry, respectively. If a surface M admits a non-trivial BI or BII or BIII-isometry, we call M a *Bonnet* I or II or III-*surface*, respectively.

Isometric deformations which were considered in [1]–[5] are BI-isometries, because an isometric deformation preserves Gaussian curvature K, so if the map preserves mean curvature H, then it preserves the two principal curvatures κ_1 and κ_2, and hence it is a BI-isometry.

In the present paper we shall study BIII-isometries and obtain some results given in Theorems 1–6, which are shown to be similar to the case of BI-isometries.

1. Lemmas and formulas

We shall let $\omega = \omega_1 + i\omega_2$ $(i^2 = -1)$ be the complex structure of the metric $\mathrm{I} = (\omega_1)^2 + (\omega_2)^2$ and let ω_{12} be the connection form associated to I, which is determined by the structural equations

$$d\omega_1 = -\omega_2 \wedge \omega_{12}, \quad d\omega_2 = \omega_1 \wedge \omega_{12}$$

or

$$d\omega = i\omega \wedge \omega_{12},$$

where ω_1 and ω_2 are two real linearly independent forms.

LEMMA 1. *Suppose ω is a complex structure.*
(i) *If $\omega^* = \bar{\omega}$, then $\omega_{12}^* = -\omega_{12}$.*
(ii) *If $\omega^* = e^{i\tau}\omega$, then $\omega_{12}^* = \omega_{12} - d\tau$.*
(iii) *If $\omega^* = A\omega$, then $\omega_{12}^* = \omega_{12} + *d\log A$.*

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Here τ, A are functions, and "$*$" is the Hodge $*$-operator with

$$*\omega_1 = \omega_2, \quad *\omega_2 = -\omega_1.$$

We consider a piece of an oriented surface M in E^3, which we assume to be sufficiently differentiable and with no umbilic points and non-zero Gaussian curvature. Over M there is a field of orthonormal frames $me_1e_2e_3$, such that $m \in M$, where unit vectors e_1 and e_2 are the principal directions of M at m, and e_3 is the unit normal vector to M at m. Suppose ω_1 and ω_2 are a basis of the 1-forms of M dual to the field of principal frames. Set $a > c$, and

$$\omega_1 = a\omega_{13}, \quad \omega_2 = c\omega_{23}, \quad ac \neq 0, \tag{1}$$
$$\omega_{12} = h\omega_{13} + k\omega_{23}. \tag{2}$$

The mean curvature and the Gaussian curvature of M are

$$H = \tfrac{1}{2}(a^{-1} + c^{-1}), \quad K = (ac)^{-1}. \tag{3}$$

The structural equations of M are

$$d\omega_1 = -\omega_2 \wedge \omega_{12}, \quad d\omega_2 = \omega_1 \wedge \omega_{12}, \tag{4}$$
$$d\omega_{12} = -K\omega_1 \wedge \omega_2 = -\omega_{13} \wedge \omega_{23}, \tag{5}$$
$$d\omega_{13} = -\omega_{23} \wedge \omega_{12}, \quad d\omega_{23} = \omega_{13} \wedge \omega_{12}. \tag{6}$$

The metric of the Gaussian image $g(M)$ of M is

$$I_g = (\omega_{13})^2 + (\omega_{23})^2. \tag{7}$$

The complex structure of this metric is

$$\omega = \omega_{13} + i\omega_{23}. \tag{8}$$

From (6), we have

$$d\omega = i\omega \wedge \omega_{12}. \tag{9}$$

We denote

$$f = a - c > 0, \quad g = a + c = 2HK^{-1}. \tag{10}$$

Taking exterior derivatives of (1) and using (4) and (6), we get the existence of functions α, β, ν, δ, such that

$$\begin{aligned} da &= \alpha\omega_{13} + \beta\omega_{23} = \alpha\omega_{13} + fh\omega_{23}, \\ f\omega_{12} &= \beta\omega_{13} + \nu\omega_{23}, \quad \beta = fh, \quad \nu = fk, \\ dc &= \nu\omega_{13} + \delta\omega_{23} = fk\omega_{13} + \delta\omega_{23}. \end{aligned} \tag{11}$$

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Taking exterior derivatives of (11) and using (5), (6) and (11), we get the existence of $A, B, \ldots, E$, such that

$$
(12)\quad
\begin{aligned}
d\alpha &= 3\beta\omega_{12} + A\omega_{13} + B\omega_{23} = (A + 3fh^2)\omega_{13} + (B + 3fhk)\omega_{23}, \\
d\beta &= -(\alpha + 2\nu)\omega_{12} + B\omega_{13} + (C + a)\omega_{23} \\
&= [B - fh(\alpha f^{-1} - 2k)]\omega_{13} + [C + a - fh(\alpha f^{-1} - 2k)]\omega_{23}, \\
d\nu &= (\delta + 2\beta)\omega_{12} + (C + c)\omega_{13} + D\omega_{23} \\
&= [C + c + fh(\delta f^{-1} - 2h)]\omega_{13} + [D + fk(\delta f^{-1} - 2h)]\omega_{23}, \\
d\delta &= -3\nu\omega_{12} + D\omega_{13} + E\omega_{23} = (D - 3fhk)\omega_{13} + (E - 3fk^2)\omega_{23}.
\end{aligned}
$$

Using (10) and (11), we get

$$(13)\qquad dg = 2d(HK^{-1}) = f(u\omega_{13} + v\omega_{23}),$$

$$(14)\qquad df = f[(u - 2k)\omega_{13} - (v - 2h)\omega_{23}],$$

where

$$(15)\qquad fu = \alpha + \nu, \quad fv = \beta + \delta.$$

Then we can determine the following 1-forms. By using u, v in (15), let

$$(16)\qquad \theta_1 = u\omega_{13} + v\omega_{23}, \quad \theta_2 = *\theta_1 = -v\omega_{13} + u\omega_{23},$$

$$(17)\qquad \alpha_1 = u\omega_{13} - v\omega_{23}, \quad \alpha_2 = *\alpha_1 = v\omega_{13} + u\omega_{23}.$$

If $HK^{-1} =$ constant, then $\theta_i = \alpha_i = 0$; if $HK^{-1} \neq$ constant, θ_1 and θ_2, or α_1 and α_2 are linearly independent. From (13) and (14), it follows that

$$(18)\qquad dg = f\theta_1,$$

$$(19)\qquad d\log f = \alpha_1 + 2 * \omega_{12}.$$

According to (3),

$$(20)\qquad 4K^{-1} = g^2 - f^2.$$

Taking derivatives of (20) and using (18) and (19), we have

$$(21)\qquad 2f^{-1}d(K^{-1}) = g\theta_1 - f(\alpha_2 + 2 * \omega_{12}).$$

Suppose $HK^{-1} \neq$ constant. We denote

$$(22)\qquad u + iv = Le^{i\psi},$$

$$(23)\qquad L^2 = u^2 + v^2 = f^{-2}[\alpha^2 + \beta^2 + \nu^2 + \delta^2 + 2(\alpha\nu + \beta\delta)].$$

$$(24)\qquad \cos\psi = uL^{-1}, \quad \sin\psi = vL^{-1}.$$

Let

$$(25)\qquad \theta = \theta_1 + i\theta_2,$$

$$(26)\qquad \alpha = \alpha_1 + i\alpha_2.$$

Using (8), (16), (17), and (22), we get

$$\theta = Le^{-i\psi}\omega, \quad \alpha = Le^{i\psi}\omega, \quad \theta = e^{-2i\psi}\alpha, \tag{27}$$

$$\begin{aligned} \theta_1 &= \alpha_1 \cos 2\psi + \alpha_2 \sin 2\psi = L(\omega_{13}\cos\psi + \omega_{23}\sin\psi), \\ \theta_2 &= -\alpha_1 \sin 2\psi + \alpha_2 \cos 2\psi = L(-\omega_{13}\sin\psi + \omega_{23}\cos\psi), \\ \alpha_1 &= L(\omega_{13}\cos\psi - \omega_{23}\sin\psi), \quad \alpha_2 = L(\omega_{13}\sin\psi + \omega_{23}\cos\psi). \end{aligned} \tag{27$'$}$$

From (10), (20), (23) and (13), it follows that

$$f^2 = 4[(HK^{-1})^2 - K^{-1}], \quad [\mathrm{grad}(g)]^2 = 4[\mathrm{grad}(HK^{-1})]^2 = f^2L^2,$$

so

$$L^2 = 4f^{-2}[\mathrm{grad}(g)]^2 = \frac{[\mathrm{grad}(HK^{-1})]^2}{(HK^{-1})^2 - K^{-1}}. \tag{28}$$

We now consider a metric which is conformal to I_g (see (7))

$$\widehat{I} = (\alpha_1)^2 + (\alpha_2)^2 = L^2 I_g. \tag{29}$$

Let θ_{12} and α_{12} be the connection forms associated to complex structures θ and α, respectively. From (27), using Lemma 1, we have

$$\theta_{12} = \omega_{12} + d\psi + *d\log L, \tag{30}$$

$$\alpha_{12} = \omega_{12} - d\psi + *d\log L, \tag{31}$$

$$\theta_{12} = \alpha_{12} + 2d\psi. \tag{32}$$

We rewrite (2) as

$$\omega_{12} = h'\alpha_1 + k'\alpha_2. \tag{2$'$}$$

From $(2)'$ and (27), we have

$$\begin{aligned} fLh' &= f^{-1}(h\cos\psi - k\sin\psi) = \alpha\beta - \nu\delta, \\ fLk' &= f^{-1}(h\sin\psi + k\cos\psi) = \alpha\nu + \beta\delta + \beta^2 + \nu^2. \end{aligned} \tag{33}$$

Taking derivatives of (23) and using (11), (12), (18) and (19), we get

$$d\log L = -\alpha_1 - 2*\omega_{12} + *\Omega + \rho\theta_1, \tag{34}$$

where

$$2fL^2\Omega = 2(B+D)\alpha_1 - (A - E - f)\alpha_2, \tag{35}$$

$$2fL^2\rho = A + 2C + E + 2HK^{-1}. \tag{36}$$

From (24), it can be seen that

$$u\sin\psi - v\cos\psi = 0. \tag{24$'$}$$

Taking derivatives of $(24)'$, using (11) and previous formulas, we get

(37) $$d\psi = -\omega_{12} + \Omega + \rho\theta_2.$$

Inserting (34) and (37) into (30) and (31), we get

(38) $$\theta_{12} = 2\omega_{12} - \alpha_2 + 2\rho\theta_2,$$

(39) $$\alpha_{12} = 4\omega_{12} - \alpha_2 - 2\Omega = 2\omega_{12} - \alpha_2 - 2d\psi + 2\rho\theta_2.$$

Let

(40) $$\alpha_{12} = P\alpha_1 + Q\alpha_2,$$

where P, Q are two functions. Using (39), (35), (36), $(2)'$ and (33) gives

(41) $$P = -2(fL)^{-2}[f(B+D) + 2(\alpha\beta - \gamma\delta)],$$
$$Q = 1 + (fL)^{-2}[f(A - E - f) - 2(\alpha^2 - \beta^2 - \nu^2 + \delta^2)].$$

By solving (39), we have

$(39)'$ $$\Omega = 2\omega_{12} - \tfrac{1}{2}[P\alpha_1 + (Q+1)\alpha_2].$$

Inserting $(39)'$ into (34) and (37), we get

(42) $$2d\log L = (Q-1)\alpha_1 - P\alpha_2 + 2\rho\theta_1,$$

(43) $$2d\psi = 2\omega_{12} - [P\alpha_1 + (Q+1)\alpha_2] + 2\rho\theta_2.$$

Set

(44) $$dP = P_i\alpha_i, \quad dQ = Q_i\alpha_i, \quad d\rho = \rho_i\alpha_i, \qquad i = 1, 2.$$

Taking exterior derivatives of (42), we have

(45) $$(d\rho - \rho * \theta_{12}) \wedge \theta_1 + J\theta_1 \wedge \theta_2 = 0,$$

where

(46) $$-2J = P + P_1 + Q_2.$$

Taking exterior derivatives of (43), we have

(47) $$(d\rho - \rho * \theta_{12}) \wedge \theta_2 - I\theta_1 \wedge \theta_2 = 0,$$

where

(48) $$2I = \widehat{K} - Q - 2L^{-2}$$

and $\widehat{K}$ is the Gaussian curvature of the metric $\widehat{I}$ (see (29)) so that

(49) $$d\alpha_{12} = -\widehat{K}\alpha_1 \wedge \alpha_2, \quad \widehat{K} = -P^2 - Q^2 + P^2 - Q_1.$$

From (45) and (47), we obtain

$(50)'$ $$d\rho - \rho * \theta_{12} = -I\theta_1 + J\theta_2$$

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or, by (32) and (48),

$$d\rho = \rho * \theta_{12} - I\theta_1 + J\theta_2 = \rho(*\alpha_{12} + 2 * d\psi) + [L^{-2} - \tfrac{1}{2}(\widehat{K} - Q)]\theta_1 + J\theta_2. \tag{50}$$

Taking exterior derivatives of [19] and using (40), we get

$$d * \omega_{12} = -\tfrac{1}{2}P\alpha_1 \wedge \alpha_2. \tag{51}$$

Applying the $*$-operator and taking exterior derivatives of (40), we get

$$d * \alpha_{12} = (P_1 + Q_2)\alpha_1 \wedge \alpha_2. \tag{52}$$

Similarly, from (31), using (51) and (52), we get

$$d * d\psi = \left(2J + \tfrac{1}{2}P\right)\alpha_1 \wedge \alpha_2 \tag{53}$$

and from (30) and (19), we have

$$d * \theta_{12} = 2J\alpha_1 \wedge \alpha_2, \tag{54}$$

$$d * d\log f = (Q + 2L^{-2})\alpha_1 \wedge \alpha_2. \tag{55}$$

Applying the $*$-operator to (34) and using (37), we get

$$*d\log L = -\omega_{12} + \alpha_{12} + d\psi. \tag{56}$$

Taking exterior derivatives of the above equation,

$$d * d\log L = (L^{-2} - \widehat{K})\alpha_1 \wedge \alpha_2. \tag{57}$$

We denote

$$d\psi = \psi_i\alpha_i, \quad dJ = J_i\alpha_i, \quad d(\widehat{K} - Q) = (\widehat{K} - Q)_i\alpha_i, \quad i = 1, 2. \tag{58}$$

Taking exterior derivatives of (50) gives

$$(-I\theta_1 + J\theta_2) \wedge *\theta_{12} + (I\theta_2 + J\theta_1) \wedge \theta_{12} + I(d * \alpha_{12} + 2d * d\psi) - [\tfrac{1}{2}d(\widehat{K} - Q) - dL^{-2}] \wedge \theta_1 + dJ \wedge \theta_2 = 0. \tag{59}$$

Let us compute the left side in (59).

(a) The sum of the first two terms is

$$\begin{aligned}(-I\theta_1 + J\theta_2) \wedge (*\alpha_{12} + 2 * d\psi) + (I\theta_2 + J\theta_1) \wedge \theta_{12} \\ = \{[-I(P + 1\psi_1) + J(Q + 2\psi_2)]\cos 2\psi \\ - [I(Q + 2\psi_2) + J(P + 2\psi_1)]\sin 2\psi\}\alpha_1 \wedge \alpha_2.\end{aligned} \tag{60}$$

(b) The third term is

$$I(d * \alpha_{12} + 2d * d\psi) = 2\rho J\alpha_1 \wedge \alpha_2. \tag{61}$$

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(c) The sum of the last two terms is

(62)
$$-[\tfrac{1}{2}d(\widehat{K}-Q)-dL^{-2}]\wedge\theta_1+dJ\wedge\theta_2$$
$$=-2L^{-2}d\log L\wedge\theta_1-\tfrac{1}{2}d(\widehat{K}-Q)\wedge\theta_1+dJ\wedge\theta_2$$
$$=\{[\tfrac{1}{2}(\widehat{K}-Q)_2+J_1]\cos 2\psi-[\tfrac{1}{2}(\widehat{K}-Q)_1-J_2]\sin 2\psi$$
$$-L^{-2}[P\cos 2\psi+(Q-1)\sin 2\psi]\}\alpha_1\wedge\alpha_2.$$

Inserting (60)–(62) into (59), we get

(63) $$\rho J-\tfrac{1}{2}L^{-2}[P\cos 2\psi_{+}(Q+1)\sin 2\psi]$$
$$+\{-I(P+2\psi_1)+J(Q+2\psi_2)+\tfrac{1}{2}[\tfrac{1}{2}(\widehat{K}-Q)_2+J_1]\}\cos 2\psi$$
$$-\{J(P+2\psi_1)+I(Q+2\psi_2)+\tfrac{1}{2}[\tfrac{1}{2}(\widehat{K}-Q)_1-J_2]\}\sin 2\psi=0.$$

We need the following lemma.

LEMMA 2. *A necessary and sufficient condition for a surface M with HK^{-1} $\neq$ constant to be a Weingarten-surface is*

(64) $$(P+2\psi_1)\cos 2\psi+(Q+2\psi_2)\sin 2\psi=0.$$

PROOF. According to (10), (18) and (21), a necessary and sufficient condition for M to be a W-surface is $da\wedge dc=0$, which can be written as

(65) $$(\alpha_1+2*\omega_{12})\wedge\theta_1=0.$$

Applying the $*$-operator to (39), we get

$$\alpha_1+2*\omega_{12}=*\alpha_{12}+2*d\psi+2\rho\theta_1.$$

Using the above equation from (65), we have

$$*(\alpha_{12}+2d\psi)\wedge\theta_1=0$$

or

$$(\alpha_{12}+2d\psi)\wedge\theta_2=0.$$

Using (40), (58), (27)$'$ and rewriting the above equation, we see that (64) follows.

2. BIII-isometry

Let $F\colon M\to M^*$ be a III-isometry from M to $\overset{*}{M}$, with $me_1e_2e_3$ and $\overset{*}{m}\overset{*}{e}_1\overset{*}{e}_2\overset{*}{e}_3$ the fields of principal frames over M and $\overset{*}{M}$, respectively. We have

(1) $$\omega_1=a\omega_{13},\quad \omega_2=c\omega_{23},\quad \overset{*}{\omega}_1=\overset{*}{a}\overset{*}{\omega}_{13},\quad \overset{*}{\omega}_2=\overset{*}{c}\overset{*}{\omega}_{23}.$$

Let

$$\omega = \omega_{13} + i\omega_{23}, \quad \overset{*}{\omega} = \overset{*}{\omega}_{13} + i\overset{*}{\omega}_{23}.$$

Since F is a III-isometry, we have

$$\overset{*}{\omega} = e^{i\tau}\omega \tag{2}$$

or

$$\overset{*}{\omega}_{13} = \omega_{13}\cos\tau - \omega_{23}\sin\tau, \quad \overset{*}{\omega}_{23} = \omega_{13}\sin\tau + \omega_{23}\cos\tau, \tag{2$'$}$$

where τ is an angle of rotation of the principal directions during the BIII-isometric deformation. On the other hand, from the invariance of principal curvatures, we get

$$\overset{*}{a} = a, \quad \overset{*}{c} = c. \tag{3}$$

Using (1.10) and (3) gives

$$\overset{*}{f} = f, \quad \overset{*}{g} = g. \tag{4}$$

From (1.16), (1.33) and (1.24),

$$\overset{*}{\theta}_1 = \theta_1 \tag{5}$$

or

$$\overset{*}{u}\overset{*}{\omega}_{13} + \overset{*}{v}\overset{*}{\omega}_{23} = u\omega_{13} + v\omega_{23},$$

which gives, in view of $(2)'$,

$$\overset{*}{u} = u\cos\tau - v\sin\tau, \quad \overset{*}{v} = u\sin\tau + v\cos\tau. \tag{6}$$

Taking derivatives of the first equation in (4) and using (1.19) we get

$$\overset{*}{\alpha}_1 + 2 * \overset{*}{\omega}_{12} = \alpha_1 + 2 * \omega_{12}. \tag{7}$$

Using Lemma 1 from (2), we have

$$\overset{*}{\omega}_{12} = \omega_{12} - \tau. \tag{8}$$

Using (7), (8), we have

$$d\tau = \tfrac{1}{2}(\alpha_2 - \overset{*}{\alpha}_2). \tag{9}$$

From (6) and $(1.27)'$

$$\overset{*}{\alpha}_2 = \alpha_1\sin 2\tau + \alpha_2\cos 2\tau. \tag{10}$$

Putting

$$t = \operatorname{ctg}\tau, \tag{11}$$

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we get from (9),

(12) $$dt = t\alpha_1 - \alpha_2.$$

This is the total differential equation satisfied by the angle τ. In order that the BIII-isometry be non-trivial it is both necessary and sufficient that (12) is integrable. Taking exterior derivatives of (12), in view of (1.40), we get an integrable condition

(13) $$tP + 1 - Q = 0.$$

Now let us distinguish three cases about BIII-isometry. Similarly, BI-isometry is classified into three types.

(1) *First type*, $HK^{-1} = \text{constant}$. Then by (1.13), $\alpha_i = 0$, $i = 1, 2$.

(2) *Second type*, $HK^{-1} \neq \text{constant}$, and $P \equiv 0$, $Q \equiv 1$. Then (13) holds identically for all t, and (12) has a continuum of solutions, each depending on an arbitrary constant. Thus we obtain a one-parameter family of surfaces BIII-isometric to M.

(3) *Third type*, $HK^{-1} \neq \text{constant}$, and $P \not\equiv 0$, $Q \not\equiv 1$. Then from (13), we have

(13)′ $$t = (Q-1)P^{-1},$$

and (12) has a single solution. Thus we obtain a single surface which is BIII-isometric to M.

THEOREM 1. *Any surface with constant HK^{-1} is a* BIII-*surface of the first type. In other words, any surface with constant HK^{-1} can be* III-*isometrically deformed, preserving the principal curvatures. During this deformation the principal directions are rotated by a fixed angle τ* (= constant).

Since in this case $\alpha_1 = \alpha_2 = 0$, $dt = 0$, $t = \text{constant}$, $\tau = \text{constant}$, Theorem 1 naturally holds. This theorem is an analogy of O. Bonnet's theorem for BI-isometries [1].

THEOREM 2. *Let M be a* BIII-*surface of the second type, that is, $HK^{-1} \neq$ constant and $P \equiv 0$, $Q \equiv 1$.*

(i) *The metric which is conformal to the metric I_g of the Gaussian image $g(M)$ of M,*

$$\hat{I} = \frac{[\operatorname{grad}(HK^{-1})]^2}{(HK^{-1})^2 - K^{-1}} I_g,$$

has Gaussian curvature equal to -1, where H and K are the mean curvature and Gaussian curvature of M, respectively.

(ii) *M is a W-surface.*

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(iii) *The non-trivial family of* BIII-*surfaces is a family of surfaces which depends on six arbitrary constants.*

PROOF. Since

$$(14) \qquad P \equiv 0, \quad Q \equiv 1,$$

(1.3) is identically true for all t. Using (1.49), we have

$$\widehat{K} = -(P^2 + Q^2 - P_2 + Q_1) = -1.$$

From (14), $P_i = Q_i = 0$ and using (1.46) and (1.48), we get

$$J = 0, \quad I = -(1 + L^{-2}), \quad J_i = 0, \quad \widehat{K}_i = 0.$$

Inserting the above equations into (1.63) we get

$$(15) \qquad 2\psi_1 \cos 2\psi + (1 + 2\psi_2) \sin 2\psi = 0.$$

This is exactly (1.64). By Lemma 2, we obtain (ii).

From (15) we have

$$(16) \qquad 2\psi_1 = p \sin 2\psi, \quad 1 + 2\psi_2 = -p \cos 2\psi,$$

where p is a function. Taking derivatives of (16), we get, for $i = 1, 2$,

$$(17) \qquad 2\psi_{1i} = p_i \sin 2\psi + 2p\psi_i \cos 2\psi, \quad 2\psi_{2i} = p_i \cos 2\psi + 2p\psi_i \sin 2\psi.$$

Inserting (17) into (1.53), using $\psi_{12} = \psi_{21}$ plus $J = 0$, $P = 0$, and by solving the equation obtained, we get

$$(18) \qquad p_1 = -2p\psi_2, \quad p_2 = 2p\psi_1.$$

It can be verified by differentiating (18) that the integrable condition for p is satisfied. From our discussion the differentials of the six functions a, c, $\log L$, ρ, ψ, p are all determined. Hence our surfaces of non-constant HK^{-1}, which can be III-isometrically deformed in a non-trivial way preserving the principal curvatures, depend on six arbitrary constants.

REMARK. Theorem 2 is analogous to S. S. Chern's Theorem for BI-isometry [4].

About the third type of BIII-surfaces, we only consider the case of a surface satisfying the equation

$$(19) \qquad P \cos 2\psi + (Q - 1) \sin 2\psi = 0.$$

First of all, we get the following.

LEMMA 3. *Let M be a* BIII-*surface, which satisfies* (19). *Then the following differentials satisfy*

(20) $$d\psi, dP, dQ, dL, d\rho, da, dc \equiv 0 \pmod{\theta_1}.$$

PROOF. By solving (13), we get

(21) $$t = (1-Q)P^{-1}.$$

Inserting (21) into (12).

(22) $$PdQ - (Q-1)dP = P(Q-1)\alpha_1 - P^2\alpha_2.$$

Taking exterior derivatives of (22), we have

(23) $$2dP \wedge dQ = (Q-1)dP \wedge \alpha_1 + PdQ \wedge \alpha_1 - 2PdP \wedge \alpha_2 - P^2\alpha_1 \wedge \alpha_2.$$

Taking the wedge product of (22) with dP, dQ, α_1 and α_2, respectively we obtain

$(24)_{1-4}$
$$\begin{aligned} dP \wedge dQ &= (Q-1)dP \wedge \alpha_1 - PdP \wedge \alpha_2, \\ (Q-1)dP \wedge dQ &= P(Q-1)dQ \wedge \alpha_1 - P^2 dQ \wedge \alpha_2, \\ P^2\alpha_1 \wedge \alpha_2 &= -(Q-1)dP \wedge \alpha_1 + PdQ \wedge \alpha_1, \\ P(Q-1)\alpha_1 \wedge \alpha_2 &= -(Q-1)dP \wedge \alpha_2 + PdQ \wedge \alpha_2. \end{aligned}$$

Taking derivatives of (19), we get

(25) $$dP\cos 2\psi + dQ\sin 2\psi + 2[-P\sin 2\psi + (Q-1)\cos 2\psi]d\psi = 0.$$

From (19),

$(19)'$ $$\cos 2\psi = \frac{Q-1}{\sqrt{P^2+(Q-1)^2}}, \quad \sin 2\psi = \frac{-P}{\sqrt{P^2+(Q-1)^2}}$$

Using $(19)'$ and (22), we get

(26) $$dP\cos 2\psi + dQ\sin 2\psi = -P\theta_1.$$

Inserting (26) into (25), we get

(27) $$2d\psi = -\theta_1 \sin 2\psi.$$

Taking exterior derivatives of (27), we have

$$\theta_2 \wedge \theta_{12} \cdot \sin 2\psi = 0.$$

Since $P \neq 0$, $\sin 2\psi \neq 0$, and it follows that $\theta_2 \wedge \theta_{12} = 0$, or

(28) $$(P + 2\psi_1)\cos 2\psi + (Q + 2\psi_2)\sin 2\psi = 0.$$

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From (28), using Lemma 2, we have that M is a W-surface. Using (28) and (19) we get

$$2\psi_1 \cos 2\psi + (2\psi_2 + 1)\sin 2\psi = 0. \tag{29}$$

From (19), (28) and (29),

$$\operatorname{tg} 2\psi = \frac{-P}{Q-1} = \frac{-(P+2\psi_1)}{Q+2\psi_2} = \frac{-2\psi_1}{2\psi_{2+1}}. \tag{30}$$

Applying the $*$-operator to (27), we have

$$2 * d\psi = -\theta_2 \sin 2\psi. \tag{31}$$

Taking exterior derivatives of (31), in view of (27), we get

$$2d * d\psi = P\alpha_1 \wedge \alpha_2. \tag{32}$$

On the other hand, from (1.53),

$$2d * d\psi = (4J + P)\alpha_1 \wedge \alpha_2.$$

By the above two equations, we get $J = 0$, or (see (1.46))

$$P_1 + Q_2 + P = 0. \tag{33}$$

We denote

$$dP = P_i\alpha_i, \quad dQ = Q_i\alpha_i, \quad i = 1, 2. \tag{34}$$

From (23), $(24)_{1-4}$ and (33), we have

$$\begin{aligned} &2(P_1Q_2 - P_2Q_1) = -(Q-1)P_2 - PQ_2 - 2PP_1 - P^2, \\ &P_1Q_2 - P_1Q_1 = -(Q-1)P_2 - PP_1, \\ &(Q-1)(P_1Q_2 - P_2Q_1) = -P[(Q-1)Q_2 + PQ_1], \\ &\qquad (Q-1)P_2 - PQ_2 = P^2, \quad P_1 + Q_2 = -P. \end{aligned} \tag{35}$$

By solving (35) for $P_1, \dots, Q_2$ we get

$$\frac{P_2}{P_1} = \frac{Q_2}{Q_1} = \frac{-P}{Q-1} = \operatorname{tg} 2\psi \tag{36}$$

or

$$\begin{aligned} &P_1 \sin 2\psi - P_2 \cos 2\psi = 0, \quad Q_1 \sin 2\psi - Q_2 \cos 2\psi = 0, \\ &PP_1 + (Q-1)P_2 = 0, \quad PQ_1 + (Q-1)Q_2 = 0. \end{aligned} \tag{37}$$

Using (19), (33) and (1.48), from (1.63), we have

$$\begin{aligned} &[\tfrac{1}{2}(\widehat{K} - Q) - L^{-2}][(P + 2\psi_1)\cos 2\psi + (Q + 2\psi_2)\sin 2\psi] \\ &\qquad + \tfrac{1}{4}[(\widehat{K} - Q)_2 \cos 2\psi - (\widehat{K} - Q)_1 \sin 2\psi] = 0. \end{aligned} \tag{38}$$

Using (28) and (38) implies

$$(\widehat{K} - Q)_2 \cos 2\psi - (\widehat{K} - Q)_1 \sin 2\psi = 0.$$

From (37) and the above equation, we have

(39) $$\widehat{K}_2 \cos 2\psi - \widehat{K}_1 \sin 2\psi = 0.$$

Using (1.49) and its differential, we get

(40)
$$\begin{aligned} -\widehat{K} &= P^2 + Q^2 - P_2 + Q_1, \\ -\widehat{K}_1 &= 2(PP_1 + QQ_1) - P_{21} + Q_{11}, \\ -\widehat{K}_2 &= 2(PP_2 + QQ_2) - P_{22} + Q_{12}. \end{aligned}$$

Using (40), (19)$'$ and (39) implies

(41)
$$\begin{aligned} &P(Q_{11} - P_{21}) + (Q-1)(Q_{12} - P_{22}) \\ &\quad = 2[P(PP_1 + QQ_1) - (Q-1)(PP_2 + QQ_2)]. \end{aligned}$$

Then (11), (19) and (21) imply

$$t = \operatorname{ctg} \tau = \frac{Q-1}{P} = -\operatorname{ctg} 2\psi.$$

Hence

(42) $$\tau = -2\psi + k\pi, \quad k = \text{integer}.$$

We wish to express the differentials on the left side of (20) in terms of θ_1 and θ_2. First, from (37) we have

(43) $$dP = P_1 \sec 2\psi \cdot \theta_1, \quad dQ = Q_1 \sec 2\psi \cdot \theta_1.$$

Furthermore,

$$\alpha_{12} = P\alpha_1 + Q\alpha_2 = \theta_1(P\cos 2\psi + Q \sin 2\psi) + \theta_2(-P\sin 2\psi + Q\cos 2\psi).$$

From (19)$'$, (27), we get

(44) $$\theta_{12} = \alpha_{12} + 2d\psi = (-P\sin 2\psi + Q\cos 2\psi)\theta_2.$$

Using (1.42) gives

(45) $$d \log L = [1 + \tfrac{1}{2}(Q-1)\sec 2\psi]\theta_1.$$

From (1.50),

(46) $$d\rho = [L^{-2} - \tfrac{1}{2}(\widehat{K} - Q) + \rho(P\sin 2\psi - Q\cos 2\psi)]\theta_1.$$

Using (27) and (45), from (1.38) it follows that

(47) $$\omega_{12} = \tfrac{1}{2}\sin 2\psi \cdot \theta_1 - \{1 + \tfrac{1}{2}[P\sin 2\psi - (Q+1)\cos 2\psi]\}\theta_2.$$

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From (1.18), (1.19) and (47), we get

$$(48) \qquad dg = f\theta_1, \quad df = f[2\rho + P\sin 2\psi - (Q-2)\cos 2\psi]\theta_1.$$

Using (1.10) and (48), we see that

$$(49) \qquad \begin{aligned} 2da &= f[2\rho + 1 + P\sin 2\psi - (Q-2)\cos 2\psi]\theta_1, \\ 2dc &= f[-2\rho + 1 - P\sin 2\psi + (Q-2)\cos 2\psi]\theta_1. \end{aligned}$$

According to (27), (43), (45), (46) and (49), we obtain (20).

Now we easily obtain the following theorem.

THEOREM 3. *Let M be a* BIII-*surface of the third type, and satisfying equation* (19). *Then M is a helicoidal surface.*

A helicoidal surface in E^3 is a surface which is invariant under a helicoidal motion:

$$c_t(x) = x', \quad x = (x_1, x_2, x_3), \quad x' = (x_1', x_2', x_3'),$$

$$\begin{aligned} x_1' &= x_1 \cos t + x_2 \sin t, \\ x_2' &= -x_1 \sin t + x_2 \cos t, \qquad -\infty < t < +\infty, \\ x_3' &= x_3 + bt, \end{aligned}$$

where the x_3-axis is taken as the axis of a helicoidal motion. Let C be a curve parametrized by s:

$$c(s) = (x_1(s), x_2(s), x_3(s)).$$

Any helicoidal surface M may be considered as the one generated by helicoidal motion of all the points of C. Thus its parametrization by s, t is

$$(50) \quad x(s, t) = (x_1(s)\cos t + x_2(s)\sin t, \\ - x_1(s)\sin t + x_2(s)\cos t, \ x_3(s) + bt),$$

where $b =$ constant. In other words, on a helicoidal surface there exists a family of helicoidal curves, which have the same helicoidal distance ($b =$ constant) and helicoidal axis.

PROOF OF THEOREM 3. Let us show that on the surface M the set of θ_2-curves (the curve along which $\theta_1 = 0$) is a family of helicoidal curves.

First of all, from Lemma 3 we conclude that

$$(51) \qquad \psi, P, Q, L, \rho, a, c = \text{constant} \pmod{\theta_1}$$

that is, they are all constant along the θ_2-curves $(\theta_1 = 0)$. Let us find the curvature κ and torsion τ of the θ_2-curves.

According to (1.1) and (1.27), along θ_2-curves, we have

$$\omega_1 = a\omega_{13} = -aL^{-1}\theta_2 \sin\psi,$$
$$\omega_2 = c\omega_{23} = cL^{-1}\theta_2 \cos\psi.$$

Hence the arc length differential of θ_2-curves is

$$ds = \sqrt{(\omega_1^2 + \omega_2^2)} = L^{-1}\sqrt{a^2 \sin^2\psi + c^2 \cos^2\psi}\,\theta_2. \tag{52}$$

Since the angle between tangent directions of θ_2-curves and the first principal directions $(\phi = 0)$ is $\phi = \psi + \pi/2$, the normal curvature of θ_2-curves, by Euler's theorem is

$$\begin{aligned}\kappa_n &= a^{-1}\cos^2\phi + c^{-1}\sin^2\phi \\ &= a^{-1}\sin^2\psi + c^{-1}\cos^2\psi.\end{aligned} \tag{53}$$

Along θ_2-curves, $\theta_1 = 0$ implies $d\psi = 0$, $d\phi = 0$. Using (47), we have

$$\omega_{12} = -\{1 + \tfrac{1}{2}[P\sin 2\psi - (Q+1)\cos 2\psi]\}\theta_2. \tag{54}$$

Using the formula for geodesic curvature $\kappa_g = d\phi/ds + \omega_{12}/ds$ and (52) and (54), we obtain the geodesic curvature of a θ_2-curve

$$\kappa_g = \frac{-L}{\sqrt{a^2\sin^2\psi + c^2\cos^2\psi}}\{1 + \tfrac{1}{2}[P\sin 2\psi - (Q+1)\cos 2\psi]\}. \tag{55}$$

From (51), (53) and (55), κ_n and κ_g are constant on each θ_2-curve, so its curvature

$$\kappa = \sqrt{\kappa_n^2 + \kappa_g^2} = \text{constant}. \tag{56}$$

Then the torsion of the θ_2-curve is given by

$$\tau = \tau_g + d\theta/ds, \tag{57}$$

where τ_g is the geodesic torsion of the θ_2-curve, θ is the angle between the principal space normal of the θ_2-curve and the normal to the surface. We have

$$\tau_g = (c^{-1} - a^{-1})\cos\phi\sin\phi = (c^{-1} - a^{-1})\sin\psi\cos\psi, \quad \operatorname{tg}\theta = \kappa_g/\kappa_n. \tag{58}$$

From (51), (57) and (58), torsion $\tau = \text{constant}$ along the θ_2-curve.

Consequently, we have that the θ_2-curves are circular helices which are distinct, in general. Thus the surface M is a helicoidal surface.

3. Some special BIII-surfaces

1. The minimal BIII-surfaces.

THEOREM 4. *Suppose M and $\overset{*}{M}$ are minimal surfaces, and $F: M \to \overset{*}{M}$ is a mapping. Then F is a* BI*-isometry if and only if F is a* BIII*-isometry.*

PROOF. For any surface M, we have

$$\mathrm{III} - 2H\mathrm{II} + K\mathrm{I} = 0,$$

where H and K are the mean curvature and Gaussian curvature, respectively, and I, II and III are the three fundamental forms of M. Since M and $\overset{*}{M}$ are minimal surfaces, $H = \overset{*}{H} = 0$ and so $\mathrm{III} = -K\mathrm{I}$, $\overset{*}{\mathrm{III}} = -\overset{*}{K}\overset{*}{\mathrm{I}}$. When F is a BI or BIII-isometry, $K = \overset{*}{K}$. Thus the above equations imply the conclusion of Theorem 4.

EXAMPLE. A BIII-isometry between the catenoid and the helicoid.

Catenoid M: $m(t, \theta) = (\cosh t \cos\theta, \cosh t \sin\theta, t)$,

Helicoid $\overset{*}{M}$: $\overset{*}{m}(u, v) = (u\cos v, u \sin v, v)$,

$$-\infty < t < \infty, \quad 0 \le \theta < 2\pi, \quad u \ge 0, \quad 0 \le v < 2\pi.$$

The fundamental forms and curvatures of M:

$$\mathrm{I} = \cosh^2 t(dt^2 + d\theta^2), \quad H = 0,$$
$$\mathrm{III} = \cosh^{-2} t(dt^2 + d\theta^2), \quad K = -\cosh^{-4} t.$$

The fundamental forms and curvatures of $\overset{*}{M}$:

$$\overset{*}{\mathrm{I}} = du^2 + (1+u^2)dv^2, \quad \overset{*}{H} = 0,$$
$$\overset{*}{\mathrm{III}} = (1+u^2)^{-2}[du^2 + (1+u^2)dv^2], \quad \overset{*}{K} = -(1+u^2)^{-2}.$$

The mapping $F(t, \theta) = (u, v)$: $u = \sinh t$, $v = \theta$ is both a BI-isometry and BIII-isometry:

$$F^*(\overset{*}{\mathrm{I}}) = \mathrm{I}, \quad \overset{*}{F}(\overset{*}{\mathrm{III}}) = \mathrm{III},$$
$$\overset{*}{H} = H = 0, \quad \overset{*}{K} = K.$$

2. The BIII-surfaces of revolution.

We consider the plane curve $x = \gamma(z) > 0$, $y = 0$ and the surface of revolution

(1) $$M: m(z, \theta) = (\gamma(z)\cos\theta, \gamma(z)\sin\theta, z).$$

Thus

$$m'_z = (\gamma' \cos\theta, \gamma' \sin\theta, 1), \quad m'_\theta = (\gamma \sin\theta, \gamma \cos\theta, 0).$$

We choose an orthonormal frame by

$$(2)\qquad \begin{aligned} e_1 &= (\gamma'^2 + 1)^{-1/2}(\gamma' \cos\theta, \gamma' \sin\theta, 1), \\ e_2 &= (-\sin\theta, \cos\theta, 0), \\ e_3 &= e_1 \times e_2 = (\gamma'^2 + 1)^{-1/2}(-\cos\theta, -\sin\theta, \gamma'), \end{aligned}$$

so that

$$(4)\qquad \begin{aligned} &\omega_1 = (\gamma'^2 + 1)^{-1/2} dz, \quad \omega_2 = \gamma d\theta, \\ &\omega_{12} = h\omega_1 + k\omega_2, \quad h = 0, \quad k = \gamma'(\gamma'^2 + 1)^{-1/2}, \end{aligned}$$

$$(5)\qquad \begin{aligned} &\omega_{13} = a^{-1}\omega_1, \quad a = -(\gamma'')^{-1}(\gamma'^2 = 1)^{3/2}, \quad \gamma'' \neq 0, \\ &\omega_{23} = c^{-1}\omega_2, \quad c = \gamma(\gamma'^2 + 1)^{1/2}, \end{aligned}$$

$$(6)\qquad \begin{aligned} f &= a - c = -(\gamma'')^{-1}(\gamma'^2 + 1)^{1/2}(1 + \gamma'^2 + \gamma\gamma''), \\ g &= a + c = (\gamma'')^{-1}(\gamma'^2 + 1)^{1/2}(-1 - \gamma'^2 + \gamma\gamma''). \end{aligned}$$

THEOREM 5. *The surfaces of revolution which are* BIII-*surfaces are exactly as follows.*

(i) *Those of the first type* ($HK^{-1} =$ constant), *which satisfy*

$$(7)\qquad (\gamma'^2 + 1)^{1/2}(\gamma\gamma'' - \gamma'^2 - 1) = c\gamma'', \qquad c = \text{constant}.$$

(ii) *Those of the second type, which satisfy*

$$(8)\qquad \left[\frac{g'(\gamma'^2 + 1)}{f\gamma''}\right]' = \left(\frac{g'}{f}\right)^2 \frac{\gamma'^2 + 1}{\gamma''}.$$

(iii) *There are no* BIII-*surfaces of the third type.*

PROOF. According to (6) and (1.13), we have

$$dg = g' dz = f(u\omega_{13} + v\omega_{23}).$$

It follows that

$$(9)\qquad u = af^{-1}g'(\gamma'^2 + 1)^{1/2}, \qquad v = 0.$$

From (1.16) and (1.17), we get

$$(10)\qquad \begin{aligned} \alpha_1 &= \theta_1 = u\omega_{13} = f^{-1}g' dz, \\ \alpha_2 &= \theta_2 = u\omega_{23} = -(f\gamma'')^{-1}g'(\gamma'^2 + 1)d\theta. \end{aligned}$$

Taking exterior derivatives of (10).

$$(11)\qquad d\alpha_1 = 0, \quad d\alpha_2 = -\left[\frac{g'(\gamma'^2+1)}{f\gamma''}\right]' dz\wedge d\theta.$$

Using (10), we have

$$(12)\qquad \alpha_1\wedge\alpha_2 = -\left[\frac{g'}{f}\right]^2 \frac{\gamma'^2+1}{\gamma''} dz\wedge d\theta.$$

From (11), (12), (1.9) and (1.40), we get

$$(13)\qquad P\equiv 0, \quad Q = \left[\frac{g'(\gamma'^2+1)}{f\gamma''}\right]' \left[\frac{f}{g}\right]^2 \frac{\gamma''}{\gamma'^2+1}.$$

For the first type, from $\alpha_1 = \alpha_2 = 0$ or $u = v = 0$, using (9), $g'' = 0$, $g = c = \text{constant}$, we have that $(6)_2$ becomes (7).

For the second type, from $P\equiv 0$, $Q\equiv 1$, using (13), we have (8).

For the third type, $P\neq 0$, and according to (13) this is not possible. So there are no surfaces in this case.

3. The BIII-surfaces with constant Gaussian curvature.

Suppose a surface M has non-zero constant Gaussian curvature K and $HK^{-1}\neq \text{constant}$. Since $dK^{-1} = 0$, from (1.19) and (1.21), we get

$$(14)\qquad df = f(\alpha_1 + 2*\omega_{12}) = g\theta_1$$

or

$$(15)\qquad \alpha_1 + 2*\omega_{12} = \sigma\theta_1,$$

where

$$(16)\qquad \sigma = gf^{-1} \neq \pm 1.$$

Note that the inequality in (16) can be concluded from $K\neq 0$. In fact, if $\sigma = (a+c)/(a-c) = \pm 1$, we get $a = 0$ or $c = 0$, and hence $K = ac = 0$. Using $(1.2)'$ and (1.27), rewrite (15) as

$$(17)\qquad 1 - 2k' = \sigma\cos 2\psi, \quad 2h' = \sigma\sin 2\psi.$$

Using (1.38), (1.27) and (15), we get

$$(18)\qquad \theta_{12} = (2\rho - \sigma)\theta_2, \quad *\theta 12 = (\sigma - 2\rho)\theta_1.$$

Taking derivatives of (16), using (1.18), and (1.19), we have

$$(19)\qquad d\sigma = (1-\sigma^2)\theta_1.$$

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According to equation (16), $1-\sigma^2 \neq 0$, and in view of $(1+\sigma)/(1-\sigma) = (f+g)/(f-g) = -a/c$, from (19) we get

$$\frac{d\sigma}{1-\sigma^2} = \frac{1}{2} d \log\left|\frac{a}{c}\right| = \theta_1. \tag{20}$$

Applying the $*$-operator to (15), we get

$$\alpha_2 - 2w_{12} = \sigma\theta_2. \tag{21}$$

Taking exterior derivatives of (21), from (1.40), (1.5) and (19), we have

$$(Q-1)/2 = \sigma(1-\sigma) - L^{-2}. \tag{22}$$

From (1.50) and (18), we obtain

$$d\rho = [\rho(\sigma - 2\rho) + L^{-2} - \tfrac{1}{2}(\widehat{K} - Q)]\theta_1 + J\theta_2. \tag{23}$$

From (1.42), we obtain

$$2d\log L = (Q-1)\alpha_1 - P\alpha_2 + 2\rho\theta_1. \tag{24}$$

Taking derivatives of (22), using (19), (23) and (24), we get

$$\tfrac{1}{2}dQ = \lambda\theta_1 + \mu\theta_2 + L^{-2}[(Q-1)\alpha_1 - P\alpha_2], \tag{25}$$

where

$$\lambda = \rho(1-2\rho\sigma) - 2\sigma(1-\sigma^2) + L^{-2}(\sigma+2\rho) - \tfrac{1}{2}\sigma(\widehat{K} - Q), \quad \mu = \sigma J. \tag{26}$$

Theorem 6. *There does not exist any* BIII-*surfaces of the second type such that* $K = \text{constant} \neq 0$, $H \neq \text{constant}$.

Proof. For the second type of BIII-surface, we have

$$P \equiv 0, \quad Q \equiv 1, \quad \widehat{K} = -1. \tag{27}$$

Since $K = \text{constant} \neq 0$, $HK^{-1} \neq \text{constant}$, using (27) and (22), we get

$$L^{-2} = \sigma(1-\sigma). \tag{28}$$

Using (27) and (28), from (23) and (24), we get

$$d\rho = \{1 - [\rho^2 + (1-\sigma)^2]\}\theta_1, \tag{29}$$

$$d\log L = \rho\theta_1. \tag{30}$$

From (27) it follows that $dQ = 0$, and from (25) we have $\lambda = \mu = 0$. Using (26), we get $(\sigma^2 - 1)(\sigma - 1) = 0$. It follows that $\sigma = 1$, in view of $\sigma^2 \neq 1$.

From (28) again we have $L^{-2} = 0$, contradicting that $HK^{-1} \neq$ constant, $L \neq 0$. So the surface cannot exist.

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